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Petr Tomášek

VISUALIZATION AND ANALYSIS OF STABILITY REGIONS OF CERTAIN DISCRETIZATION OF DIFFERENTIAL EQUATION WITH CONSTANT DELAY

Abstract. The paper discusses the asymptotic stability regions of multistep discretization of linear delay differential equation with a constant delay. Different location of delay dependent parts of stability regions with respect to parity of number of steps clarifies unexpected changes in numerical solution's behaviour under various settings of the equation's parameters and stepsize.*

2010 Mathematics Subject Classification. 34K28, 34K20.

Key words and phrases. Delay differential equation, constant delay, difference equation, numerical solution, necessary and sufficient conditions for asymptotic stability.

რეზიუმე. ნაშრომში განხილულია მრავალსაფეხურიანი დისკრეტიზაციის ასიმპტოტურად მდგრადი არეები წრფივი დაგვიანებულ-არგუმენტიანი დიფერენციალური განტოლებისთვის მუდმივი დაგვიანებით. მდგრადობის არეების დაგვიანებაზე დამოკიდებული ნაწილების განსხვავებული ადგილმდებარეობა ბიჯთა რაოდენობის ლუწ-კენტობასთან მიმართებაში ხსნის რიცხვითი ამონახსნების ქცევის მოულოდნელ ცვლილებებს განტოლების სხვადასხვა პარამეტრების და სხვადასხვა ბიჯის შემთხვევაში.

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1 Introduction

For the recent decades the delay differential equations theory has made great achievements. Consequently, appropriate numerical methods and corresponding theoretical background are being developed since 1970's. A valuable monograph, which summarize numerical methods for various delay differential equations and introduce comparison with the methods known for ordinary differential equations, is due to Bellen and Zennaro [2]. A various phenomena were observed as differences between the both mentioned classes of differential equations and their numerical discrete counterparts.

The concepts of asymptotic stability in numerical analysis are usually related to the numerical solution behaviour of the studied method applied to a certain test equation. Such equations in the delay differential case are, e.g.,

$$\begin{aligned} y'(t) &= by(t-\tau), \quad t>0, \\ y'(t) &= ay(t)+by(t-\tau), \quad t>0, \end{aligned} \tag{1.1}$$

where $a, b, \tau \in \mathbb{R}$, $\tau > 0$. In general, the coefficients a, b are considered as complex ones in various types of stability manner. In this paper, we constrain our considerations to the case of equation (1.1) with real coefficients a, b. This restriction arises from the studied numerical discretization and visualization purposes.

The numerical scheme, that we are going to analyse, can be captured by the linear difference equation

$$y_{n+2} + \alpha y_n + \gamma y_{n-\ell} = 0, \quad n = 0, 1, 2, \dots, \tag{1.2}$$

where $\alpha, \gamma \in \mathbb{R}$ and $\ell \in \mathbb{N}$. We recall that equations (1.1) and (1.2) are said to be asymptotically stable if for any of their solutions $y(t) \to 0$ as $t \to \infty$ and $y_n \to 0$ as $n \to \infty$, respectively. This terminology is usual in the theory of homogeneous linear differential (difference) equations with a constant delay.

In the case of linear difference equations with constant coefficients the asymptotic stability coincides with affiliation of all roots of a characteristic polynomial to the open unit disk in the complex plane. There exist several valuable criteria for checking this property, but these are suitable just for a computational verification for concrete given values of equation's (polynomial's) parameters. These criteria are mostly based on the analysis of signs of certain determinant sequences (see [9] or [12]). In several particular cases the necessary and sufficient conditions for asymptotic stability were derived in a closed effective form, i.e., a few conditions should be verified instead of a huge number of computations depending on the order of difference equation in the case of algebraic criterion. The asymptotic stability conditions for (1.2) in necessary and sufficient manner are introduced in [6]. We recall them in Section 2 for our consideration purposes. In addition to the previous, we remark that there are several results introducing closed form of necessary and sufficient conditions for asymptotic stability of certain difference equations, which cover many numerical schemes intended for delay differential equations, e.g.,

$$\begin{aligned} y_{n+1} + \alpha y_n + \gamma y_{n-\ell} &= 0, \quad n = 0, 1, 2, \dots, \\ y_{n+1} + \alpha y_{n-m} + \gamma y_{n-\ell} &= 0, \quad n = 0, 1, 2, \dots, \\ y_{n+1} + \alpha y_n + \beta y_{n-\ell+1} + \gamma y_{n-\ell} &= 0, \quad n = 0, 1, 2, \dots, \\ y_{n+2} + \alpha y_n + \beta y_{n-\ell+2} + \gamma y_{n-\ell} &= 0, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $m, \ell \in \mathbb{N}$, $m < \ell$. The results can be found in [3, 4, 11] and [5], respectively.

The structure of the paper is as follows. In Section 2, we recall the necessary and sufficient asymptotic stability conditions for equations (1.1) and (1.2). In Section 3, we introduce the analysed numerical scheme, description and visualization of its stability regions and discussion of some unexpected situations arising at numerical computations. We conclude the paper by final remarks in Section 4.

2 Preliminaries

Any asymptotic stability property of a numerical scheme is usually connected with asymptotic stability properties of a certain test differential equation. In this paper, we are going to analyse asymptotic stability regions (i.e., the sets of pairs $(a,b)\in\mathbb{R}^2$ such that the studied discretization is asymptotically stable considering fixed stepsize) of numerical scheme applied to delay differential equation (1.1). Therefore we recall the necessary and sufficient conditions for asymptotic stability of (1.1) itself introduced in [1] and [7].

Theorem 1. *Any solution of equation* (1.1) *is asymptotically stable if and only if one of the following two conditions holds:*

$$a \le b < -a \quad \text{for any} \quad \tau > 0; \tag{2.1}$$

$$|a| + b < 0 \quad \text{for} \quad \tau < \frac{\arccos(-a/b)}{(b^2-a^2)^{1/2}}\,. \tag{2.2}$$

As we can see, the first condition is valid for any positive delay τ. We call such case as delay independent asymptotic stability region, which is depicted in Figure 1 as S_{DI}. Condition (2.2) contains a restriction on delay τ. This condition forms a dependent stability region S_{DD} (see Figure 1). The greater value of τ, the closer the most right point of S_{DD} to the origin of the plane (a,b) is.

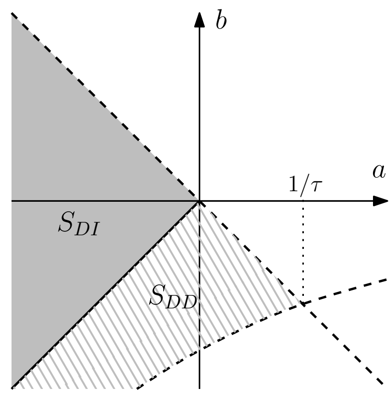

Figure 1. Asymptotic stability region of (1.1): delay independent (S_{DI}) and delay dependent (S_{DD}) case.

Next, we recall the necessary and sufficient conditions for asymptotic stability of difference equation (1.2) introduced in [6]:

Theorem 2. *Let α,γ be arbitrary reals such that $\alpha\gamma \ne 0$.*

(i) *Let ℓ be even and $\gamma(-\alpha)^{\ell/2+1} < 0$. Then* (1.2) *is asymptotically stable if and only if*

$$|\alpha| + |\gamma| < 1. \tag{2.3}$$

(ii) *Let ℓ be even and $\gamma(-\alpha)^{\ell/2+1} > 0$. Then* (1.2) *is asymptotically stable if and only if either*

$$|\alpha| + |\gamma| \le 1, \tag{2.4}$$

or

$$\big||\alpha| - |\gamma|\big| < 1 < |\alpha| + |\gamma|, \quad \ell < 2\arccos\frac{\alpha^2+\gamma^2-1}{2|\alpha\gamma|}\Big/\arccos\frac{\alpha^2-\gamma^2+1}{2|\alpha|}$$

holds.

(iii) *Let ℓ be odd and $\alpha < 0$. Then* (1.2) *is asymptotically stable if and only if* (2.3) *holds.*

(iv) *Let ℓ be odd and $\alpha > 0$. Then* (1.2) *is asymptotically stable if and only if either* (2.4)*, or*

$$\gamma^2 < 1 - \alpha < |\gamma|, \quad \ell < 2\arcsin\frac{1-\alpha^2-\gamma^2}{2|\alpha\gamma|}\Big/\arccos\frac{\alpha^2-\gamma^2+1}{2|\alpha|}$$

holds.

Actually, equivalent description of asymptotic stability regions can be found, e.g., in [10] and [13], where another proving procedures naturally lead to another form of the conditions. In the first mentioned paper, the boundary of stability region was described by straight lines and parametric curves, while in the second one the conditions contained an auxiliary nonlinear equation, which should be solved for certain choice of differential equation parameters α, γ, ℓ.

A comparison of conditions for asymptotic stability for (1.1) (see Theorem 1) and conditions for its discrete counterpart (1.2) in Theorem 2 leads us to a conclusion that such asymptotic stability analysis is much more complicated in the case of difference equation.

3 Numerical discretization and its properties

We consider an equidistant mesh with stepsize h satisfying the property $\tau = kh$ with $k \in \mathbb{N}$, $k > 2$. We denote the nodal points of the mesh as $t_n = nh$, $n = 0, 1, 2, \ldots$.

By integration of both sides of (1.1) from t_n to t_{n+2} we obtain

$$y(t_{n+2}) = y(t_n) + \int_{t_n}^{t_{n+2}} ay(s)\,\mathrm{d}s + \int_{t_n}^{t_{n+2}} by(s-\tau)\,\mathrm{d}s. \tag{3.1}$$

Numerical scheme we obtain by applying trapezoidal rule and midpoint rule to the integrals in (3.1), respectively. Denoting by y_n the approximation of value $y(t_n)$, we have

$$y_{n+2} = y_n + ah(y_n + y_{n+2}) + 2bhy_{n-k+1}. \tag{3.2}$$

The obtained formula is a $(k+1)$-step numerical method. We emphasize that there is no need of interpolation dealing with delayed term due to the appropriate stepsize $h = \tau/k$ and integration of (1.1) over two steps. Since we are going to utilize Theorem 2, we rewrite (3.2) in the form of linear difference equation

$$y_{n+2} - \frac{1+ah}{1-ah} y_n - \frac{2bh}{1-ah}\, y_{n-k+1} = 0, \quad n = 0, 1, \ldots, \tag{3.3}$$

where the stepsize h satisfies $ah \neq 1$.

3.1 Asymptotic stability conditions

Now we state the necessary and sufficient conditions for asymptotic stability of (3.3). The analysis of (3.3) falls naturally into two parts according to the parity of k. For an effective and clear formulation of the main result we introduce the symbols

$$\tau_1^*(h) = h + 2h \arcsin \frac{a + b^2 h}{(1+ah)|b|} \Big/ \arccos \frac{1 + a^2h^2 - 2b^2h^2}{a^2h^2 - 1}\,,$$

$$\tau_2^*(h) = h + 2h \arccos \frac{a + b^2 h}{|(1+ah)b|} \Big/ \arccos \frac{1 + a^2h^2 - 2b^2h^2}{|a^2h^2 - 1|}\,,$$

which are utilized in these two parts, respectively.

Theorem 3. *The asymptotic stability conditions for* (3.3) *are formulated below in two cases, considering k even and k odd, respectively.*

1. *Let $k \geq 2$ be even. Then* (3.3) *is asymptotically stable if and only if one of the following conditions holds:*

$$|bh| \leq 1, \quad |b| + a < 0, \tag{3.4}$$

$$2 < 2b^2h^2 < 1 - ah, \quad \tau < \tau_1^*(h). \tag{3.5}$$

2. *Let $k \geq 3$ be odd and $m = (k-1)/2$. Then (3.3) is asymptotically stable if and only if one of the following conditions holds:*

$$a \leq b < -a, \quad |bh| < 1, \tag{3.6}$$

$$|b| + a < 0, \quad (-1)^m bh = 1, \tag{3.7}$$

$$b + |a| < 0, \quad bh > -1, \quad \tau < \tau_2^*(h), \tag{3.8}$$

$$(-1)^m b + a < 0, \quad (-1)^m bh > 1, \quad \tau < \tau_2^*(h), \tag{3.9}$$

$$(-1)^m b + a > 0, \quad (-1)^{m+1} bh > 1, \quad \tau < \tau_2^*(h). \tag{3.10}$$

Proof. The necessary and sufficient conditions stated above follow from the application of Theorem 2 to (3.3). Considering

$$\alpha = -\frac{1+ah}{1-ah}, \quad \gamma = -\frac{2bh}{1-ah}, \quad \ell = k-1,$$

the difference equation (3.3) turns into (1.2). The complete proof (with detailed analysis) can be found in [8]. □

The above asymptotic stability conditions define in the plane (a,b) the asymptotic stability regions. Analogously to the continuous counterpart, the delay independent ((3.4), (3.6), (3.7)) and delay dependent ((3.5), (3.8)–(3.10)) stability regions can be distinguished. Figures 2 and 3 illustrate these stability region in the case of k even and odd, respectively. Moreover, in the case of k odd a position of delay dependent stability regions (in figures hatched ones) depends also on a parity of $m = (k-1)/2$. We emphasize that in the case k even the delay dependent part for $b < 0$, $b < a$ is missing.

The next part illustrates by numerical examples consequences of stability regions location diversity with respect to the change of k.

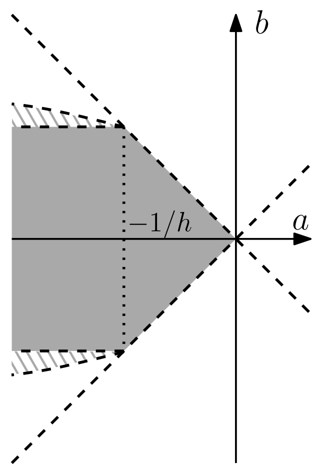

Figure 2. Asymptotic stability region for k even

3.2 Asymptotic stability discussion

Numerical solutions of delay differential equations can have some unexpected properties with respect to one's experience with numerical solving of ordinary differential equations. Several numerical phenomena are introduced in [2]. One of them is related to the following discussion:

We consider the initial value problem for (1.1) with $\tau = 1$

$$y'(t) = ay(t) + by(t-1), \quad t > 0, \tag{3.11}$$

$$y(t) = 1 \ \text{ for } \ t \in [-1, 0] \tag{3.12}$$

and we decide to use formula (3.3) to obtain numerical solution.

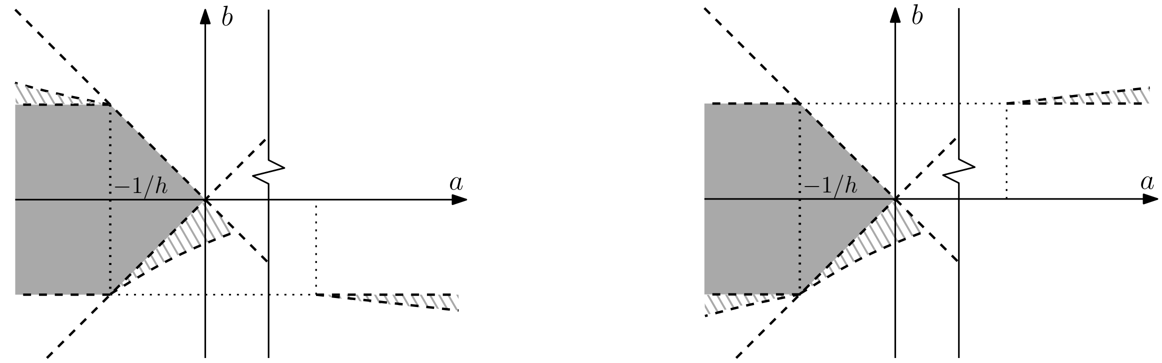

Figure 3. Asymptotic stability region for k odd and m even; k odd and m odd

Example 4. First we point the attention to the situation arising by the choice of $a = 30$, $b = -51/10$. As we can see from Theorem 1 (and as well as from Figure 1), the solution cannot be asymptotically stable. On the contrary, numerical solution with $k = 5$, i.e., $h = 0.2$, evinces asymptotically stable behaviour (see Figure 4) and the numerical formula really is asymptotically stable according to Theorem 3. Moreover, numerical solutions for any integer $k \geq 2$, $k \neq 5$, do not have this property. This extraordinary case $k = 5$ of asymptotic stable solution for given $(a, b) = (30, -51/10)$ is the only occurrence of (a, b) in delay dependent stability region within the fourth quadrant (see Figure 3(1)).

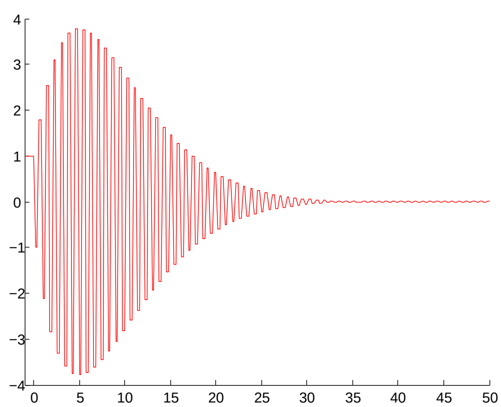

Figure 4. $a = 30$, $b = -51/10$, $k = 5$

Example 5. We consider $a = -1$, $b = -3/2$. The solution of (3.11), (3.12) is asymptotically stable in accordance with Theorem 1 (see Figure 1). The numerical solution for $k = 50$, $k = 51$ and $k = 52$, $k = 53$ is depicted on Figures 5 and 6, respectively.

As we can see, for this fixed pair of (a, b) there occurs switching of asymptotically stable (k even) and unstable (k odd) solutions for several values of k in sequence. This can be explained by Figures 3(1) and 3(2): (a, b) is included in a delay dependent stability region and (a, b) is not included in a delay dependent stability region, by rotation.

Finally, we discuss a limit form of Theorem 3 considering $h \to 0$. In the case of even k, the asymptotic stability region of (3.3) becomes $|b| + a < 0$. It corresponds to (2.1) with the exception of the boundary. In the case of odd k, the asymptotic stability conditions turn into (2.1), (2.2) letting $h \to 0$. These conditions are equivalent to the ones defining the asymptotic stability region of (1.1).

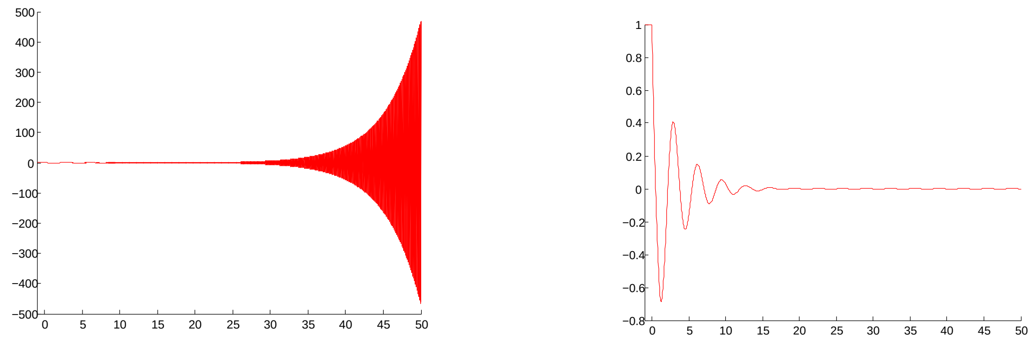

Figure 5. $a = -1$, $b = -3/2$, $k = 50$; $a = -1$, $b = -3/2$, $k = 51$

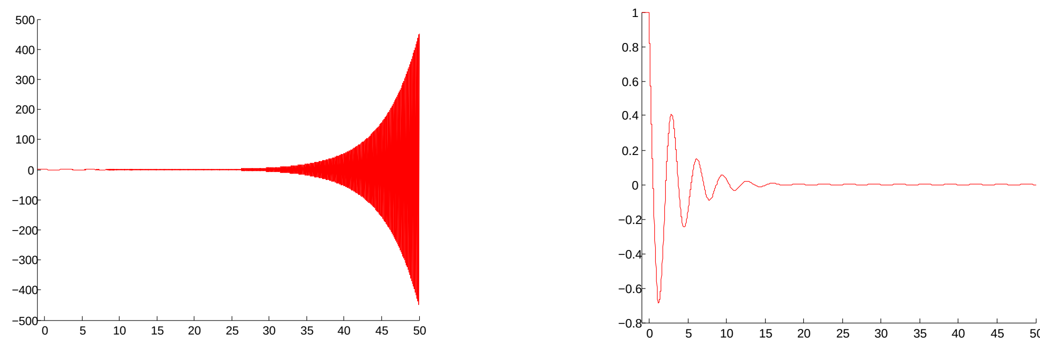

Figure 6. $a = -1$, $b = -3/2$, $k = 52$; $a = -1$, $b = -3/2$, $k = 53$

4 Conclusions and remarks

To summarize the previous, Theorem 3 describes the asymptotic stability regions of difference equation (3.3). This equation actually represents a discretization of delay differential equation (1.1) by modified midpoint rule. It was shown that the asymptotic stability regions depend not only on the value of stepsize h, but also on parity of k. We had provided the discussion with two examples, where specific situations occurred with respect to the position of delay dependent asymptotic stability regions. Deeper analysis for more complicated numerical methods is a great call because of the absence of effective form of the appropriate necessary and sufficient conditions for asymptotic stability.

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Author's address:

Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, 616 69 Brno, Czech Republic.

E-mail: tomasek@fme.vutbr.cz