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Chengjun Guo, Ravi P. Agarwal,
Chengjiang Wang, and Donal O'Regan

THE EXISTENCE OF HOMOCLINIC ORBITS FOR A CLASS OF FIRST-ORDER SUPERQUADRATIC HAMILTONIAN SYSTEMS

Abstract. Using critical point theory, we study the existence of homoclinic orbits for the first-order superquadratic Hamiltonian system

$$\dot{z} = JH_z(t, z),$$

where $H(t, z)$ depends periodically on t and is superquadratic.

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რეზიუმე. პირველი რიგის სუპერკვადრატული ჰამილტონური სისტემისათვის

$$\dot{z} = JH_z(t, z),$$

სადაც $H(t, z)$ არის სუპერკვადრატული და t-ს მიმართ პერიოდული, კრიტიკული წერტილის თეორიის გამოყენებით, გამოკვლეულია ჰომოკლინიკური ორბიტების არსებობის საკითხი.

1. Introduction

This paper is devoted to the study of the existence of homoclinic orbits for the first-order time-dependent Hamiltonian system

$$\dot{z} = JH_z(t,z), \tag{1.1}$$

where $z = (p,q) \in \mathbf{R}^N \times \mathbf{R}^N$. Here H has the form

$$H(t,z) = \frac{1}{2}B(t)z \cdot z + G(t,z) + h(t)z, \tag{1.2}$$

where $G \in C(\mathbf{R}\times\mathbf{R}^{2N}, \mathbf{R})$ is T-periodic in t, $B(t)$ is a continuous T-periodic and symmetric $2N \times 2N$ matrix function, $h : \mathbf{R} \to \mathbf{R}^{2N}$ is a continuous and bounded function and J is the standard $2N \times 2N$ symplectic matrix

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}.$$

In recent years several authors studied homoclinic orbits for Hamiltonian systems via the critical point theory. For the second order Hamiltonian systems we refer the reader to [1, 2, 7, 8, 10–13] and for for the first order to [3–6, 9, 14–17] (and the references therein).

Throughout this paper, we always assume the following:

(H_1) $G(t,z) \geq 0$, for all $(t,z) \in \mathbf{R} \times \mathbf{R}^{2N}$;

(H_2) $G(t,z) = o(|z|^2)$ as $|z| \longrightarrow 0$ uniformly in t;

(H_3) $\frac{G(t,z)}{|z|^2} \longrightarrow +\infty$ as $|z| \longrightarrow +\infty$ uniformly in t;

(H_4) There exist constants $\beta > 1$, $1 < \lambda < 1 + \frac{\beta-1}{\beta}$, $a_1 > 0$, $a_2 > 0$ and $\tau \in L^1(\mathbf{R}, \mathbf{R}^+)$ such that

$$z \cdot G_z(t,z) - 2G(t,z) \geq a_1|z|^\beta - \tau(t), \quad (t,z) \in \mathbf{R} \times \mathbf{R}^{2N} \tag{1.3}$$

and

$$|G_z(t,z)| \leq a_2|z|^\lambda, \quad \forall\, (t,z) \in \mathbf{R} \times \mathbf{R}^{2N}; \tag{1.4}$$

(H_5) there exist constants $a_3 > 0$ and $\eta > 0$ such that

$$\int\limits_{\mathbf{R}} |h(t)|\, dt \leq a_3, \quad \Bigg(\int\limits_{\mathbf{R}} |h(t)|^2\, dt\Bigg)^{\frac{1}{2}} \leq \frac{\eta}{2\varrho},$$

$$\frac{2(\eta + \varrho\|\tau\|_{L^1})}{\varrho\xi} \leq 1, \quad a_2 < \min\left\{\frac{\xi}{2}, \frac{\xi}{2\varrho^{\lambda+1}}\right\},$$

where ϱ and ξ are two positive constants which will be defined in Proposition 3.1 and in (3.13) later.

A solution $z(t)$ of (1.1) is said to be homoclinic (to 0) if $z(t) \to 0$ as $t \to \pm\infty$. In addition, if $z(t) \not\equiv 0$, then $z(t)$ is called a nontrivial homoclinic solution.

Theorem 1.1. *Let* (H_1) $-$ (H_5) *be satisfied. Then* (1.1) *possesses a nontrivial homoclinic solution such that* $z(t) \to 0$ *as* $t \to \pm\infty$.

This paper is motivated by the work of Rabinowitz [12] in which the existence of nontrivial homoclinic solutions for the second order Hamiltonian system

$$\ddot{q} + V_q(t, q) = 0$$

was established.

The paper is organized as follows. In Section 2, we establish a variational structure for (1.1) with a periodic boundary value condition. Our main result (Theorem 1.1) will be proved in Section 3.

2. Variational Structure

Let $A = -(J(d/dt + B(t))$ be a self-adjoint operator acting on $L^2(\mathbf{R}, \mathbf{R}^{2N})$ with the domain $\widetilde{D}(A) = H^1(\mathbf{R}, \mathbf{R}^{2N})$. If $E := \widetilde{D}(|A|^{\frac{1}{2}})$, then E is a Hilbert space with the inner product

$$\langle z, v\rangle = (z, v)_{L^2} + \big(|A|^{\frac{1}{2}} z, |A|^{\frac{1}{2}} v\big)_{L^2}, \quad z, v \in E,$$

and $E = H^{\frac{1}{2}}(\mathbf{R}, \mathbf{R}^{2N})$. Let $E_k := H^{\frac{1}{2}}_{2kT}(\mathbf{R}, \mathbf{R}^{2N})$ for each $k \in \mathbf{N}$. Then E_k is a Hilbert space with the norm $\|\cdot\|_{E_k}$ given by (here $z \in E_k$)

$$\|z\|_{E_k} = \Bigg(\int_{-kT}^{kT} \big(\big||A|^{\frac{1}{2}} z\big|^2 + |z|^2\big)\, dt\Bigg)^{1/2}. \tag{2.1}$$

Furthermore, let $L^{\infty}_{2kT}(\mathbf{R}, \mathbf{R}^{2N})$ denote a space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbf{R}$ into $\mathbf{R}^{2N}$ equipped with the norm

$$\|z\|_{L^{\infty}_{2kT}} := \operatorname{ess\,sup}\big\{|z(t)| : \ t \in [-kT, kT]\big\}.$$

As in [10], a homoclinic solution of (1.1) will be obtained as a limit, as $k \to \pm\infty$, of a certain sequence of functions $z_k \in E_k$. We consider a sequence of systems of differential equations

$$\dot{z} = J(B(t)z + G_z(t, z) + h_k(t)), \tag{2.2}$$

where for each $k \in \mathbf{N}$, $h_k : \mathbf{R} \to \mathbf{R}^N$ is a $2kT$-periodic extension of the restriction of h to the interval $[-kT, kT]$ and z_k, a $2kT$-periodic solution of (2.1), will be obtained via a linking theorem.

We define

$$\langle Au, v\rangle = \int_{-kT}^{kT} \Big(-\Big(J\,\frac{d}{dt} + B\Big)u, v\Big)\, dt, \quad \forall\, u, v \in E_k \tag{2.3}$$

and

$$I_k(z) = \frac{1}{2}\,\langle Az, z\rangle - \int_{-kT}^{kT} G(t, z)\, dt - \int_{-kT}^{kT} h_k(t)\cdot z(t)\, dt. \tag{2.4}$$

We have from (2.3) that A has a sequence of eigenvalues

$$\cdots \xi_k^{(-m)} \le \cdots \le \xi_k^{(-2)} \le \xi_k^{(-1)} < 0 < \xi_k^{(1)} \le \xi_k^{(2)} \le \cdots \le \xi_k^{(m)} \cdots$$

with $\xi_k^{(m)} \longrightarrow \infty$ and $\xi_k^{(-m)} \longrightarrow -\infty$ as $m \longrightarrow \infty$. Let φ_k^j be the eigenvector of A corresponding to $\xi_k^{(j)}$, $j = \pm 1, \pm 2, \dots, \pm m, \dots$. Set

$$E_k^0 = ker(A), \quad E_k^- = \text{the negative eigenspace of } A$$

and

$$E_k^+ = \text{the positive eigenspace of } A.$$

Hence there exists an orthogonal decomposition $E_k = E_k^0 \oplus E_k^- \oplus E_k^+$ with $\dim(E_k^0) < \infty$.

Lemma 2.1 ([11]). *Let E be a real Hilbert space with $E = E^{(1)} \oplus E^{(2)}$ and $E^{(1)} = (E^{(2)})^\perp$. Suppose $I \in C^1(E, \mathbf{R})$ satisfies the $(\mathbf{PS})$ condition, and*

- (C_1) *$I(u) = \frac{1}{2}(Lu, u) + b(u)$, where $Lu = L_1P_1u + L_2P_2u$, $L_i : E^{(i)} \longmapsto E^{(i)}$ is bounded and self-adjoint, P_i is the projector of E onto $E^{(i)}$, $i = 1, 2$;*
- (C_2) *b' is compact;*
- (C_3) *there exist a subspace $\widetilde{E} \subset E$, the sets $S \subset E$, $Q \subset \widetilde{E}$ and constants $\widetilde{\alpha} > \omega$ such that*
  - (i) *$S \subset E^{(1)}$ and $I|_S \geq \widetilde{\alpha}$;*
  - (ii) *Q is bounded and $I|_{\partial Q} \leq \omega$;*
  - (iii) *S and ∂Q are linked .*

Then I possesses a critical value $c \geq \widetilde{\alpha}$ given by

$$c = \inf_{g \in \Gamma} \sup_{u \in Q} I(g(1, u)),$$

where

$$\Gamma \equiv \Big\{ g \in C([0,1] \times E, E) | g \text{ satisfies } (\Gamma_1) - (\Gamma_3) \Big\},$$

- (Γ_1) *$g(0, u) = u$;*
- (Γ_2) *$g(t, u) = u$ for $u \in \partial Q$;*
- (Γ_3) *$g(t, u) = e^{\theta(t,u)L}u + \chi(t, u)$, where $\theta(t, u) \in C([0,1] \times E, \mathbf{R})$, and χ is compact.*

3. Proof of the Main Result

The following result in [11, p. 36, Proposition 6.6] will be used.

Proposition 3.1. *There is a positive constant c_μ such that for each $k \in \mathbf{N}$ and $z \in E_k$ the following inequality holds:*

$$\|z\|_{L^\mu_{2kT}} \leq c_\mu \|z\|_{E_k}, \tag{3.1}$$

where $\mu \in [1, +\infty)$. For notational purposes let $c_{\lambda+1} = \varrho$.

Lemma 3.1. *Under the conditions of Theorem 1.1, I_k satisfies the $(\mathbf{PS})$ condition.*

Proof. Assume that $\{z_{k_n}\}_{n\in\mathbf{N}}$ in E_k is a sequence such that $\{I_k(z_{k_n})\}_{n\in\mathbf{N}}$ is bounded and $I_k'(z_{k_n}) \to 0$ as $n \to +\infty$. Then there exists a constant $d_1 > 0$ such that

$$|I_k(z_{k_n})| \le d_1, \quad I_k'(z_{k_n}) \to 0 \ \text{ as } \ n \to \infty. \tag{3.2}$$

We first prove that $\{z_{k_n}\}_{n\in\mathbf{N}}$ is bounded. Let $z_{k_n} = z_{k_n}^0 + z_{k_n}^+ + z_{k_n}^- \in E_k^0 \oplus E_k^+ \oplus E_k^-$. From (1.3) of (H_4), (H_5), (2.4) and (3.1), there exists a constant $\widetilde{c}_{\widehat{\beta}} > 0$ such that (here $\frac{1}{\widehat{\beta}} + \frac{1}{\beta} = 1$)

$$\begin{aligned}
2d_1 &\ge 2I_k(z_{k_n}) - \left\langle I_k'(z_{k_n}), z_{k_n} \right\rangle = \\
&= \int_{-kT}^{kT} \Big[z_{k_n} \cdot G_{z_{k_n}}(t, z_{k_n}) - 2G(t, z_{k_n}) \Big]\, dt - \int_{-kT}^{kT} h_k(t) \cdot z_{k_n}\, dt \ge \\
&\ge \int_{-kT}^{kT} a_1 |z_{k_n}|^\beta\, dt - \int_{-kT}^{kT} \tau_k(t)\, dt - \int_{-kT}^{kT} |h_k(t)|\, |z_{k_n}|\, dt \ge \\
&\ge a_1 \|z_{k_n}\|_{L_{2kT}^\beta}^\beta - \|\tau_k\|_{L_{2kT}^1} - c_\beta \|h_k\|_{L_{2kT}^{\widehat{\beta}}} \|z_{k_n}\|_{L_{2kT}^\beta} \ge \\
&\ge a_1 \|z_{k_n}\|_{L_{2kT}^\beta}^\beta - \|\tau\|_{L^1} - c_\beta \|h\|_{L^{\widehat{\beta}}} \|z_{k_n}\|_{L_{2kT}^\beta} \ge \\
&\ge a_1 \|z_{k_n}\|_{L_{2kT}^\beta}^\beta - \|\tau\|_{L^1} - c_\beta \widetilde{c}_{\widehat{\beta}} \|h\|_{L^1} \|z_{k_n}\|_{L_{2kT}^\beta} \ge \\
&\ge a_1 \|z_{k_n}\|_{L_{2kT}^\beta}^\beta - \|\tau\|_{L^1} - c_\beta \widetilde{c}_{\widehat{\beta}} a_3 \|z_{k_n}\|_{L_{2kT}^\beta},
\end{aligned} \tag{3.3}$$

where for each $k \in \mathbf{N}$, $\tau_k : \mathbf{R} \to \mathbf{R}^N$ is a $2kT$-periodic extension of the restriction of $\tau(t)$ to the interval $[-kT, kT]$.

Since $\beta > 1$, this implies that there exists a constant $\widetilde{M_0} > 0$ with

$$\|z_{k_n}\|_{L_{2kT}^\beta} \le \widetilde{M_0}. \tag{3.4}$$

Consider $\{\|z_{k_n}^0\|_{E_k}\}_{n\in\mathbf{N}}$. Note $\dim(E_k^0) < +\infty$, and this implies that there are the constants b_1 and b_2 such that

$$b_1 \|z_{k_n}^0\|_{L_{2kT}^\beta} \le \|z_{k_n}^0||_{E_k} \le b_2 \|z_{k_n}^0\|_{L_{2kT}^\beta} \le b_2 \|z_{k_n}\|_{L_{2kT}^\beta}. \tag{3.5}$$

By (3.4) and (3.5), there exists a constant $\widetilde{M_1} > 0$ such that

$$\|z_{k_n}^0||_{E_k} \le \widetilde{M_1}. \tag{3.6}$$

Let $\alpha = \frac{\beta - 1}{\beta(\lambda - 1)}$, then

$$\begin{cases} 1 < \lambda < 1 + \dfrac{\beta - 1}{\beta}, & 0 < \dfrac{(\lambda\alpha - 1)}{\alpha} < 1, \\ \lambda\alpha - 1 = \alpha - \dfrac{1}{\beta}, & \alpha > 1. \end{cases} \tag{3.7}$$

If $0 < \|z\|_{L^{\infty}_{2kT}} \le 1$ for $z \in E_k$, we have from (1.4) of (H_4) that

$$\int\limits_{-kT}^{kT} \big|G_z(t,z(t))\big|\,dt \le a_2 \int\limits_{-kT}^{kT} |z(t)|\,dt. \tag{3.8}$$

By using (3.1) and (3.8), we have (here $\frac{1}{\alpha}+\frac{1}{\sigma}=1$)

$$\begin{aligned}
&\|z^+_{k_n}\|_{E_k} \ge \big\langle I'_k(z_{k_n}), z^+_{k_n}\big\rangle = \\
&= \langle Az^+_{k_n}, z^+_{k_n}\rangle - \int\limits_{-kT}^{kT} \big[z^+_{k_n}\cdot G_{z_{k_n}}(t,z_{k_n})\big]\,dt - \int\limits_{-kT}^{kT} h_k(t)\cdot z^+_{k_n}\,dt = \\
&= \langle Az^+_{k_n}, z^+_{k_n}\rangle - \Bigg(\int\limits_{|z_{k_n}|\ge 1} + \int\limits_{|z_{k_n}|<1}\Bigg)\big[z^+_{k_n}\cdot G_{z_{k_n}}(t,z_{k_n})\big]\,dt - \int\limits_{-kT}^{kT} h_k(t)\cdot z^+_{k_n}\,dt \ge \\
&\ge \langle Az^+_{k_n}, z^+_{k_n}\rangle - \frac{\eta}{2\varrho}\|z_{k_n}\|_{E_k} - \int\limits_{|z_{k_n}|<1} a_2|z_{k_n}|\,|z^+_{k_n}|\,dt - \\
&- \Bigg(\int\limits_{|z_{k_n}|\ge 1} \big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\Bigg)^{\frac{1}{\alpha}} \Bigg(\int\limits_{-kT}^{kT} |z^+_{k_n}|^{\sigma}\,dt\Bigg)^{\frac{1}{\sigma}} \ge \\
&\ge \langle Az^+_{k_n}, z^+_{k_n}\rangle - \frac{\eta}{2\varrho}\|z_{k_n}\|_{E_k} - a_2\|z_{k_n}\|_{E_k}\|z^+_{k_n}\|_{E_k} - \\
&- \Bigg(\int\limits_{|z_{k_n}|\ge 1} \big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\Bigg)^{\frac{1}{\alpha}} c_\sigma \|z_{k_n}\|_{E_k}
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\|z^-_{k_n}\|_{E_k} \ge -\big\langle I'_k(z_{k_n}), z^-_{k_n}\big\rangle = \\
&= -\langle Az^-_{k_n}, z^-_{k_n}\rangle + \int\limits_{-kT}^{kT} \big[z^-_{k_n}\cdot G_{z_{k_n}}(t,z_{k_n})\big]\,dt - \int\limits_{-kT}^{kT} h_k(t)\cdot z^-_{k_n}\,dt = \\
&= -\langle Az^-_{k_n}, z^-_{k_n}\rangle - \Bigg(\int\limits_{|z_{k_n}|\ge 1} + \int\limits_{|z_{k_n}|<1}\Bigg)\big[z^-_{k_n}\cdot G_{z_{k_n}}(t,z_{k_n})\big]\,dt - \\
&- \int\limits_{-kT}^{kT} h_k(t)\cdot z^-_{k_n}\,dt \ge \\
&\ge -\langle Az^-_{k_n}, z^-_{k_n}\rangle - \frac{\eta}{2\varrho}\|z_{k_n}\|_{E_k} - \int\limits_{|z_{k_n}|<1} a_2|z_{k_n}|\,|z^-_{k_n}|\,dt -
\end{aligned}$$

$$-\Big(\int\limits_{|z_{k_n}|\geq 1}\big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\Big)^{\frac{1}{\alpha}}\Big(\int\limits_{-kT}^{kT}|z_{k_n}^{-}|^{\sigma}\,dt\Big)^{\frac{1}{\sigma}}\geq$$

$$\geq -\langle Az_{k_n}^{-},z_{k_n}^{-}\rangle-\frac{\eta}{2\varrho}\,\|z_{k_n}\|_{E_k}-a_2\|z_{k_n}\|_{E_k}\|z_{k_n}^{-}\|_{E_k}-$$

$$-\Big(\int\limits_{|z_{k_n}|\geq 1}\big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\Big)^{\frac{1}{\alpha}}c_{\sigma}\|z_{k_n}\|_{E_k}. \qquad (3.10)$$

By using (1.4) of (H_4) and (3.1), there exists a constant $c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}}>0$ such that

$$\int\limits_{|z_{k_n}|\geq 1}\big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\leq\int\limits_{|z_{k_n}|\geq 1}a_2^{\alpha}|z_{k_n}|^{\lambda\alpha}\,dt\leq$$

$$\leq a_2^{\alpha}\Big(\int\limits_{|z_{k_n}|\geq 1}|z_{k_n}|^{\beta}\,dt\Big)^{\frac{1}{\beta}}\Big(\int\limits_{|z_{k_n}|\geq 1}|z_{k_n}|^{(\lambda\alpha-1)\,\frac{\beta}{\beta-1}}\,dt\Big)^{1-\frac{1}{\beta}}\leq$$

$$\leq a_2^{\alpha}(c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}})^{\lambda\alpha-1}\Big(\int\limits_{|z_{k_n}|\geq 1}|z_{k_n}|^{\beta}\,dt\Big)^{\frac{1}{\beta}}\|z_{k_n}\|_{E_k}^{\lambda\alpha-1}. \qquad (3.11)$$

Combining (3.4) with (3.9)–(3.11), we find that

$$\|z_{k_n}^{+}\|_{E_k}+\|z_{k_n}^{-}\|_{E_k}\geq$$

$$\geq\langle Az_{k_n}^{+},z_{k_n}^{+}\rangle-\langle Az_{k_n}^{-},z_{k_n}^{-}\rangle-a_2\|z_{k_n}\|_{E_k}\big(\|z_{k_n}^{+}\|_{E_k}+\|z_{k_n}^{-}\|_{E_k}\big)-$$

$$-\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k}-c_{\sigma}\Big(\int\limits_{|z_{k_n}|\geq 1}\big|G_{z_{k_n}}(t,z_{k_n})\big|^{\alpha}\,dt\Big)^{\frac{1}{\alpha}}\big(\|z_{k_n}\|_{E_k}+\|z_{k_n}\|_{E_k}\big)\geq$$

$$\geq\xi_1\|z_{k_n}^{+}\|_{E_k}^{2}-\xi_{-1}\|z_{k_n}^{-}\|_{E_k}^{2}-\frac{\eta}{\varrho}\,\|z_{k_n}\|_{E_k}-2a_2\|z_{k_n}\|_{E_k}^{2}-$$

$$-2c_{\sigma}(a_2^{\alpha}\Big[\big(c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}}\big)^{\lambda\alpha-1}\Big(\int\limits_{|z_{k_n}|\geq 1}|z_{k_n}|^{\beta}\,dt\Big)^{\frac{1}{\beta}}\Big]^{\frac{1}{\alpha}}\|z_{k_n}\|_{E_k}^{\frac{(\lambda\alpha-1)}{\alpha}}\|z_{k_n}\|_{E_k}\geq$$

$$\geq\xi_1\|z_{k_n}^{+}\|_{E_k}^{2}-\xi_{-1}\|z_{k_n}^{-}\|_{E_k}^{2}-\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k}-$$

$$-2a_2\|z_{k_n}\|_{E_k}^{2}-2c_{\sigma}\widetilde{D}_0\big(\|z_{k_n}\|_{E_k}\big)^{\frac{(\lambda\alpha-1)+\alpha}{\alpha}}, \qquad (3.12)$$

where $\widetilde{D}_0=[a_2^{\alpha}((c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}})^{\lambda\alpha-1}\widetilde{M_0}]^{\frac{1}{\alpha}}$, and ξ_1 is the smallest positive eigenvalue, ξ_{-1} is the largest negative eigenvalue of the operator A, respectively. From (3.6) and (3.12), there exists a positive constant $\widetilde{D}_1>0$ such that

$$\widetilde{D}_1\Big(\|z_{k_n}^{+}\|_{E_k}+\|z_{k_n}^{-}\|_{E_k}+\|z_{k_n}^{0}\|_{E_k}\Big)\geq$$

$$\geq\|z_{k_n}^{+}\|_{E_k}+\|z_{k_n}^{-}\|_{E_k}+\xi\widetilde{M_1}\|z_{k_n}^{0}\|_{E_k}\geq\|z_{k_n}^{+}\|_{E_k}+\|z_{k_n}^{-}\|_{E_k}+\xi\|z_{k_n}^{0}\|_{E_k}^{2}\geq$$

$$\geq \xi_1\|z_{k_n}^+\|_{E_k}^2 - \xi_{-1}\|z_{k_n}^-\|_{E_k}^2 + \xi\|z_{k_n}^0\|_{E_k}^2 - $$
$$-\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k} - 2a_2\|z_{k_n}\|_{E_k}^2 - 2c_\sigma\widetilde{D}_0\big(\|z_{k_n}\|_{E_k}\big)^{\frac{(\lambda\alpha-1)+\alpha}{\alpha}} \geq$$
$$\geq \xi\Big(\|z_{k_n}^+\|_{E_k}^2 + \|z_{k_n}^-\|_{E_k}^2 + \|z_{k_n}^0\|_{E_k}^2\Big) -$$
$$-\frac{\eta}{\varrho}\|z_{k_n}\|_{E_k} - 2a_2\|z_{k_n}\|_{E_k}^2 - 2c_\sigma\widetilde{D}_0\big(\|z_{k_n}\|_{E_k}\big)^{\frac{(\lambda\alpha-1)+\alpha}{\alpha}}, \quad (3.13)$$

where $\xi = \min\{\xi_1, -\xi_{-1}\}$. This implies that

$$\widetilde{D}_1 + \frac{\eta}{\varrho} \geq (\xi - 2a_2)\|z_{k_n}\|_{E_k} - 2c_\sigma\widetilde{D}_0\big(\|z_{k_n}\|_{E_k}\big)^{\frac{(\lambda\alpha-1)}{\alpha}}, \quad (3.14)$$

where $0 < \frac{(\lambda\alpha-1)}{\alpha} < 1$. Since $\xi_1 - 2a_2 > 0$, we have that $\{\|z_{k_n}\|_{E_k}\}_{n\in\mathbf{N}}$ is bounded. Going if necessary to a subsequence, we can assume that there exists $z \in E_k$ such that $z_{k_n} \rightharpoonup z$, as $n \longrightarrow +\infty$, in E_k, which implies $z_{k_n} \longrightarrow z$ uniformly on $[-kT, kT]$. Hence $(I_k'(z_{k_n}) - I_k'(z))(z_{k_n} - z) \to 0$ and $\|z_{k_n} - z\|_{L^2_{[-kT,kT]}} \to 0$. Set

$$\Phi = \int_{-kT}^{kT} \Big(G_{z_{k_n}}(t, z_{k_n}(t)) - G_z(t, z(t)), z_{k_n} - z\Big)\, dt.$$

It is easy to check that $\Phi \to 0$ as $n \to +\infty$. Moreover, an easy computation shows that

$$\big(I_k'(z_{k_n}) - I_k'(z)\big)(z_{k_n} - z) = \big\langle A(z_{k_n} - z), (z_{k_n} - z)\big\rangle - \Phi.$$

This implies $\|z_{k_n} - z\|_{E_k} \to 0$. □

Lemma 3.2. *Under the conditions of Theorem* 1.1, *for every $k \in \mathbf{N}$ the system* (2.2) *possesses a $2kT$-periodic solution.*

Proof. The proof will be divided into three steps.

Step 1: Assume that $0 < \|z\|_{E_k} \leq 1$ for $z \in E_k^{(1)} = E_k^+$. From (1.3) of (H_4) and (3.1), we have

$$\int_{-kT}^{kT} G(t, z(t))\, dt \leq \frac{1}{2}\Bigg[\int_{-kT}^{kT} z \cdot G_z(t, z(t))\, dt + \int_{-kT}^{kT} \tau_k(t)\, dt\Bigg] \leq$$
$$\leq \frac{1}{2}\Bigg[a_2\int_{-kT}^{kT} |z(t)|^{\lambda+1}\, dt + \|\tau\|_{L^1}\Bigg] \leq \frac{1}{2}\Big[a_2\varrho^{\lambda+1}\|z\|_{E_k}^{\lambda+1} + \|\tau\|_{L^1}\Big] \leq$$
$$\leq \frac{1}{2}\Big[a_2\varrho^{\lambda+1}\|z\|_{E_k}^2 + \|\tau\|_{L^1}\Big]. \quad (3.15)$$

From (2.4) and (3.15), for $z \in E_k^{(1)} = E_k^+$ and $0 < \|z\|_{E_k} \leq 1$, we have

$$
\begin{aligned}
I_k(z) &= \frac{1}{2}\langle Az, z\rangle - \int_{-kT}^{kT} G(t,z)\,dt - \int_{-kT}^{kT} z\cdot h_k(t)\,dt \geq \\
&\geq \frac{\xi_1}{2}\|z\|_{E_k}^2 - \frac{1}{2}\Big[a_2\varrho^{\lambda+1}\|z\|_{E_k}^2 + \|\tau\|_{L^1}\Big] - \frac{\eta}{2\varrho}\|z\|_{E_k} \geq \\
&\geq \frac{1}{4}(\xi - 2a_2\varrho^{\lambda+1})\|z\|_{E_k}^2 + \frac{\xi}{4}\|z\|_{E_k}^2 - \frac{(\frac{\eta}{\varrho} + \|\tau\|_{L^1})}{2}\,. \qquad (3.16)
\end{aligned}
$$

Note from (H_5) that $\xi - 2a_2\varrho^{\lambda+1} > 0$. Set

$$\rho = \Big(\frac{2(\frac{\eta}{\varrho} + \|\tau\|_{L^1})}{\xi}\Big)^{\frac{1}{2}} \quad \text{and} \quad \widetilde{\alpha} = \frac{\xi - 2a_2\varrho^{\lambda+1}}{4}\,.$$

Let B_ρ denote the open ball in E_k with radius ρ about 0 and let ∂B_ρ denote its boundary. Let $S_k = \partial B_\rho \cap E_k^+$. If $z \in S_k$, then $\|z\|_{E_k} = \big(\frac{2(\frac{\eta}{\varrho}+\|\tau\|_{L^1})}{\xi}\big)^{\frac{1}{2}}$ (note that $||z||_{E_k} \leq 1$ from (H_5)) and thus (3.16) gives

$$I_k(z) \geq \widetilde{\alpha} \;\; z \in S_k.$$

Then $(C_3)(i)$ of Lemma 2.1 holds.

Step 2: Let $e \in E_k^+$ with $\|e\|_{E_k} = 1$ and $\widetilde{E}_k = E_k^- \oplus E_k^0 \oplus span\{e\}$. Let now

$$
\begin{aligned}
\Theta_k &= \big\{z \in \widetilde{E}_k : \; \|z\|_{\widetilde{E}_k} = 1\big\}, \\
\mu &= \inf_{z\in E_k^-, \|z^-\|_{E_k}=1} \big|\langle Az^-, z^-\rangle\big|, \quad \kappa = \Big(\frac{2\|A\|}{\mu}\Big)^{1/2}.
\end{aligned}
$$

For $z \in \Theta_k$, we write $z = z^- + z^0 + z^+$.

I) If $\|z^-\|_{E_k} > \kappa\|z^+ + z^0\|_{E_k}$, then for any $\gamma \geq \frac{2\eta(1+\kappa^2)}{\varrho\mu\kappa^2} > 0$, we have from (H_1) that

$$
\begin{aligned}
I_k(\gamma z) &= \frac{1}{2}\langle A\gamma z^-, \gamma z^-\rangle + \frac{1}{2}\langle A\gamma z^+, \gamma z^+\rangle - \\
&\quad - \int_{-kT}^{kT} G(t,\gamma z)\,dt - \int_{-kT}^{kT} \gamma z\cdot h_k(t)\,dt \leq \\
&\leq -\frac{\mu}{2}\gamma^2\|z^-\|_{E_k}^2 + \frac{\|A\|}{2}\gamma^2\|z^+\|_{E_k}^2 + \frac{\eta}{2\varrho}\gamma \leq \\
&\leq -\frac{\mu}{2}\gamma^2\|z^-\|_{E_k}^2 + \frac{\|A\|}{2}\gamma^2\|z^+ + z^0\|_{E_k}^2 + \frac{\eta}{2\varrho}\gamma \leq \\
&\leq -\frac{\mu}{2}\gamma^2\|z^-\|_{E_k}^2 + \frac{\|A\|}{2}\gamma^2\frac{1}{\kappa^2}\|z^-\|_{E_k}^2 + \frac{\eta}{2}\gamma = \\
&= -\frac{\mu}{4}\gamma^2\|z^-\|_{E_k}^2 + \frac{\eta}{2\varrho}\gamma \leq 0; \qquad (3.17)
\end{aligned}
$$

note $\|z^-\|^2_{E_k} > \frac{\kappa^2}{1+\kappa^2},$ since

$$1 = \|z^-\|^2_{E_k} + \|z^+ + z^0\|^2_{E_k} < \frac{(1+\kappa^2)}{\kappa^2}\|z^-\|^2_{E_k}.$$

Let

$$\Delta_k = \Big\{z \in \Theta_k :\ \|z^-\|_{E_k} \le \kappa\|z^+ + z^0\|_{E_k}\Big\}.$$

II) If $\|z^-\|_{E_k} \le \kappa\|z^+ + z^0\|_{E_k}$, we have

$$1 = \|z\|^2_{E_k} = \|z^-\|^2_{E_k} + \|z^+ + z^0\|^2_{E_k} \le (1+\kappa^2)\|z^+ + z^0\|^2_{E_k}, \tag{3.18}$$

i.e.,

$$\|z^+ + z^0\|^2_{E_k} \ge \frac{1}{(1+\kappa^2)} > 0. \tag{3.19}$$

The argument in [6, pp. 6–7] guarantees that there exists $\varepsilon_1^k > 0$ such that, $\forall\, u \in \Delta_k$,

$$\text{meas}\Big\{t \in [0, 2kT] :\ |u(t)| \ge \varepsilon_1^k\Big\} \ge \varepsilon_1^k. \tag{3.20}$$

For $z = z^+ + z^0 + z^- \in \Delta_k$, let

$$\Omega_k^z = \big\{t \in [0, 2kT] :\ |z(t)| \ge \varepsilon_1^k\big\}.$$

By (H_3), for $M_k = \frac{\|A\|}{(\varepsilon_1^k)^3} > 0$, there exists L_k such that

$$G(t,z) \ge M_k|z|^2,\ \ \forall\,|z| \ge L_k,\ \ \text{uniformly in } t. \tag{3.21}$$

Let

$$\gamma_k \ge \max\Big\{\frac{L_k}{\varepsilon_1^k}, \frac{\eta}{\varrho\|A\|}\Big\}.$$

For $\gamma \ge \gamma_k$, we have from (3.20) and (3.21) that

$$G(t,\gamma z) \ge M_k|\gamma z|^2 \ge M_k\gamma^2(\varepsilon_1^k)^2,\ \ \forall\, t \in \Omega_k^z. \tag{3.22}$$

From (H_1) and (3.22), for $\gamma \ge \gamma_k$ we have for $z \in \Delta_k$ that

$$\begin{aligned} I_k(\gamma z) &= \frac{1}{2}\gamma^2\langle Az^+, z^+\rangle + \frac{1}{2}\gamma^2\langle Az^-, z^-\rangle - \\ &- \int_{-kT}^{kT} G(t,\gamma z)\,dt - \int_{-kT}^{kT} \gamma z\cdot h_k(t)\,dt \le \frac{1}{2}\|A\|\gamma^2 - \int_{\Omega_k^z} G(t,\gamma z)\,dt + \frac{\eta}{2\varrho}\gamma \le \\ &\le \frac{1}{2}\|A\|\gamma^2 - M_k\gamma^2(\varepsilon_1^k)^3 + \frac{\eta}{2\varrho}\gamma = -\frac{1}{2}\|A\|\gamma^2 + \frac{\eta}{2\varrho}\gamma \le 0. \end{aligned} \tag{3.23}$$

Therefore we have shown that

$$I_k(\gamma z) \le 0\ \text{ for any }\ z \in \Delta_k\ \text{ and }\ \gamma \ge \gamma_k. \tag{3.24}$$

Let

$$E_k^{(2)} = E_k^- \oplus E_k^0,$$

$$Q_k = \big\{\gamma e :\ 0 \le \gamma \le 2\gamma_k\big\} \oplus \big\{z \in E_k^{(2)} :\ \|z\|_{E_k} \le 2\gamma_k\big\}.$$

By (H_2), (3.16)–(3.17) and (3.24) we have $I_k|_{\partial Q_k} \leq 0$, i.e., I_k satisfies $(C_2)(ii)$ of the Lemma 2.1.

Step 3: $(C_3)(iii)$ (i.e. S_k links ∂Q_k) holds from the definition of S_k and Q_k and [11, p. 32]. Thus $(C_3)(iii)$ holds.

From (H_2)–(H_5) and (2.3), (C_1) and (C_2) of Lemma 2.1 are true, so by Lemma 2.1, I_k possesses a critical value c_k given by

$$c_k = \inf_{g_k \in \Upsilon_k} \sup_{u_k \in Q_k} I_k(g_k(1, u_k)), \tag{3.25}$$

where Υ_k satisfies $(\Gamma_1) - (\Gamma_3)$. Hence, for every $k \in \mathbf{N}$, there is $z_k^* \in E_k$ such that

$$I_k(z_k^*) = c_k, \quad I_k'(z_k^*) = 0. \tag{3.26}$$

The function z_k^* is a desired classical $2kT$-periodic solution of (2.2). Since $c_k \geq \widetilde{\alpha} = \frac{\xi - 2a_2\varrho^{\lambda+1}}{4} > 0$, z_k^* is a nontrivial solution. □

Lemma 3.3. *Let $\{z_k^*\}_{k\in\mathbf{N}}$ be the sequence given by Lemma 3.3. There exists a $z_0 \in C(\mathbf{R}, \mathbf{R}^{2N})$ such that $z_k^* \to z_0$ in $C_{loc}(\mathbf{R}, \mathbf{R}^{2N})$ as $k \to +\infty$.*

Proof. The first step in the proof is to show that the sequences $\{c_k\}_{k\in\mathbf{N}}$ and $\{\|z_k^*\|_{E_k}\}_{k\in\mathbf{N}}$ are bounded. There exists $\widehat{z}_1^* \in E_1$ with $\widehat{z}_1^*(\pm T) = 0$ such that

$$c_1 \leq I_1(\widehat{z}_1^*) = \inf_{g_1 \in \Upsilon_1} \sup_{u_1 \in Q_1, u_1(\pm T)=0} I_1(g_1(1, u_1)). \tag{3.27}$$

For every $k \in \mathbf{N}$, let

$$\widehat{z}_k^*(t) = \begin{cases} \widehat{z}_1^*(t) & \text{for } |t| \leq T \\ 0 & \text{for } T < |t| \leq kT \end{cases} \tag{3.28}$$

and $\widetilde{g}_k : [0,1] \times E_k \longrightarrow E_k$ be a curve given by $\widetilde{g}_k(t, z) \equiv z$, where $z \in E_k$. Then $\widetilde{g}_k \in \Upsilon_k$ and $I_k(\widetilde{g}_k(1, \widehat{z}_k^*)) = I_1(\widetilde{g}_1(1, z_1^*)) = I_1(z_1^*)$ for all $k \in \mathbf{N}$. Therefore, from (3.25), (3.27) and (3.28),

$$c_k \leq I_k\big(\widetilde{g}_k(1, \widehat{z}_k^*)\big) = I_1\big(\widetilde{g}_1(1, z_1^*)\big) = I_1(z_1^*) \equiv M_0. \tag{3.29}$$

We now prove that $\{z_k^*\}_{k\in\mathbf{N}}$ is bounded.

Let $z_k^* = (z_k^*)^0 + (z_k^*)^+ + (z_k^*)^- \in E_k^0 \oplus E_k^+ \oplus E_k^-$. From (1.3) of (H_4), (H_5), (2.4), (3.1) and (3.29), there exists a constant $\widehat{c}_{\widehat{\beta}} > 0$ such that (here $\frac{1}{\widehat{\beta}} + \frac{1}{\beta} = 1$)

$$\begin{aligned} 2M_0 &\geq 2I_k(z_k^*) - \langle I_k'(z_k^*), z_k^* \rangle \\ &= \int_{-kT}^{kT} \Big[z_k^* \cdot G_{z_k^*}(t, z_k^*) - 2G(t, z_k^*) \Big]\, dt - \int_{-kT}^{kT} h_k(t) \cdot z_k^*\, dt \geq \\ &\geq \int_{-kT}^{kT} a_1 |z_k^*|^\beta\, dt - \int_{-kT}^{kT} \tau_k(t)\, dt - \int_{-kT}^{kT} |h_k(t)|\, |z_k^*|\, dt \geq \end{aligned}$$

$$\geq a_1 \|z_k^*\|_{L_{2kT}^\beta}^\beta - \|\tau_k\|_{L_{2kT}^1} - c_\beta \|h_k\|_{L_{2kT}^{\widehat{\beta}}} \|z_k^*\|_{L_{2kT}^\beta} \geq$$
$$\geq a_1 \|z_k^*\|_{L_{2kT}^\beta}^\beta - \|\tau\|_{L^1} - c_\beta \widehat{c}_{\widehat{\beta}} \|h\|_{L^1} \|z_k^*\|_{L_{2kT}^\beta} \geq$$
$$\geq a_1 \|z_k^*\|_{L_{2kT}^\beta}^\beta - \|\tau\|_{L^1} - c_\beta \widehat{c}_{\widehat{\beta}} a_3 \|z_k^*\|_{L_{2kT}^\beta}. \tag{3.30}$$

Since $\beta > 1$, this implies that there exists a constant $\widetilde{M}_0^* >$ with

$$\|z_k^*\|_{L_{2kT}^\beta} \leq \widetilde{M}_0^*. \tag{3.31}$$

Note $\dim(E_k^0) < +\infty$, therefore there exists a constant $\widetilde{M}_1^* > 0$ such that

$$\|(z_k^*)^0\|_{E_k} \leq \widetilde{M}_1^*. \tag{3.32}$$

By using (3.1) and (3.8), we have (here $\frac{1}{\alpha} + \frac{1}{\sigma} = 1$)

$$\|(z_k^*)^+\|_{E_k} \geq \left\langle I_k'(z_k^*), (z_k^*)^+ \right\rangle =$$
$$= \langle A(z_k^*)^+, (z_k^*)^+ \rangle - \int_{-kT}^{kT} \left[(z_k^*)^+ \cdot G_{z_k^*}(t, z_k^*) \right] dt - \int_{-kT}^{kT} h_k(t) \cdot (z_k^*)^+ \, dt =$$
$$= \langle A(z_k^*)^+, (z_k^*)^+ \rangle - \left(\int_{|z_k^*| \geq 1} + \int_{|z_k^*| < 1} \right) \left[(z_k^*)^+ \cdot G_{z_k^*}(t, z_k^*) \right] dt -$$
$$- \int_{-kT}^{kT} h_k(t) \cdot (z_k^*)^+ \, dt \geq$$
$$\geq \langle A(z_k^*)^+, (z_k^*)^+ \rangle - \frac{\eta}{2\varrho} \|z_k^*\|_{E_k} - \int_{|z_k^*| < 1} a_2 |z_k^*| \, |(z_k^*)^+| \, dt -$$
$$- \left(\int_{|z_k^*| \geq 1} |G_{z_k^*}(t, z_k^*)|^\alpha \, dt \right)^{\frac{1}{\alpha}} \left(\int_{-kT}^{kT} |z_{k_n}^+|^\sigma \, dt \right)^{\frac{1}{\sigma}} \geq$$
$$\geq \langle A(z_k^*)^+, (z_k^*)^+ \rangle - \frac{\eta}{2\varrho} \|z_k^*\|_{E_k} - a_2 \|z_k^*\|_{E_k} \|(z_k^*)^+\|_{E_k} -$$
$$- \left(\int_{|z_k^*| \geq 1} |G_{z_k^*}(t, z_k^*)|^\alpha \, dt \right)^{\frac{1}{\alpha}} c_\sigma \|z_k^*\|_{E_k} \tag{3.33}$$

and

$$\|(z_k^*)^-\|_{E_k} \geq \langle I_k'(z_k^*), (z_k^*)^- \rangle$$
$$= \langle A(z_k^*)^-, (z_k^*)^- \rangle - \int_{-kT}^{kT} \left[(z_k^*)^- \cdot G_{z_k^*}(t, z_k^*) \right] dt - \int_{-kT}^{kT} h_k(t) \cdot (z_k^*)^- \, dt =$$

$$= \langle A(z_k^*)^-, (z_k^*)^- \rangle - \Bigg(\int\limits_{|z_k^*|\geq 1} + \int\limits_{|z_k^*|<1} \Bigg) \big[(z_k^*)^- \cdot G_{z_k^*}(t, z_k^*) \big]\, dt -$$

$$- \int\limits_{-kT}^{kT} h_k(t) \cdot (z_k^*)^- \, dt \geq$$

$$\geq \langle A(z_k^*)^-, (z_k^*)^- \rangle - \frac{\eta}{2\varrho} \|z_k^*\|_{E_k} - \int\limits_{|z_k^*|<1} a_2 |z_k^*| \, |(z_k^*)^-| \, dt -$$

$$- \Bigg(\int\limits_{|z_k^*|\geq 1} |G_{z_k^*}(t, z_k^*)|^\alpha \, dt \Bigg)^{\frac{1}{\alpha}} \Bigg(\int\limits_{-kT}^{kT} |z_{k_n}^-|^\sigma \, dt \Bigg)^{\frac{1}{\sigma}} \geq$$

$$\geq \langle A(z_k^*)^-, (z_k^*)^- \rangle - \frac{\eta}{2\varrho} \|z_k^*\|_{E_k} - a_2 \|z_k^*\|_{E_k} \|(z_k^*)^-\|_{E_k} -$$

$$- \Bigg(\int\limits_{|z_k^*|\geq 1} |G_{z_k^*}(t, z_k^*)|^\alpha \, dt \Bigg)^{\frac{1}{\alpha}} c_\sigma \|z_k^*\|_{E_k}. \tag{3.34}$$

Combining (3.11), (3.31) with (3.33)–(3.34), we have

$$\|(z_k^*)^-\|_{E_k} + \|(z_k^*)^+\|_{E_k} \geq$$
$$\geq \xi_1 \|(z_k^*)^+\|_{E_k}^2 - \xi_{-1} \|(z_k^*)^-\|_{E_k}^2 - \frac{\eta}{\varrho} \|z_k^*\|_{E_k} -$$
$$- 2a_2 \|z_k^*\|_{E_k}^2 - 2c_\sigma \widetilde{D}_0^* \|z_k^*\|_{E_k}^{\frac{(\lambda\alpha-1)}{\alpha}} \|z_k^*\|_{E_k}, \tag{3.35}$$

where

$$\widetilde{D}_0^* = \Big[a_2^\alpha \big(c_{\frac{\beta(\lambda\alpha-1)}{\beta-1}} \big)^{\lambda\alpha-1} \widetilde{M_0^*} \Big]^{\frac{1}{\alpha}}.$$

From (3.32) and (3.35), there exists a positive constant $\widetilde{D}_1^* > 0$ such that

$$\widetilde{D}_1^* (\|(z_k^*)^+\|_{E_k} + \|(z_k^*)^-\|_{E_k} + \|(z_k^*)^0\|_{E_k}) \geq$$
$$\geq \|(z_k^*)^+\|_{E_k} + \|(z_k^*)^-\|_{E_k} + \xi \widetilde{M_1^*} \|(z_k^*)^0\|_{E_k} \geq$$
$$\geq \|(z_k^*)^+\|_{E_k} + \|(z_k^*)^-\|_{E_k} + \xi \|(z_k^*)^0\|_{E_k}^2 \geq$$
$$\geq \xi \Big(\|(z_k^*)^+\|_{E_k}^2 + \|(z_k^*)^-\|_{E_k}^2 + |(z_k^*)^0\|_{E_k}^2 \Big) -$$
$$- \frac{\eta}{\varrho} \|z_k^*\|_{E_k} - 2a_2 \|z_k^*\|_{E_k}^2 - 2c_\sigma \widetilde{D}_0^* \big(\|z_k^*\|_{E_k} \big)^{\frac{(\lambda\alpha-1)+\alpha}{\alpha}}. \tag{3.36}$$

This implies that

$$\widetilde{D}_1^* + \frac{\eta}{\varrho} \geq (\xi - 2a_2) \|z_k^*\|_{E_k} - 2c_\sigma \widetilde{D}_0^* \big(\|z_k^*\|_{E_k} \big)^{\frac{(\lambda\alpha-1)}{\alpha}}, \tag{3.37}$$

where $0 < \frac{(\lambda\alpha-1)}{\alpha} < 1$. Since $\xi - 2a_2 > 0$, we have that $\{\|z_{k_n}\|_{E_k}\}_{n\in\mathbf{N}}$ is bounded. Hence (3.37) shows that there exists a constant $M_1 > 0$ such that

$$\|z_k^*\|_{E_k} \le M_1. \tag{3.38}$$

We now show that for a large enough k,

$$\|z_k^*\|_{L^\infty_{2kT}} \le M_2. \tag{3.39}$$

If not (note (2.1) and (3.38)), by passing to a subsequence, without loss of generality, for each $k \in N$, there exist z_k^*, ℓ_k and $\widetilde{\ell}_k$ such that $|z_k^*(\ell_k)| = M_k^*$, $|z_k^*(\widetilde{\ell}_k)| = 1$ and $1 \le |z_k^*(t)| \le M_k^*$ for $t \in (\widetilde{\ell}_k, \ell_k) \subseteq [-kT, kT]$ (and $M_k^* \to \infty$ as $k \to \infty$). Hence, we have from (1.3) of (H_4), (H_5) and (3.31) that

$$M_k^* - 1 = |z_k^*(\ell_k)| - |z_k^*(\widetilde{\ell}_k)| = \int_{\widetilde{\ell}_k}^{\ell_k} \frac{d}{ds}|z_k^*(s)|\, ds =$$

$$= \int_{\widetilde{\ell}_k}^{\ell_k} z_k^*(s) \cdot \frac{\dot{z}_k^*(s)}{|z_k^*(s)|}\, ds \le \int_{\widetilde{\ell}_k}^{\ell_k} |\dot{z}_k^*(s)|\, ds$$

$$\le \int_{\widetilde{\ell}_k}^{\ell_k} \left|G_{z_k^*}(t, z_k^*(s))\right| ds + \int_{\widetilde{\ell}_k}^{\ell_k} \left|B(s) z_k^*(s)\right| ds + \int_{\widetilde{\ell}_k}^{\ell_k} |h_k(s)|\, ds \le$$

$$\le \left(a_2 + \|B\|_{L^\infty_{2kT}}\right) \int_{\widetilde{\ell}_k}^{\ell_k} |z_k^*(s)|^\lambda\, ds + \|h_k\|_{L^1_{2kT}} \le$$

$$\le \left(a_2 + \|B\|_{L^\infty_{2kT}}\right) \int_{\widetilde{\ell}_k}^{\ell_k} |z_k^*(s)|^\beta\, ds + \|h\|_{L^1} \le \quad (\text{since } 1 < \lambda < 1 + \frac{\beta-1}{\beta} < \beta)$$

$$\le \left(a_2 + \|B\|_{L^\infty_{2kT}}\right) (\widetilde{M_0^*})^\beta + a_3, \tag{3.40}$$

where a_2, a_3, $\|B\|_{L^\infty_{2kT}}$ and $\widetilde{M_0^*}$ are k-independent constants. However, we have $M_k^* \to \infty$ as $k \to \infty$, which leads to a contradiction. Hence there exists a constant $M_2 > 0$ such that

$$\|z_k^*\|_{L^\infty_{2kT}} \le (a_2 + \|B\|_{L^\infty_{2kT}}) (\widetilde{M_0^*})^\beta + a_3 + 1 = M_2. \tag{3.41}$$

This shows that (3.39) holds.

It remains now to show that $\{z_k^*\}_{k\in N}$ is equicontinuous. It suffices to prove that the sequence satisfies a Lipschitz condition with a constant, independent of k.

From (1.1) and (3.39), there exists a constant $M_3 > 0$, independent of k such that

$$|\dot{z}_k^*(t)| = |J(G_{z_k^*}(t, z_k^*(t)) + B(t) z_k^*(t) + h_k(t))| \le$$
$$\le M_3 \quad (\text{since } \|z_k^*\|_{L^\infty_{2kT}} \le M_2)$$

which implies

$$\|\dot{z}_k^*\|_{L^\infty_{2kT}} \le M_3. \tag{3.42}$$

Let $k \in \mathbf{N}$ and $t, t_0 \in R$, then

$$\left|z_k^*(t) - z_k^*(t_0)\right| = \left|\int_{t_0}^{t} \dot{z}_k^*(s)\, ds\right| \le \int_{t_0}^{t} |\dot{z}_k^*(s)|\, ds \le M_3(t - t_0).$$

Since $\{z_k^*\}_{k\in\mathbf{N}}$ is bounded in $L^\infty_{2kT}(\mathbf{R}, \mathbf{R}^{2N})$ and equicontinuous, we obtain that the sequence $\{z_k^*\}_{k\in\mathbf{N}}$ converges to a certain $z_0 \in C_{loc}(\mathbf{R}, \mathbf{R}^{2N})$ by using the Arzelà–Ascoli theorem. □

Lemma 3.4. *The function* z_0 *determined by Lemma* 3.4 *is the desired homoclinic solution of* (1.1).

Proof. The proof will be divided into three steps.

Step 1: We prove that $z_0(t) \to 0$ as $t \to \pm\infty$.

We have

$$\int_{-\infty}^{+\infty} |z_0(t)|^2\, dt = \lim_{j\to+\infty} \int_{-jT}^{jT} |z_0(t)|^2\, dt = \lim_{j\to+\infty} \lim_{k\to+\infty} \int_{-jT}^{jT} |z_{n_k}^*(t)|^2\, dt.$$

Clearly, by (2.1) and (3.38), for every $j \in \mathbf{N}$ there exists $n_j \in \mathbf{N}$ such that for all $k \ge n_j$ we have

$$\int_{-jT}^{jT} |z_{n_k}^*(t)|^2\, dt \le \|z_{n_k}^*\|_{E_k}^2 \le M_1^2,$$

and now, letting $j \to +\infty$, we have

$$\int_{-\infty}^{+\infty} |z_0(t)|^2\, dt \le \widetilde{M}_1^2,$$

and hence

$$\int_{|t|\ge m} |z_0(t)|^2\, dt \to 0 \text{ as } m \to +\infty. \tag{3.43}$$

Then (3.43) shows that our claim holds.

Step 2: We show that $z_0 \not\equiv 0$ when $h(t) \equiv 0$.

Now, up to a subsequence, we have either

$$\int_{-\infty}^{+\infty} |z_0(t)|^2\, dt = \lim_{j\to+\infty} \int_{-jT}^{jT} |z_0(t)|^2\, dt =$$

$$= \lim_{j\to+\infty} \lim_{k\to+\infty} \int_{-jT}^{jT} |z_{n_k}^*(t)|^2\, dt = 0, \tag{3.44}$$

or there exist $\widehat{\alpha} > 0$ such that

$$\int_{-\infty}^{+\infty} |z_0(t)|^2\,dt = \lim_{j\to+\infty} \int_{-jT}^{jT} |z_0(t)|^2\,dt = \\ = \lim_{j\to+\infty}\lim_{k\to+\infty} \int_{-jT}^{jT} |z^*_{n_k}(t)|^2\,dt \ge \widehat{\alpha} > 0. \tag{3.45}$$

In the first case we shall say that z_0 is vanishing and in the second that z_0 is nonvanishing.

By assumptions (H_2), (H_3) and (1.4) of (H_4), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left|G(t, z^*_{n_k})\right| \le \varepsilon |z^*_{n_k}|^2 + C_\varepsilon |z^*_{n_k}|^{\lambda+1}. \tag{3.46}$$

Hence, we have from (1.4) of (H_4) and (3.46) that

$$\begin{cases} \displaystyle\int_{-kT}^{kT} \left|(z^*_{n_k})^{\pm}\right| \left|G_{z^*_{n_k}}(t, z^*_{n_k})\right| dt \le \\ \qquad \le \varepsilon \|z^*_{n_k}\|_{L^2_{2kT}} \left\|(z^*_{n_k})^{\pm}\right\|_{L^2_{2kT}} + a_2 \|z^*_{n_k}\|^{\lambda+1}_{L^{\lambda+1}_{2kT}}, \\ \displaystyle\int_{-kT}^{kT} G(t, z^*_{n_k})\,dt \le \varepsilon \|z^*_{n_k}\|^2_{L^2_{2kT}} + C_\varepsilon \|z^*_{n_k}\|^{\lambda+1}_{L^{\lambda+1}_{2kT}}. \end{cases} \tag{3.47}$$

Arguing indirectly, we suppose that $\{z^*_{n_k}\}_{k=1}^{\infty}$ is bounded and vanishing. We have from (3.44) and (3.47) that

$$\lim_{k\to\infty} \int_{-kT}^{kT} (z^*_k)^{\pm} \cdot G_{z^*_k}(t, z^*_k)\,dt = \lim_{k\to\infty} \int_{-kT}^{kT} G(t, z^*_k)\,dt = 0. \tag{3.48}$$

Since $\langle I'_k(z^*_{n_k}), (z^*_{n_k})^{\pm}\rangle = 0$, for some positive constant $\widetilde{C}$, using (3.1) and (3.47), we find that

$$\xi_1 \left\|(z^*_{n_k})^+\right\|^2_{E_k} \le \left\langle A(z^*_{n_k})^+, (z^*_{n_k})^+\right\rangle = \int_{-kT}^{kT} (z^*_{n_k})^+ \cdot G_{z^*_{n_k}}(t, z^*_{n_k})\,dt \le \\ \le \varepsilon \|z^*_{n_k}\|_{E_k} \left\|(z^*_{n_k})^+\right\|_{E_k} + \widetilde{C} \|z^*_{n_k}\|^{\lambda+1}_{E_k} \le \frac{\xi}{8} \|z^*_{n_k}\|^2_{E_k} + \widetilde{C} \|z^*_{n_k}\|^{\lambda+1}_{E_k} \tag{3.49}$$

and

$$-\xi_{-1} \left\|(z^*_{n_k})^-\right\|^2_{E_k} \le -\left\langle A(z^*_{n_k})^-, (z^*_{n_k})^-\right\rangle = -\int_{-kT}^{kT} (z^*_{n_k})^-_k \cdot G_{z^*_{n_k}}(t, z^*_{n_k})\,dt \le \\ \le \varepsilon \|z^*_{n_k}\|_{E_k} \left\|(z^*_{n_k})^-\right\|_{E_k} + \widetilde{C} \|z^*_{n_k}\|^{\lambda+1}_{E_k} \le \frac{\xi}{8} \left\|z^*_{n_k}\right\|^2_{E_k} + \widetilde{C} \|z^*_{n_k}\|^{\lambda+1}_{E_k}. \tag{3.50}$$

Note that $\dim(E_k^0) < +\infty$, there exist two positive constants $\widetilde{b}_1$, and $\widetilde{b}_2$ such that

$$\widetilde{b}_1 \big\|(z_{n_k}^*)^0\big\|_{L^2_{2kT}} \le \big\|(z_{n_k}^*)^0\big\|_{E_k} \le \widetilde{b}_2 \big\|(z_{n_k}^*)^0\big\|_{L^2_{2kT}} \le \widetilde{b}_2 \|z_{n_k}^*\|_{L^2_{2kT}}. \quad (3.51)$$

From (3.44) and (3.51) we have

$$\xi \big\|(z_{n_k}^*)^0\big\|_{E_k}^2 \le \xi \widetilde{b}_2 \big\|(z_{n_k}^*)^0\big\|_{L^2_{2kT}} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \quad (3.52)$$

Now (3.52) implies that there exists a positive constant $b_\varepsilon (0 < b_\varepsilon \le \frac{\xi}{4})$ such that

$$\xi \big\|(z_{n_k}^*)^0\big\|_{E_k}^2 \le b_\varepsilon \|z_{n_k}^*\|_{E_k}^2. \quad (3.53)$$

Hence, from (3.49), (3.50) and (3.53) we obtain that

$$\begin{aligned} &\xi\Big(\big\|(z_{n_k}^*)^+\big\|_{E_k}^2 + \big\|(z_{n_k}^*)^-\big\|_{E_k}^2 + \big\|(z_{n_k}^*)^0\big\|_{E_k}^2\Big) \le \\ &\le \xi_1 \big\|(z_{n_k}^*)^+\big\|_{E_k}^2 + \xi_{-1} \big\|(z_{n_k}^*)^-\big\|_{E_k}^2 + \xi \big\|(z_{n_k}^*)^0\big\|_{E_k}^2 \le \\ &\le \frac{\xi}{2} \|z_{n_k}^*\|_{E_k}^2 + 2\widetilde{C} \|z_{n_k}^*\|_{E_k}^{\lambda+1}, \end{aligned}$$

and $\|z_{n_k}^*\|_{E_k} \ge \widetilde{\zeta}$ for some $\widetilde{\zeta} > 0$.

On the other hand, from (3.44), (3.48) and (3.53), we have

$$\big\|(z_{n_k}^*)^\pm\big\|_{E_k}^2 \to 0 \text{ and } \big\|(z_{n_k}^*)^0\big\|_{E_k}^2 \to 0 \text{ as } k \to \infty.$$

This means that $\|z_{n_k}^*\|_{E_k} \to 0$ as $k \to \infty$, which leads to a contradiction. Hence $\{z_k^*\}$ is nonvanishing, so (3.45) holds, and this shows that our claim holds.

Step 3: We show that $z_0(t)$ is a nontrivial homoclinic solution of (1.1).
Proof. According to step 2, $z_0(t) \not\equiv 0$, it suffices to prove that for any $\varphi \in C_0^\infty(\mathbf{R}, \mathbf{R}^{2N})$,

$$\int_{-\infty}^{+\infty} (\dot{z}_0(t) - JH_{z_0}(t, z_0(t))) \cdot \varphi(t)\, dt = 0. \quad (3.54)$$

By step 1, we can choose k_0 such that $\operatorname{supp}\varphi \subseteq [-k_iT, k_iT]$ for all $k_i \ge k_0$, and we have for $k_i \ge k_0$

$$\int_{-\infty}^{+\infty} \Big\{\dot{z}_{k_i}^*(t) - J\Big[B(t)z_{k_i}^*(t) + G_{z_{k_i}^*}(t, z_{k_i}^*(t)) + h_{k_i}(t)\Big]\Big\} \cdot \varphi(t)\, dt = 0. \quad (3.55)$$

By (3.43) and (3.55), letting $k_i \to \infty$ we get (3.54), which shows $z_0(t)$ is a nontrivial homoclinic solution of (1.1). □

Proof of Theorem 1.1. The result follows from Lemma 3.4. □

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Authors' addresses:

Chengjun Guo and Chengjiang Wang
School of Applied Mathematics, Guangdong University of Technology, Guangzhou, 510006, China.

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland.
E-mail: donal.oregan@nuigalway.ie

Ravi P. Agarwal
Department of Mathematics, Texas A and M University-Kingsville, Texas, 78363, USA.
E-mail: Ravi.Agarwal@tamuk.edu