Memoirs on Differential Equations and Mathematical Physics

Volume 72, 2017, 141–150

Margarita Zvereva

A STRING OSCILLATIONS SIMULATION WITH NONLINEAR CONDITIONS

Abstract. In the present paper we investigate the initial-boundary value problems describing oscillation processes with hysteresis type conditions. Analogues of the d'Alembert formula are obtained.*

2010 Mathematics Subject Classification. Primary: 34B45, 34H05; Secondary: 34C55, 34B15.

Key words and phrases. Wave equation, string oscillations, hysteresis, geometric graph.

რეზიუმე. წარმოდგენილ ნაშრომში გამოკვლეულია საწყის-სასაზღვრო ამოცანები, რომლებიც აღწერს რხევად პროცესებს ჰისტერეზისის ტიპის პირობებით. მიღებულია დალამბერის ტიპის ფორმულები.

*Reported on Conference "Differential Equation and Applications", September 4-7, 2017, Brno

1 Introduction

There are many papers devoted to the control problems for distributed systems, not to mention the works of V. A. Il'in, E. I. Moiseev, L. N. Znamenskaya, A. I. Egorov, A. V. Borovskikh (see [1–4]). In these works, a control, allowing to govern by the oscillation process was obtained. The d'Alembert formula is very important for the search of the control. In the present paper, we investigate the initial-boundary value problems describing oscillation processes with conditions of hysteresis type. This kind of problems arise in a simulation of string oscillations, where the movement is restricted by a sleeve concentrated at one point. We consider the cases when the sleeve is located at the end of a segment and at the node of a graph-star. Analogues of the d'Alembert formula are obtained.

2 Preliminaries

In this section, we recall some notions and definitions which we will need in the sequel (details can be found in [5]).

Let H be a Hilbert space. The inner product in H is denoted by $\langle \cdot , \cdot \rangle$.

For a closed convex $C \subset H$ and $x \in C$, the set

$$N_C(x) = \big\{\xi \in H : \ \langle \xi, c - x\rangle \le 0 \ \ \forall\, c \in C\big\}$$

denotes the outward normal cone to C at x.

Note that we always have $0 \in N_C(x)$, $N_{\{x\}}(x) = H$, and $N_C(x) = \{0\}$ for $x \in \operatorname{int} C$, the interior of C, if provided $\operatorname{int} C \ne \varnothing$. The last relation shows that the outward normal cone is non-trivial only for $x \in \partial C$, the boundary of C.

Recall that the Hausdorff distance $d_H(C_1, C_2)$ between closed sets C_1 and C_2 is given by the formula

$$d_H(C_1, C_2) = \max\Big\{ \sup_{x \in C_2} \operatorname{dist}(x, C_1), \ \sup_{x \in C_1} \operatorname{dist}(x, C_2)\Big\}.$$

Consider the so-called "sweeping process"

$$-u'(t) \in N_{C(t)}(u(t)) \ \text{ for a.e. } \ t \in [0, T], \tag{2.1}$$

$$u(0) = u_0 \in C(0). \tag{2.2}$$

A function $u : [0, T] \to H$ is called a solution of the initial problem (2.1), (2.2) if

(a) $u(0) = u_0$;

(b) $u(t) \in C(t)$ for all $t \in [0, T]$;

(c) u is differentiable for almost every point $t \in [0, T]$;

(d) $-u'(t) \in N_{C(t)}(u(t))$ for almost every $t \in [0, T]$.

There are many papers devoted to sweeping processes (see, e.g., [5–13]).

Later we will use the next theorems.

Theorem 2.1 (Existence [5, Theorem 2]). *Assume that the map $t \to C(t)$ satisfies*

$$d_H(C(t), C(s)) \le L|t - s|$$

and $C(t) \subset H$ is nonempty, closed and convex for every $t \in [0, T]$. Let $u_0 \in C(0)$. Then there exists a solution $u : [0, T] \to H$ of (2.1), (2.2) which is Lipschitz continuous with the constant L. In particular, $|u'(t)| \le L$ for almost every $t \in [0, T]$.

Theorem 2.2 (Uniqueness [5, Theorem 3]). *The solution of (2.1), (2.2) is unique in the class of absolutely continuous functions.*

3 A string with a hysteresis type boundary value condition

Suppose a string is located along the segment $[0,l]$. Let $u(x,t)$ be a deviation from the equilibrium position at the time t. Assume that the left end of the string has the elastic support (a spring), so we have $u'_x(0,t)=\gamma u(0,t)$. The right end of the string moves along a vertical needle (without friction) inside a sleeve, represented by $[-h,h]$, where $h>0$. While $|u(l,t)|<h$, the right end of the string inside of the sleeve remains free, i.e., $u'_x(l,t)=0$. If the string reaches the boundary points of the sleeve, then the conditions $u(l,t)=h$ or $u(l,t)=-h$, respectively, are satisfied at a certain moment. Notice that we consider the case where the sleeve move in perpendicular to the axis Ox and its movement is given by

$$C(t)=[-h,h]+\xi(t). \tag{3.1}$$

Suppose that the string velocity is zero at the initial time $t=0$ and the string form is determined by a function $\varphi(x)\in W_2^1[0,l]$, where $\varphi'_x(0)=\gamma\varphi(0)$, $\varphi(l)\in C(0)$.

The mathematical model of such problem can be described as

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}\,, \quad 0<x<l, \ \ 0<t<T,\\ u(x,0)=\varphi(x),\\ \dfrac{\partial u}{\partial t}(x,0)=0,\\ u'_x(0,t)=\gamma u(0,t),\\ u(l,t)\in C(t)\\ -u'_x(l,t)\in N_{C(t)}(u(l,t)), \end{cases} \tag{3.2}$$

where the set $N_C(a)$ is an outward normal cone to C at a defined by

$$N_C(a)=\big\{\xi\in R^1:\ \xi\cdot(c-a)\le 0\ \ \forall\, c\in C\big\}.$$

Notice that if a is an interior point of C, then $N_C(a)=\{0\}$; if $a=-h+\xi(t)$, then $N_C(a)=(-\infty,0]$; if $a=h+\xi(t)$, then $N_C(a)=[0,+\infty)$.

The condition $-u'_x(l,t)\in N_{C(t)}(u(l,t))$ means that if $u(l,t)$ is an interior point of $C(t)$, then $u'_x(l,t)=0$, i.e., the oscillation process is the same as for a string with a free right end (see [14]); when the right end of the string is tangent to the boundary sleeve point, the right end of the string is not free anymore: there is a force $f(t)$, which blocks this end, so $-u'_x(l,t)=-f(t)\in N_{C(t)}(u(l,t))$.

We consider a solution of (3.2) belonging to a special class of functions introduced for the first time by V. A. Il'in in [15,16]. Let Q_T be the rectangle $Q_T=[0\le x\le l]\times[0\le t\le T]$. As in [15,16], we suppose that u belongs to the class $\widehat{W}_2^1(Q_T)$ if the function $u(x,t)$ is continuous in the closed rectangle Q_T and in this rectangle has both generalized partial derivatives $u'_x(x,t)$ and $u'_t(x,t)$, which belong to the class $L_2(Q_T)$ and, moreover, $u'_x(\,\cdot\,,t)$ belongs to the class $L_2[0\le x\le l]$ for every fixed t of the segment $[0,T]$, and $u'_t(x,\,\cdot\,)$ belongs to the class $L_2[0\le t\le T]$, for any fixed x of the segment $[0,l]$. By a solution of (3.2) we call a function $u(x,t)\in\widehat{W}_2^1(Q_T)$ such that $u(l,t)\in C(t)$ for all t, the condition $-u'_x(l,t)\in N_{C(t)}(u(l,t))$ holds for almost every t, and the integral identity

$$\int\limits_0^l\int\limits_0^T u(x,t)\big[\Psi_{tt}(x,t)-\Psi_{xx}(x,t)\big]\,dx\,dt+\int\limits_0^l \Psi'_t(x,0)\varphi(x)\,dx \\ -\int\limits_0^T \Psi(l,t)u'_x(l,t)\,dt+\int\limits_0^T \Psi'_x(l,t)u(l,t)\,dt=0 \tag{3.3}$$

holds for any function $\Psi(x,t)\in C^2(Q_T)$, which satisfies the conditions $\Psi'_x(0,t)=\gamma\Psi(0,t)$, $\Psi(x,T)=0$, $\Psi'_t(x,T)=0$.

Theorem 3.1. *Assume that the function $\xi(t)$ satisfies the Lipschitz condition and the function $\varphi \in W_2^1[0,l]$. If $0 \le t \le l$, then the solution of problem (3.2) can be represented as*

$$u(x,t) = \frac{\Phi(x-t)+\Phi(x+t)}{2}\,, \tag{3.4}$$

where

$$\Phi(x) = \begin{cases} \varphi(x), & x \in [0,l], \\ 2g(x-l)+\varphi(2l-x)-2\varphi(l), & x \in [l,2l], \\ \varphi(-x) - 2\gamma e^{\gamma x} \int\limits_0^{-x} \varphi(s) e^{\gamma s}\, ds, & x \in [-l,0], \end{cases}$$

and g is a solution of the problem

$$-v_1'(t) \in N_{D(t)}(v_1(t)), v_1(0) = \varphi(l) \in D(0), \tag{3.5}$$

$$D(t) = C(t) + \int\limits_0^t \varphi'(l-s)\, ds.$$

Proof. Consider the problem

$$-v_1'(t) \in N_{D(t)}(v_1(t)), \quad v_1(0) = \varphi(l) \in D(0).$$

We will use Theorems 2.1 and 2.2. Since $D(t)$ is a nonempty, closed, convex set, and the mapping $t \mapsto D(t)$ satisfies the Lipschitz condition with a constant L^*, i.e.,

$$d_H(D(t), D(s)) \le L^*|t-s|, \quad t,s \in [0,T],$$

there is a unique absolutely continuous function $g(t)$ defined on all $[0,l]$, which is the solution of problem (3.5), and $|g'(t)| \le L^*$ for almost all $t \in [0,l]$. Since $g(t) \in D(t)$, where $D(t) = C(t)+\varphi(l)-\varphi(l-t)$ and $u(l,t) = g(t)+\varphi(l-t)-\varphi(l)$, we have $u(l,t) \in C(t)$.

Let us show that $-u_x'(l,t) \in N_{C(t)}(u(l,t))$. Notice that $u_x'(l,t) = g'(t)$. Let us show that $-g'(t) \in N_{C(t)}(g(t)+\varphi(l-t)-\varphi(l))$. Since $-g'(t) \in N_{D(t)}(g(t))$, we get $-g'(t)(c(t)-\varphi(l-t)+\varphi(l)-g(t)) \le 0$ for all $c(t) \in C(t)$. So, $-g'(t) \in N_{C(t)}(g(t)+\varphi(l-t)-\varphi(l))$.

Our aim now is to prove equality (3.3). We have

$$\begin{aligned} &\int\limits_0^l \Big(\int\limits_0^T u(x,t)\Psi_{tt}(x,t)\, dt \Big)\, dx - \int\limits_0^T \Big(\int\limits_0^l u(x,t)\Psi_{xx}(x,t)\, dx \Big)\, dt \\ &\qquad + \int\limits_0^l \Psi_t'(x,0)\varphi(x)\, dx - \int\limits_0^T \Psi(l,t)u_x'(l,t)\, dt + \int\limits_0^T \Psi_x'(l,t)u(l,t)\, dt \\ &= \int\limits_0^l \big(u(x,T)\Psi_t'(x,T) - u(x,0)\Psi_t'(x,0)\big)\, dx - \int\limits_0^l \int\limits_0^T u_t'\Psi_t'\, dt\, dx - \int\limits_0^T \big(\Psi_x'(l,t)u(l,t) - \Psi_x'(0,t)u(0,t)\big)\, dt \\ &\qquad + \int\limits_0^T \int\limits_0^l u_x'\Psi_x'\, dx\, dt + \int\limits_0^l \Psi_t'(x,0)\varphi(x)\, dx - \int\limits_0^T \Psi(l,t)u_x'(l,t)\, dt + \int\limits_0^T \Psi_x'(l,t)u(l,t)\, dt. \end{aligned}$$

We have to prove that

$$\int\limits_0^T \int\limits_0^l u_x'\Psi_x'\, dx\, dt - \int\limits_0^T \int\limits_0^l u_t'\Psi_t'\, dx\, dt = \int\limits_0^T \Psi(l,t)u_x'(l,t)\, dt - \gamma \int\limits_0^T \Psi(0,t)u(0,t)\, dt.$$

According to (3.4), we obtain

$$\frac{1}{2}\int_0^T\int_0^l (\Phi'(x-t)+\Phi'(x+t))\Psi'_x\,dx\,dt - \frac{1}{2}\int_0^l\int_0^T (\Phi'(x+t)-\Phi'(x-t))\Psi'_t\,dt\,dx$$

$$=\frac{1}{2}\int_0^l \big(\Psi'_x(x,T)(\Phi(x+T)-\Phi(x-T))-\Psi'_x(x,0)(\Phi(x)-\Phi(x))\big)\,dx-\frac{1}{2}\int_0^l\int_0^T(\Phi(x+t)-\Phi(x-t))\Psi_{xt}\,dt\,dx$$

$$-\frac{1}{2}\int_0^T\big(\Psi'_t(l,t)(\Phi(l+t)-\Phi(l-t))-\Psi'_t(0,t)(\Phi(t)-\Phi(-t))\big)\,dt+\frac{1}{2}\int_0^l\int_0^T(\Phi(x+t)-\Phi(x-t))\Psi_{xt}\,dt\,dx$$

$$=-\frac{1}{2}\int_0^T \Psi'_t(l,t)(\Phi(l+t)-\Phi(l-t))\,dt+\frac{1}{2}\int_0^T\Psi'_t(0,t)(\Phi(t)-\Phi(-t))\,dt$$

$$=\int_0^T \Psi'_t(l,t)(\varphi(l)-g(t)))\,dt+\gamma\int_0^T\Psi'_t(0,t)e^{-\gamma t}\int_0^t\varphi(s)e^{\gamma s}\,ds\,dt$$

$$=\int_0^T \Psi'_t(l,t)(\varphi(l)-g(t)))\,dt-\gamma\int_0^T\Psi(0,t)\left(\varphi(t)-\gamma e^{-\gamma t}\int_0^t\varphi(s)e^{\gamma s}\,ds\right)dt.$$

On the other hand,

$$\int_0^T\Psi(l,t)u'_x(l,t)\,dt-\gamma\int_0^T\Psi(0,t)u(0,t)\,dt$$

$$=\int_0^T \Psi'_t(l,t)(\varphi(l)-g(t))\,dt-\gamma\int_0^T\Psi(0,t)\left(\varphi(t)-\gamma e^{-\gamma t}\int_0^t\varphi(s)e^{\gamma s}\,ds\right)dt.$$

This completes the proof of the theorem. □

Remark 3.1. The presentation of the solution in the form of (3.4) is true for all T. The initial condition $u(x,0)=\Phi(x)=\varphi(x)$ determines the value of Φ on the interval $[0,l]$. According to the boundary condition at the point l, we can represent the derivative $u'_t(l,t)$ through $u'_x(l,t)$. We obtain

$$-u'_t(l,t)\in N_{C(t)}(u(l,t))+\Phi'(l-t),$$

where the function $\Phi'(l-t)$ for all $0\le t\le l$ is known, namely, $\Phi'(l-t)=\varphi'(l-t)$. Denote $w(t)=u(l,t)$, $\Phi'(l-t)=\eta(t)$. So we get the problem

$$-w'(t)\in N_{C(t)}(w(t))+\eta(t),\quad w(0)=\varphi(l)\in C(0).$$

Put the function $v_1(t)=w(t)+\int_0^t\eta(s)\,ds$, and the set $D(t)=C(t)+\int_0^t\eta(s)\,ds$. Notice that $N_{C(t)}(w(t))=N_{D(t)}(v_1(t))$. We get the problem

$$-v_1'(t)\in N_{D(t)}(v_1(t)),\quad v_1(0)=\varphi(l)\in D(0).$$

According to Theorems 2.1 and 2.2, this problem has a unique solution $v_1(t)$, which is defined on the whole interval $[0,l]$. This function $v_1(t)$ is absolutely continuous, and its derivative is bounded almost everywhere. Hence we have that the function $\Phi(x)$ on the interval $[l,2l]$ should be defined as

$$\Phi(x)=2v_1(x-l)+\varphi(2l-x)-2\varphi(l).$$

Using the condition $u'_x(0,t) = 0$, we can extend $\Phi(x)$ on the interval $[-l, 0]$. So, if $T \le l$, then the problem is solved. Otherwise, the function $\Phi(x)$ is known for all $x \in [-l, 2l]$. Repeating the above scheme several times, we obtain the required representation.

Remark 3.2. Notice that problem (3.2) has a unique solution. Assume that $\varphi(l) \in (-h + \xi(0), h + \xi(0))$. Then the oscillation process occurs likewise for the string with a free end for all $t \in [0, t_1]$, and the string form is the solution of the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < l, \ \ 0 < t < t_1, \\ u(x,0) = \varphi(x), \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \\ u'_x(0,t) = \gamma u(0,t), \\ u'_x(l,t) = 0. \end{cases}$$

Notice that the last problem has a unique solution $u(x,t)$. If $t_1 < T$, then the relation $u(l,t_1) = \pm h + \xi(t)$ holds at the moment t_1, and for all $t \in [t_1, t_2]$ a string form is a solution of the problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial^2 v}{\partial t^2}, \quad 0 < x < l, \ \ t_1 < t < t_2, \\ v(x,t_1) = u(x,t_1), \\ \dfrac{\partial v}{\partial t}(x,t_1) = u'_t(x,t_1), \\ v'_x(0,t) = \gamma v(0,t), \\ v(l,t) = -h + \xi(t) \end{cases}$$

or

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial^2 v}{\partial t^2}, \quad 0 < x < l, \ \ t_1 < t < t_2, \\ v(x,t_1) = u(x,t_1), \\ \dfrac{\partial v}{\partial t}(x,t_1) = u'_t(x,t_1), \\ v'_x(0,t) = \gamma v(0,t), \\ v(l,t) = h + \xi(t). \end{cases}$$

Each of the above problems have a unique solution for every $t \in [t_1, t_2]$. By a similar reasoning, we find that the original problem has a unique solution.

4 A problem on a geometric graph

Let the points O, $A_1, A_2, \dots, A_n$ belong to the horizontal plane π. Consider a mechanical system consisting of n strings, which in equilibrium are the segments $OA_1, OA_2, \dots, OA_n$. The ends of the strings have elastic supports (springs) at the points $A_1, A_2, \dots, A_n$ and interconnected at the point O. There is a perpendicular needle inside a sleeve passing through the point O. The graph Γ consists of edges (intervals) $OA_1, OA_2, \dots, OA_n$ and vertices $O, A_1, A_2, \dots, A_n$. We will use the notions and the terminology from [17]. Under the influence of a distributed force perpendicular to the plane π, the strings deviate from the equilibrium position. We assume that a deviation of all points is parallel to the same straight line, which is perpendicular to the plane, and consider small deviations from the equilibrium position. Take the system of coordinates to describe string deformations. The X-axis Ox_i

for the i-th string $(i = 1, 2, \ldots, n)$ contains the segment OA_i and is directed from A_i to O. Thus the graph is directed to the node. The Y-axis OY passes perpendicularly to the plane π. Consider the oscillation process. Let $u_i(x,t)$ be the deviation of i-th string from the equilibrium position at time t. We assume that the length of all strings equals l, i.e. $0 \le x \le l$.

Thus, the oscillations of each string of the system can be described by the wave equation

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 u_i}{\partial t^2}\,.$$

The connection between the strings at the node means that $u_1(l,t) = u_i(l,t)$ $(i = 1, 2, \ldots, n)$. The conditions of the elastic fixing mean that $\frac{\partial u_i}{\partial x}(0,t) = \gamma u_i(0,t)$ $(i = 1, 2, \ldots, n)$. At the point $x = l$ there is the sleeve, whose movement in the perpendicular direction to the plane π is given by (3.1), where the function $\xi(t)$ satisfies the Lipschitz condition.

Assume that at the initial moment the initial form and the initial velocity of strings are $u_i(x,0) = \varphi_i(x)$, $\frac{\partial u_i}{\partial t}(x,0) = 0$, where $\varphi_i \in W_2^1[0,l]$, $\varphi_1(l) = \varphi_2(l) = \cdots = \varphi_n(l) = \varphi(l)$, $\frac{\partial \varphi_i}{\partial x}(0) = \gamma\varphi_i(0)$, $\varphi(l) \in C(0)$.

Then a mathematical model of such a problem can be described as

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial x^2} = \dfrac{\partial^2 u_i}{\partial t^2}\,, \quad 0 < x < l, \;\; 0 < t < T \;\; (i = 1, 2, \ldots, n), \\ u_i(x,0) = \varphi_i(x), \\ \dfrac{\partial u_i}{\partial t}(x,0) = 0, \\ -\displaystyle\sum_{i=1}^{n} \dfrac{\partial u_i}{\partial x}(l-0,t) \in N_{C(t)}(u(l,t)), \\ u(l,t) = u_1(l,t) = u_2(l,t) = \cdots = u_n(l,t), \\ u(l,t) \in C(t), \\ \dfrac{\partial u_i}{\partial x}(0,t) = \gamma u_i(0,t). \end{cases}$$

By a solution of this problem we mean the function $u(x,t)$, whose restrictions to the edges coincide with $u_i(x,t)$ $(i = 1, 2, \ldots, n)$. The functions $u_i(x,t) \in \widehat{W}_2^1(Q_T)$ satisfy the conditions $u_1(l,t) = u_2(l,t) = \cdots = u_n(l,t) = u(l,t)$, $u(l,t) \in C(t)$ for all t. The condition

$$-\sum_{i=1}^{n} \frac{\partial u_i}{\partial x}(l-0,t) \in N_{C(t)}(u(l,t))$$

holds for almost all $t \in [0,T]$. The integral equalities

$$\sum_{i=1}^{n} \int_0^l \int_0^T u_i(x,t)\Big[\frac{\partial^2 \Psi_i}{\partial t^2}(x,t) - \frac{\partial^2 \Psi_i}{\partial x^2}(x,t)\Big]\,dx\,dt + \sum_{i=1}^{n} \int_0^l \frac{\partial \Psi_i}{\partial t}(x,0)\varphi_i(x)\,dx$$
$$+ \sum_{i=1}^{n} \int_0^T \Big(u(l,t)\,\frac{\partial \Psi_i}{\partial x}(l-0,t) - \Psi(l,t)\frac{\partial u_i}{\partial x}(l-0,t)\Big)\,dt = 0, \quad i = 1, \ldots, n,$$

hold for all $\Psi_i(x,t) \in C^2(Q_T)$, such as

$$\frac{\partial \Psi_i}{\partial x}(0,t) = \gamma\Psi_i(0,t), \quad \Psi_i(x,T) = 0, \quad \frac{\partial \Psi_i}{\partial t}(x,T) = 0, \quad \Psi_1(l,t) = \cdots = \Psi_n(l,t) = \Psi(l,t).$$

An analogue of the d'Alembert formula for the representation of the solution of this problem can be obtained by bringing the problem on the segment. Suppose that the solution of the last problem exists. Denote it by $v(x,t)$, where on each edge of the graph Γ the function $v(x,t)$ is defined as $v_i(x,t)$.

Let $\widetilde{u}(x,t) = \frac{1}{n}\sum\limits_{i=1}^{n} v_i(x,t)$. Then $\widetilde{u}(x,t)$ is the solution of the problem which has been studied above

$$\begin{cases} \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = \dfrac{\partial^2 \widetilde{u}}{\partial t^2}, \quad 0 < x < l, \ \ 0 < t < T, \\ \widetilde{u}(x,0) = \frac{1}{n}\sum\limits_{i=1}^{n} \varphi_i(x), \\ \dfrac{\partial \widetilde{u}}{\partial t}(x,0) = 0, \\ -\dfrac{\partial \widetilde{u}}{\partial x}(l,t) \in N_{C(t)}(\widetilde{u}(l,t)), \\ \widetilde{u}(l,t) \in C(t), \\ \widetilde{u}'_x(0,t) = \gamma \widetilde{u}(0,t). \end{cases}$$

So,

$$\widetilde{u}(x,t) = \frac{\Phi(x-t) + \Phi(x+t)}{2}.$$

Suppose $\omega_i(x,t) = v_i(x,t) - \widetilde{u}(x,t)$ $(i = 1, 2, \ldots, n)$. Notice that $\omega_i(x,t)$ $(i = 1, 2, \ldots, n)$ are solutions of the problem

$$\begin{cases} \dfrac{\partial^2 \omega_i}{\partial x^2} = \dfrac{\partial^2 \omega_i}{\partial t^2}, \quad 0 < x < l, \ \ 0 < t < T \ \ (i = 1, 2, \ldots, n), \\ \omega_i(x,0) = \varphi_i(x) - \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \varphi_j(x), \\ \dfrac{\partial \omega_i}{\partial t}(x,0) = 0, \\ \dfrac{\partial \omega_i}{\partial x}(0,t) = \gamma \omega_i(0,t), \\ \omega_i(l,t) = 0. \end{cases}$$

For ω_i, we have

$$\omega_i(x,t) = \frac{\Phi_i(x-t) + \Phi_i(x+t)}{2},$$

where

$$\Phi_i(x) = \begin{cases} \varphi_i(x) - \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \varphi_j(x), & x \in [0,l] \\ \varphi_i(-x) - \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \varphi_j(-x) - 2\gamma e^{\gamma x} \int\limits_{0}^{-x} (\varphi_i(s) - \dfrac{1}{n}\sum_{j=1}^{n} \varphi_j(s)) e^{\gamma s}\, ds, & x \in [-l, 0], \\ \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \varphi_j(2l - x) - \varphi_i(2l - x), & x \in [l, 2l]. \end{cases}$$

Notice that $v_i(x,t) = \omega_i(x,t) + \widetilde{u}(x,t)$. So,

$$v_i(x,t) = \frac{\Phi(x-t) + \Phi(x+t)}{2} + \frac{\Phi_i(x-t) + \Phi_i(x+t)}{2}.$$

Acknowledgement

This research was supported by Russian Science Foundation grant № 16-11-10125, performed in Voronezh State University.

References

[1] V. A. Il'in and E. I. Moiseev, Optimization of boundary controls of string vibrations. (Russian) *Uspekhi Mat. Nauk* **60** (2005), no. 6(366), 89–114; translation in *Russian Math. Surveys* **60** (2005), no. 6, 1093–1119.

[2] V. A. Il'in and E. I. Moiseev, Optimization of a boundary control by an elastic force at one end of a string, based on finding the minimum of the integral of the modulus of an elastic force raised to an arbitrary power $p \geq 1$. (Russian) *Dokl. Akad. Nauk* **412** (2007), no. 6, 732–735; translation in *Dokl. Math.* **75** (2007), no. 1, 123–126.

[3] A. I. Egorov and L. N. Znamenskaya, On the boundary observability of the elastic oscillations of coupled objects with distributed and lumped parameters. (Russian) *Avtomat. i Telemekh.* **2007**, no. 2, 95–102; translation in *Autom. Remote Control* **68** (2007), no. 2, 296–302.

[4] A. V. Borovskikh, Formulas for the boundary control of an inhomogeneous string. I. (Russian) *Differ. Uravn.* **43** (2007), no. 1, 64–89; translation in *Differ. Equ.* **43** (2007), no. 1, 69–95.

[5] M. Kunze and M. D. P. Monteiro Marques, *An Introduction to Moreau's Sweeping Process.* Impacts in mechanical systems (Grenoble, 1999), 1–60, Lecture Notes in Phys., 551, Springer, Berlin, 2000.

[6] Ch. Castaing and M. D. P. Monteiro Marques, BV periodic solutions of an evolution problem associated with continuous moving convex sets. *Set-Valued Anal.* **3** (1995), no. 4, 381–399.

[7] M. Valadier, Rafle et viabilité. (French) *Sém. Anal. Convexe* **22** (1992), Exp. No. 17, 10 pp.

[8] S. Adly and Ba Khiet Le, Unbounded second-order state-dependent Moreau's sweeping processes in Hilbert spaces. *J. Optim. Theory Appl.* **169** (2016), no. 2, 407–423.

[9] M. Brokate and P. Krejčí, Optimal control of ODE systems involving a rate independent variational inequality. *Discrete Contin. Dyn. Syst. Ser. B* **18** (2013), no. 2, 331–348.

[10] L. Adam and J. Outrata, On optimal control of a sweeping process coupled with an ordinary differential equation. *Discrete Contin. Dyn. Syst. Ser. B* **19** (2014), no. 9, 2709–2738.

[11] F. Bullo, Averaging and vibrational control of mechanical systems. *SIAM J. Control Optim.* **41** (2002), no. 2, 542–562.

[12] J. F. Edmond and L. Thibault, Relaxation of an optimal control problem involving a perturbed sweeping process. *Math. Program.* **104** (2005), no. 2-3, Ser. B, 347–373.

[13] P. Krejci and T. Roche, Lipschitz continuous data dependence of sweeping processes in BV spaces. *Discrete Contin. Dyn. Syst. Ser. B* **15** (2011), no. 3, 637–650.

[14] N. A. Tikhonov and A. A. Samarskii, Equations of mathematical physics. *Courier Corporation*, 2013.

[15] V. A. Il'in, Boundary control of oscillations on two ends in terms of the generalized solution of the wave equation with finite energy. (Russian) *Differ. Uravn.* **36** (2000), no. 11, 1513–1528, 1583; translation in *Differ. Equ.* **36** (2000), no. 11, 1659–1675.

[16] V. A. Il'in, Boundary control of oscillations at one end and the other end fixed in terms of the generalized solution of the wave equation with finite energy. (Russian) *Differ. Uravn.* **36** (2000), no. 12, 1670–1686, 1727–1728; translation in *Differ. Equ.* **36** (2000), no. 12, 1832–1849.

[17] Yu. V. Pokornyi and A. V. Borovskikh, Differential equations on networks (geometric graphs). Differential equations on networks. *J. Math. Sci. (N.Y.)* **119** (2004), no. 6, 691–718.

(Received 21.10.2017)

Author's address:

Faculty of Mathematics, Voronezh State University, 1 Universitetskaya pl., Voronezh 394036, Voronezhskaya oblast', Russia.

E-mail: margz@rambler.ru