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# FRACTIONAL HARDY TYPE INEQUALITIES VIA CONFORMABLE CALCULUS

**Abstract.** $\alpha$-fractional analogs of of Hardy's classical integral inequalities are established.

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**Key words and phrases.** Hardy inequality, Conformable fractional derivative, Conformable fractional integral, Hölder inequality.

**რეზიუმე.** დადგენილია ჰარდის კლასიკური ინტეგრალურ უტოლობათა $\alpha$-წილადური ანალოგები.

# 1 Introduction

In 1925, Hardy [4] used the calculus of variations to prove the inequality

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p\,dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f^p(x)\,dx, \tag{1.1}$$

where $f \ge 0$ is integrable over any finite interval $(0,x)$ and $f^p$ is integrable and convergent over $(0,\infty)$ and $p>1$. The constant $(p/(p-1))^p$ is the best possible.

In 1928, Hardy [5] generalized inequality (1.1) and proved that if $p>1$ and $f$ is non-negative for $x \ge 0$, then

$$\int_0^\infty x^{-c}\Big(\int_0^x f(t)\,dt\Big)^p\,dx \le \Big(\frac{p}{c-1}\Big)^p \int_0^\infty x^{p-c} f^p(x)\,dx \quad \text{for } \ c>1, \tag{1.2}$$

and

$$\int_0^\infty x^{-c}\Big(\int_x^\infty f(t)\,dt\Big)^p\,dx \le \Big(\frac{p}{1-c}\Big)^p \int_0^\infty x^{p-c} f^p(x)\,dx \quad \text{for } \ c<1. \tag{1.3}$$

The constants $(p/(c-1))^p$ and $(p/(1-c))^p$ are the best possible.

In recent years, fractional inequalities were studied by using the fractional Caputo and Riemann–Liouville derivative; for details, we refer the reader to [3] and [17]. In [1] and [7], the authors presented conformable calculus and classical inequalities with the use of conformable fractional calculus such as Opial's inequality (see [11] and [12]), Hermite–Hadamard's inequality (see [8] and [10]), Chebyshev's inequality (see [2]) and Steffensen's inequality (see [13]). In this paper, using a somewhat different approach we present new Hardy type inequalities via conformable fractional calculus. Also, one can see from our approach and presentation that the conformable fractional inequalities encountered in the literature are, in fact, special cases of weighted inequalities (for an appropriate weight function). Our goal in this paper is, first, to show how naturally weights work in inequalities and, second, to indicate and correct some slight mistakes (usually when one integrates by parts) in the literature.

The paper is organized as follows. In Section 2, we present some concepts on conformable fractional calculus and also Hölder's inequality for $\alpha$-fractional differentiable functions which we will use to prove our main results. In Section 3, we prove some Hardy type inequalities for $\alpha$-fractional differentiable functions and obtain the classical ones as special cases when $\alpha = 1$.

# 2 Basic concepts and lemmas

In this section, we present some basic definitions concerning conformable fractional calculus. For more details, we refer the reader to [1] and [7].

**Definition 2.1.** Let $f:[0,\infty)\to\mathbb{R}$. Then the conformable fractional derivative of order $\alpha$ of $f$ is defined by

$$D_\alpha f(t) = \lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t>0$ and $0<\alpha\le 1$, and

$$D_\alpha f(0) = \lim_{t\to 0^+} D_\alpha f(t).$$

Let $\alpha\in(0,1]$ and $f$, $g$ be $\alpha$-differentiable at a point $t$. Then

$$D_\alpha(fg) = fD_\alpha g + gD_\alpha f. \tag{2.1}$$

Further, let $\alpha\in(0,1]$ and $f$, $g$ be $\alpha$-differentiable at a point $t$, with $g(t)\neq 0$. Then

$$D_\alpha\Big(\frac{f}{g}\Big) = \frac{gD_\alpha f - fD_\alpha g}{g^2}\,. \tag{2.2}$$

**Remark 2.1.** If $f$ is a differentiable function, then

$$D_\alpha f(t) = t^{1-\alpha}\,\frac{df(t)}{dt}\,.$$

**Definition 2.2.** Let $f : [0,\infty) \to \mathbb{R}$. Then the conformable fractional integral of order $\alpha$ of $f$ is defined by

$$I_\alpha f(t) = \int_0^t f(x)\; d_\alpha x = \int_0^t x^{\alpha-1} f(x)\, dx \tag{2.3}$$

for all $t > 0$ and $0 < \alpha \le 1$.

Now, we state an integration by parts formula (see [1] and [7]) which is immediate.

**Lemma 2.1.** *Assume that $w, g : [0,\infty) \to \mathbb{R}$ are two functions such that $w$, $g$ are differentiable and $0 < \alpha \le 1$. Then for any $b > 0$,*

$$\int_0^b w(x) D_\alpha g(x)\, d_\alpha x = w(x)g(x)|_0^b - \int_0^b g(x) D_\alpha w(x)\; d_\alpha x. \tag{2.4}$$

Next, we prove the Hölder type inequality needed in the next section (of course, it is the usual Hölder inequality for the functions under consideration (i.e., $x^{\frac{(\alpha-1)}{p}} f(x)$ and $x^{\frac{(\alpha-1)}{q}} g(x)$); for completeness we include its proof).

**Lemma 2.2.** *Let $f, g : [0,\infty) \to \mathbb{R}$ and $0 < \alpha \le 1$. Then for any $b > 0$,*

$$\int_0^b |f(x)g(x)|\, d_\alpha x \le \Big(\int_0^b |f(x)|^p\, d_\alpha x\Big)^{\frac{1}{p}} \Big(\int_0^b |g(x)|^q\, d_\alpha x\Big)^{\frac{1}{q}}, \tag{2.5}$$

*where $1/p + 1/q = 1$ (provided the integrals exist (and are finite)).*

*Proof.* For nonnegative real numbers $\beta$, $\gamma$, the classical Young inequality is

$$\beta^{\frac{1}{p}} \gamma^{\frac{1}{q}} \le \frac{\beta}{p} + \frac{\gamma}{p}\,.$$

Suppose now, without loss of generality, that

$$\int_0^b |f(x)|^p\, d_\alpha x \ne 0 \quad \text{and} \quad \int_0^b |g(x)|^q\, d_\alpha x \ne 0.$$

Applying Young's inequality with

$$\beta = \frac{|f(x)|^p}{\int_0^b |f(x)|^p\, d_\alpha x}, \qquad \gamma = \frac{|g(x)|^q}{\int_0^b |g(x)|^q\; d_\alpha x}\,,$$

and integrating the obtained inequality from 0 to $b$, we get

$$\begin{aligned}\int_0^b \frac{|f(x)|}{\big(\int_0^b |f(s)|^p\, d_\alpha s\big)^{\frac{1}{p}}}\, \frac{|g(x)|}{\big(\int_0^b |g(s)|^q\, d_\alpha s\big)^{\frac{1}{q}}}\, d_\alpha x& \\ = \int_0^b \beta^{\frac{1}{p}}(x) \gamma^{\frac{1}{q}}(x)\, d_\alpha x \le \int_0^b \Big(\frac{\beta}{p} + \frac{\gamma}{q}\Big)\, d_\alpha x&\end{aligned}$$

$$= \int_0^b \left( \frac{|f(x)|^p}{p\big(\int_0^b |f(s)|^p \, d_\alpha s\big)} + \frac{|g(x)|^q}{q\big(\int_0^b |g(s)|^q \, d_\alpha s\big)} \right) d_\alpha x$$

$$= \frac{\int_0^b |f(x)|^p \, d_\alpha x}{p\big(\int_0^b |f(s)|^p \, d_\alpha s\big)} + \frac{\int_0^b |g(x)|^q \, d_\alpha x}{q\big(\int_0^b |g(s)|^q \, d_\alpha s\big)} = \frac{1}{p} + \frac{1}{q} = 1,$$

which is the desired inequality (2.5). □

# 3 Hardy type inequalities of $\alpha$-fractional order

In this section, we state and prove the main results of this paper and we begin with the fractional version of the classical Hardy type inequality. Throughout the paper, we will assume that the functions are nonnegative locally $\alpha$-integrable and the integrals throughout are assumed to exist (and are finite, i.e., convergent).

**Theorem 3.1.** *Let $f$ be a nonnegative function on $(0,\infty)$, and $0 < \alpha \le 1$ and $p > 1$. Also assume $x^{\alpha-1} f(x)$ is continuous on $[0,\infty)$. Then*

$$\int_0^\infty \left( \frac{1}{x} \int_0^x f(s) \, d_\alpha s \right)^p d_\alpha x \le \left( \frac{p}{p-\alpha} \right)^p \int_0^\infty (x^{\alpha-1} f(x))^p \, d_\alpha x. \tag{3.1}$$

*Proof.* Let

$$F(x) := \frac{1}{x} \int_0^x f(s) \, d_\alpha s. \tag{3.2}$$

Integrating by parts, see formula (2.4) with $w(x) = F^p(x)$ and $D_\alpha g(x) = 1$ (note here $g(x) = \frac{x^\alpha}{\alpha}$), and using Remark 2.1, we obtain (here $t > 0$)

$$\int_0^t F^p(x) \, d_\alpha x = \frac{F^p(x) x^\alpha}{\alpha} \bigg|_0^t - \int_0^t \frac{x^\alpha}{\alpha} D_\alpha F^p(x) \, d_\alpha x = \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t \frac{x^\alpha}{\alpha} x^{1-\alpha} F^{p-1}(x) F'(x) \, d_\alpha x$$

$$= \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t x F^{p-1}(x) F'(x) \, d_\alpha x; \tag{3.3}$$

note

$$\lim_{x \to 0^+} x^{\frac{\alpha}{p}} F(x) = \lim_{x \to 0^+} \frac{\int_0^x s^{\alpha-1} f(s) \, ds}{x^{\frac{p-\alpha}{p}}} = \lim_{x \to 0^+} \frac{x^{\alpha-1} f(x)}{\big(\frac{p-\alpha}{p}\big) x^{-\frac{\alpha}{p}}} = \lim_{x \to 0^+} \left( \frac{p}{p-\alpha} \right) x^{\alpha-1} f(x) x^{\frac{\alpha}{p}} = 0.$$

From the definition of $F$, we see that

$$x F'(x) = x^{\alpha-1} f(x) - F(x),$$

and substituting it into (3.3), we obtain

$$\int_0^t F^p(x) \, d_\alpha x = \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t F^{p-1}(x) \big( x^{\alpha-1} f(x) - F(x) \big) \, d_\alpha x$$

$$= \frac{t^\alpha F^p(t)}{\alpha} - \frac{p}{\alpha} \int_0^t x^{\alpha-1} F^{p-1}(x) f(x) \, d_\alpha x + \frac{p}{\alpha} \int_0^t F^p(x) \, d_\alpha x,$$

and so

$$\Big(1-\frac{p}{\alpha}\Big)\int\limits_0^t F^p(x)\,d_\alpha x=\frac{t^\alpha F^p(t)}{\alpha}-\frac{p}{\alpha}\int\limits_0^t x^{\alpha-1}F^{p-1}(x)f(x)\,d_\alpha x.$$

Thus

$$\int\limits_0^t F^p(x)\,d_\alpha x=\frac{t^\alpha F^p(t)}{\alpha-p}+\frac{p}{p-\alpha}\int\limits_0^t x^{\alpha-1}F^{p-1}(x)f(x)\,d_\alpha x.$$

Applying Hölder's inequality with indices $p$ and $p/(p-1)$, and using the fact that $t^\alpha F^p(x)/(\alpha-p)$ is negative, we get

$$\int\limits_0^t F^p(x)\,d_\alpha x\le\frac{p}{p-\alpha}\Bigg(\int\limits_0^t F^p(x)\,d_\alpha x\Bigg)^{\frac{p-1}{p}}\Bigg(\int\limits_0^t \big(x^{\alpha-1}f(x)\big)^p\,d_\alpha x\Bigg)^{\frac{1}{p}},$$

and so,

$$\Bigg(\int\limits_0^t F^p(x)\,d_\alpha x\Bigg)^{\frac{1}{p}}\le\frac{p}{p-\alpha}\Bigg(\int\limits_0^t \big(x^{\alpha-1}f(x)\big)^p\,d_\alpha x\Bigg)^{\frac{1}{p}}.$$

Hence,

$$\int\limits_0^t F^p(x)\,d_\alpha x\le\Big(\frac{p}{p-\alpha}\Big)^p\int\limits_0^t \big(x^{\alpha-1}f(x)\big)^p\,d_\alpha x.$$

Let $t\to\infty$, and then

$$\int\limits_0^\infty F^p(x)\,d_\alpha x\le\Big(\frac{p}{p-\alpha}\Big)^p\int\limits_0^\infty \big(x^{\alpha-1}f(x)\big)^p\,d_\alpha x,$$

which is the desired inequality (3.1). □

**Remark 3.1.** From the proof of Theorem 3.1 we see that if the condition "$x^{\alpha-1}f(x)$ is continuous on $[0,\infty)$" is replaced either by

(i) $x^{\alpha-1}f(x)$ is continuous on $(0,\infty)$ and $\lim\limits_{x\to0^+}x^{\alpha-1+\frac{\alpha}{p}}f(x)=0$,

or

(ii) $\lim\limits_{x\to0^+}x^\alpha F^p(x)=0$,

then (3.1) is again true.

**Corollary 3.1.** *In Theorem* 3.1, *if* $\alpha=1$, *then we obtain the classical Hardy inequality* (1.1).

**Theorem 3.2.** *Let $f$ be a nonnegative function on $(0,\infty)$ and $0<\alpha\le1$. Let $c>1$ and $p>1$. Also assume that $x^{\alpha-1}f(x)$ is continuous on $[0,\infty)$ and $p>c-\alpha$. Then*

$$\int\limits_0^\infty x^{-c}\Bigg(\int\limits_0^x f(t)\,d_\alpha t\Bigg)^p d_\alpha x\le\Big(\frac{p}{c-\alpha}\Big)^p\int\limits_0^\infty(x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x. \tag{3.4}$$

*Proof.* Let

$$F(x):=\int\limits_0^x f(s)\,d_\alpha s.$$

Integrating by parts $\int\limits_0^t x^{-c}F^p(x)\,d_\alpha x$ (here $t>0$) with

$$w(x)=x^{-c}F^p(x),\quad D_\alpha g(x)=1\quad \Big(g(x)=\frac{x^\alpha}{\alpha}\Big),$$
$$D_\alpha w(x)=x^{1-\alpha}\big(-cx^{-c-1}F^p(x)+px^{-c}F^{p-1}(x)F'(x)\big),$$

we obtain

$$\begin{aligned}\int\limits_0^t x^{-c}F^p(x)\,d_\alpha x&=\frac{x^{-c}F^p(x)x^\alpha}{\alpha}\Big|_0^t-\int\limits_0^t\frac{x^\alpha}{\alpha}x^{1-\alpha}\big(-cx^{-c-1}F^p(x)+px^{-c}F^{p-1}(x)F'(x)\big)\,d_\alpha x\\&=\frac{t^{\alpha-c}F^p(t)}{\alpha}+\frac{c}{\alpha}\int\limits_0^t x^{-c}F^p(x)\,d_\alpha x-\frac{p}{\alpha}\int\limits_0^t x^{1-c}F^{p-1}(x)F'(x)\,d_\alpha x;\end{aligned}$$

note

$$\begin{aligned}\lim_{x\to0^+}x^{\frac{\alpha-c}{p}}F(x)&=\lim_{x\to0^+}\frac{\int\limits_0^x s^{\alpha-1}f(s)\,ds}{x^{\frac{c-\alpha}{p}}}\\&=\lim_{x\to0^+}\frac{x^{\alpha-1}f(x)}{(\frac{c-\alpha}{p})x^{\frac{c-\alpha}{p}-1}}=\lim_{x\to0^+}\Big(\frac{p}{c-\alpha}\Big)x^{\alpha-1}f(x)x^{1+\frac{\alpha-c}{p}}=0.\end{aligned}$$

Thus

$$\int\limits_0^t x^{-c}F^p(x)\,d_\alpha x=\frac{t^{\alpha-c}F^p(t)}{\alpha-c}+\frac{p}{c-\alpha}\int\limits_0^t x^{1-c}F^{p-1}(x)F'(x)\,d_\alpha x.$$

Since $F'(x)=x^{\alpha-1}f(x)$, we obtain

$$\begin{aligned}\int\limits_0^t x^{-c}F^p(x)\,d_\alpha x&=\frac{t^{\alpha-c}F^p(t)}{\alpha-c}+\frac{p}{c-\alpha}\int\limits_0^t x^{1-c}F^{p-1}(x)x^{\alpha-1}f(x)\,d_\alpha x\\&\le\frac{p}{c-\alpha}\int\limits_0^t x^{\alpha-c}F^{p-1}(x)f(x)\,d_\alpha x\le\frac{p}{c-\alpha}\int\limits_0^t x^{\alpha-c}F^{p-1}(x)f(x)\,d_\alpha x\\&\le\frac{p}{c-\alpha}\int\limits_0^t x^{\alpha-c}\frac{F^{p-1}(x)}{(x^{-c})^{\frac{p-1}{p}}(x^c)^{\frac{p-1}{p}}}f(x)\,d_\alpha x\le\frac{p}{c-\alpha}\int\limits_0^t\frac{x^{\alpha-c}}{(x^{-c})^{\frac{p-1}{p}}}\big((x^{-c}F^p(x))\big)^{\frac{p-1}{p}}f(x)\,d_\alpha x\\&\le\frac{p}{c-\alpha}\int\limits_0^t x^{\alpha-\frac{c}{p}}\big(x^{-c}F^p(x)\big)^{\frac{p-1}{p}}f(x)\,d_\alpha x.\end{aligned}$$

Applying Hölder's inequality with indices $p$ and $p/(p-1)$, we obtain (note $\alpha-c<0$)

$$\begin{aligned}\int\limits_0^t x^{-c}F^p(x)\,d_\alpha x&\le\frac{p}{c-\alpha}\Bigg(\int\limits_0^t\Big((x^{-c}F^p(x))^{\frac{p-1}{p}}\Big)^{\frac{p}{p-1}}d_\alpha x\Bigg)^{\frac{p-1}{p}}\Bigg(\int\limits_0^t(x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x\Bigg)^{\frac1p}\\&\le\frac{p}{c-\alpha}\Bigg(\int\limits_0^t x^{-c}F^p(x)\,d_\alpha x\Bigg)^{\frac{p-1}{p}}\Bigg(\int\limits_0^t(x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x\Bigg)^{\frac1p}.\end{aligned}$$

Thus

$$\Bigg(\int\limits_0^t x^{-c}F^p(x)\,d_\alpha x\Bigg)^{\frac1p}\le\Big(\frac{p}{c-\alpha}\Big)\Bigg(\int\limits_0^t(x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x\Bigg)^{\frac1p},$$

and so

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \Big(\frac{p}{c-\alpha}\Big)^p \int_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x.$$

Let $t\to\infty$, and then

$$\int_0^\infty x^{-c}F^p(x)\,d_\alpha x \le \Big(\frac{p}{c-\alpha}\Big)^p \int_0^\infty (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x,$$

which is the desired inequality (3.4). □

**Remark 3.2.** From the proof of Theorem 3.2 we see that if the condition "$x^{\alpha-1}f(x)$ is continuous on $[0,\infty)$" is replaced either by

(i) $x^{\alpha-1}f(x)$ is continuous on $(0,\infty)$ and $\lim\limits_{x\to0^+} x^{\alpha+\frac{\alpha-c}{p}}f(x)=0$,

or

(ii) $\lim\limits_{x\to0^+} x^{\alpha-c}F^p(x)=0$,

then (3.4) is again true.

**Corollary 3.2.** *In Theorem* 3.2*, if $\alpha=1$, then we have the weighted Hardy inequality* (1.2)*.*

**Corollary 3.3.** *In Theorem* 3.2*, if $c=p$ and $\alpha=1$, then we have the classical Hardy inequality* (1.1)*.*

**Theorem 3.3.** *Let $f$ be a nonnegative function on $(0,\infty)$ and $0<c<\alpha\le1$. Let $p>1$. In addition, assume that $x^{\alpha-1}f(x)$ is continuous on $(0,\infty)$ and $\lim\limits_{t\to\infty} t^{\alpha+\frac{\alpha-c}{p}}f(t)=0$. Then*

$$\int_0^\infty x^{-c}\Big(\int_x^\infty f(t)\,d_\alpha t\Big)^p d_\alpha x \le \Big(\frac{p}{\alpha-c}\Big)^p \int_0^\infty (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x. \tag{3.5}$$

*Proof.* Let $F(x):=\int_x^\infty f(s)\,d_\alpha s=\int_x^\infty s^{\alpha-1}f(s)\,ds$ and integrate by parts the term $\int_\epsilon^t x^{-c}F^p(x)\,d_\alpha x$ (here $t>0$ and $0<\epsilon<t$ small) with

$$w(x)=x^{-c}F^p(x),\quad D_\alpha g(x)=1\quad \Big(g(x)=\frac{x^\alpha}{\alpha}\Big),$$
$$D_\alpha w(x)=x^{1-\alpha}\big(-cx^{-c-1}F^p(x)+px^{-c}F^{p-1}(x)F'(x)\big).$$

Then we obtain

$$\begin{aligned}\int_\epsilon^t x^{-c}F^p(x)\,d_\alpha x &= \frac{x^{-c}F^p(x)x^\alpha}{\alpha}\Big|_\epsilon^t - \int_\epsilon^t \frac{x^\alpha}{\alpha}x^{1-\alpha}\big(-cx^{-c-1}F^p(x)+px^{-c}F^{p-1}(x)F'(x)\big)\,d_\alpha x\\ &= \frac{t^{\alpha-c}F^p(t)}{\alpha}-\frac{\epsilon^{\alpha-c}F^p(\epsilon)}{\alpha}+\frac{c}{\alpha}\int_\epsilon^t x^{-c}F^p(x)\,d_\alpha x-\frac{p}{\alpha}\int_\epsilon^t x^{1-c}F^{p-1}(x)F'(x)\,d_\alpha x\\ &\le \frac{t^{\alpha-c}F^p(t)}{\alpha}+\frac{c}{\alpha}\int_\epsilon^t x^{-c}F^p(x)\,d_\alpha x-\frac{p}{\alpha}\int_\epsilon^t x^{1-c}F^{p-1}(x)F'(x)\,d_\alpha x,\end{aligned}$$

and therefore (letting $\epsilon\to0^+$), since $\alpha-c>0$, we have

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c}-\frac{p}{\alpha-c}\int_0^t x^{1-c}F^{p-1}(x)F'(x)\,d_\alpha x.$$

Since $F'(x) = -x^{\alpha-1}f(x)$, we obtain

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{\alpha-c}\int_0^t x^{1-c}F^{p-1}(x)x^{\alpha-1}f(x)\,d_\alpha x$$

$$= \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{\alpha-c}\int_0^t x^{\alpha-c}F^{p-1}(x)f(x)\,d_\alpha x = \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \int_0^t x^{\alpha-c}\frac{F^{p-1}(x)}{(x^{-c})^{\frac{p-1}{p}}.(x^c)^{\frac{p-1}{p}}}f(x)\,d_\alpha x$$

$$= \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \int_0^t \frac{x^{\alpha-c}}{(x^{-c})^{\frac{p-1}{p}}}\left((x^{-c}F(x))^p\right)^{\frac{p-1}{p}}f(x)\,d_\alpha x$$

$$= \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \int_0^t x^{\alpha-\frac{c}{p}}(x^{-c}F^p(x))^{\frac{p-1}{p}}f(x)\,d_\alpha x.$$

Applying Hölder's inequality with indices $p$ and $p/(p-1)$, we obtain

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{\alpha-c}\left(\int_0^t \left((x^{-c}F^p(x))^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}}d_\alpha x\right)^{\frac{p-1}{p}}\left(\int_0^t \left(x^{\alpha-\frac{c}{p}}f(x)\right)^p d_\alpha x\right)^{\frac{1}{p}}$$

$$\le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \frac{p}{\alpha-c}\left(\int_0^t x^{-c}F^p(x)\,d_\alpha x\right)^{\frac{p-1}{p}}\left(\int_0^t \left(x^{\alpha-\frac{c}{p}}f(x)\right)^p d_\alpha x\right)^{\frac{1}{p}},$$

so,

$$\left(\int_0^t x^{-c}F^p(x)\,d_\alpha x\right)^{\frac{1}{p}} \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \left(\frac{p}{\alpha-c}\right)\left(\int_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x\right)^{\frac{1}{p}}.$$

Thus

$$\int_0^t x^{-c}F^p(x)\,d_\alpha x \le \frac{t^{\alpha-c}F^p(t)}{\alpha-c} + \left(\frac{p}{\alpha-c}\right)^p\int_0^t (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x.$$

Let $t \to \infty$ and note

$$\lim_{t\to\infty} t^{\frac{\alpha-c}{p}}F(t) = \lim_{t\to\infty}\frac{\int_t^\infty s^{\alpha-1}f(s)}{t^{\frac{c-\alpha}{p}}}\,ds = \lim_{t\to\infty} -\frac{t^{\alpha-1}f(t)}{(\frac{c-\alpha}{p})t^{\frac{c-\alpha}{p}-1}} = -\lim_{t\to\infty}\left(\frac{p}{c-\alpha}\right)f(t)t^{\alpha+\frac{\alpha-c}{p}} = 0,$$

so,

$$\int_0^\infty x^{-c}F^p(x)\,d_\alpha x \le \left(\frac{p}{\alpha-c}\right)^p\int_0^\infty (x^{\alpha-\frac{c}{p}}f(x))^p\,d_\alpha x,$$

which is the desired inequality (3.5). □

**Corollary 3.4.** *In Theorem* 3.3*, if $\alpha = 1$, then we have the weighted Hardy inequality* (1.3)*.*

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