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EXISTENCE AND STABILITY RESULTS FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSES

Abstract. In this paper, we establish the existence and uniqueness of solutions for a class of boundary value problems for nonlinear implicit fractional differential equations with impulse and Caputo's fractional derivatives, the stability of this class of problems is considered, as well. The arguments are based upon the Banach contraction principle and the Schaefer's fixed point theorem. We present two examples to show the applicability of our results.

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1. Introduction

In this paper, we establish existence, uniqueness and stability results to the following boundary value problems (BVPs) for nonlinear implicit fractional differential equations with impulses

$$ {}^cD^{\alpha}_{t_k} y(t) = f\big(t, y, {}^cD^{\alpha}_{t_k} y(t)\big) \text{ for each } t \in (t_k, t_{k+1}],\ \ k = 0, \dots, m,\ \ 0 < \alpha \le 1, \tag{1} $$

$$ \Delta y\big|_{t=t_k} = I_k(y(t_k^-)),\ \ k = 1, \dots, m, \tag{2} $$

$$ ay(0) + by(T) = c, \tag{3} $$

where ${}^cD^{\alpha}_{t_k}$ is the Caputo's fractional derivative, $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $I_k : \mathbb{R} \to \mathbb{R}$ are given functions, and a, b, c are real constants with $a+b \neq 0$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y\big|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim\limits_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

In recent years, there has been a significant development in the theory of fractional differential equations. It is caused by its applications in the modeling of many phenomena in various fields of science and engineering such as acoustic, control theory, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, biosciences, bioengineering, etc. See, for example, [1, 6, 7, 15, 20, 27], and the references therein. On the other hand, impulsive differential equations have received much attention, we refer the reader to the books [2, 10, 16, 22, 24, 26], and the papers [13, 19, 29], and the references therein. Very recently, boundary value problems of fractional differential equations have received a considerable attention because they occur in the mathematical modeling of a variety of physical processes; see, for example, [3, 4, 8, 9, 14, 28, 31]. In [11, 12], the authors give some existence and uniqueness results for some classes of implicit fractional order differential equations. In [23], the authors consider the existence of multiple positive solutions of systems of nonlinear Caputo's fractional differential equations with general separated boundary conditions.

Motivated by the works mentioned above, in this paper we present some existence and uniqueness results for a class of boundary value problems for implicit fractional differential equations. The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminaries about fractional calculus and auxiliary results. In Section 3, two results for the problem (1)–(3) are presented: the first one is based on the Banach contraction principle, and the second one on Schaefer's fixed point theorem. In Section 4, we present Ulam–Hyers stability result for the problem (1)–(2). Finally, in the last Section, we give two examples to illustrate the applicability of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $T > 0$, $J = [0, T]$. By $C(J, \mathbb{R})$ we denote the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$ \|y\|_\infty = \sup\big\{|y(t)| :\ t \in J\big\}; $$

$L^1(J, \mathbb{R})$ is the space of Lebesgue-integrable functions $w : J \to \mathbb{R}$ with the norm

$$ \|w\|_1 = \int\limits_0^T |w(s)|\, ds, $$

$$ AC^n(J) = \big\{h : J \to \mathbb{R} : h, h', \dots h^{(n-1)} \in C(J, \mathbb{R}) \text{ and } h^{(n-1)} \text{ is absolutely continuous}\big\}. $$

In what follows, $\alpha > 0$. Consider the set of functions

$$ PC(J, \mathbb{R}) = \Big\{y :\ J \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \dots, m \\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+),\ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\Big\}. $$

$PC(J, \mathbb{R})$ is a Banach space with the norm

$$ \|y\|_{PC} = \sup_{t\in J} |y(t)|. $$

Let $J_0 = [t_0, t_1]$ and $J_k = (t_k, t_{k+1}]$ where $k = 1, \dots, m$.

Definition 2.1 ([21, 25])**.** The fractional (arbitrary) order integral of the function $h \in L^1([0,T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the Euler's gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$, $\alpha > 0$.

Definition 2.2 ([21, 25])**.** For a function $h \in AC^n(J)$, the Caputo's fractional-order derivative of order α is defined by

$$({}^cD_0^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3 ([21, 25])**.** *Let $\alpha \geq 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^cD_0^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k.$$

Lemma 2.4 ([21])**.** *Let $\alpha > 0$. Then the differential equation*

$${}^cD_0^\alpha k(t) = 0$$

has solutions $k(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Lemma 2.5 ([21])**.** *Let $\alpha > 0$. Then*

$$I^{\alpha c}D_0^\alpha k(t) = k(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

D. Bainov and S. Hristova [5] introduced the following integral inequality of Gronwall type for piecewise continuous functions which can be used in the sequel.

Lemma 2.6. *Let for $t \geq t_0 \geq 0$ the inequality*

$$x(t) \leq a(t) + \int_{t_0}^t g(t,s) x(s)\, ds + \sum_{t_0 < t_k < t} \beta_k(t) x(t_k)$$

holds, where $\beta_k(t)$ ($k \in \mathbb{N}$) are nondecreasing functions for $t \geq t_0$, $a \in PC([t_0, \infty), \mathbb{R}_+)$, a is nondecreasing and $g(t,s)$ is a continuous nonnegative function for $t, s \geq t_0$ and nondecreasing with respect to t for any fixed $s \geq t_0$. Then, for $t \geq t_0$, the following inequality is valid:

$$x(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k(t)) \exp\left(\int_{t_0}^t g(t,s)\, ds \right).$$

Definition 2.7. A function $y \in PC(J, \mathbb{R}) \cap AC(J_k)$ is said to be a solution of (1)–(3) if y satisfies the equation ${}^cD_{t_k}^\alpha y(t) = f(t, y(t), {}^cD_{t_k}^\alpha y(t))$ on J_k and the conditions

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m,$$
$$ay(0) + by(T) = c.$$

Here, we adopt the concepts from Wang *et al.* [30] and introduce Ulam's type stability concepts for the problem (1)–(3). Let $z \in PC(J,\mathbb{R})$, $\varepsilon > 0$, $\psi > 0$, and $\varphi \in PC(J,\mathbb{R}_+)$ be nondecreasing. We consider the set of inequalities

$$\begin{cases} \left|{}^cD^\alpha z(t) - f\big(t, z(t), {}^cD^\alpha z(t)\big)\right| \le \varepsilon, & t \in (t_k, t_{k+1}], \ \ k = 1, \dots, m, \\ \left|\Delta y(t_k) - I_k\big(y(t_k^-)\big)\right| \le \varepsilon, & k = 1, \dots, m, \end{cases} \tag{4}$$

$$\begin{cases} \left|{}^cD^\alpha z(t) - f\big(t, z(t), {}^cD^\alpha z(t)\big)\right| \le \varphi(t), & t \in (t_k, t_{k+1}], \ \ k = 1, \dots, m, \\ \left|\Delta y(t_k) - I_k\big(y(t_k^-)\big)\right| \le \psi, & k = 1, \dots, m, \end{cases} \tag{5}$$

and

$$\begin{cases} \left|{}^cD^\alpha z(t) - f\big(t, z(t), {}^cD^\alpha z(t)\big)\right| \le \varepsilon\varphi(t), & t \in (t_k, t_{k+1}], \ \ k = 1, \dots, m, \\ \left|\Delta y(t_k) - I_k\big(y(t_k^-)\big)\right| \le \varepsilon\psi, & k = 1, \dots, m. \end{cases} \tag{6}$$

Definition 2.8. The problem (1)–(3) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ of (4) there exists a solution y of the problem (1)–(3) with

$$|z(t) - y(t)| \le c_{f,m}\varepsilon, \ \ t \in J.$$

Definition 2.9. The problem (1)–(3) is generalized Ulam–Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}_+,\mathbb{R}_+)$, $\theta_{f,m}(0) = 0$ such that for each solution $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ of (4) there exists a solution y of the problem (1)–(3) with

$$|z(t) - y(t)| \le \theta_{f,m}(\varepsilon), \ \ t \in J.$$

Definition 2.10. The problem (1)–(3) is Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ of (6) there exists a solution y of the problem (1)–(3) with

$$|z(t) - y(t)| \le c_{f,m,\varphi}\varepsilon(\varphi(t) + \psi), \ \ t \in J.$$

Definition 2.11. The problem (1)–(3) is generalized Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each solution $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ of (5) there exists a solution y of the problem (1)–(3) with

$$|z(t) - y(t)| \le c_{f,m,\varphi}(\varphi(t) + \psi), \ \ t \in J.$$

Remark 2.12. It is clear that:

(i) Definition 2.8 implies Definition 2.9;

(ii) Definition 2.10 implies Definition 2.11;

(iii) Definition 2.10 for $\varphi(t) = \psi = 1$ implies Definition 2.8.

Remark 2.13. A function $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ is a solution of (6) if and only if there are $\sigma \in PC(J,\mathbb{R})$ and a sequence σ_k, $k = 1, \dots, m$ (which depend on z), such that

(i) $|\sigma(t)| \le \varepsilon\varphi(t)$, $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, and $|\sigma_k| \le \varepsilon\psi$, $k = 1, \dots, m$;

(ii) ${}^cD^\alpha z(t) = f(t, z(t), {}^cD^\alpha z(t)) + \sigma(t)$, $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$;

(iii) $\Delta z(t_k) = I_k(z(t_k^-)) + \sigma_k$, $k = 1, \dots, m$.

One can have similar remarks for inequalities (4) and (5).

Theorem 2.14 ([18]) (Ascoli–Arzela theorem)**.** *Let $A \subset C(J,\mathbb{R})$. A is relatively compact (i.e., $\overline{A}$ is compact) if:*

1. *A is uniformly bounded, i.e., there exists $M > 0$ such that*

$$|f(x)| < M \ \text{ for every } \ f \in A \ \text{ and } \ x \in J.$$

2. *A is equicontinuous, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each x, $\overline{x} \in J$, $|x - \overline{x}| \le \delta$ implies $|f(x) - f(\overline{x})| \le \varepsilon$, for every $f \in A$.*

Theorem 2.15 ([17]) (The Banach fixed point theorem)**.** *Let C be a non-empty closed subset of a Banach space X. Then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 2.16 ([17]) (The Schaefer's fixed point theorem)**.** *Let X be a Banach space and $N : X \longrightarrow X$ be a completely continuous operator. If the set $\mathcal{E} = \{y \in X : \ y = \lambda N y$ for some $\lambda \in (0,1)\}$ is bounded, then N has fixed points.*

3. The Existence of Solutions

To prove the existence of solutions to (1)–(3), we need the following auxiliary Lemma.

Lemma 3.1. *Let $0 < \alpha \leq 1$ and let $\sigma : J \to \mathbb{R}$ be continuous. A function $y \in PC(J, \mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \dfrac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)\,ds + \dfrac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}\sigma(s)\,ds - c\right] \\ \qquad + \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)\,ds, \quad \text{if } t \in [0,t_1] \\ \dfrac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)\,ds + \dfrac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}\sigma(s)\,ds - c\right] \\ \qquad + \sum_{i=1}^{k} I_i(y(t_i^-)) + \dfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)\,ds \\ \qquad + \dfrac{1}{\Gamma(\alpha)}\int_{t_k}^t (t-s)^{\alpha-1}\sigma(s)\,ds, \quad \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{7}$$

where $k = 1, \dots, m$, if and only if $y \in PC(J, \mathbb{R}) \cap AC(J_k)$ is a solution of the fractional BVP

$$^cD^\alpha y(t) = \sigma(t), \quad t \in J_k, \tag{8}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{9}$$

$$ay(0) + by(T) = c. \tag{10}$$

Proof. Assume that y satisfies (8)–(10). If $t \in [0, t_1]$, then

$$^cD^\alpha y(t) = \sigma(t).$$

By Lemma 2.5

$$y(t) = c_0 + I^\alpha \sigma(t) = c_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)\,ds$$

for $c_0 \in \mathbb{R}$. If $t \in (t_1, t_2]$, then Lemma 2.5 implies

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)\,ds = \Delta y\big|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)\,ds \\ &= I_1(y(t_1^-)) + \left[c_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)\,ds\right] + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)\,ds \\ &= c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)\,ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 2.5 we get

$$y(t) = y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds = \Delta y\big|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds$$

$$= I_2(y(t_2^-)) + \left[c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)\,ds\right]$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds$$

$$= c_0 + \left[I_1(y(t_1^-)) + I_2(y(t_2^-))\right] + \left[\frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)\,ds\right]$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds.$$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$, where $k = 1, \ldots, m$, can be written as

$$y(t) = c_0 + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds.$$

Applying the boundary condition $ay(0) + by(T) = c$ we get

$$c = c_0(a+b) + b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)\,ds.$$

Then

$$c_0 = \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)\,ds - c\right].$$

Thus, if $t \in (t_k, t_{k+1}]$, where $k = 1, \ldots, m$, then

$$y(t) = \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)\,ds - c\right]$$

$$+ \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)\,ds.$$

Conversely, assume that y satisfies the impulsive fractional integral equation (7). If $t \in [0, t_1]$, then $ay(0) + by(T) = c$ and, using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha y(t) = \sigma(t) \ \text{ for each } \ t \in [0, t_1].$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$${}^cD^\alpha y(t) = \sigma(t) \ \text{ for each } \ t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \ \ k = 1, \ldots, m.$$

We are now in a position to state and prove our existence result for the problem (1)–(3) based on the Banach fixed point theorem. □

Theorem 3.2. *Assume*

$(H1)$ *the function* $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous;*

$(H2)$ *there exist constants* $K > 0$ *and* $0 < L < 1$ *such that*

$$\big|f(t,u,v) - f(t,\overline{u},\overline{v})\big| \leq K|u-\overline{u}| + L|v-\overline{v}|$$

for any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ *and* $t \in J$*;*

$(H3)$ *there exists a constant* $\widetilde{l} > 0$ *such that*

$$\big|I_k(u) - I_k(\overline{u})\big| \leq \widetilde{l}\,|u-\overline{u}|$$

for each $u, \overline{u} \in \mathbb{R}$ *and* $k = 1, \dots, m$.

If

$$\Big(\frac{|b|}{|a+b|} + 1\Big)\Big[m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\Big] < 1, \tag{11}$$

then there exists a unique solution for the BVP (1)–(3).

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the operator $N : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$\begin{aligned} N(y)(t) = \frac{-1}{a+b}\Bigg[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)\,ds + \frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)\,ds - c\Bigg] \\ + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \end{aligned} \tag{12}$$

where $g \in C(J,\mathbb{R})$ is such that

$$g(t) = f\big(t, y(t), g(t)\big).$$

Clearly, the fixed points of operator N are solutions of problem (1)–(3).

Let $u, w \in PC(J,\mathbb{R})$. Then for $t \in J$ we have

$$\begin{aligned} \big|N(u)(t) - N(w)(t)\big| \leq \frac{|b|}{|a+b|}\Bigg[\sum_{i=1}^{m}\big|I_i(u(t_i^-)) - I_i(w(t_i^-))\big| \\ + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|g(s)-h(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|g(s)-h(s)|\,ds\Bigg] \\ + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|g(s)-h(s)|\,ds \\ + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g(s)-h(s)|\,ds + \sum_{0<t_k<t}\big|I_k(u(t_k^-)) - I_k(w(t_k^-))\big|, \end{aligned}$$

where $g, h \in C(J,\mathbb{R})$ are such that

$$g(t) = f\big(t, u(t), g(t)\big)$$

and

$$h(t) = f\big(t, w(t), h(t)\big).$$

By $(H2)$, we have

$$|g(t)-h(t)| = \big|f(t,u(t),g(t)) - f(t,w(t),h(t))\big| \leq K|u(t)-w(t)| + L|g(t)-h(t)|.$$

Then

$$|g(t)-h(t)| \leq \frac{K}{1-L}\,|u(t)-w(t)|.$$

Therefore, for each $t \in J$,

$$\big|N(u)(t)-N(w)(t)\big| \le \frac{|b|}{|a+b|}\Bigg[\sum_{k=1}^{m}\widetilde{l}\,\big|u(t_k^-)-w(t_k^-)\big|$$
$$+\frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|u(s)-w(s)|\,ds+\frac{K}{(1-L)\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|u(s)-w(s)|\,ds\Bigg]$$
$$+\frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|u(s)-w(s)|\,ds$$
$$+\frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|u(s)-w(s)|\,ds+\sum_{k=1}^{m}\widetilde{l}\,\big|u(t_k^-)-w(t_k^-)\big|$$
$$\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big[m\widetilde{l}+\frac{mKT^{\alpha}}{(1-L)\Gamma(\alpha+1)}+\frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\Big]\|u-w\|_{PC}.$$

Thus
$$\|N(u)-N(w)\|_{PC}\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big[m\widetilde{l}+\frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\Big]\|u-w\|_{PC}.$$

By (11), the operator N is a contraction. Hence, by the Banach contraction principle, N has a unique fixed point which is a unique solution of the problem (1)–(3). □

Our second result is based on the Schaefer's fixed point theorem.

Theorem 3.3. *Assume that* $(H1)$, $(H2)$ *and the following conditions are fulfilled:*

$(H4)$ *there exist* $p,q,r\in C(J,\mathbb{R}_+)$ *with* $r^*=\sup_{t\in J}r(t)<1$ *such that*
$$|f(t,u,w)|\le p(t)+q(t)|u|+r(t)|w| \ \text{ for } \ t\in J \ \text{ and } \ u,w\in\mathbb{R};$$

$(H5)$ *the functions* $I_k:\mathbb{R}\to\mathbb{R}$ *are continuous and there exist constants* $M^*,N^*>0$ *such that*
$$|I_k(u)|\le M^*|u|+N^* \ \text{ for each } \ u\in\mathbb{R}, \ \ k=1,\dots,m.$$

If
$$\Big(\frac{|b|}{|a+b|}+1\Big)\Big(mM^*+\frac{(m+1)q^*T^{\alpha}}{(1-r^*)\Gamma(\alpha+1)}\Big)<1, \tag{13}$$
then the BVP (1)–(3) *has at least one solution on* J.

Proof. Let the operator N be defined by (12). We shall use the Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1: *N is continuous.* Let $\{u_n\}$ be a sequence such that $u_n\to u$ in $PC(J,\mathbb{R})$. Then for each $t\in J$

$$\big|N(u_n)(t)-N(u)(t)\big|\le\frac{|b|}{|a+b|}\Bigg[\sum_{i=1}^{m}\big|I_k(u_n(t_k^-))-I_k(u(t_k^-))\big|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|g_n(s)-g(s)|\,ds+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}|g_n(s)-g(s)|\,ds\Bigg]$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|g_n(s)-g(s)|\,ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g_n(s)-g(s)|\,ds+\sum_{0<t_k<t}\big|I_k(u_n(t_k^-))-I_k(u(t_k^-))\big|, \tag{14}$$

where $g_n,g\in C(J,\mathbb{R})$ are such that
$$g_n(t)=f\big(t,u_n(t),g_n(t)\big)$$

and

$$g(t) = f\big(t, u(t), g(t)\big).$$

By $(H2)$, we have

$$|g_n(t) - g(t)| = \big|f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))\big| \le K|u_n(t) - u(t)| + L|g_n(t) - g(t)|.$$

Then

$$|g_n(t) - g(t)| \le \frac{K}{1-L}\,|u_n(t) - u(t)|.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \le \eta$ and $|g(t)| \le \eta$. Then we have

$$(t-s)^{\alpha-1}|g_n(s) - g(s)| \le (t-s)^{\alpha-1}\big[|g_n(s)| + |g(s)|\big] \le 2\eta(t-s)^{\alpha-1}$$

and

$$(t_k-s)^{\alpha-1}|g_n(s) - g(s)| \le (t_k-s)^{\alpha-1}\big[|g_n(s)| + |g(s)|\big] \le 2\eta(t_k-s)^{\alpha-1}.$$

For each $t \in J$, the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k-s)^{\alpha-1}$ are integrable on $[0,t]$, then the Lebesgue Dominated Convergence Theorem and (14) imply that

$$\big|N(u_n)(t) - N(u)(t)\big| \longrightarrow 0 \ \text{ as } \ n \to \infty$$

and hence

$$\big\|N(u_n) - N(u)\big\|_{PC} \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

Consequently, N is continuous.

Step 2: *F maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.* Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in PC(J,\mathbb{R}) : \|u\|_{PC} \le \eta^*\}$, $\|N(u)\|_{PC} \le \ell$. For each $t \in J$ we have

$$N(u)(t) = \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(u(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)\,ds + \frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)\,ds - c\right]$$

$$+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)\,ds + \sum_{0<t_k<t} I_k(u(t_k^-)), \qquad (15)$$

where $g \in C(J,\mathbb{R})$ is such that

$$g(t) = f\big(t, u(t), g(t)\big).$$

By $(H4)$, for each $t \in J$ we have

$$\begin{aligned}|g(t)| &= \big|f(t, u(t), g(t))\big| \le p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\le p(t) + q(t)\eta^* + r(t)|g(t)| \le p^* + q^*\eta^* + r^*|g(t)|,\end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$ and $q^* = \sup_{t\in J} q(t)$. Then

$$|g(t)| \le \frac{p^* + q^*\eta^*}{1-r^*} := M.$$

Thus (15) implies

$$\begin{aligned}|N(u)(t)| \le{}& \frac{|b|}{|a+b|}\left[m(M^*|u| + N^*) + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)}\right] \\ &+ \frac{|c|}{|a+b|} + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m\big(M^*|u| + N^*\big) \\ \le{}& \left(\frac{|b|}{|a+b|} + 1\right)\left[m(M^*|u| + N^*) + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)}\right] + \frac{|c|}{|a+b|}\,.\end{aligned}$$

Therefore

$$\|N(u)\|_{PC} \le \left(\frac{|b|}{|a+b|} + 1\right)\left[m(M^*\eta^* + N^*) + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)}\right] + \frac{|c|}{|a+b|} := \ell.$$

Step 3: *F maps bounded sets into equicontinuous sets of* $PC(J,\mathbb{R})$. Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$, B_{η^*} be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, and let $u\in B_{\eta^*}$. Then

$$\begin{aligned}\big|N(u)(\tau_2)-N(u)(\tau_1)\big| &\le \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\big|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\big|\,|g(s)|\,ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\big|(\tau_2-s)^{\alpha-1}\big|\,|g(s)|\,ds+\sum_{0<t_k<\tau_2-\tau_1}\big|I_k(u(t_k^-))\big|\\ &\le \frac{M}{\Gamma(\alpha+1)}\big[2(\tau_2-\tau_1)^\alpha+(\tau_2^\alpha-\tau_1^\alpha)\big]+(\tau_2-\tau_1)\big(M^*|u|+N^*\big)\\ &\le \frac{M}{\Gamma(\alpha+1)}\big[2(\tau_2-\tau_1)^\alpha+(\tau_2^\alpha-\tau_1^\alpha)\big]+(\tau_2-\tau_1)\big(M^*\eta^*+N^*\big).\end{aligned}$$

As $\tau_1\to\tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Ascoli–Arzela theorem, we can conclude that $N:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous.

Step 4: *A priori bounds.* Now it remains to show that the set

$$E=\big\{u\in PC(J,\mathbb{R}):\ u=\lambda N(u)\text{ for some }0<\lambda<1\big\}$$

is bounded. Let $u\in E$, then $u=\lambda N(u)$ for some $0<\lambda<1$. Thus for each $t\in J$

$$\begin{aligned}u(t)&=\frac{-1}{a+b}\left[b\lambda\sum_{i=1}^m I_i(u(t_i^-))+\frac{b\lambda}{\Gamma(\alpha)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)\,ds+\frac{b\lambda}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)\,ds-c\lambda\right]\\ &\quad+\frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)\,ds+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)\,ds+\lambda\sum_{0<t_k<t}I_k(u(t_k^-)).\end{aligned}\tag{16}$$

By $(H4)$, for each $t\in J$ we have

$$|g(t)|=\big|f(t,u(t),g(t))\big|\le p(t)+q(t)|u(t)|+r(t)|g(t)|\le p^*+q^*|u(t)|+r^*|g(t)|.$$

Thus

$$|g(t)|\le\frac{1}{1-r^*}\big(p^*+q^*|u(t)|\big)\le\frac{1}{1-r^*}\big(p^*+q^*\|u\|_{PC}\big).$$

This implies, by (16) and $(H5)$, that for each $t\in J$

$$\begin{aligned}|u(t)|&\le\frac{|b|}{|a+b|}\Big[m\big(M^*\|u\|_{PC}+N^*\big)+\frac{mT^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}+\frac{T^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}\Big]\\ &\quad+\frac{|c|}{|a+b|}+\frac{mT^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}+\frac{T^\alpha(p^*+q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}+m\big(M^*\|u(t)\|_{PC}+N^*\big).\end{aligned}$$

Then

$$\begin{aligned}\|u\|_{PC}&\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big[m\big(M^*\|u(t)\|_{PC}+N^*\big)+\frac{(m+1)(p^*+q^*\|u\|_{PC})T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big]+\frac{|c|}{|a+b|}\\ &\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big(mN^*+\frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big)\\ &\quad+\frac{|c|}{|a+b|}+\Big(\frac{|b|}{|a+b|}+1\Big)\Big(mM^*+\frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big)\|u\|_{PC}.\end{aligned}$$

Thus

$$\begin{aligned}&\Big[1-\Big(\frac{|b|}{|a+b|}+1\Big)\Big(mM^*+\frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big)\Big]\|u\|_{PC}\\ &\qquad\le\Big(\frac{|b|}{|a+b|}+1\Big)\Big[\frac{|c|}{|a+b|}+mN^*+\frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big].\end{aligned}$$

Finally, by (13), we obtain

$$\|u\|_{PC} \le \frac{(\frac{|b|}{|a+b|}+1)\big[mN^* + \frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{|c|}{|a+b|}\big]}{\big[1-(\frac{|b|}{|a+b|}+1)(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)})\big]} := R.$$

This shows that the set E is bounded. As a consequence of the Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (1)–(3). □

4. Ulam–Hyers Rassias Stability

Now, we state the following Ulam–Hyers–Rassias stable result.

Theorem 4.1. *Assume that* $(H1)$*–*$(H3)$*,* (11) *and the following condition are satisfied:*

$(H6)$ *there exists a nondecreasing function* $\varphi \in PC(J,\mathbb{R}_+)$ *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$*:*

$$I^\alpha \varphi(t) \le \lambda_\varphi \varphi(t).$$

Then the problem (1)*–*(2) *is Ulam–Hyers–Rassias stable with respect to* (φ,ψ)*.*

Proof. Let $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ be a solution of (6). Denote by y the unique solution of the BVP

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f\big(t, y(t), {}^cD^\alpha_{t_k} y(t)\big), & t \in (t_k, t_{k+1}], \ \ k = 1,\dots,m,\\ \Delta y(t_k) = I_k(y(t_k^-)), & k = 1,\dots,m,\\ ay(0) + by(T) = c, \\ y(0) = z(0). \end{cases}$$

Using Lemma 3.1, for each $t \in (t_k, t_{k+1}]$ we obtain

$$y(t) = y(0) + \sum_{i=1}^k I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s)\,ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)\,ds, \ \ t \in (t_k, t_{k+1}],$$

where $g \in C(J,\mathbb{R})$ is such that

$$g(t) = f(t, y(t), g(t)).$$

Since z is a solution of (6), by Remark 2.13, we have

$$\begin{cases} {}^cD^\alpha_{t_k} z(t) = f\big(t, z(t), {}^cD^\alpha_{t_k} z(t)\big) + \sigma(t), & t \in (t_k, t_{k+1}], \ \ k = 1,\dots,m,\\ \Delta z(t_k) = I_k(z(t_k^-)) + \sigma_k, & k = 1,\dots,m. \end{cases} \tag{17}$$

Clearly, the solution of (17) is given by

$$z(t) = z(0) + \sum_{i=1}^k I_i(z(t_i^-)) + \sum_{i=1}^k \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)\,ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s)\,ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \sigma(s)\,ds, \ \ t \in (t_k, t_{k+1}],$$

where $h \in C(J,\mathbb{R})$ is such that

$$h(t) = f\big(t, z(t), h(t)\big).$$

Hence for each $t \in (t_k, t_{k+1}]$ it follows that

$$
\begin{aligned}
|z(t)-y(t)| \leq & \sum_{i=1}^{k} |\sigma_i| + \sum_{i=1}^{k} \left| I_i(z(t_i^-)) - I_i(y(t_i^-)) \right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} |h(s)-g(s)|\, ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} |\sigma(s)|\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s)-g(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |\sigma(s)|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|z(t)-y(t)| \leq & \, m\varepsilon\psi + (m+1)\varepsilon\lambda_\varphi \varphi(t) + \sum_{i=1}^{k} \widetilde{l} \left| z(t_i^-) - y(t_i^-) \right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} |h(s)-g(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s)-g(s)|\, ds.
\end{aligned}
$$

By $(H2)$, we get

$$
|h(t)-g(t)| = \left| f\big(t, z(t), h(t)\big) - f\big(t, y(t), g(t)\big) \right| \leq K|z(t)-y(t)| + L|g(t)-h(t)|.
$$

Then

$$
|h(t)-g(t)| \leq \frac{K}{1-L}\, |z(t)-y(t)|.
$$

Therefore, for each $t \in J$,

$$
\begin{aligned}
|z(t)-y(t)| \leq & \, m\varepsilon\psi + (m+1)\varepsilon\lambda_\varphi \varphi(t) + \sum_{i=1}^{k} \widetilde{l} \left| z(t_i^-) - y(t_i^-) \right| \\
& + \frac{K}{(1-L)\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} |z(s)-y(s)|\, ds \\
& + \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |z(s)-y(s)|\, ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|z(t)-y(t)| \leq & \sum_{i=1}^{k} \widetilde{l} \left| z(t_i^-) - y(t_i^-) \right| + \varepsilon(\psi+\varphi(t))\big(m+(m+1)\lambda_\varphi\big) \\
& + \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} |z(s)-y(s)|\, ds.
\end{aligned}
$$

Applying Lemma 2.6, we get

$$
\begin{aligned}
|z(t)-y(t)| \leq & \, \varepsilon(\psi+\varphi(t))(m+(m+1)\lambda_\varphi) \\
& \times \left[\prod_{0<t_k<t} (1+\widetilde{l}) \exp\left(\int_{0}^{t} \frac{K(m+1)}{(1-L)\Gamma(\alpha)} (t-s)^{\alpha-1}\, ds \right) \right] \leq c_\varphi \varepsilon(\psi+\varphi(t)),
\end{aligned}
$$

where

$$c_\varphi = (m + (m+1)\lambda_\varphi)\Big[\prod_{k=1}^{m}(1+\widetilde{l})\exp\Big(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\Big)\Big]$$
$$= \big(m + (m+1)\lambda_\varphi\big)\Big[(1+\widetilde{l})\exp\Big(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\Big)\Big]^m.$$

Thus, the problem (1)–(2) is Ulam–Hyers–Rassias stable with respect to (φ, ψ). □

Next, we present the following Ulam–Hyers stability result.

Theorem 4.2. *Assume that* $(H1)$*–*$(H3)$ *and* (11) *are satisfied. Then the problem* (1)*–*(2) *is Ulam–Hyers stable.*

Proof. Let $z \in PC(J,\mathbb{R}) \cap AC(J_k)$ be a solution of (4). Denote by y the unique solution of the BVP

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f\big(t, y(t), {}^cD^\alpha_{t_k} y(t)\big), \quad t \in (t_k, t_{k+1}], \quad k = 1,\dots,m,\\ \Delta y(t_k) = I_k(y(t_k^-)), \quad k = 1,\dots,m,\\ ay(0) + by(T) = c,\\ y(0) = z(0). \end{cases}$$

Similarly as in the proof of Theorem 4.1 we get the inequality

$$|z(t) - y(t)| \le \sum_{i=1}^{k} \widetilde{l}\,\big|(z(t_i^-)) - (y(t_i^-))\big|$$
$$+ m\varepsilon + \frac{T^\alpha\varepsilon(m+1)}{\Gamma(\alpha+1)} + \frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|\,ds.$$

Applying Lemma 2.6, we obtain

$$|z(t) - y(t)| \le \varepsilon\Big(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)}\Big)$$
$$\times \Big[\prod_{0<t_k<t}(1+\widetilde{l})\exp\Big(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}\,ds\Big)\Big] \le c_\varphi\varepsilon,$$

where

$$c_\varphi = \Big(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\Big)\Big[\prod_{k=1}^{m}(1+\widetilde{l})\exp\Big(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\Big)\Big]$$
$$= \Big(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{\Gamma(\alpha+1)}\Big)\Big[(1+\widetilde{l})\exp\Big(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\Big)\Big]^m.$$

which completes the proof of the theorem. □

Moreover, if we set $\gamma(\varepsilon) = c\varepsilon$, $\gamma(0) = 0$, then the problem (1)–(2) is generalized Ulam–Hyers stable.

Remark 4.3. Our results for the boundary value problem (1)–(3) are appropriate for the following problems:

- Initial value problem: $a = 1$, $b = 0$, $c = 0$.
- Terminal value Problem: $a = 0$, $b = 1$, c is arbitrary.
- Anti-periodic problem: $a = 1$, $b = 1$, $c = 0$.

However, our results are not applicable for the periodic problem, i.e., for $a = 1$, $b = -1$, $c = 0$.

5. Examples

Example 1. Consider the following impulsive boundary value problem:

$$
{}^cD_{t_k}^{\frac{1}{2}}y(t)=\frac{1}{5e^{t+2}(1+|y(t)|+|{}^cD_{t_k}^{\frac{1}{2}}y(t)|)} \quad \text{for each } \ t\in J_0\cup J_1, \tag{18}
$$

$$
\Delta y\big|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2}^-)|}{10+|y(\frac{1}{2}^-)|}\,, \tag{19}
$$

$$
2y(0)-y(1)=3, \tag{20}
$$

where $J_0=[0,\frac{1}{2}]$, $J_1=(\frac{1}{2},1]$, $t_0=0$ and $t_1=\frac{1}{2}$. Set

$$
f(t,u,v)=\frac{1}{5e^{t+2}(1+|u|+|v|)}\,, \quad t\in[0,1], \quad u,v\in\mathbb{R}.
$$

Clearly, the function f is jointly continuous.

For each $u,v,\overline{u},\overline{v}\in\mathbb{R}$ and $t\in[0,1]$

$$
\big|f(t,u,v)-f(t,\overline{u},\overline{v})\big|\le\frac{1}{5e^2}\,\big(|u-\overline{u}|+|v-\overline{v}|\big).
$$

Hence the condition $(H2)$ is satisfied with $K=L=\frac{1}{5e^2}$.

Let

$$
I_1(u)=\frac{u}{10+u}\,, \quad u\in[0,\infty).
$$

Let $u,v\in[0,\infty)$. Then we have

$$
\big|I_1(u)-I_1(v)\big|=\Big|\frac{u}{10+u}-\frac{v}{10+v}\Big|=\frac{10|u-v|}{(10+u)(10+v)}\le\frac{1}{10}\,|u-v|.
$$

Thus the condition

$$
\Big(\frac{|b|}{|a+b|}+1\Big)\Big[m\widetilde{l}+\frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\Big]=2\Big[\frac{1}{10}+\frac{\frac{2}{5e^2}}{(1-\frac{1}{5e^2})\Gamma(\frac{3}{2})}\Big]=2\Big[\frac{4}{(5e^2-1)\sqrt{\pi}}+\frac{1}{10}\Big]<1
$$

is satisfied with $T=1$, $a=2$, $b=-1$, $c=3$, $m=1$ and $\widetilde{l}=\frac{1}{10}$. From Theorem 3.2 it follows that the problem (18)–(20) has a unique solution on J.

Set for any $t\in[0,1]$, $\varphi(t)=t$, $\psi=1$. Since

$$
I^{\frac{1}{2}}\varphi(t)=\frac{1}{\Gamma(\frac{1}{2})}\int\limits_0^t(t-s)^{\frac{1}{2}-1}s\,ds\le\frac{2t}{\sqrt{\pi}}\,,
$$

the condition $(H6)$ is satisfied with $\lambda_\varphi=\frac{2}{\sqrt{\pi}}$. From this it follows that the problem (18)–(19) is Ulam–Hyers–Rassias stable with respect to (φ,ψ).

Example 2. Consider the following impulsive anti-periodic problem:

$$
{}^cD_{t_k}^{\frac{1}{2}}y(t)=\frac{2+|y(t)|+|{}^cD_{t_k}^{\frac{1}{2}}y(t)|}{108e^{t+3}(1+|y(t)|+|{}^cD_{t_k}^{\frac{1}{2}}y(t)|)} \quad \text{for each } \ t\in J_0\cup J_1, \tag{21}
$$

$$
\Delta y\big|_{t=\frac{1}{3}} = \frac{|y(\frac{1}{3}^-)|}{6+|y(\frac{1}{3}^-)|}\,, \tag{22}
$$

$$
y(0)=-y(1), \tag{23}
$$

where $J_0=[0,\frac{1}{3}]$, $J_1=(\frac{1}{3},1]$, $t_0=0$, and $t_1=\frac{1}{3}$. Set

$$
f(t,u,v)=\frac{2+|u|+|v|}{108e^{t+3}(1+|u|+|v|)}\,, \quad t\in[0,1], \quad u,v\in\mathbb{R}.
$$

Clearly, the function f is jointly continuous. For any $u,v,\overline{u},\overline{v}\in\mathbb{R}$ and $t\in[0,1]$

$$
\big|f(t,u,v)-f(t,\overline{u},\overline{v})\big|\le\frac{1}{108e^3}\,\big(|u-\overline{u}|+|v-\overline{v}|\big).
$$

Hence the condition $(H2)$ is satisfied with $K = L = \frac{1}{108e^3}$. For each $t \in [0,1]$ we have

$$|f(t,u,v)| \leq \frac{1}{108e^{t+3}}\big(2 + |u| + |v|\big).$$

Thus the condition $(H4)$ is satisfied with $p(t) = \frac{1}{54e^{t+3}}$ and $q(t) = r(t) = \frac{1}{108e^{t+3}}$.

Let

$$I_1(u) = \frac{u}{6+u}, \quad u \in [0,\infty).$$

For each $u \in [0,\infty)$ we have

$$|I_1(u)| \leq \frac{1}{6}u + 1.$$

Thus the condition $(H5)$ is satisfied with $M^* = \frac{1}{6}$ and $N^* = 1$. Therefore the condition

$$\Big(\frac{|b|}{|a+b|} + 1\Big)\Big(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\Big) = \frac{3}{2}\Big(\frac{1}{6} + \frac{4}{(108e^3-1)\sqrt{\pi}}\Big) < 1,$$

is satisfied with $T = 1$, $a = 1$, $b = 1$, $c = 0$, $m = 1$ and $q^*(t) = r^*(t) = \frac{1}{108e^3}$. From Theorem 3.3 it follows that the problem (21)–(23) has at least one solution on J.

References

1. S. Abbas, M. Benchohra and G. M. N'Guérékata, Topics in fractional differential equations. *Developments in Mathematics*, 27. *Springer, New York*, 2012.
2. S. Abbas, M. Benchohra and G. M. N'Guérékata, Advanced fractional differential and integral equations. Mathematics Research Developments Series. *Nova Science Publishers, Inc., New York*, 2015.
3. R. P. Agarwal and B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. *Comput. Math. Appl.* **62** (2011), no. 3, 1200–1214.
4. B. Ahmad and J. J. Nieto, Existence of solutions for impulsive anti-periodic boundary value problems of fractional order. *Taiwanese J. Math.* **15** (2011), no. 3, 981–993.
5. D. D. Bainov and S. G. Hristova, Integral inequalities of Gronwall type for piecewise continuous functions. *J. Appl. Math. Stochastic Anal.* **10** (1997), no. 1, 89–94.
6. D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, Fractional calculus. Models and numerical methods. *Series on Complexity, Nonlinearity and Chaos* 3. *World Scientific, Hackensack, NJ*, 2012.
7. D. Baleanu, Z. B. Güvenç and J. A. Tenreiro Machado (eds.), New trends in nanotechnology and fractional calculus applications. *Selected papers based on the presentations at the workshop new trends in science and technology (NTST 08), and the workshop fractional differentiation and its applications (FDA 09), Ankara, Türkei, November* 2008. *Springer, Dordrecht*, 2010.
8. M. Benchohra, F. Berhoun, N. Hamidi and J. J. Nieto, Fractional differential inclusions with anti-periodic boundary conditions. *Nonlinear Anal. Forum* **19** (2014), 27–35.
9. M. Benchohra, N. Hamidi and J. Henderson, Fractional differential equations with anti-periodic boundary conditions. *Numer. Funct. Anal. Optim.* **34** (2013), no. 4, 404–414.
10. M. Benchohra, J. Henderson and S. Ntouyas, Impulsive differential equations and inclusions. Contemporary Mathematics and Its Applications, 2. *Hindawi Publishing Corporation, New York*, 2006.
11. M. Benchohra and J. E. Lazreg, Nonlinear fractional implicit differential equations. *Commun. Appl. Anal.* **17** (2013), no. 3-4, 471–482.
12. M. Benchohra and J. E. Lazreg, Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. *Rom. J. Math. Comput. Sci.* **4** (2014), no. 1, 60–72.
13. Y.-K. Chang, A. Anguraj and P. Karthikeyan, Existence results for initial value problems with integral condition for impulsive fractional differential equations. *J. Fract. Calc. Appl.* **2** (2012), no. 7, 1-–10.
14. A. Chen and Yi Chen, Existence of solutions to anti-periodic boundary value problem for nonlinear fractional differential equations with impulses. *Adv. Difference Equ.* **2011**, Art. ID 915689, 17 pp.
15. K. Diethelm, The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type. *Lecture Notes in Mathematics*, 2004. *Springer-Verlag, Berlin*, 2010.
16. J. R. Graef, J. Henderson and A. Ouahab, Impulsive differential inclusions. A fixed point approach. *De Gruyter Series in Nonlinear Analysis and Applications*, 20. *De Gruyter, Berlin*, 2013.
17. A. Granas and J. Dugundji, Fixed point theory. *Springer Monographs in Mathematics. Springer-Verlag, New York*, 2003.
18. J. K. Hale and S. M. Verduyn Lunel, Introduction to functional-differential equations. *Applied Mathematical Sciences*, 99. *Springer-Verlag, New York*, 1993.
19. J. Henderson and A. Ouahab, Impulsive differential inclusions with fractional order. *Comput. Math. Appl.* **59** (2010), no. 3, 1191–1226.
20. R. Hilfer (ed.), Applications of fractional calculus in physics. *World Scientific Publishing Co., Inc., River Edge, NJ*, 2000.

21. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations. *North-Holland Mathematics Studies*, 204. *Elsevier Science B.V., Amsterdam*, 2006.
22. V. Lakshmikantham, D. D. Baĭnov and P. S. Simeonov, Theory of impulsive differential equations. *Series in Modern Applied Mathematics*, 6. *World Scientific Publishing Co., Inc., Teaneck, NJ*, 1989.
23. K. Q. Lan and W. Lin, Positive solutions of systems of Caputo fractional differential equations. *Commun. Appl. Anal.* **17** (2013), no. 1, 61–85.
24. N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko and N. V. Skripnik, Differential equations with impulse effects. Multivalued right-hand sides with discontinuities. *de Gruyter Studies in Mathematics*, 40. *Walter de Gruyter & Co., Berlin*, 2011.
25. I. Podlubny, Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. *Mathematics in Science and Engineering*, 198. *Academic Press, Inc., San Diego, CA*, 1999.
26. A. M. Samoĭlenko and N. A. Perestyuk, Impulsive differential equations. *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, 14. *World Scientific Publishing Co., Inc., River Edge, NJ*, 1995.
27. V. E. Tarasov, Fractional dynamics. Applications of fractional calculus to dynamics of particles, fields and media. Nonlinear Physical Science. *Springer, Heidelberg; Higher Education Press, Beijing*, 2010.
28. G. Wang, B. Ahmad, and L. Zhang, Impulsive anti-periodic boundary value problem for nonlinear differential equations of fractional order. *Nonlinear Anal.* **74** (2011), no. 3, 792–804.
29. G. Wang, B. Ahmad and L. Zhang, Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions. *Comput. Math. Appl.* **62** (2011), no. 3, 1389–1397.
30. J. Wang, M. Fečkan and Y. Zhou, Ulam's type stability of impulsive ordinary differential equations. *J. Math. Anal. Appl.* **395** (2012), no. 1, 258–264.
31. L. Zhang and G. Wang, Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2011**, no. 7, 11 pp.

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