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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF ONE CLASS OF n-th ORDER DIFFERENTIAL EQUATIONS

Abstract. We obtain the existence conditions and asymptotic representations of a certain class of power-mode solutions of a binomial non-autonomous n-th order ordinary differential equation with regularly varying nonlinearities and their derivatives of order up to $n-1$.

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რეზიუმე. n-ური რიგის ბინომიალური არაავტონომიური ჩვეულებრივი დიფერენციალური განტოლებებისათვის რეგულარულად ცვლადი არაწრფივობებით დადგენილია ამონახსნთა ერთი კლასის არსებობის პირობები და ნაპოვნია მათი ასიმპტოტური წარმოდგენები.

1 Introduction

Consider the differential equation

$$y^{(n)} = \alpha p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1.1}$$

where $n \geq 2$, $\alpha \in \{-1, 1\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, $\varphi_j : \Delta Y_j \to]0, +\infty[$ are the continuous functions regularly varying, as $y^{(j)} \to Y_j$, of order σ_j, $j = \overline{0, n-1}$, ΔY_j is a one-sided neighborhood of the point Y_j, $Y_j \in \{0, \pm\infty\}$[1].

Equation (1.1) is a particular case of the equation

$$y^{(n)} = \sum_{k=1}^{m} \alpha_k p_k(t) \prod_{j=0}^{n-1} \varphi_{kj}(y^{(j)}),$$

which is comprehensively studied by V. M. Evtukhov and A. M. Klopot [1, 2], M. M. Klopot [3, 4]. Here $n \geq 2$, $\alpha_k \in \{-1, 1\}$ $(k = \overline{1, m})$, $p_k : [a, \omega[\to]0, +\infty[$ $(k = \overline{1, m})$ are continuous functions, $-\infty < a < \omega \leq +\infty$, $\varphi_{kj} : \Delta Y_j \to]0, +\infty[$ $(k = \overline{1, m},\ j = \overline{0, n-1})$ are continuous functions regularly varying, as $y^{(j)} \to Y_j$, of order σ_j, ΔY_j is a one-sided neighborhood of the point Y_j, which is equal either to 0 or to $\pm\infty$.

From the above-mentioned results, the necessary and sufficient existence conditions of the so-called $\mathcal{P}_{+\infty}(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of equation (1.1) can be obtained for all λ_0 $(-\infty \leq \lambda_0 \leq +\infty)$. Moreover, asymptotic representations as $t \to +\infty$ of such solutions and their derivatives of order up to $n-1$ can be established.

It follows directly from the definition of these solutions that the conditions

$$\lim_{t \to +\infty} y^{(j)}(t) = Y_j \ \ (j = \overline{0, n-1}), \quad \lim_{t \to +\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0 \tag{1.2}$$

hold.

However, the set of monotonous solutions of equation (1.1), defined in some neighborhood of $+\infty$, can also have the solutions for each of which there exists a number $k \in \{1, \dots, n\}$ such that

$$y^{(n-k)}(t) = c + o(1) \ \ (c \neq 0) \ \text{ as } \ t \to +\infty. \tag{1.3}$$

When $k = 1, 2$, or the functions $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2})$ tend to the positive constants, as $y^{(i)} \to Y_i$, a question on the existence of solutions of type (1.3) of equation (1.1) can be resolved without any assumption like the last condition in (1.2). Otherwise, we will not be able to get asymptotic formulas of these solutions and their derivatives of order up to $n-1$ directly from equation (1.1).

Some results concerning the existence of solutions of type (1.3) have been obtained in Corollary 8.2 of the monograph by I. T. Kiguradze and T. A. Chanturiya [5, Ch. II, § 8, p. 207] for the equations of general type. But these results provide for a considerably strict restriction to the $(n-k+1)$-st derivative of a solution. In order to get new results with less strict restrictions to the behaviour of this and the subsequent derivatives of order $\leq n-1$ in case $k \in \{3, \dots, n\}$ and not all $\varphi_i(y^{(i)})$ $(i = \overline{n-k+1, n-2})$ tend to a positive constant, as $y^{(i)} \to Y_i$, we formulate the following definition.

Definition 1.1. A solution y of the differential equation (1.1) is called (for $k \in \{3, \dots, n\}$) a $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on the interval $[t_0, +\infty[\subset [a, +\infty[$ and satisfies the conditions

$$\lim_{t \to +\infty} y^{(n-k)}(t) = c \ \ (c \neq 0), \quad \lim_{t \to +\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n-2)}(t) y^{(n)}(t)} = \lambda_0. \tag{1.4}$$

It is obvious that by virtue of the first relation in (1.4), for these solutions the following representations

$$y^{(l-1)}(t) = \frac{ct^{n-l-k+1}}{(n-l-k+1)!} \, [1 + o(1)] \ \ (l = \overline{1, n-k}) \ \text{ as } \ t \to +\infty \tag{1.5}$$

[1]For $Y_j = \pm\infty$ here and in the sequel, all numbers in the neighborhood of ΔY_j are assumed to have constant sign.

hold, and $c \in \Delta Y_{n-k}$.

It readily follows from the form of equation (1.1) that $y^{(n)}(t)$ has a constant sign in some neighborhood of $+\infty$. Then $y^{(n-l)}(t)$ $(l = \overline{1, k-1})$ are strictly monotone functions in the neighborhood of $+\infty$ and, by virtue of (1.3), can tend only to zero, as $t \to +\infty$. Therefore, it is necessary that

$$Y_{j-1} = 0 \ \text{ for } \ j = \overline{n-k+2, n}. \tag{1.6}$$

Let us introduce the numbers μ_j $(j = \overline{0, n-1})$,

$$\mu_j = \begin{cases} 1 & \text{if } Y_j = +\infty, \text{ or } Y_j = 0 \text{ and } \Delta Y_j \text{ is a right neighborhood of the point } 0, \\ -1 & \text{if } Y_j = -\infty, \text{ or } Y_j = 0 \text{ and } \Delta Y_j \text{ is a left neighborhood of the point } 0, \end{cases}$$

and assume that they satisfy the following conditions:

$$\begin{aligned} &\mu_j \mu_{j+1} > 0 \ \text{ for } \ j = \overline{0, n-k-1}, \\ &\mu_j \mu_{j+1} < 0 \ \text{ for } \ j = \overline{n-k+1, n-2}, \end{aligned} \tag{1.7}$$

$$\alpha \mu_{n-1} < 0. \tag{1.8}$$

These conditions on μ_j $(j = \overline{0, n-1})$ and α are necessary for the existence of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1.1) as long as for each of them in some neighborhood of $+\infty$

$$\operatorname{sign} y^{(j)}(t) = \mu_j \ \ (j = \overline{0, n-1}), \ \ \operatorname{sign} y^{(n)}(t) = \alpha.$$

Besides, for such solutions it follows from (1.5) that

$$Y_{j-1} = \begin{cases} +\infty & \text{if } \ \mu_{n-k} > 0, \\ -\infty & \text{if } \ \mu_{n-k} < 0 \end{cases} \quad \text{for } \ j = \overline{1, n-k}. \tag{1.9}$$

The aim of the present paper is to obtain the necessary and sufficient existence conditions of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions $(k \in \{3, \dots, n\})$ of equation (1.1) for $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$, and to establish asymptotic, as $t \to +\infty$, formulas of their derivatives of order $\le n-1$. Moreover, a question on the quantity of the studied by us solutions will be solved.

It is significant to note that by virtue of the results obtained by V. M. Evtukhov [6], the solutions of equation (1.1) satisfy the following a priori asymptotic conditions.

Lemma 1.1. *Let $k \in \{3, \dots, n\}$ and $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$. Then for each $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution $y : [t_0, +\infty[\to \mathbb{R}$ of equation (1.1) the following asymptotic, as $t \to +\infty$, relations hold:*

$$y^{(l-1)}(t) \sim \frac{[(\lambda_0 - 1)t]^{n-l}}{\prod\limits_{i=l}^{n-1} [(n-i)\lambda_0 - (n-i-1)]} \, y^{(n-1)}(t) \ \ (l = \overline{n-k+2, n-1}). \tag{1.10}$$

2 Auxiliary notations and the main results

In equation (1.1), each of the functions φ_j $(j = \overline{0, n-1})$, being a regularly varying function of order σ_j, as $y^{(j)} \to Y_j$, can be represented (see [7, Ch. I, § 1, p. 10]) in the form

$$\varphi_j(y^{(j)}) = |y^{(j)}|^{\sigma_j} L_j(y^{(j)}) \ \ (j = \overline{0, n-1}), \tag{2.1}$$

where $L_j : \Delta Y_j \to]0, +\infty[$ $(j = \overline{0, n-1})$ is a slowly varying function, as $y^{(j)} \to Y_j$. According to the definition and properties of slowly varying functions,

$$\lim_{\substack{y^{(j)} \to Y_j \\ y^{(j)} \in \Delta Y_j}} \frac{L_j(\lambda y^{(j)})}{L_j(y^{(j)})} = 1 \ \text{ for each } \ \lambda > 0 \ \ (j = \overline{0, n-1}), \tag{2.2}$$

and these limit relations hold uniformly with respect to λ on an arbitrary interval $[c,d] \subset]0,+\infty[$. Moreover, by virtue of Theorem 1.2 (see [7, Ch. I, § 2, p. 10]), there exist continuously differentiable functions $L_{0j} : \Delta Y_j \to]0,+\infty[$ $(j=\overline{0,n-1})$, slowly varying as $y^{(j)} \to Y_j$, such that

$$\lim_{\substack{y^{(j)}\to Y_j \\ y^{(j)}\in\Delta Y_j}} \frac{L_j(y^{(j)})}{L_{0j}(y^{(j)})} = 1, \quad \lim_{\substack{y^{(j)}\to Y_j \\ y^{(j)}\in\Delta Y_j}} \frac{y^{(j)} L'_{0j}(y^{(j)})}{L_{0j}(y^{(j)})} = 0. \tag{2.3}$$

Examples of functions, slowly varying as $y \to Y_0$, are the functions

$$|\ln|y||^{\gamma_1}, \ \ \ln^{\gamma_2}|\ln|y||, \ \ \gamma_1,\gamma_2 \in \mathbb{R},$$
$$\exp\big(|\ln|y||^{\gamma_3}\big), \ \ 0<\gamma_3<1, \quad \exp\Big(\frac{\ln|y|}{\ln|\ln|y||}\Big),$$

as well as the functions that have a nonzero finite limit as $y \to Y_0$, and others.

We say that a continuous function $L : \Delta Y_0 \to]0,+\infty[$, slowly varying as $y \to Y_0$, satisfies the condition S_0 if

$$L(\mu e^{[1+o(1)]\ln|y|}) = L(y)[1+o(1)] \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta Y_0),$$

where $\mu = \operatorname{sign} y$.

The condition S_0 is necessarily satisfied for functions L that have a nonzero finite limit, as $y \to Y_0$, for functions of the form

$$L(y) = |\ln|y||^{\gamma_1}, \quad L(y) = |\ln|y||^{\gamma_1}\big|\ln|\ln|y||\big|^{\gamma_2},$$

where $\gamma_1,\gamma_2 \neq 0$, and for many others.

Remark 2.1. If a function $L : \Delta Y_0 \to]0,+\infty[$, slowly varying as $y \to Y_0$, satisfies the condition S_0, then for each function $l : \Delta Y_0 \to]0,+\infty[$, slowly varying as $y \to Y_0$, we have

$$L(yl(y)) = L(y)[1+o(1)] \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta Y_0).$$

Remark 2.2 (see [8])**.** If a function $L : \Delta Y_0 \to]0,+\infty[$, slowly varying as $y \to Y_0$, satisfies the condition S_0 and $y : [t_0,+\infty[\to \Delta Y_0$ is a continuously differentiable function such that

$$\lim_{t\to+\infty} y(t) = Y_0, \quad \frac{y'(t)}{y(t)} = \frac{\xi'(t)}{\xi(t)}\,[r+o(1)] \ \text{ as } \ t \to +\infty,$$

where r is a nonzero real constant, ξ is a real function, continuously differentiable in some neighborhood of $+\infty$ and such that $\xi'(t) \neq 0$, then

$$L(y(t)) = L(\mu|\xi(t)|^r)[1+o(1)] \ \text{ as } \ t \to +\infty,$$

where $\mu = \operatorname{sign} y(t)$ in some neighborhood of $+\infty$.

Remark 2.3 (see [2])**.** If a function $L : \Delta Y_0 \to]0,+\infty[$, slowly varying as $y \to Y_0$, satisfies the condition S_0 and a function $r : \Delta Y_0 \times K \to \mathbb{R}$, where K is compact in $\mathbb{R}^n$, is such that

$$\lim_{\substack{y\to\Delta Y_0 \\ y\in\Delta Y_0}} r(z,v) = 0 \ \text{ uniformly with respect to } \ v \in K,$$

then

$$\lim_{\substack{y\to\Delta Y_0 \\ y\in\Delta Y_0}} \frac{L(ve^{[1+r(z,v)]\ln|z|})}{L(z)} = 1 \ \text{ uniformly with respect to } \ v \in K,$$

where $v = \operatorname{sign} z$.

Besides these facts about the functions, regularly and slowly varying as $y^{(j)} \to Y_j$ $(j = \overline{0, n-1})$, we need the following auxiliary notations:

$$\gamma = 1 - \sum_{j=n-k+1}^{n-1} \sigma_j, \quad \nu = \sum_{j=n-k+1}^{n-2} \sigma_j (n-j-1), \quad a_{0j} = (n-j)\lambda_0 - (n-j-1) \ \ (j = \overline{1, n}),$$

$$C = \prod_{j=n-k+1}^{n-2} \left| \frac{(\lambda_0 - 1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} \right|^{\sigma_j}, ^2 \quad M(c) = \prod_{j=1}^{n-k} \left| \frac{c}{(n-j-k+1)!} \right|^{\sigma_{j-1}},$$

$$I(t) = \varphi_{n-k}(c) M(c) \int_A^t p(\tau) \tau^\nu \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1} \tau) \, d\tau,$$

where

$$A = \begin{cases} a_1 & \text{if } \int\limits_{a_1}^{+\infty} p(\tau) \tau^\nu \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1} \tau) \, d\tau = +\infty, \\ +\infty & \text{if } \int\limits_{a_1}^{+\infty} p(\tau) \tau^\nu \varphi_0(\mu_0 \tau^{n-k}) \cdots \varphi_{n-k-1}(\mu_{n-k-1} \tau) \, d\tau < +\infty, \end{cases}$$

$a_1 \geq a$ such that $\mu_{j-1} t^{n-k-j+1} \in \Delta Y_{j-1}$ $(j = \overline{1, n-k})$ for $t \geq a_1$.

The following assertions hold for equation (1.1).

Theorem 2.1. *Let $\gamma \neq 0$, $k \in \{3, \dots, n\}$ and $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$. Then, for the existence of $\mathcal{P}_{+\infty}^k(\lambda_0)$-solutions of equation* (1.1)*, it is necessary that $c \in \Delta Y_{n-k}$ and along with* (1.6)–(1.9) *the conditions*

$$\lambda_0 < 1, \quad a_{0j+1} > 0 \ \ (j = \overline{n-k+1, n-2}), \tag{2.4}$$

$$\lim_{t \to +\infty} \frac{t I'(t)}{I(t)} = \frac{\gamma}{\lambda_0 - 1} \tag{2.5}$$

hold. Moreover, each solution of that kind admits along with (1.3) *and* (1.5) *the asymptotic representations* (1.10) *as $t \to +\infty$ and*

$$\frac{|y^{(n-1)}(t)|^\gamma}{\prod\limits_{j=n-k+1}^{n-1} L_j \Big(\frac{[(\lambda_0 - 1)t]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1} a_{0i}} y^{(n-1)}(t) \Big)} = \alpha \mu_{n-1} \gamma C I(t) [1 + o(1)]. \tag{2.6}$$

Here we have the asymptotic, as $t \to +\infty$, representations (1.10) and (2.6), written out implicitly. Let us define conditions under which asymptotic, as $t \to +\infty$, representations of $\mathcal{P}_{+\infty}^k(\lambda_0)$-solutions of equation (1.1) and their derivatives of order $\leq n-1$ can be written out in explicit form.

Theorem 2.2. *Let $\gamma \neq 0$, $k \in \{3, \dots, n\}$, $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$ and the functions L_j $(j = \overline{n-k+1, n-1})$, slowly varying as $y^{(j)} \to Y_j$, satisfy the condition S_0. Then, in case of the existence of $\mathcal{P}_{+\infty}^k(\lambda_0)$-solutions of equation* (1.1)*, the following condition*

$$\int_{a_2}^{+\infty} \tau^{k-2} |I(\tau) \prod_{j=n-k+1}^{n-1} L_j(\mu_j \tau^{\frac{a_{0j+1}}{\lambda_0 - 1}})|^{\frac{1}{\gamma}} \, d\tau < +\infty \tag{2.7}$$

[2]Here and in the sequel, it is assumed that $\prod\limits_m^l = 1$ if $m > l$.

holds, where $a_2 \geq a_1$ *such that* $\mu_{j-1} t^{\frac{a_{0j}}{\lambda_0 - 1}} \in \Delta Y_{j-1}$ $(j = \overline{n-k+2, n})$ *for* $t \geq a_2$, *and each solution of that kind admits along with* (1.5) *the following asymptotic, as* $t \to +\infty$, *representations:*

$$y^{(n-k)}(t) = c + \frac{\mu_{n-1}(\lambda_0 - 1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1} a_{0i}} W(t)[1 + o(1)], \tag{2.8$_1$}$$

$$y^{(l-1)}(t) = \frac{\mu_{n-1}(\lambda_0 - 1)^{n-l} t^{n-l-k+2}}{\prod\limits_{i=l}^{n-1} a_{0i}} W'(t)[1 + o(1)] \quad (l = \overline{n-k+2, n-1}), \tag{2.8$_2$}$$

$$y^{(n-1)}(t) = \mu_{n-1} \frac{W'(t)}{t^{k-2}} [1 + o(1)], \tag{2.8$_3$}$$

where

$$W(t) = \int\limits_{+\infty}^{t} \tau^{k-2} \Big| \gamma C I(\tau) \prod_{j=n-k+1}^{n-1} L_j\Big(\mu_j \tau^{\frac{a_{0j+1}}{\lambda_0 - 1}}\Big) \Big|^{\frac{1}{\gamma}} d\tau.$$

Theorem 2.3. *Let* $\gamma \neq 0$, $k \in \{3, \dots, n\}$, $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}, 1\}$, $c \in \Delta Y_{n-k}$, *the conditions* (1.6)–(1.9), (2.4), (2.5), (2.7) *hold and the functions* L_j $(j = \overline{n-k+1, n-1})$, *slowly varying as* $y^{(j)} \to Y_j$, *satisfy the condition* S_0. *In addition, let the inequality* $\sigma_{n-1} \neq 1$ *hold and the algebraic relative to* ρ *equation*

$$\sum_{j=2}^{k-1} \frac{\sigma_{n-j}}{\lambda_0 - 1} \prod_{l=1}^{j-1} \frac{a_{0n-l}}{\lambda_0 - 1} \prod_{l=j}^{k-2} \Big(\rho + \frac{a_{0n-l}}{\lambda_0 - 1}\Big) = \Big(\rho - \frac{\sigma_{n-1} - 1}{\lambda_0 - 1}\Big) \prod_{l=1}^{k-2} \Big(\rho + \frac{a_{0n-l}}{\lambda_0 - 1}\Big) \tag{2.9}$$

have no roots with a zero real part. Then for $\lambda_0 \in \,]-\infty, \frac{k-2}{k-1}[\, \setminus \{0, \frac{1}{2}, \dots, \frac{k-3}{k-2}\}$ $(\lambda_0 \in [\frac{k-2}{k-1}, 1[)$, *equation* (1.1) *has a* $(n-k+m+1)$*-parameter* ($(n-k+m)$*-parameter, respectively*) *family of* $\mathcal{P}^k_{+\infty}(\lambda_0)$*-solutions that admit asymptotic, as* $t \to +\infty$, *representations* (1.5) *and* (2.8$_i$) $(i = 1, 2, 3)$, *where* m *is a number of roots* (*taking into account divisible*) *with a negative real part of the algebraic equation* (2.9).

Proof of Theorems 2.1–2.2. Let $y : [t_0, +\infty[\to \Delta Y_0$ be an arbitrary $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation (1.1). Then, as it has been proved before formulations of the theorems, $c \in \Delta Y_{n-k}$, the conditions (1.6)–(1.9) hold and the asymptotic relations (1.3) and (1.5) are true. It follows from (1.5) that

$$\frac{y^{(j+1)}(t)}{y^{(j)}(t)} = \frac{n-j-k}{t} [1 + o(1)] \quad (j = \overline{0, n-k-1}) \ \text{ as } \ t \to +\infty.$$

Now, by taking into account representations (2.1) of the functions $\varphi_j(y^{(j)})$ $(j = \overline{0, n-k-1})$, regularly varying as $t \to +\infty$, and the fact that relations (2.2) hold uniformly with respect to λ on an arbitrary interval $[d_1, d_2] \subset \,]0, +\infty[$, we have

$$\begin{aligned}
&\varphi_{j-1}\Big(\frac{c t^{n-j-k+1}}{(n-j-k+1)!} [1 + o(1)]\Big) \\
&\quad = \Big| \frac{c t^{n-j-k+1}}{(n-j-k+1)!} [1 + o(1)] \Big|^{\sigma_{j-1}} L_{j-1}\Big(\frac{c t^{n-j-k+1}}{(n-j-k+1)!} [1 + o(1)]\Big) \\
&\quad = \Big| \frac{c}{(n-j-k+1)!} \Big|^{\sigma_{j-1}} t^{n-j-k+1} L_{j-1}(\mu_{j-1} t^{n-j-k+1}) [1 + o(1)] \\
&\quad = \Big| \frac{c}{(n-j-k+1)!} \Big|^{\sigma_{j-1}} \varphi_{j-1}(\mu_{j-1} t^{n-j-k+1}) [1 + o(1)] \quad (j = \overline{1, n-k}) \ \text{ as } \ t \to +\infty.
\end{aligned}$$

Therefore, by virtue of (1.1), we obtain

$$\begin{aligned}
&\frac{y^{(n)}(t)}{\varphi_{n-1}(y^{(n-1)}(t)) \cdots \varphi_{n-k+1}(y^{(n-k+1)}(t))} \\
&\qquad = \alpha M(c) p(t) \varphi_0(\mu_0 t^{n-k}) \varphi_1(\mu_1 t^{n-k-1}) \cdots \varphi_{n-k}(c) [1 + o(1)] \ \text{ as } \ t \to +\infty.
\end{aligned} \tag{2.10}$$

It follows from the second relation in (1.4) that

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{1}{(\lambda_0-1)t}\,[1+o(1)] \;\text{ as }\; t\to+\infty. \tag{2.11}$$

Then, by virtue of (1.7), the first inequality in (2.4) is true, namely, $\lambda_0 < 1$.

Furthermore, Lemma 1.1 implies that the asymptotic relations (1.10) hold, and therefore

$$\frac{y^{(j+1)}(t)}{y^{(j)}(t)} = \frac{a_{0j+1}}{(\lambda_0-1)t}\,[1+o(1)] \;\; (j=\overline{n-k+1,n-2}) \;\text{ as }\; t\to+\infty. \tag{2.12}$$

Hence, by virtue of (1.7) and the first inequality in (2.4), the second one in (2.4) is true.

Taking into account (2.1) and (1.10), we rewrite (2.10) as

$$\frac{y^{(n)}(t)|y^{(n-1)}(t)|^{\gamma-1}}{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(t))} = \alpha M(c)Cp(t)t^{\nu}\varphi_{n-k}(c)\prod_{j=0}^{n-k-1}\varphi_j(\mu_j t^{n-k-j})[1+o(1)]. \tag{2.13}$$

Integrating this relation from t_0 to t if $A=a_1$ and from t to $+\infty$ if $A=+\infty$, we have

$$\begin{aligned}\int_B^t \frac{y^{(n)}(\tau)|y^{(n-1)}(\tau)|^{\gamma-1}}{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(\tau))}\,d\tau &= \alpha M(c)C\varphi_{n-k}(c)\int_B^t p(\tau)\tau^{\nu}\prod_{j=0}^{n-k-1}\varphi_j(\mu_j\tau^{n-k-j})[1+o(1)]\,d\tau \\ &= \alpha M(c)C\varphi_{n-k}(c)\int_A^t p(\tau)\tau^{\nu}\prod_{j=0}^{n-k-1}\varphi_j(\mu_j\tau^{n-k-j})\,d\tau\,[1+o(1)] \\ &= \alpha CI(t)[1+o(1)] \;\text{ as }\; t\to+\infty, \end{aligned} \tag{2.14}$$

where $B\in\{t_0,+\infty\}$.

Let us compare the integral occurring on the left-hand side with the expression $\frac{|y^{(n-1)}(t)|^{\gamma}}{\prod_{j=n-k+1}^{n-1} L_{0j}(y^{(j)}(t))}$.

Taking into account (2.3), the second condition in (1.4) and (2.11), by the l'Hospital rule in the Stolz form, we have

$$\begin{aligned}&\lim_{t\to+\infty}\frac{\dfrac{|y^{(n-1)}(t)|^{\gamma}}{\prod_{j=n-k+1}^{n-1} L_{0j}(y^{(j)}(t))}}{\int_B^t \frac{y^{(n)}(\tau)|y^{(n-1)}(\tau)|^{\gamma-1}}{\prod_{j=n-k+1}^{n-1} L_j(y^{(j)}(\tau))}\,d\tau} \\ &= \mu_{n-1}\lim_{t\to+\infty}\frac{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(t))}{\prod\limits_{j=n-k+1}^{n-1} L_{0j}(y^{(j)}(t))}\left[\gamma-\sum_{j=n-k+1}^{n-1}\Big(\frac{y^{(j)}(t)L'_{0j}(y^{(j)}(t))}{L_{0j}(y^{(j)}(t))}\,\frac{y^{(j+1)}(t)}{y^{(j)}(t)}\,\frac{y^{(n-1)}(t)}{y^{(n)}(t)}\Big)\right] \\ &= \mu_{n-1}\gamma.\end{aligned}$$

By virtue of this limit relation and (2.3), from (2.14) we obtain

$$\frac{|y^{(n-1)}(t)|^{\gamma}}{\prod\limits_{j=n-k+1}^{n-1} L_j(y^{(j)}(t))} = \alpha\mu_{n-1}\gamma CI(t)[1+o(1)] \;\text{ as }\; t\to+\infty.$$

Hence, taking into account (1.10) and the properties of regularly varying functions, we establish the asymptotic representations (2.6), as $t\to+\infty$. In addition, they, together with (2.13), imply that

$$\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{I'(t)}{\gamma I(t)}\,[1+o(1)] \;\text{ as }\; t\to+\infty,$$

and, by virtue of (2.11), the limit relation (2.5) holds. Thus assertions of Theorem 2.1 are true.

Let us additionally suppose that the functions L_j $(j=\overline{n-k+1,n-1})$, slowly varying as $t\to+\infty$, satisfy the condition S_0. Then, by virtue of (2.11) and (2.12), the assertions

$$\frac{y^{(j+1)}(t)}{y^{(j)}(t)}=\frac{1}{t}\left[\frac{a_{0j+1}}{\lambda_0-1}+o(1)\right]\ \text{ as }\ t\to+\infty\ \ (j=\overline{n-k+1,n-1})$$

hold, and therefore, by Remark 2.2 and the second inequality in (2.4), we have

$$L_j\left(\frac{[(\lambda_0-1)t]^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}y^{(n-1)}(t)\right)=L_j(\mu_j t^{\frac{a_{0j+1}}{\lambda_0-1}})[1+o(1)]\ \text{ as }\ t\to+\infty\ \ (j=\overline{n-k+1,n-1}).$$

It follows from the obtained relations and (2.6) that for $t\to+\infty$

$$y^{(n-1)}(t)=\mu_{n-1}\left|\gamma CI(t)\prod_{j=n-k+1}^{n-1}L_j\left(\mu_j t^{\frac{a_{0j+1}}{\lambda_0-1}}\right)\right|^{\frac{1}{\gamma}}[1+o(1)].$$

This, together with (1.10), implies that

$$\begin{aligned}y^{(l-1)}(t)=&\frac{\mu_{n-1}[(\lambda_0-1)t]^{n-l}}{\prod\limits_{i=l}^{n-1}a_{0i}}\\&\times\left|\gamma CI(t)\prod_{j=n-k+1}^{n-1}L_j\left(\mu_j t^{\frac{a_{0j+1}}{\lambda_0-1}}\right)\right|^{\frac{1}{\gamma}}[1+o(1)]\ \ (l=\overline{n-k+2,n-1})\ \text{ as }\ t\to+\infty.\end{aligned}$$

Integrating this relation for $l=n-k+2$ from t_* to t, where $t_*=\max\{a_2,t_0\}$, we have

$$\begin{aligned}y^{(n-k)}(t)=&\,y^{(n-k)}(t_*)\\&+\frac{\mu_{n-1}[(\lambda_0-1)]^{k-2}}{\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\int_{t_*}^{t}\tau^{k-2}\left|\gamma CI(\tau)\prod_{j=n-k+1}^{n-1}L_j\left(\mu_j\tau^{\frac{a_{0j+1}}{\lambda_0-1}}\right)\right|^{\frac{1}{\gamma}}[1+o(1)]\,d\tau.\end{aligned}$$

By virtue of the first condition in (1.4), we find that

$$\lim_{t\to+\infty}\int_{t_*}^{t}\tau^{k-2}\left|I(\tau)\prod_{j=n-k+1}^{n-1}L_j\left(\mu_j\tau^{\frac{a_{0j+1}}{\lambda_0-1}}\right)\right|^{\frac{1}{\gamma}}[1+o(1)]\,d\tau=const$$

and therefore, by the comparison criterion, the assertion (2.7) holds. Using Proposition 6 of the monograph [9, Ch. V, § 3, p. 293] on the asymptotic calculation of integrals, for the $(n-k)$-th derivative of a solution we get the representation form (2.8_1).

Consequently, the asymptotic relations (1.3), (1.10) and (2.6), as $t\to+\infty$, can be rewritten in the form (2.8_i) $(i=1,2,3)$. The proof of Theorems 2.1–2.2 is complete. □

Proof of Theorem 2.3. Let us show that, for this c from the hypothesis of the theorem, equation (1.1) has at least one $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution that is defined on some interval $[t_0,+\infty[\,\subset[a,+\infty[$ and admits the asymptotic representations (1.5) and (2.8_i) $(i=1,2,3)$, as $t\to+\infty$. Moreover, consider the problem on evaluating a number of such solutions. At the same time note that by virtue of the first inequality in (2.4), in case $\lambda_0>1$, the differential equation (1.1) does not have $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions.

Applying the transformation

$$
\begin{aligned}
&y^{(l-1)}(t)=\frac{ct^{n-l-k+1}}{(n-l-k+1)!}\,[1+v_l(t)]\quad (l=\overline{1,n-k}),\\
&y^{(n-k)}(t)=c+\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)[1+v_{n-k+1}(t)],\\
&y^{(l-1)}(t)=\frac{\mu_{n-1}(\lambda_0-1)^{n-l}t^{n-l-k+2}}{\prod\limits_{i=l}^{n-1}a_{0i}}\,W'(t)[1+v_l(t)]\quad (l=\overline{n-k+2,n-1}),\\
&y^{(n-1)}(t)=\mu_{n-1}\,\frac{W'(t)}{t^{k-2}}\,[1+v_n(t)],
\end{aligned}
\tag{2.15}
$$

to equation (1.1), we obtain the system of differential equations

$$
\begin{cases}
v_l'=\dfrac{n-l-k+1}{t}\,[-v_l+v_{l+1}]\quad (l=\overline{1,n-k-1}),\\[2mm]
v_{n-k}'=\dfrac{1}{t}\left[\dfrac{\mu_{n-1}(\lambda_0-1)^{k-2}}{c\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)[1+v_{n-k+1}]-v_{n-k}\right],\\[2mm]
v_{n-k+1}'=\dfrac{W'(t)}{W(t)}\big[-v_{n-k+1}+v_{n-k+2}\big],\\[2mm]
v_l'=\dfrac{1}{t}\,\dfrac{a_{0l}}{\lambda_0-1}\,[1+v_{l+1}]\\
\qquad\quad -\dfrac{1}{t}\,(n-l-k+2)[1+v_l]-\dfrac{W''(t)}{W'(t)}\,[1+v_l]\quad (l=\overline{n-k+2,n-1}),\\[2mm]
v_n'=\dfrac{1}{t}\bigg[\Big(-2+k-\dfrac{W''(t)t}{W'(t)}\Big)[1+v_n]\\
\qquad\quad +\dfrac{\alpha p(t)\varphi_0\big(\frac{ct^{n-k}}{(n-k)!}\,[1+v_1]\big)\cdots\varphi_{n-1}(\mu_{n-1}t^{2-k}W'(t)[1+v_n])}{\mu_{n-1}t^{1-k}W'(t)}\bigg].
\end{cases}
\tag{2.16}
$$

Consider the resulting system on the set $\Omega^n=[t_0,+\infty[\,\times\mathbb{R}^n_{\frac12}$, where $\mathbb{R}^n_{\frac12}=\{(v_1,\dots,v_n)\in\mathbb{R}^n:|v_j|\le\frac12\,,\ j=\overline{1,n}\}$ and $t_0\ge a_2$ is chosen, by virtue of (2.7), so that for $t>t_0$ and $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac12}$ the conditions hold:

$$
\begin{gathered}
\frac{ct^{n-j-k+1}}{(n-j-k+1)!}\,[1+v_j(t)]\in\Delta Y_{j-1}\quad (j=\overline{1,n-k}),\\
c+\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)[1+v_{n-k+1}(t)]\in\Delta Y_{n-k},\\
\frac{\mu_{n-1}(\lambda_0-1)^{n-j}t^{n-j-k+2}}{\prod\limits_{i=j}^{n-1}a_{0i}}\,W'(t)[1+v_j(t)]\in\Delta Y_{j-1}\quad (j=\overline{n-k+2,n-1}),\\
\mu_{n-1}\frac{W'(t)}{t^{k-2}}\,[1+v_n(t)]\in\Delta Y_{n-1}.
\end{gathered}
$$

As the functions $\varphi_j(y^{(j)})$ $(j\in\{0,\dots,n-1\}\setminus\{n-k\})$ are representable as (2.1) and the relations (2.2) hold uniformly with respect to λ on an arbitrary interval $[d_1,d_2]\subset\,]0,+\infty[$, and in addition, by virtue of the continuity of the function $\varphi_{n-k}(y^{(n-k)})$, (2.7) and the fact that the functions L_j

$(j=\overline{n-k+1,n-1})$, slowly varying as $t\to+\infty$, satisfy the condition S_0, we have

$$\begin{aligned}
&\varphi_j\Big(\frac{ct^{n-k-j}}{(n-k-j)!}\,[1+v_{j+1}]\Big)=\varphi_j\Big(\frac{ct^{n-k-j}}{(n-k-j)!}\Big)(1+v_{j+1})^{\sigma_j}\big(1+R_j(t,v_{j+1})\big)\\
&\qquad=\Big|\frac{c}{(n-k-j)!}\Big|^{\sigma_j}\varphi_j(\mu_j t^{n-k-j})(1+v_{j+1})^{\sigma_j}\big(1+R_j(t,v_{j+1})\big)\ \ (j=\overline{0,n-k-1}),\\
&\varphi_j\Bigg(\frac{\mu_{n-1}(\lambda_0-1)^{n-j-1}t^{n-j-k+1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\,W'(t)[1+v_{j+1}]\Bigg)\\
&\qquad=\Bigg|\frac{(\lambda_0-1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\Bigg|^{\sigma_j}\varphi_j\big(\mu_j t^{n-k-j+1}W'(t)\big)(1+v_{j+1})^{\sigma_j}\big(1+R_j(t,v_{j+1})\big)\\
&\qquad=\Bigg|\frac{(\lambda_0-1)^{n-j-1}}{\prod\limits_{i=j+1}^{n-1}a_{0i}}\Bigg|^{\sigma_j}\varphi_j(\mu_j t^{\frac{a_{0j+1}}{\lambda_0-1}})(1+v_{j+1})^{\sigma_j}\big(1+R_j(t,v_{j+1})\big)\ \ (j=\overline{n-k+1,n-2}),\\
&\varphi_{n-1}\big(\mu_{n-1}t^{2-k}W'(t)[1+v_n]\big)=\varphi_{n-1}(\mu_{n-1}t^{2-k}W'(t))(1+v_n)^{\sigma_{n-1}}\big(1+R_{n-1}(t,v_n)\big)\\
&\qquad=\varphi_{n-1}(\mu_{n-1}t^{\frac{1}{\lambda_0-1}})(1+v_n)^{\sigma_{n-1}}\big(1+R_{n-1}(t,v_n)\big),\\
&\varphi_{n-k}\Bigg(c+\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)[1+v_{n-k+1}(t)]\Bigg)=\varphi_{n-k}(c)\big(1+R_{n-k}(t,v_{n-k+1})\big),
\end{aligned}$$

where the functions $R_j(t,v_{j+1})$ $(j=\overline{0,n-1})$ tend to zero, as $t\to+\infty$ uniformly with respect to $v_{j+1}\in[-\frac12,\frac12]$.

It follows from the form of $W(t)$ and (2.7) that

$$\lim_{t\to+\infty}\frac{W'(t)t}{W(t)}=k-1+\frac{1}{\lambda_0-1},\qquad \lim_{t\to+\infty}\frac{W''(t)t}{W'(t)}=k-2+\frac{1}{\lambda_0-1},$$

and both of these limits are nonzero in case $\lambda_0\in]-\infty,1[\setminus\{0,\frac12,\dots,\frac{k-2}{k-1}\}$. Therefore, using the aforementioned representations and (2.5), the system of equations (2.16) can be rewritten in the form

$$\begin{cases}
v_l'=\dfrac{n-l-k+1}{t}\,[-v_l+v_{l+1}]\ \ (l=\overline{1,n-k-1}),\\[2mm]
v_{n-k}'=\dfrac1t\,\big[-v_{n-k}+Y_{n-k,1}(t,v_1,\dots,v_n)\big],\\[2mm]
v_l'=\dfrac1t\Big[-\dfrac{a_{0l}}{\lambda_0-1}\,v_l+\dfrac{a_{0l}}{\lambda_0-1}\,v_{l+1}+Y_{l,1}(t,v_1,\dots,v_n)\Big]\ \ (l=\overline{n-k+1,n-1}),\\[2mm]
v_n'=\dfrac1t\Big[\sum\limits_{j=1}^{n-k}\dfrac{\sigma_{j-1}}{\lambda_0-1}\,v_j+\sum\limits_{j=n-k+2}^{n-1}\dfrac{\sigma_{j-1}}{\lambda_0-1}\,v_j+\dfrac{\sigma_{n-1}-1}{\lambda_0-1}\,v_n+\sum\limits_{i=1}^{2}Y_{n,i}(t,v_1,\dots,v_n)\Big],
\end{cases}\tag{2.17}$$

where

$$\begin{aligned}
Y_{n-k,1}(t,v_1,\dots,v_n)&=\frac{\mu_{n-1}(\lambda_0-1)^{k-2}}{c\prod\limits_{i=n-k+2}^{n-1}a_{0i}}\,W(t)(1+v_{n-k+1}),\\
Y_{n-k+1,1}(t,v_1,\dots,v_n)&=\frac{W'(t)t}{W(t)}-k+1-\frac{1}{\lambda_0-1}\,,
\end{aligned}$$

$$Y_{l,1}(t,v_1,\dots,v_n)=\frac{W''(t)t}{W'(t)}-k+2-\frac{1}{\lambda_0-1}\ \ (l=\overline{n-k+2,n-1}),$$
$$Y_{n1}(t,v_1,\dots,v_n)=\frac{1}{\lambda_0-1}\Big(\prod_{j=0}^{n-1}\big(1+R_j(t,v_{j+1})\big)-1\Big)\prod_{\substack{j=1\\ j\neq n-k+1}}^{n}(1+v_j)^{\sigma_{j-1}}$$
$$+\Big(-2+k-\frac{W''(t)t}{W'(t)}+\frac{1}{\lambda_0-1}\Big)[1+v_n],$$
$$Y_{n2}(t,v_1,\dots,v_n)=\frac{1}{\lambda_0-1}\Big(\prod_{\substack{j=1\\ j\neq n-k+1}}^{n}(1+v_j)^{\sigma_{j-1}}-\prod_{\substack{j=1\\ j\neq n-k+1}}^{n}v_j\sigma_{j-1}-1\Big).$$

At the same time we note here that

$$\lim_{t\to+\infty}Y_{j,1}(t,v_1,\dots,v_n)=0\ \ (j=\overline{n-k,n})$$

uniformly with respect to $(v_1,\dots,v_n)\in\mathbb{R}^n_{\frac12}$, and

$$\lim_{|v_1|+\dots+|v_n|\to0}\frac{Y_{n,2}(t,v_1,\dots,v_n)}{|v_1|+\dots+|v_n|}=0$$

uniformly with respect to $t\in[t_0,+\infty[$.

The characteristic equation of the matrix consisting of coefficients of $v_1,\dots,v_n$ in system (2.17),

$$\prod_{l=k}^{n-1}(\rho+(n-l))\Big(\rho+\frac{a_{0n-k+1}}{\lambda_0-1}\Big)$$
$$\times\Big[\sum_{j=2}^{k-1}\frac{\sigma_{n-j}}{\lambda_0-1}\prod_{l=1}^{j-1}\frac{a_{0n-l}}{\lambda_0-1}\prod_{l=j}^{k-2}\Big(\rho+\frac{a_{0n-l}}{\lambda_0-1}\Big)-\Big(\rho-\frac{\sigma_{n-1}-1}{\lambda_0-1}\Big)\prod_{l=1}^{k-2}\Big(\rho+\frac{a_{0n-l}}{\lambda_0-1}\Big)\Big]=0,$$

has a zero root if $\frac{a_{0n-k+1}}{\lambda_0-1}=0$ (in case $\lambda_0=\frac{k-2}{k-1}$), $n-k$ negative roots $\rho_l=-(n-l)$ $(l=\overline{k,n-1})$ and $k-1$ roots of the algebraic equation (2.9), among which there are no any roots (according to the hypothesis of the theorem) with a zero real part.

Consequently, we get the system of differential equations that for $\lambda_0\in]-\infty,1[\setminus\{0,\frac12,\dots,\frac{k-2}{k-1}\}$ satisfies all assumptions of Theorem 2.2 in [10]. This theorem implies that the system (2.17) has at least one solution $(v_j)_{j=1}^n:[t_1,+\infty[\to\mathbb{R}^n_{\frac12}$ $(t_1\in[t_0,+\infty[)$ that tends to zero as $t\to+\infty$. By virtue of the transformation (2.15), each solution of this kind corresponds to a $\mathcal{P}^k_{+\infty}(\lambda_0)$-solution of equation (1.1) that admits the asymptotic representations (1.5) and (2.8$_i$) $(i=1,2,3)$ as $t\to+\infty$.

Moreover, in accordance with this theorem, if there are m (taking into account divisible) roots with a negative real part of the algebraic equation (2.9), then in case $\lambda_0\in]-\infty,\frac{k-2}{k-1}[\setminus\{0,\frac12,\dots,\frac{k-3}{k-2}\}$ $(\lambda_0\in]\frac{k-2}{k-1};1[)$ there exists an $(n-k+m+1)$-parameter ($(n-k+m)$-parameter, respectively) family of $\mathcal{P}^k_{+\infty}(\lambda_0)$-solutions of equation (1.1) with the found representations.

Consider now the case $\lambda_0=\frac{k-2}{k-1}$. Applying the change of variables

$$\begin{cases}v_j=z_j\ \ (j=\overline{1,n-k}),\\ v_{n-k+1}=z_n,\\ v_{j+1}=z_j\ \ (j=\overline{n-k+1,n-1}),\end{cases}\tag{2.18}$$

we reduce (2.16) to the system of differential equations

$$\begin{cases} z_l' = \dfrac{n-l-k+1}{t}\,[-z_l + z_{l+1}] \ \ (l = \overline{1, n-k-1}), \\ z_{n-k}' = \dfrac{1}{t}\big[-z_{n-k} + Z_{n-k,1}(t, z_1, \dots, z_n)\big], \\ z_l' = \dfrac{1-k}{t}\big[-a_{0l}z_l + a_{0l}z_{l+1} + Z_{l,1}(t, z_1, \dots, z_n)\big] \ \ (l = \overline{n-k+1, n-2}), \\ z_{n-1}' = \dfrac{1-k}{t}\Big[\displaystyle\sum_{j=1}^{n-k} \sigma_{j-1}z_j + \sum_{j=n-k+2}^{n-1} \sigma_{j-1}z_{j-1} \\ \qquad\qquad + (\sigma_{n-1} - 1)z_{n-1} + \displaystyle\sum_{i=1}^{2} Z_{n,i}(t, z_1, \dots, z_n)\Big], \\ z_n' = \dfrac{W'(t)}{W(t)}\,[-z_n + z_{n-k+1}], \end{cases} \tag{2.19}$$

where

$$Z_{j,m}(t, z_1, \dots, z_n) = Y_{j,m}(t, v_1, \dots, v_{n-k}, v_{n-k+2}, \dots, v_n, v_{n-k+1}) \ \ (m = 1, 2, \ \ j = \overline{n-k, n})$$

are such that

$$\lim_{t\to+\infty} Z_{j,1}(t, z_1, \dots, z_n) = 0$$

uniformly with respect to $(z_1, \dots, z_n) \in \mathbb{R}^n_{\frac{1}{2}}$, and

$$\lim_{|z_1|+\cdots+|z_n|\to 0} \frac{\partial Z_{n,2}(t, z_1, \dots, z_n)}{\partial z_k} = 0 \ \ (k = \overline{1, n})$$

uniformly with respect to $t \in [t_0, +\infty[$.

It follows from the form of $W(t)$ and (2.7) that $\lim\limits_{t\to+\infty} W(t) = 0$,

$$\lim_{t\to+\infty} \frac{W'(t)t}{W(t)} = 0, \quad \int_{t_0}^{+\infty} \frac{W'(t)dt}{W(t)} = \pm\infty \ \text{ and } \ \frac{W'(t)}{W(t)} < 0 \ \text{ as } \ t > t_0.$$

The characteristic equation of the matrix consisting of coefficients of $z_1, \dots, z_{n-1}$ (the coefficient of z_n differs from 0) in system (2.19),

$$\prod_{l=k}^{n-1}(\rho + (n-l))\Big[\sum_{j=2}^{k-1}(1-k)\sigma_{n-j}\prod_{l=1}^{j-1}((1-k)a_{0n-l})\prod_{l=j}^{k-2}(\rho + (1-k)a_{0n-l}) \\ - \big(\rho - (1-k)(\sigma_{n-1} - 1)\big)\prod_{l=1}^{k-2}(\rho + (1-k)a_{0n-l})\Big] = 0,$$

has $n-k$ negative roots $\rho_l = -(n-l)$ $(l = \overline{k, n-1})$ and $k-1$ roots of the algebraic equation (2.9), as $\lambda_0 = \frac{k-2}{k-1}$, among which there are no any roots (according to the hypothesis of the theorem) with a zero real part.

Consequently, system (2.19) satisfies all assumptions of Theorem 2.6 in [10]. Hence it has at least one solution $(z_j)_{j=1}^n : [t_1, +\infty[\to \mathbb{R}^n_{\frac{1}{2}}$ $(t_1 \in [t_0, +\infty[)$ that tends to zero as $t \to +\infty$. By virtue of transformations (2.15) and (2.18), each solution of this kind corresponds to the $\mathcal{P}^k_{+\infty}(\frac{k-2}{k-1})$-solution of equation (1.1) that admits asymptotic representations (1.5) and (2.8_i) $(i = 1, 2, 3)$ as $t \to +\infty$.

As $\rho_l = -(n-l)$ $(l = \overline{k, n-1})$ are negative roots, then, in accordance with this theorem, there certainly exists an $(n-k)$-parameter family of such solutions. Moreover, there exists an $(n-k+m)$-parameter family of solutions with the above found representations, where m is a number of roots (taking into account divisible) with a negative real part of the algebraic equation (2.9), as $\lambda_0 = \frac{k-2}{k-1}$. The proof of the theorem is complete. □

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