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BOUNDARY VALUE PROBLEMS OF THE THEORY OF THERMOELASTICITY WITH MICROTEMPERATURES FOR DOMAINS BOUNDED BY A SPHERICAL SURFACE

Abstract. We consider the stationary oscillation case of the theory of linear thermoelasticity of materials with microtemperatures. The representation formula of a general solution of the homogeneous system of differential equations obtained in the paper is expressed by means of seven metaharmonic functions. This formula is very convenient and useful in many particular problems for domains with concrete geometry. Here we demonstrate an application of this formulas to the Dirichlet and Neumann type boundary value problem for a ball. The uniqueness theorems are proved. An explicit solutions in the form of absolutely and uniformly convergent series are constructed.

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Key words and phrases. Microtemperature, thermoelasticity, Fourier-Laplace series, stationary oscillation.

რეზიუმე. ნაშრომში განხილულია თერმოდრეკადობის წრფივი თეორიის სტაციონარული რხევის ამოცანები მიკროტემპერატურის გათვალისწინებით. მიღებულია ერთგვაროვან დიფერენციალურ განტოლებათა სისტემის ამონახსნის ზოგადი წარმოდგენის ფორმულა გამოსახული შვიდი მეტაჰარმონიული ფუნქციის საშუალებით. მიღებული წარმოდგენა არის მეტად მოხერხებული კონკრეტული გეომეტრიის მქონე არეების შემთხვევაში სასაზღვრო ამოცანების ამოსახსნელად. ამ ნაშრომში შესწავლილია დირიხლესა და ნეიმანის ტიპის სასაზღვრო ამოცანები ბირთვისათვის. დამტკიცებულია ერთადერთობის თეორემები. ამოცანების ამოხსნები მიღებულია აბსოლუტურად და თანაბრად კრებადი მწკრივების სახით.

1. Introduction

Mathematical model describing the chiral properties of the linear thermoelasticity of materials with microtemperatures have been proposed by Iesan [6], [8] and recently it has been extended to a more general case, when the material points admit micropolar structure [7].

The Dirichlet, Neumann and mixed type boundary value problems corresponding to this model are well investigated for general domains of arbitrary shape, the uniqueness and existence theorems are proved, and regularity results for solutions are established by potential and variational methods (see [1, 10, 14, 15] and the references therein).

The main goal of this paper is to derive general representation formulas for the displacement vector of microtemperatures and temperature function by means of metaharmonic functions. That is, we can represent solutions to a very complicated coupled system of simultaneous differential equations of thermoelasticity with the help of solutions of simpler canonical equations.

In particular, here we apply these representation formulas to construct explicit solutions to the Dirichlet and Neumann type boundary value problems for a ball. We represent the solution in the form of Fourier–Laplace series and show their absolute and uniform convergence along with their derivatives of the first order if the boundary data satisfy appropriate smoothness conditions. One of the methods to satisfy the boundary conditions is given in A. Ulitko [17], F. Mors and G Feshbah [12], L. Giorgashvili [2,3], L. Giorgashvili, D. Natroshvili [4], L. Giorgashvili, A. Jaghmaidze, K. Skhvitaridze [5], D. Natroshvili, L. Giorgashvili, I. Stratis [13] and other papers.

2. Basic Equations and Auxiliary Theorems

A system of homogeneous differential equations of the stationary oscillation of the thermoelasticity with microtemperatures is written in the form [7]

$$\mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x) - \gamma\,\mathrm{grad}\,\theta(x) + \rho\sigma^2 u(x) = 0, \tag{2.1}$$

$$\varkappa_6\Delta w(x) + (\varkappa_5+\varkappa_4)\,\mathrm{grad}\,\mathrm{div}\,w(x) - \varkappa_3\,\mathrm{grad}\,\theta(x) + \tau w(x) = 0, \tag{2.2}$$

$$\varkappa\Delta\theta(x) + i\sigma\gamma T_0\,\mathrm{div}\,u(x) + \varkappa_1\,\mathrm{div}\,w(x) + i\sigma a T_0\theta(x) = 0, \tag{2.3}$$

where Δ is the three-dimensional Laplace operator, $u=(u_1,u_2,u_3)^\top$ is the displacement vector, $w=(w_1,w_2,w_3)^\top$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 ($T_0>0$), $\top$ is the transposition symbol, λ, μ, γ, $\varkappa$, $\varkappa_j$, $j=1,2,\dots,6$, are constitutive coefficients, satisfying the conditions [7]

$$\mu>0,\ \ 3\lambda+2\mu>0,\ \ \varkappa>0,\ \ 3\varkappa_4+\varkappa_5+\varkappa_6>0,\ \ \varkappa_6+\varkappa_5>0,$$
$$\varkappa_6-\varkappa_5>0,\ \ (\varkappa_1+T_0\varkappa_3)^2<4T_0\varkappa\varkappa_2,\ \ \gamma>0,\ \ a>0,$$

$\tau=-\varkappa_2+i\sigma\delta$, $\delta>0$, $\rho>0$ is the mass density of the elastic material. In the sequel we assume that $\sigma=\sigma_1+i\sigma_2$, $\sigma_2>0$, $\sigma_1\in\mathbb{R}$.

Let $U=(u,w,\theta)^\top$. The stress vector, which we denote by the symbol $P(\partial,n)U$, has the form

$$P(\partial,n)U=\Big(P^{(1)}(\partial,n)U',\ \ P^{(2)}(\partial,n)U'',\ \ P^{(3)}(\partial,n)U''\Big)^\top,$$

where $U'=(u,\theta)^\top$, $U''=(w,\theta)^\top$, $n=(n_1,n_2,n_3)^\top$ is the unit vector,

$$\begin{aligned}
P^{(1)}(\partial,n)U'&=T^{(1)}(\partial,n)u-\gamma n\theta,\\
P^{(2)}(\partial,n)U''&=T^{(2)}(\partial,n)w-\varkappa_3 n\theta,\\
P^{(3)}(\partial,n)U''&=\varkappa\frac{\partial\theta}{\partial n}+(\varkappa_1+\varkappa_3)(n\cdot w),\\
T^{(1)}(\partial,n)u&=2\mu\frac{\partial u}{\partial n}+\lambda n\operatorname{div}u+\mu[n\times\operatorname{rot}u],\\
T^{(2)}(\partial,n)w&=(\varkappa_6+\varkappa_5)\frac{\partial w}{\partial n}+\varkappa_4 n\operatorname{div}w+\varkappa_5[n\times\operatorname{rot}w].
\end{aligned}\tag{2.4}$$

Definition. The vector $U=(u,w,\theta)^\top$ is said to be regular in a domain $\Omega\subset\mathbb{R}^3$ if $U\in C^2(\Omega)\cap C^1(\overline{\Omega})$.

Theorem 2.1. *A vector $U=(u,w,\theta)^\top$ is a regular solution of system (2.1)–(2.3) in a domain $\Omega\subset\mathbb{R}^3$, if and only if it is represented in the form*

$$\begin{aligned}
u(x)&=\sum_{j=1}^{3}\operatorname{grad}\Phi_j(x)+\operatorname{rot}\operatorname{rot}(x\Phi_4(x))+\operatorname{rot}(x\Phi_5(x)),\\
w(x)&=\sum_{j=1}^{3}\alpha_j\operatorname{grad}\Phi_j(x)+\operatorname{rot}\operatorname{rot}(x\Phi_6(x))+\operatorname{rot}(x\Phi_7(x)),\\
\theta(x)&=-\sum_{j=1}^{3}\beta_j k_j^2\Phi_j(x),
\end{aligned}\tag{2.5}$$

where

$$(\Delta+k_j^2)\Phi_j(x)=0,\ \ j=1,2,3,\quad (\Delta+k_4^2)\Phi_j(x)=0,\ \ j=4,5,$$
$$(\Delta+k_5^2)\Phi_j(x)=0,\ \ j=6,7,$$

$k_4^2=\rho\sigma^2/\mu$, $k_5^2=\tau/\varkappa_6$, $-k_j^2$, $j=1,2,3$, *are the roots of the equation*

$$z^3+a_1z^2+a_2z+a_3=0\tag{2.6}$$

with

$$\begin{aligned}
a_1 &= \frac{1}{\Delta_1}\Big\{l\big[i\sigma T_0(a(\lambda+2\mu)+\gamma^2)+\varkappa\rho\sigma^2\big]+(\lambda+2\mu)(i\sigma aT_0\varkappa\tau+\varkappa_1\varkappa_3)\Big\},\\
a_2 &= \frac{1}{\Delta_1}\Big\{\rho\sigma^2(\varkappa_1\varkappa_3+i\sigma aT_0l+\varkappa\tau)+\tau\big[i\sigma T_0\gamma^2+i\sigma aT_0(\lambda+2\mu)\big]\Big\},\\
a_3 &= \frac{i}{\Delta_1}\,aT_0\rho\sigma^3\tau,\quad \Delta_1=\varkappa(\lambda+2\mu)l>0,\quad l=\varkappa_4+\varkappa_5+\varkappa_6>0,\\
\alpha_j &= \frac{\varkappa_3[\rho\sigma^2-(\lambda+2\mu)k_j^2]}{\gamma(\tau-lk_j^2)}\,,\quad \beta_j=\frac{i\sigma\gamma T_0+\varkappa_1\alpha_j}{\varkappa k_j^2-i\sigma aT_0}\,,\quad j=1,2,3.
\end{aligned} \tag{2.7}$$

Proof. Assume that a vector $U=(u,w,\theta)^\top$ is a solution of system (2.1)–(2.3). From equations (2.1)–(2.2) we have

$$u(x)=u'(x)+u''(x),\quad w(x)=w'(x)+w''(x),$$

where

$$\begin{aligned}
u'(x) &= \frac{1}{\rho\sigma^2}\ \mathrm{grad}\big[-(\lambda+2\mu)\,\mathrm{div}\,u(x)+\gamma\theta(x)\big],\\
w'(x) &= \frac{1}{\tau}\ \mathrm{grad}\big[-l\,\mathrm{div}\,w(x)+\varkappa_3\theta(x)\big];
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
u''(x) &= \frac{\mu}{\rho\sigma^2}\,\mathrm{rot}\,\mathrm{rot}\,u(x),\\
w''(x) &= \frac{\varkappa_6}{\rho}\,\mathrm{rot}\,\mathrm{rot}\,w(x).
\end{aligned} \tag{2.9}$$

If we apply the operator div to both parts of equalities (2.1) and (2.2), and take into account equalities (2.3), then we obtain

$$\begin{aligned}
\big[(\lambda+2\mu)\Delta+\rho\sigma^2\big]\,\mathrm{div}\,u(x)-\gamma\Delta\theta(x)&=0,\\
(l\Delta+\tau)\,\mathrm{div}\,w(x)-\varkappa_3\Delta\theta(x)&=0,\\
i\sigma\gamma T_0\,\mathrm{div}\,u(x)+\varkappa_1\,\mathrm{div}\,w(x)+(\varkappa\Delta+i\sigma aT_0)\theta(x)&=0.
\end{aligned}$$

From these equations we get

$$(\Delta+k_1^2)(\Delta+k_2^2)(\Delta+k_3^2)(\mathrm{div}\,u,\mathrm{div}\,w,\theta)^\top=0, \tag{2.10}$$

where $-k_j^2$, $j=1,2,3$, are the roots of equation (2.6).

In view of equalities (2.8) and (2.10), we obtain

$$(\Delta+k_1^2)(\Delta+k_2^2)(\Delta+k_3^2)(u',w')^\top=0,\quad \mathrm{rot}\,u'=0,\quad \mathrm{rot}\,w'=0. \tag{2.11}$$

We represent the vectors $u'(x)$, $w'(x)$ and the function $\theta(x)$ as:

$$u'(x)=\sum_{j=1}^{3}u^{(j)}(x),\quad w'(x)=\sum_{j=1}^{3}w^{(j)}(x),\quad \theta(x)=\sum_{j=1}^{3}\theta^{(j)}(x). \tag{2.12}$$

Naturally,

$$(u^{(j)},\ w^{(j)},\ \theta^{(j)})^\top=\Big[\prod_{j\neq q=1}^{3}\frac{\Delta+k_q^2}{k_q^2-k_j^2}\Big](u',\ w',\ \theta)^\top,\quad j=1,2,3. \tag{2.13}$$

From (2.10)–(2.11) and (2.13) we derive

$$\begin{gathered}(\Delta+k_j^2)u^{(j)}(x)=0,\ \ \operatorname{rot}u^{(j)}(x)=0,\ \ j=1,2,3,\\ (\Delta+k_j^2)w^{(j)}(x)=0,\ \ \operatorname{rot}w^{(j)}(x)=0,\ \ j=1,2,3,\\ (\Delta+k_j^2)\theta^{(j)}(x)=0,\ \ j=1,2,3.\end{gathered}\tag{2.14}$$

Since $\operatorname{div}u=\operatorname{div}u'$, $\operatorname{div}w=\operatorname{div}w'$, $\operatorname{rot}u'=0$, $\operatorname{rot}w'=0$, with the help of (2.14) and the identity

$$\operatorname{grad}\operatorname{div}u'=\Delta u'+\operatorname{rot}\operatorname{rot}u'=\Delta u',\ \ \operatorname{grad}\operatorname{div}w'=\Delta w',$$

from (2.8) and (2.3) we get

$$\big[\rho\sigma^2-(\lambda+2\mu)k_j^2\big]u^{(j)}(x)-\gamma\operatorname{grad}\theta^{(j)}(x)=0,\tag{2.15}$$

$$(\tau-lk_j^2)w^{(j)}(x)-\varkappa_3\operatorname{grad}\theta^{(j)}(x)=0,\tag{2.16}$$

$$i\sigma\gamma T_0\operatorname{div}u^{(j)}(x)+\varkappa_1\operatorname{div}w^{(j)}(x)+(i\sigma aT_0-\varkappa k_j^2)\theta^{(j)}(x)=0,\quad j=1,2,3.\tag{2.17}$$

From (2.15) and (2.16) we have

$$w^{(j)}(x)=\alpha_j u^{(j)}(x),\ \ j=1,2,3,\tag{2.18}$$

where

$$\alpha_j=\frac{\varkappa_3[\rho\sigma^2-(\lambda+2\mu)k_j^2]}{\gamma(\tau-lk_j^2)},\ \ j=1,2,3.$$

If we substitute the expressions of $w^{(j)}(x)$ from (2.18) into (2.17), we get

$$\theta^{(j)}(x)=\beta_j\operatorname{div}u^{(j)}(x),\ \ j=1,2,3,\tag{2.19}$$

where

$$\beta_j=\frac{i\sigma\gamma T_0+\varkappa_1\alpha_j}{\varkappa k_j^2-i\sigma aT_0},\ \ j=1,2,3.$$

Substitute the expressions of $w^{(j)}(x)$ and $\theta^{(j)}(x)$, $j=1,2,3$, given by (2.18)–(2.19) into (2.12) to obtain

$$\begin{gathered}u'(x)=\sum_{j=1}^{3}u^{(j)}(x),\ \ w'(x)=\sum_{j=1}^{3}\alpha_j u^{(j)}(x),\\ \theta(x)=\sum_{j=1}^{3}\beta_j\operatorname{div}u^{(j)}(x),\ \ \operatorname{rot}u^{(j)}(x)=0,\ \ j=1,2,3.\end{gathered}\tag{2.20}$$

On the other hand, since $\operatorname{rot}u=\operatorname{rot}u''$, $\operatorname{rot}w=\operatorname{rot}w''$, $\operatorname{div}u''=0$, $\operatorname{div}w''=0$ and $\operatorname{rot}\operatorname{rot}u''=-\Delta u''$, $\operatorname{rot}\operatorname{rot}w''=-\Delta w''$, from (2.9) we get

$$\begin{gathered}(\Delta+k_4^2)u''(x)=0,\quad \operatorname{div}u''(x)=0,\\ (\Delta+k_5^2)w''(x)=0,\quad \operatorname{div}w''(x)=0,\end{gathered}\tag{2.21}$$

where $k_4^2=\rho\sigma^2/\mu$, $k_5^2=\tau/\varkappa_6$.

The following lemmas are valid [3,12].

Lemma 2.2. *If a vector $v = (v_1, v_2, v_3)^\top$ in the domain $\Omega \subset \mathbb{R}^3$ satisfies the following system of differential equations*

$$(\Delta + k^2)v(x) = 0, \quad \operatorname{rot} v(x) = 0,$$

then v can be represented as

$$v(x) = \operatorname{grad} \Phi(x),$$

where $\Phi(x)$ is a solution of the Helmholtz equation $(\Delta + k^2)\Phi(x) = 0$; here k is an arbitrary constant.

Lemma 2.3. *If a vector $v = (v_1, v_2, v_3)^\top$ in the domain $\Omega \subset \mathbb{R}^3$ satisfies the following system of differential equations*

$$(\Delta + k^2)v(x) = 0, \quad \operatorname{div} v(x) = 0,$$

then v can be represented as

$$v(x) = \operatorname{rot}\operatorname{rot}(x\Psi_1(x)) + \operatorname{rot}(x\Psi_2(x)),$$

where $\Psi_j(x)$, $j=1,2$, are solutions of the Helmholtz equation $(\Delta+k^2)\Psi_j(x)=0$, $j = 1, 2$; here k is an arbitrary constant.

Due to Lemma 2.2 and Lemma 2.3, a solution of systems (2.14) and (2.21) can be represented as

$$\begin{aligned} u'(x) &= \operatorname{grad} \Phi_j(x), \quad j = 1, 2, 3, \\ u''(x) &= \operatorname{rot}\operatorname{rot}(x\Phi_4(x)) + \operatorname{rot}(x\Phi_5(x)), \\ w''(x) &= \operatorname{rot}\operatorname{rot}(x\Phi_6(x)) + \operatorname{rot}(x\Phi_7(x)), \end{aligned} \tag{2.22}$$

where

$$(\Delta + k_j^2)\Phi_j(x) = 0, \;\; j = 1, 2, 3, \quad (\Delta + k_4^2)\Phi_j(x) = 0, \;\; j = 4, 5,$$
$$(\Delta + k_5^2)\Phi_j(x) = 0, \;\; j = 6, 7.$$

Substitution of the expressions (2.22) into (2.20) proves the first part of the theorem. As to the second part, it is proved by a straightforward verification that the vector $U = (u, w, \theta)^\top$ represented in the form (2.5) is a solution of system (2.1)–(2.3). □

Remark 2.4. Hereinafter, we will assume that $k_j \neq k_p$, $j \neq p$, $\Im k_j > 0$, $j = 1, 2, 3, 4, 5$.

Let $\Omega^+ = B(R) \subset \mathbb{R}^3$ be a ball with center at the origin, of radius R, and $\Sigma_R = \partial\Omega$. We denote $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$.

Theorem 2.5. *A vector $U = (u, w, \theta)^\top$ represented by (2.5) will be uniquely defined in the area Ω^+ by the functions $\Phi_j(x)$, $j = 1, 2, \dots, 7$, if the following conditions are fulfilled:*

$$\int\limits_{\Sigma_r} \Phi_j(x)\, d\Sigma_r = 0, \;\; j = 4, 5, 6, 7, \;\; r = |x| < R. \tag{2.23}$$

Proof. From formulas (2.5) we get

$$\sum_{j=1}^{3} k_j^2\Phi_j(x) = -\operatorname{div} u, \quad \sum_{j=1}^{3} \alpha_j k_j^2\Phi_j(x) = -\operatorname{div} w,$$

$$\sum_{j=1}^{3} \beta_j k_j^2\Phi_j(x) = -\theta(x),$$

$$r^2\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} + k_{3+j}^2\Big)\Phi_{3+2j}(x) = x\cdot(\delta_{1j}\operatorname{rot} u + \delta_{2j}\operatorname{rot} w), \quad j=1,2,$$

$$r^2\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} + k_{3+j}^2\Big)\Phi_{2+2j}(x) =$$
$$= -\frac{1}{k_{3+j}^2}\,x\cdot(\delta_{1j}\operatorname{rot}\operatorname{rot} u + \delta_{2j}\operatorname{rot}\operatorname{rot} w), \quad j=1,2,$$

δ_{lj} is the Kronecker function.

If $u(x)=0$, $w(x)=0$, $\theta(x)=0$, $x\in\Omega^+$, we have $\Phi_j(x)=0$, $j=1,2,3$, $x\in\Omega^+$,

$$\begin{aligned} &r^2\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} + k_4^2\Big)\Phi_j(x) = 0, \quad j=4,5, \quad x\in\Omega^+,\\ &r^2\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r} + k_5^2\Big)\Phi_j(x) = 0, \quad j=6,7, \quad x\in\Omega^+. \end{aligned} \tag{2.24}$$

Thus it remains to show that $\Phi_j(x)=0$, $j=4,5,6,7$. Applying the well known representation of metaharmonic functions in the form of series, we can write

$$\Phi_j(x) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_l r)A_{mk}^{(j)}Y_k^{(m)}(\vartheta,\varphi), \quad j=4,5,6,7, \quad x\in\Omega^+,$$

where $A_{mk}^{(j)}$ are constants, $Y_k^{(m)}(\vartheta,\varphi)$ is a spherical function

$$Y_k^{(m)}(\vartheta,\varphi) = \sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\;P_k^{(m)}(\cos\vartheta)e^{im\varphi},$$

$P_k^{(m)}(\cos\vartheta)$ is the associated Legendre polinomial of the first kind of degree k and order m,

$$g_k(k_l r) = r^{-1/2}\mathcal{J}_{k+1/2}(k_l r), \quad k_l = \begin{cases} k_4, & j=4,5,\\ k_5, & j=6,7,\end{cases}$$

$\mathcal{J}_{k+1/2}(k_l r)$ are the Bessel functions.

With the help of the equality

$$\Big(\frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr} + k_l^2\Big)g_k(k_l r) = \frac{k(k+1)}{r^2}g_k(k_l r),$$

from (2.24) we get

$$\sum_{k=0}^{\infty}\sum_{m=-k}^{k} k(k+1)g_k(k_l r)A_{mk}^{(j)}Y_k^{(m)}(\vartheta,\varphi)=0, \quad j=4,5,6,7,$$

whence the equations $A_{mk}^{(j)}=0$ follow for $k\ge 1$ and $j=4,5,6,7$. Therefore

$$\Phi_j(x)=\frac{1}{2\sqrt{\pi}}g_0(k_l r)A_{00}^{(j)}, \quad j=4,5,6,7.$$

Further, from (2.23) we easily conclude $A_{00}^{(j)}=0$ for $j=4,5,6,7$, which completes the proof. □

3. Orthonormal System of Spherical Vectors

Let r, ϑ, φ ($0\le r<+\infty$, $0\le\vartheta\le\pi$, $0\le\varphi<2\pi$) be the spherical coordinates of $x\in\mathbb{R}^3$. Denote by Σ_1 the unit sphere.

In the space $L_2(\Sigma_1)$ consider the following complete orthonormal vectors system (see [2, 12, 17])

$$\begin{aligned}
&X_{mk}(\vartheta,\varphi)=e_r Y_k^{(m)}(\vartheta,\varphi), \quad k\ge 0,\\
&Y_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(e_\vartheta\frac{\partial}{\partial\vartheta}+\frac{e_\varphi}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\Big)Y_k^{(m)}(\vartheta,\varphi), \quad k\ge 1,\\
&Z_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(\frac{e_\vartheta}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}-e_\varphi\,\frac{\partial}{\partial\vartheta}\Big)Y_k^{(m)}(\vartheta,\varphi), \quad k\ge 1,
\end{aligned}\tag{3.1}$$

where $|m|\le k$, e_r, e_ϑ, e_φ are the orthonormal vectors in $\mathbb{R}^3$,

$$\begin{aligned}
e_r&=(\cos\varphi\sin\vartheta,\sin\varphi\sin\vartheta,\cos\vartheta)^\top,\\
e_\vartheta&=(\cos\varphi\cos\vartheta,\sin\varphi\cos\vartheta,-\sin\vartheta)^\top,\\
e_\varphi&=(-\sin\varphi,\cos\varphi,0)^\top,\\
Y_k^{(m)}(\vartheta,\varphi)&=\sqrt{\frac{2k+1}{4\pi}\,\frac{(k-m)!}{(k+m)!}}P_k^{(m)}(\cos\theta)e^{im\varphi},
\end{aligned}$$

$P_k^{(m)}(\cos\vartheta)$ is the adjoint Legendre function.

Let us assume that a vector-function $f^{(j)}=(f_1^{(j)},\ f_2^{(j)},\ f_3^{(j)})^\top$ and a function f_4 are represented as

$$\begin{aligned}
f^{(j)}(\vartheta,\varphi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{&\alpha_{mk}X_{mk}(\vartheta,\varphi)+\\
&+\sqrt{k(k+1)}\Big[\beta_{mk}^{(j)}Y_{mk}(\vartheta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\vartheta,\varphi)\Big]\Big\},
\end{aligned}\tag{3.2}$$

$$f_4(\vartheta,\varphi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\alpha_{mk}Y_k^m(\vartheta,\varphi),\tag{3.3}$$

where

$$\begin{aligned}
\alpha_{mk}^{(j)} &= \int_0^{2\pi} d\varphi \int_0^{\pi} f^{(j)}(\vartheta,\varphi)\cdot \overline{X}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta, \quad k\geq 0,\\
\beta_{mk}^{(j)} &= \frac{1}{\sqrt{k(k+1)}}\int_0^{2\pi} d\varphi \int_0^{\pi} f^{(j)}(\vartheta,\varphi)\cdot \overline{Y}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta, \quad k\geq 1,\\
\gamma_{mk}^{(j)} &= \frac{1}{\sqrt{k(k+1)}}\int_0^{2\pi} d\varphi \int_0^{\pi} f^{(j)}(\vartheta,\varphi)\cdot \overline{Z}_{mk}(\vartheta,\varphi)\sin\vartheta\, d\vartheta, \quad k\geq 1,\\
\alpha_{mk} &= \int_0^{2\pi} d\varphi \int_0^{\pi} f_4(\vartheta,\varphi)\cdot \overline{Y}_{k}^{(m)}(\vartheta,\varphi)\sin\vartheta\, d\vartheta, \quad k\geq 0.
\end{aligned} \tag{3.4}$$

The symbol $a\cdot\overline{b}$ denotes the scalar product of two vectors, $\overline{b}$ is complex conjugate of b.

Note that in formula (3.2) and, in the sequel, in the summands of analogous series, which contain the vectors $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$, the summation index k varies from 1 to $+\infty$.

Let us introduce a few important lemmas [3, 11].

Lemma 3.1. *Let $f^{(j)}\in C^l(\Sigma_1)$, $l\geq 1$; then the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$ defined by* (3.4) *admit the following estimates*

$$\alpha_{mk}^{(j)} = O(k^{-l}), \quad \beta_{mk}^{(j)} = O(k^{-l-1}), \quad \gamma_{mk}^{(j)} = O(k^{-l-1}).$$

Lemma 3.2. *Let $f_4\in C^l(\Sigma_1)$, $l\geq 1$; then the coefficients α_{mk} defined by* (3.4) *admit the following estimates*

$$\alpha_{mk} = O(k^{-l}).$$

Lemma 3.3. *The vectors $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ defined by equalities* (3.1) *admit the estimates:*

$$\begin{aligned}
\big|X_{mk}(\vartheta,\varphi)\big| &\leq \sqrt{\frac{2k+1}{4\pi}}, \quad k\geq 0,\\
\big|Y_{mk}(\vartheta,\varphi)\big| &< \sqrt{\frac{k(k+1)}{2k+1}}, \quad k\geq 1,\\
\big|Z_{mk}(\vartheta,\varphi)\big| &< \sqrt{\frac{k(k+1)}{2k+1}}, \quad k\geq 1,
\end{aligned}$$

Hereinafter we make use the following equalities [6]

$$\begin{gathered}
e_r\cdot X_{mk}(\vartheta,\varphi) = Y_k^{(m)}(\vartheta,\varphi), \quad e_r\cdot Y_{mk}(\vartheta,\varphi)=0,\\
e_r\cdot Z_{mk}(\vartheta,\varphi)=0,\\
e_r\times X_{mk}(\vartheta,\varphi)=0, \quad e_r\times Y_{mk}(\vartheta,\varphi) = -Z_{mk}(\vartheta,\varphi),\\
e_r\times Z_{mk}(\vartheta,\varphi) = Y_{mk}(\vartheta,\varphi);
\end{gathered} \tag{3.5}$$

$$
\begin{aligned}
&\operatorname{grad}\big[a(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\\
&=\frac{da(r)}{dr}\,X_{mk}(\vartheta,\varphi)+\frac{\sqrt{k(k+1)}}{r}\,a(r)Y_{mk}(\vartheta,\varphi),\\
&\operatorname{rot}\big[xa(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\sqrt{k(k+1)}\,a(r)Z_{mk}(\vartheta,\varphi),\\
&\operatorname{rot}\operatorname{rot}\big[xa(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\\
&=\frac{k(k+1)}{r}\,a(r)X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&\operatorname{div}\big[a(r)X_{mk}(\vartheta,\varphi)\big]=\Big(\frac{d}{dr}+\frac{2}{r}\Big)a(r)Y_k^{(m)}(\vartheta,\varphi),\\
&\operatorname{div}\big[a(r)Y_{mk}(\vartheta,\varphi)\big]=-\frac{\sqrt{k(k+1)}}{r}\,a(r)Y_k^{(m)}(\vartheta,\varphi),\\
&\operatorname{div}\big[a(r)Z_{mk}(\vartheta,\varphi)\big]=0,\\
&\operatorname{rot}\big[a(r)X_{mk}(\vartheta,\varphi)\big]=\frac{\sqrt{k(k+1)}}{r}\,a(r)Z_{mk}(\vartheta,\varphi),\\
&\operatorname{rot}\big[a(r)Y_{mk}(\vartheta,\varphi)\big]=-\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Z_{mk}(\vartheta,\varphi),\\
&\operatorname{rot}\big[a(r)Z_{mk}(\vartheta,\varphi)\big]=\frac{\sqrt{k(k+1)}}{r}\,a(r)X_{mk}(\vartheta,\varphi)+\\
&\qquad\qquad+\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi).
\end{aligned}
\tag{3.7}
$$

4. Statement of the Problem. The Uniqueness Theorem

Problem. Find, in the domain Ω^+, a regular vector $U=(u,w,\theta)^\top$ satisfying in this domain the system of differential equations (2.1)–(2.3) and, on the boundary $\partial\Omega$, one of the following boundary conditions:

$(\mathrm{I}^{(\sigma)})^+$ (the Dirichlet problem)

$$\{u(z)\}^+=f^{(1)}(z),\quad \{w(z)\}^+=f^{(2)}(z),\quad \{\theta(z)\}^+=f_4(z);$$

$(\mathrm{II}^{(\sigma)})^+$ (the Neumann problem)

$$
\begin{gathered}
\big\{P^{(1)}(\partial,n)U'(z)\big\}^+=f^{(1)}(z),\quad \big\{P^{(2)}(\partial,n)U''(z)\big\}^+=f^{(2)}(z),\\
\big\{P^{(3)}(\partial,n)U''(z)\big\}^+=f_4(z),
\end{gathered}
\tag{4.1}
$$

where the vectors $f^{(j)}=(f_1^{(j)},f_2^{(j)},f_3^{(j)})$, $j=1,2$, and the function f_4 are given on the boundary $\partial\Omega$, $n(z)$ is the outward normal unit vector at the point $z\in\partial\Omega$.

Theorem 4.1. *Problems* $(\mathrm{I}^{(\sigma)})^+$ *and* $(\hat{\mathrm{II}}^{(\sigma)})^+$ *have, in the domain* Ω^+, *a unique solution in the class of regular functions.*

Proof. The theorem will be proved if we show that the homogeneous problems ($f^{(j)}=0$, $j=1,2$, $f_4=0$) have only the trivial solution.

Let the vector $U=(u,w,\theta)^\top$ be a solution of the homogeneous problem either $(\mathrm{I}^{(\sigma)})^+$ or $(\mathrm{II}^{(\sigma)})^+$. We multiply both sides of (2.1) by the vector $i\,\overline{\sigma}\,T_0\,\overline{u}$, (2.2) by $\overline{w}$ and the complex-conjugate of (2.3) by the function θ. The integration of these expressions over the domain Ω^+ and summation give

$$\int\limits_{\partial\Omega}\Big[i\overline{\sigma}T_0\overline{u}(z)\cdot P^{(1)}(\partial,n)U'(z)+ + \overline{w}(z)\cdot P^{(2)}(\partial,n)U''(z)+\theta(z)\cdot P^{(3)}(\partial,n)\overline{U}''(z)\Big]^+ ds- - \int\limits_{\Omega^+}\Big[i\overline{\sigma}T_0E^{(1)}(u,\overline{u})-i\rho\sigma|\sigma|^2|u(x)|^2+E^{(2)}(w,\overline{w})-\tau|w(x)|^2+ + \varkappa|\operatorname{grad}\theta(x)|^2+(\varkappa_1+\varkappa_3)\overline{w}(x)\cdot\operatorname{grad}\theta(x)+i\overline{\sigma}aT_0|\theta(x)|^2\Big]\,dx=0, \quad (4.2)$$

where [9, 15]

$$E^{(1)}(u,\overline{u})=\frac{3\lambda+2\mu}{3}\,|\operatorname{div}u|^2+\frac{\mu}{2}\sum_{k\neq j=1}^{3}\Big|\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\Big|^2+ + \frac{\mu}{3}\sum_{k,j=1}^{3}\Big|\frac{\partial u_k}{\partial x_k}-\frac{\partial u_j}{\partial x_j}\Big|^2,$$

$$E^{(2)}(w,\overline{w})=\frac{3\varkappa_4+\varkappa_5+\varkappa_6}{3}\,|\operatorname{div}w|^2+\frac{\varkappa_6-\varkappa_5}{2}|\operatorname{rot}w|^2+ + \frac{\varkappa_5+\varkappa_6}{4}\sum_{k\neq j=1}^{3}\Big|\frac{\partial w_k}{\partial x_j}+\frac{\partial w_j}{\partial x_k}\Big|^2+\frac{\varkappa_5+\varkappa_6}{6}\sum_{k,j=1}^{3}\Big|\frac{\partial w_k}{\partial x_k}-\frac{\partial w_j}{\partial x_j}\Big|^2.$$

Since $U=(u,w,\theta)^\top$ is a solution of the homogeneous problem, equality (4.2) implies

$$\int\limits_{\Omega^+}\Big[i\overline{\sigma}T_0E^{(1)}(u,\overline{u})-i\rho\sigma|\sigma|^2|u(x)|^2+E^{(2)}(w,\overline{w})-\tau|w(x)|^2+ + \varkappa|\operatorname{grad}\theta(x)|^2+(\varkappa_1+\varkappa_3)\overline{w}(x)\cdot\operatorname{grad}\theta(x)+i\overline{\sigma}aT_0|\theta(x)|^2\Big]\,dx=0.$$

If in this equality we separate the real part, we will get

$$\int\limits_{\Omega^+}\Big[\sigma_2T_0E^{(1)}(u,\overline{u})+E^{(2)}(w,\overline{w})+\rho\sigma_2|\sigma|^2|u(x)|^2+\sigma_2\delta|w(x)|^2+ + aT_0\sigma_2|\theta(x)|^2+\frac{4\varkappa\varkappa_2-(\varkappa_1+\varkappa_3)^2}{4\varkappa}|w(x)|^2+ + \frac{1}{4\varkappa}\,\big|(\varkappa_1+\varkappa_3)w(x)+2\varkappa\operatorname{grad}\theta(x)\big|^2\Big]\,dx=0.$$

Hence it follows that $u(x)=0$, $w(x)=0$, $\theta(x)=0$, $x\in\Omega^+$. □

5. Solution of the Boundary Value Problems

We seek a solution of the Dirichlet and Neumann Problems by formulas (2.5), where

$$\begin{aligned}
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_j r) Y_k^{(m)}(\vartheta,\varphi) A_{mk}^{(j)}, \ \ j=1,2,3,\\
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_4 r) Y_k^{(m)}(\vartheta,\varphi) A_{mk}^{(j)}, \ \ j=4,5, \qquad (5.1)\\
\Phi_j(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(k_5 r) Y_k^{(m)}(\vartheta,\varphi) A_{mk}^{(j)}, \ \ j=6,7.
\end{aligned}$$

Here $A_{mk}^{(j)}$, $j=\overline{1,7}$, are the sought constants, $Y_k^{(m)}(\vartheta,\varphi)$ is a spherical function and

$$g_k(k_j r) = \sqrt{\frac{R}{r}}\,\frac{\mathcal{J}_{k+\frac12}(k_j r)}{\mathcal{J}_{k+\frac12}(k_j R)}\,,$$

$\mathcal{J}_{k+\frac12}(x)$ is a Bessel function.

Substituting the expressions of $\Phi_j(x)$ $j=4,5,6,7$, from (5.1), into (2.23), we get $A_{00}^{(j)}=0$, $j=4,5,6,7$. If we substitute the expressions of the functions $\Phi_j(x)$ $j=\overline{1,7}$, from (2.5) and take into account equalities (3.6), we obtain

$$\begin{aligned}
u(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(1)}(r) X_{mk}(\vartheta,\varphi) +\\
&\quad + \sqrt{k(k+1)}\Big[v_{mk}^{(1)}(r) Y_{mk}(\vartheta,\varphi) + w_{mk}^{(1)}(r) Z_{mk}(\vartheta,\varphi)\Big]\Big\},\\
w(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(2)}(r) X_{mk}(\vartheta,\varphi) + \qquad (5.2)\\
&\quad + \sqrt{k(k+1)}\Big[v_{mk}^{(2)}(r) Y_{mk}(\vartheta,\varphi) + w_{mk}^{(2)}(r) Z_{mk}(\vartheta,\varphi)\Big]\Big\},\\
\theta(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} u_{mk}(r) Y_k^{(m)}(\vartheta,\varphi),
\end{aligned}$$

where

$$\begin{aligned}
u_{mk}^{(1)}(r) &= \sum_{j=1}^{3}\frac{d}{dr}\, g_k(k_j r) A_{mk}^{(j)} + \frac{k(k+1)}{r}\, g_k(k_4 r) A_{mk}^{(4)}, \ \ k\geq 0,\\
v_{mk}^{(1)}(r) &= \sum_{j=1}^{3}\frac{1}{r}\, g_k(k_j r) A_{mk}^{(j)} + \Big(\frac{d}{dr}+\frac{1}{r}\Big) g_k(k_4 r) A_{mk}^{(4)}, \ \ k\geq 1,\\
w_{mk}^{(1)}(r) &= g_k(k_4 r) A_{mk}^{(5)}, \ \ k\geq 1,
\end{aligned}$$

$$u_{mk}^{(2)}(r) = \sum_{j=1}^{3} \alpha_j \frac{d}{dr}\, g_k(k_j r) A_{mk}^{(j)} + \frac{k(k+1)}{r}\, g_k(k_5 r) A_{mk}^{(6)}, \quad k \geq 0,$$

$$v_{mk}^{(2)}(r) = \sum_{j=1}^{3} \alpha_j \frac{1}{r}\, g_k(k_j r) A_{mk}^{(j)} + \Big(\frac{d}{dr} + \frac{1}{r}\Big) g_k(k_5 r) A_{mk}^{(6)}, \quad k \geq 1,$$

$$w_{mk}^{(2)}(r) = g_k(k_5 r) A_{mk}^{(7)}, \quad k \geq 1,$$

$$u_{mk}(r) = -\sum_{j=1}^{3} \beta_j k_j^2 g_k(k_j r) A_{mk}^{(j)}, \quad k \geq 0.$$

If we substitute the expressions of the vectors $u(x)$, $w(x)$ and the function $\theta(x)$ into (2.4) and use equalities (3.5) and (3.7), we get

$$\begin{aligned}
P^{(1)}(\partial, n) U'(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ a_{mk}^{(1)}(r) X_{mk}(\vartheta,\varphi) + \sqrt{k(k+1)} \times \\
&\qquad \times \Big[b_{mk}^{(1)}(r) Y_{mk}(\vartheta,\varphi) + c_{mk}^{(1)}(r) Z_{mk}(\vartheta,\varphi) \Big] \Big\}, \\
P^{(2)}(\partial, n) U''(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ a_{mk}^{(2)}(r) X_{mk}(\vartheta,\varphi) + \sqrt{k(k+1)} \times \\
&\qquad \times \Big[b_{mk}^{(2)}(r) Y_{mk}(\vartheta,\varphi) + c_{mk}^{(2)}(r) Z_{mk}(\vartheta,\varphi) \Big] \Big\}, \\
P^{(3)}(\partial, n) U''(x) &= \sum_{k=0}^{\infty} \sum_{m=-k}^{k} a_{mk}(r) Y_k^{(m)}(\vartheta,\varphi),
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
a_{mk}^{(1)}(r) &= \sum_{j=1}^{3} \Big[2\mu \frac{d^2}{dr^2} + (\gamma\beta_j - \lambda) k_j^2 \Big] g_k(k_j r) A_{mk}^{(j)} + \\
&\qquad + \frac{2\mu k(k+1)}{r} \Big(\frac{d}{dr} - \frac{1}{r}\Big) g_k(k_4 r) A_{mk}^{(4)}, \quad k \geq 0, \\
b_{mk}^{(1)}(r) &= \sum_{j=1}^{3} 2\mu \frac{1}{r} \Big(\frac{d}{dr} - \frac{1}{r}\Big) g_k(k_j r) A_{mk}^{(j)} + \\
&\qquad + \mu \Big[2 \frac{d}{r} \Big(\frac{d}{dr} + \frac{1}{r}\Big) + k_4^2 \Big] g_k(k_4 r) A_{mk}^{(4)}, \quad k \geq 1, \\
c_{mk}^{(1)}(r) &= \mu \Big(\frac{d}{dr} - \frac{1}{r}\Big) g_k(k_4 r) A_{mk}^{(5)}, \quad k \geq 1, \\
a_{mk}^{(2)}(r) &= \sum_{j=1}^{3} \Big[(\varkappa_5 + \varkappa_6)\alpha_j \frac{d^2}{dr^2} + (\varkappa_3\beta_j - \varkappa_4\alpha_j) k_j^2 \Big] g_k(k_j r) A_{mk}^{(j)} + \\
&\qquad + \frac{(\varkappa_5 + \varkappa_6) k(k+1)}{r} \Big(\frac{d}{dr} - \frac{1}{r}\Big) g_k(k_5 r) A_{mk}^{(6)}, \quad k \geq 0,
\end{aligned}$$

$$b_{mk}^{(2)}(r)=\sum_{j=1}^{3}\alpha_j(\varkappa_5+\varkappa_6)\,\frac{1}{r}\left(\frac{d}{dr}-\frac{1}{r}\right)g_k(k_jr)A_{mk}^{(j)}+$$
$$+\left[(\varkappa_5+\varkappa_6)\,\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)+\varkappa_5k_5^2\right]g_k(k_5r)A_{mk}^{(6)},\ \ k\geq 1,$$
$$c_{mk}^{(2)}(r)=\left(\varkappa_6\,\frac{d}{dr}-\varkappa_5\,\frac{1}{r}\right)g_k(k_5r)A_{mk}^{(7)},\ \ k\geq 1,$$
$$a_{mk}(r)=\sum_{j=1}^{3}\left(\alpha_j(\varkappa_1+\varkappa_3)-\varkappa\beta_jk_j^2\right)\frac{d}{dr}\,g_k(k_jr)A_{mk}^{(j)}+$$
$$+\frac{(\varkappa_1+\varkappa_3)k(k+1)}{r}\,g_k(k_5r)A_{mk}^{(6)},\ \ k\geq 0.$$

Let us first consider the Neumann problem.

Assume that the vectors $f^{(j)}(\vartheta,\varphi)$, $j=1,2$, and the function $f_4(\vartheta,\varphi)$ can be represented in the form (3.2) and (3.3).

Passing to the limit on both sides of (5.3) as $x\to z\in\partial\Omega$ and using both the Neumann boundary conditions (4.1) and equalities (3.2)–(3.3), for the sought constants $A_{mk}^{(j)}$, $j=\overline{1,7}$, we obtain the following system of linear algebraic equations:

(1) for $k=0$, $m=0$ (three simultaneous equations with the three unknowns $A_{00}^{(j)}$, $j=1,2,3$),

$$a_{00}^{(1)}(R)=\alpha_{00}^{(1)},\quad a_{00}^{(2)}(R)=\alpha_{00}^{(2)},\quad a_{00}(R)=\alpha_{00};\tag{5.4}$$

(2) for $k\geq 1$, $-k\leq m\leq k$

(a)

$$\begin{gathered}\mu\left(\frac{d}{dR}-\frac{1}{R}\right)g_k(k_4R)A_{mk}^{(5)}=\gamma_{mk}^{(1)},\\ \left(\varkappa_6\,\frac{d}{dR}-\varkappa_5\,\frac{1}{R}\right)g_k(k_5R)A_{mk}^{(7)}=\gamma_{mk}^{(2)};\end{gathered}\tag{5.5}$$

(b) (five simultaneous equations with the five unknowns $A_{mk}^{(j)}$, $j=1,2,3,4,6$)

$$a_{mk}^{(j)}(R)=\alpha_{mk}^{(1)},\quad b_{mk}^{(j)}(R)=\beta_{mk}^{(j)},\ \ j=1,2,\ \ a_{mk}(R)=\alpha_{mk}.\tag{5.6}$$

Due to Theorems 4.1 and 2.5, system (5.4)–(5.6) is uniquely solvable with respect to the unknowns $A_{mk}^{(j)}$, $j=\overline{1,7}$. Thus we can construct explicitly a formal solution of the Neumann problem in the form of series. Further we have to investigate the convergence of these formal series and their derivatives.

The asymptotic representations

$$g_k(k_jr)\approx\left(\frac{r}{R}\right)^k,\quad g_k'(k_jr)\approx\frac{k}{r}\left(\frac{r}{R}\right)^k,\ \ r<R\tag{5.7}$$

are valid for $k\to+\infty$ [16].

If $x \in \Omega^+$ $(r < R)$, then by asymptotics (5.7), the series (5.2)–(5.3) converge absolutely and uniformly.

If $x \in \partial\Omega$ $(r = R)$, then by Lemmas 3.1–3.3 and asymptotics (5.7), series (5.2)–(5.3) will be absolutely and uniformly convergent provided that the majorized series

$$\sum_{k=k_0}^{\infty} k^{3/2} \sum_{j=1}^{2} \Big(|\alpha_{mk}^{(j)}| + k|\beta_{mk}^{(j)}| + k|\gamma_{mk}^{(j)}| + |\alpha_{mk}| \Big) \tag{5.8}$$

are convergent. Series (5.8) will be convergent if the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, α_{mk}, $j = 1, 2$, admit the following estimates

$$\begin{gathered} \alpha_{mk}^{(j)} = O(k^{-3}), \quad \beta_{mk}^{(j)} = O(k^{-4}), \\ \gamma_{mk}^{(j)} = O(k^{-4}), \quad \alpha_{mk} = O(k^{-3}), \ \ j = 1, 2. \end{gathered} \tag{5.9}$$

According to Lemmas 3.1 and 3.2, estimates (5.9) will hold if we require that

$$f^{(j)}(z) \in C^3(\partial\Omega), \ \ j = 1, 2, \ \ f_4(z) \in C^3(\partial\Omega). \tag{5.10}$$

Therefore if the boundary vector-functions satisfy conditions (5.10), then the vector $U = (u, w, \theta)^\top$ represented by equalities (5.2) will be a regular solution of Problem $(II^{(\sigma)})^+$.

Problem $(I^{(\sigma)})^+$ can be treated analogously.

6. Appendix: Properties of the Characteristic Roots and Wave Numbers

Let us introduce the blockwise 7×7 matrix differential operator corresponding to system (2.1)–(2.3)

$$L(\partial, \sigma) := \begin{bmatrix} L^{(1)}(\partial, \sigma) & L^{(2)}(\partial, \sigma) & L^{(5)}(\partial, \sigma) \\ L^{(3)}(\partial, \sigma) & L^{(4)}(\partial, \sigma) & L^{(6)}(\partial, \sigma) \\ L^{(7)}(\partial, \sigma) & L^{(8)}(\partial, \sigma) & L^{(9)}(\partial, \sigma) \end{bmatrix}_{7\times 7},$$

where

$$\begin{gathered} L^{(1)}(\partial, \sigma) := [\mu\Delta + \rho\sigma^2] I_3 + (\lambda + \mu) Q(\partial), \\ L^{(2)}(\partial, \sigma) := L^{(3)}(\partial, \sigma) = [O]_{3\times 3}, \\ L^{(4)}(\partial, \sigma) := [\varkappa_6\Delta + \tau] I_3 + (\varkappa_4 + \varkappa_5) Q(\partial), \\ L^{(5)}(\partial, \sigma) := -\gamma\nabla^\top, \quad L^{(6)}(\partial, \sigma) := -\varkappa_3\nabla^\top, \quad L^{(7)}(\partial, \sigma) := i\sigma\gamma T_0 \nabla, \\ L^{(8)}(\partial, \sigma) := \varkappa_1\nabla, \quad L^{(9)}(\partial, \sigma) := \varkappa\nabla + i\sigma a T_0, \quad Q(\partial) = [\partial_k\partial_j]_{3\times 3}, \end{gathered}$$

$\nabla = \nabla(\partial) = [\partial_1, \partial_2, \partial_3]$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$, I_3 stands for the 3×3 unit matrix.

Due to the above notation, system (2.1)–(2.3) can be rewritten in matrix form as

$$L(\partial, \sigma) U(x) = 0, \quad U = (u, w, \theta)^\top. \tag{6.1}$$

Denote by $\mathfrak{F}_{x\to\xi}$ the Fourier transforms

$$\mathfrak{F}_{x\to\xi}[f]=\int\limits_{R^3} f(x)e^{ix\cdot\xi}\,dx=\widehat{f}(\xi),$$

where $x=(x_1,x_2,x_3)$, $\xi=(\xi_1,\xi_2,\xi_3)$.

The Fourier transform has the following property:

$$L(\partial^\alpha f)=(-i\xi)^\alpha\mathfrak{F}[f], \tag{6.2}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $\xi^\alpha=\xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\xi_3^{\alpha_3}$.

Let us perform Fourier transforms of (6.1) and take into consideration (6.2); we obtain

$$L(-i\xi,\sigma)\widehat{U}(\xi)=0, \tag{6.3}$$

where

$$L^{(1)}(-i\xi,\sigma):=\big(-\mu|\xi|^2+\rho\sigma^2\big)I_3-(\lambda+\mu)Q(\xi),$$
$$L^{(2)}(-i\xi,\sigma):=L^{(3)}(-i\xi,\sigma)=[O]_{3\times3},$$
$$L^{(4)}(-i\xi,\sigma):=\big(-\varkappa_6|\xi|^2+\tau\big)I_3-(\varkappa_4+\varkappa_5)Q(\xi),$$
$$L^{(5)}(-i\xi,\sigma):=i\gamma\xi^\top,\ \ L^{(6)}(-i\xi,\sigma):=i\varkappa_3\xi^\top,\ \ L^{(7)}(-i\xi,\sigma):=\sigma\gamma T_0\xi,$$
$$L^{(8)}(-i\xi,\sigma):=-i\varkappa_1\xi,\ \ L^{(9)}(-i\xi,\sigma):=-\varkappa|\xi|^2+i\sigma aT_0,$$
$$Q(\xi)=[\xi_k\,\xi_j]_{3\times3}.$$

The determinant of system (6.3) reads as

$$\det L(-i\xi,\sigma)=$$
$$=\mu(\lambda+2\mu)l\varkappa_6\big(\mu|\xi|^2-\rho\sigma^2\big)^2(\varkappa_6|\xi|^2-\tau)^2\big(|\xi|^6-a_1|\xi|^4+a_2|\xi|^2-a_3\big),$$

where a_1, a_2, a_3 are given by (2.7), $l=\varkappa_4+\varkappa_5+\varkappa_6$.

The numbers k_j^2, $j=\overline{1,5}$, are the roots of the equation $\det L(-i\xi,\sigma)=0$ with respect to $|\xi|$.

Lemma 6.1. *Let us assume that $\sigma=\sigma_1+i\sigma_2$ is a complex parameter, where $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Then*

$$\det L(-i\xi,\sigma)\neq0$$

for arbitrary $\xi\in\mathbb{R}^3$.

Proof. We prove the lemma by contradiction. Let $\det L(-i\xi,\sigma)=0$, $\xi\in\mathbb{R}^3$. Then the system of equations $L(-i\xi,\sigma)X=0$ has a nontrivial solution. Denote this solution by $X=(X^{(1)},X^{(2)},X^{(3)})^\top$, where $X^{(j)}=(X_1^{(j)},X_2^{(j)},X_3^{(j)})^\top\in\mathbb{C}^3$ $j=1,2$, and $X^{(3)}\in\mathbb{C}$. Taking into consideration (6.3), the system $L(-i\xi,\sigma)=0$ can be rewritten as follows:

$$\Big[\big(\rho\sigma^2-\mu|\xi|^2\big)I_3-(\lambda+\mu)Q(\xi)\Big]X^{(1)}+i\gamma\xi^\top X^{(3)}=0, \tag{6.4}$$

$$\Big[\big(\tau-\varkappa_6|\xi|^2\big)I_3-(\varkappa_4+\varkappa_5)Q(\xi)\Big]X^{(2)}+i\varkappa_3\xi^\top X^{(3)}=0, \tag{6.5}$$

$$\sigma\gamma T_0(\xi\cdot X^{(1)})-i\varkappa_1(\xi\cdot X^{(2)})+(-\varkappa|\xi|^2+i\sigma aT_0)X^{(3)}=0. \tag{6.6}$$

Assume that $|\xi|\neq 0$.

Let us multiply equation (6.4) by the vector $i\overline{\sigma}T_0\overline{X^{(1)}}$, equation (6.5) by $\overline{X}^{(2)}$ and the complex-conjugate of equation (6.6) by the function $X^{(3)}$ and add the obtained results. After simplification, we obtain

$$\begin{gathered} i\overline{\sigma}T_0\big(\rho\sigma^2-\mu|\xi|^2\big)|X^{(1)}|^2-i\overline{\sigma}T_0(\lambda+\mu)|\xi\cdot X^{(1)}|^2+\\ +\big(\tau-\varkappa_6|\xi|^2\big)|X^{(2)}|^2-(\varkappa_4+\varkappa_5)|\xi\cdot X^{(2)}|^2+\\ +i(\varkappa_1+\varkappa_3)(\xi\cdot\overline{X}^{(2)})X^{(3)}+\big(-\varkappa|\xi|^2-i\overline{\sigma}aT_0\big)|X^{(3)}|^2=0. \end{gathered}$$

Recall that the central dot denotes the scalar product, $a\cdot b=\sum\limits_{j=1}^{3}a_jb_j$ for the vectors a and b. Let us separate the real part:

$$\begin{gathered} T_0\sigma_2\Big[\big(\rho|\sigma|^2+\mu|\xi|^2\big)|X^{(1)}|^2+(\lambda+\mu)|\xi\cdot X^{(1)}|^2\Big]+\\ +\big(\sigma_2\delta+\varkappa_6|\xi|^2\big)|X^{(2)}|^2+(\varkappa_4+\varkappa_5)|\xi\cdot X^{(2)}|^2+\sigma_2aT_0|X^{(3)}|^2+\\ +\frac{4\varkappa\varkappa_2-(\varkappa_1+\varkappa_3)^2}{4\varkappa}\,|X^{(2)}|^2+\frac{1}{4\varkappa}\Big|(\varkappa_1+\varkappa_3)X^{(2)}-2i\varkappa\xi X^{(3)}\Big|^2=0. \end{gathered} \tag{6.7}$$

Here we have used the following relation:

$$\begin{gathered} \varkappa|\xi|^2|X^{(3)}|^2-(\varkappa_1+\varkappa_3)\operatorname{Re}\big[i(\xi\cdot\overline{X}^{(2)})X^{(3)}\big]+\varkappa_2|X^{(2)}|^2=\\ =\frac{4\varkappa\varkappa_2-(\varkappa_1+\varkappa_3)^2}{4\varkappa}\,|X^{(2)}|^2+\frac{1}{4\varkappa}\Big|(\varkappa_1+\varkappa_3)X^{(2)}-2i\varkappa\xi X^{(3)}\Big|^2\geq 0. \end{gathered}$$

From equation (6.7) we obtain that $X^{(j)}=0$, $j=1,2,3$. For $\xi=0$ equation (6.7) recasts as

$$\rho|\sigma|^2\sigma_2T_0|X^{(1)}|^2+(\varkappa_2+\sigma_2\delta)|X^{(2)}|^2+\sigma_2aT_0|X^{(3)}|^2=0,$$

hence, $X^{(j)}=0$, $j=1,2,3$.

Thus, we obtain that the system $L(-i\xi,\sigma)X=0$ has only the trivial solution for arbitrary $\xi\in\mathbb{R}^3$. This contradiction proves the lemma. □

Corollary 6.2. *Let $\sigma=\sigma_1+i\sigma_2$ be a complex parameter with $\sigma_1\in\mathbb{R}$ and $\sigma_2>0$. Consider the equation*

$$\det L(-i\xi,\sigma)=0 \tag{6.8}$$

with respect to $|\xi|$. The roots $\pm k_j$, $j=\overline{1,5}$, of equation (6.8) *are complex with $\Im k_j>0$, $j=\overline{1,5}$.*

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