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**BOUNDARY VALUE PROBLEMS
IN WEIGHTED SOBOLEV SPACES
ON LIPSCHITZ MANIFOLDS**

Dedicated to Victor Kupradze on his 110-th birthday anniversary

Abstract. We explore the extent to which well-posedness results for the Poisson problem with a Dirichlet boundary condition hold in the setting of weighted Sobolev spaces in rough settings. The latter includes both the case of (strongly and weakly) Lipschitz domains in an Euclidean ambient, as well as compact Lipschitz manifolds with boundary.

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1. INTRODUCTION

One of the fundamental issues in analysis is that of correlating the regularity of a geometric ambient to the well-posedness of boundary value problems arising naturally in that setting. For example, the treatment of elliptic boundary value problems formulated on scales of Sobolev/Besov spaces for differential operators with smooth coefficients is rather complete in the setting of \mathcal{C}^∞ manifolds. See, e.g., [7], [10], [17]. By way of contrast, there are many interesting open questions formulated in the presence of less regular structures (see [8]).

Very often, a basic result which is used to jump-start the theory is the classical Lax–Milgram lemma. However, while this requires very little regularity for the objects involved, one is forced to stay within the constraints of Hilbert space structures, which enter typically through the considerations of L^2 (and various L^2 -based) spaces.

In this paper we explore the extent to which it is possible to depart from this basic case and consider L^p -based Sobolev spaces with p not necessarily equal to 2. We do so without having to strengthen the original assumptions pertaining to the nature of the coefficients (which are assumed to be only bounded and measurable), and this naturally imposes limitations on the parameters intervening in the spaces involved. On the geometric side, the main novelty is the fact that we succeed in formulating our main well-posedness results in the rather general setting of Lipschitz manifolds. Not only does this category of manifolds encompass many particular cases of great interest for applications, but this also constitutes the minimally smooth setting where our problems may be formulated and solved. As such, our results are sharp from a multitude of perspectives.

The organization of the paper is as follows. In Section 2 we consider weighted Sobolev spaces of arbitrary smoothness in Euclidean Lipschitz domains and prove that Stein’s extension operator continues to work in this setting. In turn, this is used to establish a very useful interpolation result (cf. Theorem 2.6). In Section 3 we study the trace theorem for such weighted Sobolev spaces, while in Section 4 we construct a boundary extension operator (which serves as an inverse from the right for the trace mapping). In Section 5 we treat boundary value problems for elliptic systems with bounded measurable coefficients in Euclidean Lipschitz domains. Our main well-posedness result in this regard is contained in Theorem 5.1. By means of counterexamples this is shown to be sharp. The scope of the theory developed up to this point is enlarged in Section 6 through the consideration of the class of weakly Lipschitz domains. Finally, in Section 7, we further generalize these results to the setting of compact Lipschitz manifolds with boundary. This portion of our paper may be regarded as a natural continuation of the work initiated in [4].

2. WEIGHTED SOBOLEV SPACES AND STEIN'S EXTENSION OPERATOR

We shall also work with the following weighted version of classical Sobolev spaces, which have been previously considered in [12].

Definition 2.1. If $p \in [1, \infty]$, $a \in (-1/p, 1 - 1/p)$ and $m \in \mathbb{N}_0$ are given and Ω is a nonempty, proper, open subset of \mathbb{R}^n , consider the weighted Sobolev space $W_a^{m,p}(\Omega)$, defined as the space of locally integrable functions u in Ω for which $\partial^\alpha u \in L_{loc}^1(\Omega)$ (with derivatives taken in the sense of distributions) whenever $\alpha \in \mathbb{N}_0^n$ has $|\alpha| \leq m$, and

$$\|u\|_{W_a^{m,p}(\Omega)} := \left(\sum_{|\alpha| \leq m} \int_{\Omega} |(\partial^\alpha u)(x)|^p \text{dist}(x, \partial\Omega)^{ap} dx \right)^{1/p} < \infty. \quad (2.1)$$

Finally, in the case when Ω is understood from the context, we shall employ the notation

$$W_a^{m,p}(\mathbb{R}^n) := \left\{ u \in L_{loc}^1(\mathbb{R}^n) : \partial^\alpha u \in L_{loc}^1(\mathbb{R}^n) \text{ whenever } |\alpha| \leq m, \text{ and} \right. \\ \left. \|u\|_{W_a^{m,p}(\mathbb{R}^n)} := \sum_{|\alpha| \leq m} \left(\int_{\mathbb{R}^n} |(\partial^\alpha u)(x)|^p \text{dist}(x, \partial\Omega)^{ap} dx \right)^{1/p} < \infty \right\}. \quad (2.2)$$

We wish to stress that $W_a^{m,p}(\mathbb{R}^n)$ is *not* $W_a^{m,p}(\Omega)$ corresponding to $\Omega = \mathbb{R}^n$ (which, incidentally, is not a permissible choice since Ω is assumed to be a proper subset of \mathbb{R}^n). Instead, the named space should always be understood in the sense of (2.2).

Hence, the case when $a = 0$ in Definition 2.1 describes the standard Sobolev spaces (L^p -based, of order m) defined intrinsically in the open set Ω . In such a scenario, we omit including $a (= 0)$ in the notation for these spaces and simply write $W^{m,p}(\Omega)$.

Fix a Lipschitz domain Ω in \mathbb{R}^n and recall from [1, Theorem 3.22, p. 68] that, since Ω satisfies the so-called segment condition, the inclusion operator

$$\mathcal{C}_b^\infty(\overline{\Omega}) \hookrightarrow W^{m,p}(\Omega) \text{ has dense range, if } p \in [1, \infty), \ m \in \mathbb{N}_0. \quad (2.3)$$

On the other hand, in the weighted case, given any Lipschitz domain Ω ,

$$\mathcal{C}_b^\infty(\overline{\Omega}) \hookrightarrow W_a^{m,p}(\Omega) \text{ has dense range,} \\ \text{if } p \in (1, \infty), \ m \in \mathbb{N}_0, \text{ and } a \in (-1/p, 1 - 1/p). \quad (2.4)$$

This is proved much as in (2.3), the new key technical ingredient being the fact that, given any Lipschitz domain $\Omega \subseteq \mathbb{R}^n$,

$$\text{dist}(\cdot, \partial\Omega)^{ap} \text{ is a Muckenhoupt } A_p\text{-weight in } \mathbb{R}^n \\ \text{whenever } p \in (1, \infty) \text{ and } a \in (-1/p, 1 - 1/p). \quad (2.5)$$

See [15] for more details in somewhat similar circumstances.

Let \mathcal{L}^n denote the Lebesgue measure in \mathbb{R}^n .

Definition 2.2. Assume that $p \in (1, \infty)$ and $a \in (-1/p, 1 - 1/p)$ are given, and that Ω is a nonempty, proper, open subset of \mathbb{R}^n . In this context, let $L^p(\Omega, \text{dist}(\cdot, \partial\Omega)^{ap} \mathcal{L}^n)$ denote the weighted Lebesgue space consisting of \mathcal{L}^n -measurable functions whose p -th power is absolutely integrable with respect to the weighted measure $\text{dist}(\cdot, \partial\Omega)^{ap} \mathcal{L}^n$. Also, for each $m \in \mathbb{N}_0$, define the weighted Sobolev space of negative order $W_a^{-m,p}(\Omega)$ as the subspace of the space of distributions $\mathcal{D}'(\Omega)$ given by

$$W_a^{-m,p}(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) : \text{there exist} \right. \\ \left. \{f_\alpha\}_{|\alpha| \leq m} \subset L^p(\Omega, \text{dist}(\cdot, \partial\Omega)^{ap} \mathcal{L}^n) \right. \\ \left. \text{such that } u = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha \text{ in } \mathcal{D}'(\Omega) \right\}. \quad (2.6)$$

Equip this space with the norm

$$\|u\|_{W_a^{-m,p}(\Omega)} := \\ := \inf_{u = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha} \left(\sum_{|\alpha| \leq m} \int_{\Omega} |f_\alpha(x)|^p \text{dist}(x, \partial\Omega)^{ap} dx \right)^{1/p}. \quad (2.7)$$

Finally, introduce

$$\mathring{W}_a^{m,p}(\Omega) := \text{the completion of } \mathcal{C}_c^\infty(\Omega) \text{ in } W_a^{m,p}(\Omega), \quad (2.8)$$

and endow this space with the norm inherited from $W_a^{m,p}(\Omega)$.

The scales of spaces introduced above enjoy a number of useful properties, some of which are discussed in the proposition below.

Proposition 2.3. Let $p \in (1, \infty)$, $a \in (-1/p, 1 - 1/p)$, and $m \in \mathbb{N}_0$ be given, and suppose Ω is a nonempty open subset of \mathbb{R}^n . Then $W_a^{m,p}(\Omega)$, $\mathring{W}_a^{m,p}(\Omega)$, $W_a^{-m,p}(\Omega)$ are reflexive Banach spaces and

$$(\mathring{W}_a^{m,p}(\Omega))^* = W_{-a}^{-m,p'}(\Omega), \quad (2.9)$$

where $1/p + 1/p' = 1$.

Proof. Fix a, p as in the statement and let N be the number of multi-indices $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| \leq m$. Define the injection $j : W_a^{m,p}(\Omega) \rightarrow [L^p(\Omega, \text{dist}(\cdot, \partial\Omega)^{ap} \mathcal{L}^n)]^N$ by setting $j(u) := \{\partial^\alpha u\}_{|\alpha| \leq m}$. Then j is an isometry identifying $W_a^{m,p}(\Omega)$ with a closed subspace of $[L^p(\Omega, \text{dist}(\cdot, \partial\Omega)^{ap} \mathcal{L}^n)]^N$. Since the latter is a reflexive Banach space, it follows that so is $W_a^{m,p}(\Omega)$. Having established this, it follows from (2.8) that $\mathring{W}_a^{m,p}(\Omega)$ is also a reflexive Banach space. Finally, that $W_a^{-m,p}(\Omega)$ is a reflexive Banach space will follow from what we have just established, once we justify the duality formula (2.9). This, in turn, is a consequence of the aforementioned isometric embedding of $W_a^{m,p}(\Omega)$ into a direct sum of weighted Lebesgue spaces, the Hahn–Banach theorem, and Riesz representation formula. \square

Our next goal is to discuss the action of Stein's extension operator in the context of weighted Sobolev spaces. This requires some preparations and we begin by recalling that the function $\psi : [1, \infty) \rightarrow \mathbb{R}$ given by

$$\psi(\lambda) := \frac{e}{\pi\lambda} \cdot \operatorname{Im}\{e^{-e^{-i\pi/4} \cdot (\lambda-1)^{1/4}}\}, \quad \forall \lambda \geq 1, \quad (2.10)$$

has, according to [16, Lemma 1, p. 182], the following properties:

$$\psi \in \mathcal{C}^0([1, \infty)), \quad (2.11)$$

$$\int_1^\infty \psi(\lambda) d\lambda = 1, \quad (2.12)$$

$$\int_1^\infty \lambda^k \psi(\lambda) d\lambda = 0, \quad \forall k \in \mathbb{N}, \quad (2.13)$$

$$\psi(\lambda) = \mathcal{O}(\lambda^{-N}), \quad \forall N \in \mathbb{N} \text{ as } \lambda \rightarrow \infty. \quad (2.14)$$

In particular, (2.14) guarantees that $|\psi|$ decays at infinity faster than the reciprocal of any polynomial.

On a different topic, recall from [16, Theorem 2, p. 171] that for any closed set $F \subseteq \mathbb{R}^n$ there exists a function $\rho_{reg} : \mathbb{R}^n \rightarrow [0, \infty)$ such that

$$\rho_{reg} \in \mathcal{C}^\infty(\mathbb{R}^n \setminus F), \quad \rho_{reg} \approx \operatorname{dist}(\cdot, F) \text{ on } \mathbb{R}^n, \quad (2.15)$$

and, with $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$|\partial^\alpha \rho_{reg}(x)| \leq C_\alpha [\operatorname{dist}(x, F)]^{1-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n \text{ and } \forall x \in \mathbb{R}^n \setminus F. \quad (2.16)$$

To proceed, let Ω be a graph Lipschitz domain in \mathbb{R}^n and denote by $\mathcal{C}_b^\infty(\overline{\Omega})$ the vector space of restrictions to Ω of functions from $\mathcal{C}_c^\infty(\mathbb{R}^n)$. Also, if ρ_{reg} stands for the regularized distance function associated with $\overline{\Omega}$, we set $\rho := C\rho_{reg}$, where $C > 0$ is a fixed constant chosen large enough so that

$$\rho(z - s\mathbf{e}_n) > 2s, \quad \forall z \in \partial\Omega \text{ and } \forall s > 0, \quad (2.17)$$

where $\{\mathbf{e}_j\}_{1 \leq j \leq n}$ denotes the standard orthonormal basis in \mathbb{R}^n (hence, $\mathbf{e}_n := (0, \dots, 0, 1) \in \mathbb{R}^n$). The above normalization condition on ρ ensures that

$$x + \lambda\rho(x)\mathbf{e}_n \in \Omega, \quad \forall x \in \mathbb{R}^n \setminus \overline{\Omega} \text{ and } \forall \lambda \geq 1. \quad (2.18)$$

Let us also note that in the current case (i.e., when $F := \overline{\Omega}$ where Ω is a graph Lipschitz domain in \mathbb{R}^n), there holds

$$\rho \in \operatorname{Lip}(\mathbb{R}^n), \quad (2.19)$$

where $\operatorname{Lip}(\mathbb{R}^n)$ stands for the set of Lipschitz functions in \mathbb{R}^n .

The role of ρ is to permit us to define Stein's extension operator (cf. [16, (24), p. 182]) acting on $u \in \mathcal{C}_b^\infty(\overline{\Omega})$ according to

$$(\mathcal{E}_{\Omega \rightarrow \mathbb{R}^n} u)(x) := \int_1^\infty u(x + \lambda\rho(x)\mathbf{e}_n)\psi(\lambda) d\lambda, \quad \forall x \in \mathbb{R}^n. \quad (2.20)$$

Incidentally, the fact that

$$\mathcal{E}_{\Omega \rightarrow \mathbb{R}^n} u \in \text{Lip}(\mathbb{R}^n) \text{ and } (\mathcal{E}_{\Omega \rightarrow \mathbb{R}^n} u)|_{\Omega} = u, \quad \forall u \in \mathcal{C}_b^\infty(\overline{\Omega}), \quad (2.21)$$

is a direct consequence of (2.19), (2.20) and (2.12).

We are now in a position to state the following extension result.

Theorem 2.4. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n . Then there exists a linear mapping*

$$\mathcal{E}_{\Omega \rightarrow \mathbb{R}^n} : \mathcal{C}^\infty(\overline{\Omega}) \longrightarrow \text{Lip}_c(\mathbb{R}^n) \quad (2.22)$$

with the property that for each $m \in \mathbb{N}_0$ the mapping $\mathcal{E}_{\Omega \rightarrow \mathbb{R}^n}$ extends to a bounded linear operator

$$\begin{aligned} \mathcal{E}_{\Omega \rightarrow \mathbb{R}^n} : W_a^{m,p}(\Omega) &\longrightarrow W_a^{m,p}(\mathbb{R}^n) \\ \text{such that } (\mathcal{E}_{\Omega \rightarrow \mathbb{R}^n} u)|_{\Omega} &= u, \quad \forall u \in W_a^{m,p}(\Omega), \end{aligned} \quad (2.23)$$

provided

$$\begin{aligned} \text{either } p \in (1, \infty) \text{ and } a &\in (-1/p, 1 - 1/p), \\ \text{or } p = 1 \text{ and } a &= 0. \end{aligned} \quad (2.24)$$

Proof. In the case when Ω is a graph Lipschitz domain, it has been proved in [3] that Stein's extension operator (2.20) does the job. This result may then be adjusted to the case when Ω is an arbitrary bounded Lipschitz domain. One way to see this is to glue together the extension operators constructed for various graph Lipschitz domains via arguments very similar to those in [16, Section 3.3, p. 189–192]. Another, perhaps more elegant argument is to change formula (2.20) to

$$(\mathcal{E}_{\Omega \rightarrow \mathbb{R}^n} u)(x) := \int_1^\infty u(x + \lambda \rho(x) h(x)) \psi(\lambda) d\lambda, \quad \forall x \in \mathbb{R}^n, \quad (2.25)$$

where $h \in \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is a suitably chosen vector field. In particular, it is assumed that h is transversal to $\partial\Omega$ in a uniform fashion, i.e., that for some constant $\kappa > 0$ there holds

$$\nu \cdot h \geq \kappa \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega, \quad (2.26)$$

where ν is the outward unit normal to Ω , and \mathcal{H}^{n-1} is the $(n - 1)$ -dimensional Hausdorff measure in \mathbb{R}^n . The vector field h is a replacement of \mathbf{e}_n and this permits us to avoid considering a multitude of special local systems of coordinates. \square

We conclude this section by discussing an important interpolation formula for weighted Sobolev spaces of arbitrary order in Lipschitz domains in Theorem 2.6 below. As a preamble, we first record the following folklore interpolation result. Here and elsewhere $[\cdot, \cdot]_\theta$ denotes the usual complex interpolation bracket.

Lemma 2.5. *Assume that X_0, X_1 and Y_0, Y_1 are two compatible pairs of Banach spaces such that $\{Y_0, Y_1\}$ is a retract of $\{X_0, X_1\}$ (here and elsewhere the “extension” and “restriction” operators are denoted by E and R , respectively). Then for each $\theta \in (0, 1)$ one has*

$$[Y_0, Y_1]_\theta = R([X_0, X_1]_\theta). \quad (2.27)$$

Here is the theorem advertised earlier, asserting that our class of weighted Sobolev spaces is stable under complex interpolation. In this regard, we wish to stress that the extension result from Theorem 2.4 plays a key role.

Theorem 2.6. *Let Ω be a Lipschitz domain in \mathbb{R}^n and assume that, for $i \in \{0, 1\}$, we have $1 < p_i < \infty$ and $-1/p_i < a_i < 1 - 1/p_i$. Fix $\theta \in (0, 1)$ and suppose that $p \in (0, \infty)$ and $a \in \mathbb{R}$ are such that $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $a = (1 - \theta)a_0 + \theta a_1$. Then for each $m \in \mathbb{N}_0$ there holds*

$$[W_{a_0}^{m, p_0}(\Omega), W_{a_1}^{m, p_1}(\Omega)]_\theta = W_a^{m, p}(\Omega). \quad (2.28)$$

Proof. The outline of the proof is as follows. First, from the well-known interpolation results for Lebesgue spaces with change of measure (cf. [2, Theorem 5.5.3, p. 120]) it follows that formula (2.28) holds in the particular case when $\Omega = \mathbb{R}^n$ and $m = 0$. Making use of [14, Theorem 3.3] we then allow $m \in \mathbb{N}_0$ arbitrary via convolution with an appropriate Bessel potential. With this in hand, (2.28) follows from (2.23) in Theorem 2.4 and the abstract retract-type result from Lemma 2.5. \square

3. THE TRACE THEOREM FOR WEIGHTED SOBOLEV SPACES

For each $k \in \mathbb{N}_0 \cup \{\infty\}$, we denote by $\mathcal{C}_b^k(\overline{\mathbb{R}_+^n})$ the restrictions to $\overline{\mathbb{R}_+^n}$ of compactly supported functions of class \mathcal{C}^k in \mathbb{R}^n . Recall that \mathcal{L}^n denotes the n -dimensional Lebesgue measure in \mathbb{R}^n and, for each $x \in \mathbb{R}_+^n$, abbreviate $\delta(x) := \text{dist}(x, \partial\mathbb{R}_+^n)$. Next, for each $p \in (1, \infty)$ and each $a \in (-\frac{1}{p}, 1 - \frac{1}{p})$, define the weighted Lebesgue space

$$L^p(\mathbb{R}_+^n, \delta^{ap} \mathcal{L}^n) = L^p(\mathbb{R}_+^n, \delta^{ap} dx) = L^p(\mathbb{R}_+^n, x_n^{ap} dx) \quad (3.1)$$

as the space of \mathcal{L}^n -measurable functions $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$ such that

$$\|f\|_{L^p(\mathbb{R}_+^n, \delta^{ap} \mathcal{L}^n)} := \left(\int_{\mathbb{R}_+^n} |f|^p \delta^{ap} d\mathcal{L}^n \right)^{1/p} < \infty. \quad (3.2)$$

Moving on, given $p \in (1, \infty)$ and $a \in (-\frac{1}{p}, 1 - \frac{1}{p})$, define the homogeneous weighted Sobolev space (of order one) in \mathbb{R}_+^n by setting

$$\dot{W}_a^{1, p}(\mathbb{R}_+^n) := \left\{ u \in L_{loc}^1(\mathbb{R}_+^n) : \partial_j u \in L^p(\mathbb{R}_+^n, \delta^{ap} dx), 1 \leq j \leq n \right\}, \quad (3.3)$$

where each $\partial_j u$ above is understood in the sense of distributions.

Finally, for $p \in [1, \infty]$ and $s \in (0, 1)$, define the homogeneous Besov norm $\|\cdot\|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})}$ as

$$\|f\|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})} := \left(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x') - f(y')|^p}{|x' - y'|^{n-1+sp}} dx' dy' \right)^{1/p}. \quad (3.4)$$

After this preamble, we are ready to deal with the main technical step in establishing the well-definiteness and boundedness of the trace operator for weighted Sobolev spaces in the upper half-space.

Proposition 3.1. *Let $p \in (1, \infty)$, pick $a \in (-\frac{1}{p}, 1 - \frac{1}{p})$, and set $s := 1 - a - 1/p \in (0, 1)$. Then for every $u \in \mathcal{C}_b^1(\overline{\mathbb{R}_+^n})$ there holds*

$$\begin{aligned} & \|u|_{\partial\mathbb{R}_+^n}\|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})} \leq \\ & \leq C_{p,a,n} \|\partial_n u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)}^{a+1/p} \|\nabla_{n-1} u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)}^{1-a-1/p}, \end{aligned} \quad (3.5)$$

where $\nabla_{n-1} u := (\partial_1 u, \dots, \partial_{n-1} u)$, and the constant $C_{p,a,n} \in (0, \infty)$ is given by

$$\begin{aligned} C_{p,a,n} = & \left[2^{2p+a-2+1/p} \cdot p^{ap+2} \cdot (ap+1)^{-a-1/p} \times \right. \\ & \left. \times (p(1-a)-1)^{a-2-ap+1/p} \cdot \omega_{n-2} \right]^{1/p}. \end{aligned} \quad (3.6)$$

In particular, $C_{p,a,n}$ satisfies

$$a \in (-1, 0] \implies C_{p,a,n} \longrightarrow (-a)^{-1} \left(\frac{2}{a+1} \right)^{a+1} \omega_{n-2} \text{ as } p \rightarrow 1^+, \quad (3.7)$$

and

$$a \in [0, 1) \implies C_{p,a,n} \rightarrow \infty \text{ as } p \rightarrow \infty. \quad (3.8)$$

As a consequence of (3.5), for every $u \in \mathcal{C}_b^1(\overline{\mathbb{R}_+^n})$ there holds

$$\begin{aligned} & \|u|_{\partial\mathbb{R}_+^n}\|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})} \leq \\ & \leq C_{p,a,n} \|\nabla u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)} = C_{p,a,n} \|u\|_{\dot{W}_a^{1,p}(\mathbb{R}_+^n)}. \end{aligned} \quad (3.9)$$

Proof. Identifying $\partial\mathbb{R}_+^n \equiv \mathbb{R}^{n-1}$, by definition we have

$$\|u|_{\partial\mathbb{R}_+^n}\|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})}^p = \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{|u(x', 0) - u(y', 0)|^p}{|x' - y'|^{n-1+sp}} dy' dx'. \quad (3.10)$$

Fix $x', y' \in \mathbb{R}^{n-1}$ and let $\lambda \in (0, \infty)$ be a fixed constant to be determined later. By the triangle inequality and the fact that $p \in (1, \infty)$, we write

$$|u(x', 0) - u(y', 0)|^p \leq 2^{2(p-1)} (I_1 + I_2 + I_3), \quad (3.11)$$

where

$$\begin{aligned} I_1 &:= \left| u(x', 0) - u(x', \lambda|x' - y'|) \right|^p, \\ I_2 &:= \left| u(x', \lambda|x' - y'|) - u(y', \lambda|x' - y'|) \right|^p, \\ I_3 &:= \left| u(y', \lambda|x' - y'|) - u(y', 0) \right|^p. \end{aligned} \quad (3.12)$$

Using this notation, we now have

$$\begin{aligned} \|u\|_{\partial\mathbb{R}_+^n}^p &\|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})}^p \leq \\ &\leq 2^{2(p-1)} \sum_{j=1}^3 \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{I_j}{|x' - y'|^{n-1+sp}} dy' dx'. \end{aligned} \quad (3.13)$$

From here, we wish to estimate the individual contributions from I_1, I_2 , and I_3 . In this vein, consider first

$$\begin{aligned} \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{I_1}{|x' - y'|^{n-1+sp}} dy' dx' &= \\ &= \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{|u(x', 0) - u(x', \lambda|x' - y'|)|^p}{|x' - y'|^{n-1+sp}} dy' dx'. \end{aligned} \quad (3.14)$$

Invoking the integral version of the (one-dimensional) mean value theorem in the n^{th} component then gives

$$\begin{aligned} \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{|u(x', 0) - u(x', \lambda|x' - y'|)|^p}{|x' - y'|^{n-1+sp}} dy' dx' &= \\ &= \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{1}{|x' - y'|^{n-1+sp}} \times \\ &\times \left| \int_0^1 \lambda|x' - y'| (\partial_n u)(x', (1-t)\lambda|x' - y'|) dt \right|^p dy' dx' \leq \\ &\leq \lambda^p \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{1}{|x' - y'|^{n-1+p(s-1)}} \times \\ &\times \left(\int_0^1 |(\partial_n u)(x', t\lambda|x' - y'|)| dt \right)^p dy' dx', \end{aligned} \quad (3.15)$$

after changing $t \mapsto 1-t$ and bringing the absolute value inside the integral. For each fixed $x' \in \mathbb{R}^{n-1}$, we will use polar coordinates to write $y' = x' + \rho\omega$, where $\omega \in S^{n-2}$ and $\rho \in (0, +\infty)$. Then, since $y' \in \mathbb{R}^{n-1}$, this implies

$dy' = \rho^{n-2} d\rho d\omega$. Thus,

$$\begin{aligned} & \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{I_1}{|x' - y'|^{n-1+sp}} dy' dx' \leq \\ & \leq \lambda^p \int_{x' \in \mathbb{R}^{n-1}} \int_{\omega \in S^{n-2}} \int_0^\infty \frac{\rho^{n-2}}{\rho^{n-1+p(s-1)}} \left(\int_0^1 |(\partial_n u)(x', \lambda \rho t)| dt \right)^p d\rho d\omega dx' = \\ & = \lambda^p \omega_{n-2} \int_{x' \in \mathbb{R}^{n-1}} \int_0^\infty \frac{1}{\rho^{1+p(s-1)}} \left(\int_0^1 |(\partial_n u)(x', \lambda \rho t)| dt \right)^p d\rho dx', \end{aligned} \tag{3.16}$$

where ω_{n-2} represents the area of the unit sphere in \mathbb{R}^{n-1} . Let us make the change of variables $\theta := (\lambda \rho)t$. This entails $d\theta = (\lambda \rho) dt$ and the interval of integration changes from $[0, 1]$ to $[0, \lambda \rho]$. Therefore, the last integral in (3.16) may be written as

$$\begin{aligned} & \lambda^p \omega_{n-2} \int_{x' \in \mathbb{R}^{n-1}} \int_0^\infty \rho^{-1+p(1-s)} \left(\int_0^{\lambda \rho} |(\partial_n u)(x', \theta)| \frac{1}{\lambda \rho} d\theta \right)^p d\rho dx' = \\ & = \omega_{n-2} \int_{x' \in \mathbb{R}^{n-1}} \int_0^\infty \rho^{-1-sp} \left(\int_0^{\lambda \rho} |(\partial_n u)(x', \theta)| d\theta \right)^p d\rho dx'. \end{aligned} \tag{3.17}$$

Make another change of variables by letting $\eta := \lambda \rho$. This yields $d\eta = \lambda d\rho$ and the interval of integration changes from $[0, \lambda \rho]$ to $[0, \eta]$. Consequently, the last integral above becomes

$$\begin{aligned} & \omega_{n-2} \int_{x' \in \mathbb{R}^{n-1}} \int_0^\infty \left(\frac{\eta}{\lambda} \right)^{-1-sp} \left(\int_0^\eta |(\partial_n u)(x', \theta)| d\theta \right)^p \frac{1}{\lambda} d\eta dx' = \\ & = \lambda^{sp} \omega_{n-2} \int_{x' \in \mathbb{R}^{n-1}} \left\{ \int_0^\infty \eta^{-1-sp} \left(\int_0^\eta |(\partial_n u)(x', \theta)| d\theta \right)^p d\eta \right\} dx'. \end{aligned} \tag{3.18}$$

At this point we wish to apply Hardy's inequality inside the curly brackets. Recall (cf., e.g., [16, p. 272, A.4]) that this states that for $q \in [1, \infty)$, $r \in (0, \infty)$, and $f : [0, \infty) \rightarrow [0, \infty)$ measurable,

$$\int_0^\infty \eta^{-1-r} \left(\int_0^\eta f(\theta) d\theta \right)^q d\eta \leq \left(\frac{q}{r} \right)^q \int_0^\infty f(\theta)^q \theta^{q-r-1} d\theta. \tag{3.19}$$

Since $u \in \mathcal{C}_b^1(\overline{\mathbb{R}_+^n})$ it follows that $|(\partial_n u)(x', \cdot)|$ is measurable and non-negative. Moreover, $s \in (0, 1)$ hence $r := sp \in (0, \infty)$. Thus, we are indeed

in a position to use Hardy's inequality with $q := p \in (1, \infty)$. Doing so gives

$$\begin{aligned} \lambda^{sp} \omega_{n-2} \int_{x' \in \mathbb{R}^{n-1}} \int_0^\infty \eta^{-1-sp} \left(\int_0^\eta |(\partial_n u)(x', \theta)| d\theta \right)^p d\eta dx' &\leq \\ &\leq \lambda^{sp} \frac{\omega_{n-2}}{s^p} \int_{x' \in \mathbb{R}^{n-1}} \int_0^\infty |(\partial_n u)(x', \theta)|^p \theta^{pa} d\theta dx' = \\ &= \lambda^{sp} \frac{\omega_{n-2}}{s^p} \int_{\mathbb{R}_+^n} |(\partial_n u)(x)|^p \delta(x)^{ap} dx, \end{aligned} \quad (3.20)$$

where the last equality is due to Fubini. Putting everything together, we have established

$$\begin{aligned} \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{I_1}{|x' - y'|^{n-1+sp}} dy' dx' &\leq \\ &\leq \lambda^{sp} \frac{\omega_{n-2}}{s^p} \int_{\mathbb{R}_+^n} |(\partial_n u)(x)|^p \delta(x)^{ap} dx. \end{aligned} \quad (3.21)$$

By interchanging the roles of x' and y' , a similar argument shows

$$\begin{aligned} \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{I_3}{|x' - y'|^{n-1+sp}} dy' dx' &\leq \\ &\leq \lambda^{sp} \frac{\omega_{n-2}}{s^p} \int_{\mathbb{R}_+^n} |(\partial_n u)(x)|^p \delta^{ap} dx. \end{aligned} \quad (3.22)$$

At this stage, we are left with estimating the contribution from I_2 . With this goal in mind, apply the integral version of the mean value theorem in \mathbb{R}^{n-1} in order to write

$$\begin{aligned} \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{I_2}{|x' - y'|^{n-1+sp}} dy' dx' &= \\ &= \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{|u(x', \lambda|x' - y'|) - u(y', \lambda|x' - y'|)|^p}{|x' - y'|^{n-1+sp}} dy' dx' = \\ &= \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{1}{|x' - y'|^{n-1+sp}} \left| \int_0^1 \left((x', \lambda|x' - y'|) - (y', \lambda|x' - y'|) \right) \times \right. \\ &\quad \left. \times (\nabla u) \left(t(x', \lambda|x' - y'|) + (1-t)(y', \lambda|x' - y'|) \right) dt \right|^p dy' dx' = \end{aligned}$$

$$\begin{aligned}
&= \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{1}{|x' - y'|^{n-1+sp}} \times \\
&\times \left| \int_0^1 (x' - y', 0) \cdot (\nabla u)(tx' + (1-t)y', \lambda|x' - y'|) dt \right|^p dy' dx' \leq \\
&\leq \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{1}{|x' - y'|^{n-1+sp}} \times \\
&\times \left(\int_0^1 |x' - y'| \left| (\nabla_{n-1} u)(tx' + (1-t)y', \lambda|x' - y'|) \right| dt \right)^p dy' dx', \quad (3.23)
\end{aligned}$$

where the last step is based on the Cauchy–Schwarz inequality. In turn, the last expression in (3.23) may be dominated by

$$\begin{aligned}
&\int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{1}{|x' - y'|^{n-1+p(s-1)}} \times \\
&\times \left[\int_0^1 \left| (\nabla_{n-1} u)(tx' + (1-t)y', \lambda|x' - y'|) \right| dt \right]^p dy' dx' = \\
&= \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \left[\int_0^1 \left(\frac{1}{|x' - y'|^{n-1+p(s-1)}} \right)^{1/p} \times \right. \\
&\quad \left. \times \left| (\nabla_{n-1} u)(tx' + (1-t)y', \lambda|x' - y'|) \right| dt \right]^p dy' dx'. \quad (3.24)
\end{aligned}$$

We proceed by invoking the generalized Minkowski inequality which permits us to estimate the last expression above by

$$\begin{aligned}
&\left[\int_0^1 \left(\int_{y' \in \mathbb{R}^{n-1}} \int_{x' \in \mathbb{R}^{n-1}} \frac{1}{|x' - y'|^{n-1+p(s-1)}} \times \right. \right. \\
&\quad \left. \left. \times \left| (\nabla_{n-1} u)(y' + t(x' - y'), \lambda|x' - y'|) \right|^p dx' dy' \right)^{1/p} dt \right]^p. \quad (3.25)
\end{aligned}$$

Introducing $z' := x' - y'$, for each fixed $y' \in \mathbb{R}^{n-1}$, and then using Fubini further transforms this expression into

$$\begin{aligned}
&\left[\int_0^1 \left(\int_{z' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{1}{|z'|^{n-1+p(s-1)}} \times \right. \right. \\
&\quad \left. \left. \times \left| (\nabla_{n-1} u)(y' + tz', \lambda|z'|) \right|^p dy' dz' \right)^{1/p} dt \right]^p. \quad (3.26)
\end{aligned}$$

Let us perform another change of variables by letting $\xi' := y' + tz'$ for fixed $t \in [0, 1]$ and fixed $z' \in \mathbb{R}^{n-1}$. This implies $d\xi' = dy'$ and (3.26) now becomes

$$\begin{aligned} & \left[\int_0^1 \left(\int_{z' \in \mathbb{R}^{n-1}} \int_{\xi' \in \mathbb{R}^{n-1}} \frac{1}{|z'|^{n-1+p(s-1)}} \times \right. \right. \\ & \quad \left. \left. \times \left| (\nabla_{n-1} u)(\xi', \lambda|z'|) \right|^p d\xi' dz' \right)^{1/p} dt \right]^p = \\ & = \int_{z' \in \mathbb{R}^{n-1}} \int_{\xi' \in \mathbb{R}^{n-1}} \frac{1}{|z'|^{n-1+p(s-1)}} \left| (\nabla_{n-1} u)(\xi', \lambda|z'|) \right|^p d\xi' dz'. \quad (3.27) \end{aligned}$$

From here, pass to polar coordinates in the variable z' . Specifically, set $z' := (\rho\omega)/\lambda$ where $\rho \in (0, \infty)$ and $\omega \in S^{n-2}$. This entails $dz' = \rho^{n-2}/\lambda^{n-1} d\rho d\omega$, so we may write (3.27) as

$$\begin{aligned} & \int_{z' \in \mathbb{R}^{n-1}} \int_{\xi' \in \mathbb{R}^{n-1}} \frac{1}{|z'|^{n-1+p(s-1)}} \left| (\nabla_{n-1} u)(\xi', \lambda|z'|) \right|^p d\xi' dz' = \\ & = \lambda^{1-n} \lambda^{n-1+p(s-1)} \int_0^\infty \int_{S^{n-2}} \int_{\xi' \in \mathbb{R}^{n-1}} \frac{\rho^{n-2}}{\rho^{n-1+p(s-1)}} \left| (\nabla_{n-1} u)(\xi', \rho) \right|^p d\xi' d\omega d\rho = \\ & = \lambda^{p(s-1)} \omega_{n-2} \int_0^\infty \int_{\xi' \in \mathbb{R}^{n-1}} \left| (\nabla_{n-1} u)(\xi', \rho) \right|^p \rho^{ap} d\xi' d\rho = \\ & = \lambda^{p(s-1)} \omega_{n-2} \int_{\mathbb{R}_+^n} \left| (\nabla_{n-1} u)(x) \right|^p \delta(x)^{ap} dx, \quad (3.28) \end{aligned}$$

where the last equality uses Fubini.

At this stage, combining (3.28), (3.27), (3.26), (3.25), (3.24), and (3.23) establishes

$$\begin{aligned} & \int_{x' \in \mathbb{R}^{n-1}} \int_{y' \in \mathbb{R}^{n-1}} \frac{I_2}{|x' - y'|^{n-1+sp}} dy' dx' \leq \\ & \leq \lambda^{p(s-1)} \omega_{n-2} \int_{\mathbb{R}_+^n} \left| (\nabla_{n-1} u)(x) \right|^p \delta(x)^{ap} dx. \quad (3.29) \end{aligned}$$

In concert, (3.29), (3.22), (3.21), and (3.13), then yield

$$\begin{aligned} & \|u\|_{\partial\mathbb{R}_+^n}^p \Big|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})} \leq 2^{2(p-1)} \left(\lambda^{sp} \frac{2\omega_{n-2}}{s^p} \times \right. \\ & \quad \left. \times \int_{\mathbb{R}_+^n} \left| (\partial_n u)(x) \right|^p \delta(x)^{ap} dx + \lambda^{p(s-1)} \omega_{n-2} \int_{\mathbb{R}_+^n} \left| (\nabla_{n-1} u)(x) \right|^p \delta(x)^{ap} dx \right) = \end{aligned}$$

$$\begin{aligned}
&= \frac{2^{2p-1} \omega_{n-2}}{s^p} \|\partial_n u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)}^p \lambda^{sp} + \\
&+ 2^{2p-2} \omega_{n-2} \|\nabla_{n-1} u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)}^p \lambda^{p(s-1)} = A \lambda^{sp} + B \lambda^{p(s-1)}, \quad (3.30)
\end{aligned}$$

where we have set

$$A := \frac{2^{2p-1} \omega_{n-2}}{s^p} \|\partial_n u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)}^p \in [0, \infty) \quad (3.31)$$

and

$$B := 2^{2p-2} \omega_{n-2} \|\nabla_{n-1} u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)}^p \in [0, \infty). \quad (3.32)$$

We need to consider several cases for the constants A and B . If $A = 0$ and $B \in [0, \infty)$, then $\|\partial_n u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)} = 0$ which forces u to be constant in the last component; i.e., for each fixed $x' \in \mathbb{R}^{n-1}$, there exists $C_{x'} \in \mathbb{R}$ such that $u(x', t) = C_{x'}$ for every $t \in (0, \infty)$. Since $u \in \mathcal{C}_b^1(\mathbb{R}_+^n)$ (in particular, u has compact support), this implies that $C_{x'} = 0$ for every $x' \in \mathbb{R}^{n-1}$. Hence, $u \equiv 0$ on the closure of the upper half-space and (3.5) is trivially valid in this case. The case when $B = 0$ and $A \in [0, \infty)$ is handled in a similar fashion. Finally, when $A \in (0, \infty)$ and $B \in (0, \infty)$ define $f : (0, \infty) \rightarrow \mathbb{R}$ by setting

$$f(x) := A x^{sp} + B x^{p(s-1)} = A x^{p(1-a)-1} + B x^{-ap-1}, \quad \forall x \in (0, \infty).$$

We wish to minimize f . To this end, we begin by noting that $f \in \mathcal{C}^\infty((0, \infty))$ and

$$\begin{aligned}
\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} (A x^{p(1-a)-1} + B x^{-ap-1}) = \infty, \\
\lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (A x^{p(1-a)-1} + B x^{-ap-1}) = \infty.
\end{aligned} \quad (3.33)$$

Moreover, since $-2 - ap \in (-p - 1, -1)$ implies $-2 - ap < 0$, we have

$$\begin{aligned}
f'(x) = 0 &\iff x^{-ap-2} [(p(1-a) - 1) A x^p - (ap + 1) B] = 0 \iff \\
&\iff (p(1-a) - 1) A x^p - (ap + 1) B = 0.
\end{aligned} \quad (3.34)$$

Solving the latter equation for x and denoting this solution as λ gives

$$\lambda = \left[\frac{(ap + 1)B}{(p(1-a) - 1)A} \right]^{1/p} \in (0, \infty) \quad (3.35)$$

is the only local extreme point of f . To determine whether λ is a local maximum or local minimum for f , consider the second derivative of f , i.e.,

$$\begin{aligned}
f''(x) &= (p(1-a) - 1)(p(1-a) - 2) A x^{p(1-a)-3} + \\
&+ (ap + 1)(ap + 2) B x^{-ap-3}.
\end{aligned} \quad (3.36)$$

Evaluating f'' at λ then gives (after some elementary algebra)

$$f''(\lambda) = B^{1-a-3/p} A^{a+3/p} (p(1-a) - 1)^{a+3/p} (ap + 1)^{1-a-3/p} p > 0. \quad (3.37)$$

As such, by the second derivative test, λ is a local minimum for f . Combining (3.33) with the fact that λ is the only local extreme point for f gives that λ is a global minimum for f . Recall that $\|u\|_{\partial \mathbb{R}_+^n} \|u\|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})}$ does not depend on λ . Therefore, we may minimize the right-hand side of (3.30) by choosing

λ as in (3.35). After a somewhat lengthy but elementary computation, this choice yields

$$\begin{aligned} & \|u|_{\partial\mathbb{R}_+^n}\|_{\dot{B}_s^{p,p}(\mathbb{R}^{n-1})}^p \leq \\ & \leq 2^{2p-2+a+1/p} \omega_{n-2} \frac{p^{ap+2}}{(ap+1)^{a+1/p}} (p(1-a)-1)^{a-2-ap+1/p} \times \\ & \quad \times \|\partial_n u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)}^{p(a+1/p)} \|\nabla_{n-1} u\|_{L^p(\mathbb{R}_+^n, \delta^{ap} dx)}^{p(1-a-1/p)}, \end{aligned} \quad (3.38)$$

as desired. \square

We are now ready to state and prove the main result in this section.

Theorem 3.2. *Assume that Ω is a bounded Lipschitz domain in \mathbb{R}^n and abbreviate $\delta(x) := \text{dist}(x, \partial\Omega)$ for each $x \in \mathbb{R}^n$. Also, let $p \in (1, \infty)$, pick $a \in (-\frac{1}{p}, 1 - \frac{1}{p})$, and set $s := 1 - a - 1/p \in (0, 1)$. Then the restriction to the boundary operator*

$$\mathcal{C}^\infty(\bar{\Omega}) \ni u \longmapsto u|_{\partial\Omega} \in \mathcal{C}^0(\partial\Omega) \quad (3.39)$$

extends to a mapping, henceforth called the trace operator,

$$\text{Tr} : W_a^{1,p}(\Omega) \longrightarrow B_s^{p,p}(\partial\Omega) \quad (3.40)$$

which is well-defined, linear, and bounded. Concretely, Tr satisfies the estimate

$$\|\text{Tr} u\|_{B_s^{p,p}(\partial\Omega)} \leq C \|u\|_{W_a^{1,p}(\Omega)}, \quad \forall u \in W_a^{1,p}(\Omega), \quad (3.41)$$

where the constant $C \in (0, \infty)$ depends only on Ω , n , p , and a .

Furthermore, the kernel of the trace operator (3.40) may be described as

$$\left\{ u \in W_a^{1,p}(\Omega) : \text{Tr} u = 0 \text{ in } B_s^{p,p}(\partial\Omega) \right\} = \dot{W}_a^{1,p}(\Omega). \quad (3.42)$$

Proof. Via a localization argument (involving a partition of unity consisting of smooth, compactly supported functions), and by locally flattening the boundary of Ω via bi-Lipschitz maps (which preserve both the category of Besov spaces and the class of weighted Sobolev spaces presently considered), matters may be reduced to treating the case when $\Omega = \mathbb{R}_+^n$ and when the Besov and Sobolev spaces in question are homogeneous. In such a scenario, the desired conclusions in the first part of the statement follow from (3.9) and a density argument (cf. (2.4)).

The right-to-left inclusion in (3.42) is clear, so we focus on the opposite one. Specifically, pick $u \in W_a^{1,p}(\Omega)$ such that $\text{Tr} u = 0$ in $B_s^{p,p}(\partial\Omega)$, with the goal of showing that $u \in \dot{W}_a^{1,p}(\Omega)$. Let \tilde{u} be the extension of u to \mathbb{R}^n taken to be zero outside Ω . Then $\tilde{u} \in L^p(\mathbb{R}^n, \delta^{ap} dx)$ and we claim that

$$\partial_j(\tilde{u}) = \widetilde{\partial_j u} \text{ in } \mathcal{D}'(\mathbb{R}^n), \quad \forall j \in \{1, \dots, n\}. \quad (3.43)$$

To this end, fix an arbitrary $j \in \{1, \dots, n\}$ and arbitrary $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then,

$$\begin{aligned} \langle \partial_j(\tilde{u}), \varphi \rangle &= -\langle \tilde{u}, \partial_j \varphi \rangle = \\ &= -\int_{\mathbb{R}^n} \tilde{u}(x)(\partial_j \varphi)(x) dx = -\int_{\Omega} u(x)(\partial_j \varphi)(x) dx. \end{aligned} \quad (3.44)$$

From (2.4) we know that $\mathcal{C}_b^\infty(\bar{\Omega}) \subseteq W_a^{1,p}(\Omega)$ densely. Hence, there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{C}_b^\infty(\bar{\Omega})$ convergent to u in $W_a^{1,p}(\Omega)$. This makes it possible to write

$$\int_{\Omega} u(x)(\partial_j \varphi)(x) dx = \lim_{k \rightarrow \infty} \int_{\Omega} u_k(x)(\partial_j \varphi)(x) dx, \quad (3.45)$$

hence, with σ denoting the surface measure on $\partial\Omega$, and $\nu = (\nu_j)_{1 \leq j \leq n}$ standing for the outward unit normal to Ω , we have

$$\begin{aligned} \langle \partial_j(\tilde{u}), \varphi \rangle &= -\lim_{k \rightarrow \infty} \int_{\Omega} u_k(x)(\partial_j \varphi)(x) dx = \\ &= \lim_{k \rightarrow \infty} \left[\int_{\Omega} (\partial_j u_k)(x) \varphi(x) dx - \int_{\partial\Omega} u_k \varphi \nu_j d\sigma \right] = \\ &= \int_{\Omega} (\partial_j u)(x) \varphi(x) dx - \lim_{k \rightarrow \infty} \int_{\partial\Omega} u_k \varphi \nu_j d\sigma = \\ &= \int_{\mathbb{R}^n} (\widetilde{\partial_j u})(x) \varphi(x) dx - \lim_{k \rightarrow \infty} \int_{\partial\Omega} (u_k|_{\partial\Omega}) \varphi \nu_j d\sigma = \\ &= \langle \widetilde{\partial_j u}, \varphi \rangle - \lim_{k \rightarrow \infty} \int_{\partial\Omega} \text{Tr } u_k \varphi \nu_j d\sigma. \end{aligned} \quad (3.46)$$

As far as the last limit above is concerned, note that

$$\begin{aligned} \left| \int_{\partial\Omega} \text{Tr } u_k \varphi \nu_j d\sigma \right| &\leq \|\varphi\|_{L^{p'}(\partial\Omega)} \|\text{Tr } u_k\|_{L^p(\partial\Omega)} \leq \\ &\leq \|\varphi\|_{L^{p'}(\partial\Omega)} \|\text{Tr } u_k\|_{B_s^{p,p}(\partial\Omega)} \longrightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned} \quad (3.47)$$

since, by the continuity of the trace operator, $\text{Tr } u_k \rightarrow \text{Tr } u = 0$ in $B_s^{p,p}(\partial\Omega)$ as $k \rightarrow \infty$. Now, (3.43) follows from (3.46). In turn, (3.43) proves that

$$\tilde{u} \in W_a^{1,p}(\mathbb{R}^n). \quad (3.48)$$

Moreover, using a partition of unity there is no loss of generality in assuming that

$$\begin{aligned} \text{supp } \tilde{u} &\text{ is contained in a neighborhood } \mathcal{O} \text{ of a point } x_* \in \partial\Omega, \\ &\text{near which } \partial\Omega \text{ coincides with a Lipschitz graph.} \end{aligned} \quad (3.49)$$

In particular, we may assume that there is a truncated circular cone Γ with vertex at the origin with the property that

$$x + \Gamma \subseteq \Omega, \quad \forall x \in \mathcal{O} \cap \partial\Omega. \quad (3.50)$$

To proceed, select $\eta \in C_c^\infty(\mathbb{R}^n)$ such that

$$\text{supp } \eta \subseteq \Gamma, \quad 0 \leq \eta \leq 1, \quad \int_{\mathbb{R}^n} \eta d\mathcal{L}^n = 1, \quad (3.51)$$

and, for each $\varepsilon > 0$, define $\eta_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\eta_\varepsilon(x) := \varepsilon^{-n} \eta(x/\varepsilon)$ for all $x \in \mathbb{R}^n$. Finally, for every $\varepsilon \in (0, 1/2)$, define

$$u_\varepsilon := [\tilde{u} * \eta_\varepsilon] \Big|_{\Omega}.$$

Then, clearly, $u_\varepsilon \in \mathcal{C}_b^\infty(\bar{\Omega})$, and we claim that

$$\exists \varepsilon_* > 0 \text{ such that } \text{supp } u_\varepsilon \subseteq \Omega, \quad \forall \varepsilon \in (0, \varepsilon_*). \quad (3.52)$$

Indeed,

$$\text{supp } u_\varepsilon = \text{supp}(\tilde{u} * \eta_\varepsilon) \subseteq \text{supp}(\tilde{u}) + \text{supp } \eta_\varepsilon \subseteq (\mathcal{O} \cap \bar{\Omega}) + \varepsilon \text{supp } \eta \subseteq \Omega, \quad (3.53)$$

where the last inclusion (which uses the fact that $\text{supp } \eta \subseteq \Gamma$) is valid for $\varepsilon > 0$ small enough.

From (3.52) we may therefore conclude that $u_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ for $\varepsilon > 0$ small, and the proof of the membership $u \in \dot{W}_a^{1,p}(\Omega)$ is finished once we show that

$$u_\varepsilon \rightarrow u \text{ in } W_a^{1,p}(\Omega) \text{ as } \varepsilon \rightarrow 0^+. \quad (3.54)$$

Since distributional derivatives commute with restrictions to Ω , the claim in (3.54) follows from the usual approximation to the identity argument bearing in mind (3.43), (2.5), and the fact that the Hardy–Littlewood maximal operator is bounded on weighted L^p spaces when the weight in question belongs to the Muckenhoupt A_p class. \square

4. THE BOUNDARY EXTENSION THEOREM FOR WEIGHTED SOBOLEV SPACES

The bulk of this section is devoted to proving the extension result stated in Theorem 4.1 below. In the last part we make use of this theorem in order to establish an interpolation formula which plays a basic role.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain, let $p \in (1, \infty)$, $a \in (-\frac{1}{p}, 1 - \frac{1}{p})$, and set $s := 1 - a - 1/p \in (0, 1)$. Then there exists a mapping*

$$\text{Ex} : B_s^{p,p}(\partial\Omega) \longrightarrow W_a^{1,p}(\Omega) \quad (4.1)$$

that is linear, bounded, and satisfies

$$\text{Tr}(\text{Ex}(f)) = f, \quad \forall f \in B_s^{p,p}(\partial\Omega). \quad (4.2)$$

Proof. We first focus on the case when $\Omega = \mathbb{R}_+^n$. To this end, let $\eta \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a function such that $\text{supp } \eta \subseteq B(0, 4)$, $\eta \equiv 1$ on $B(0, 2)$, and $0 \leq \eta \leq 1$ on \mathbb{R}^n . Next, define the kernel

$$k : \mathbb{R}_+^n \times \overline{\mathbb{R}_+^n} \longrightarrow \mathbb{R} \quad (4.3)$$

by setting

$$k(x, y) := \eta\left(\frac{x-y}{x_n}\right) \left[\int_{\mathbb{R}^{n-1}} \eta\left(\frac{x-(z', 0)}{x_n}\right) dz' \right]^{-1}, \quad (4.4)$$

$$\forall x = (x_1, \dots, x_n) \in \mathbb{R}_+^n, \quad \forall y \in \overline{\mathbb{R}_+^n}.$$

We claim that k is a well-defined, non-negative function belonging to $\mathcal{C}^\infty(\mathbb{R}_+^n \times \overline{\mathbb{R}_+^n})$. Indeed, for each fixed point $x = (x', x_n) \in \mathbb{R}_+^n$, we have

$$\frac{x-(z', 0)}{x_n} \in (0, 2) \iff |x-(z', 0)| < 2x_n \iff z' \in B_{n-1}(x', \sqrt{3}x_n). \quad (4.5)$$

Since $\mathcal{L}^{n-1}(B_{n-1}(x', \sqrt{3}x_n)) = c_n x_n^{n-1}$ (where B_{n-1} is an $(n-1)$ -dimensional ball) and $\eta \equiv 1$ on $B(0, 2)$, we have a strictly positive lower bound for the integral in the right-hand side of (4.4), namely

$$\int_{\mathbb{R}^{n-1}} \eta\left(\frac{x-(z', 0)}{x_n}\right) dz' \geq c_n x_n^{n-1}. \quad (4.6)$$

In particular, it is meaningful to discuss the reciprocal of this number, for which we have

$$\left[\int_{\mathbb{R}^{n-1}} \eta\left(\frac{x-(z', 0)}{x_n}\right) dz' \right]^{-1} \leq c_n x_n^{1-n}. \quad (4.7)$$

Having established this, the well-definedness and non-negativity of k follow immediately. Also, by design,

$$\int_{\mathbb{R}^{n-1}} k(x, (y', 0)) dy' = 1, \quad \forall x \in \mathbb{R}_+^n. \quad (4.8)$$

Concerning the regularity of k , this follows from the regularity of η and the Leibniz rule, which give that for every multi-index α

$$\begin{aligned} & \partial_x^\alpha k(x, y) = \\ & = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial_x^\beta \left\{ \eta\left(\frac{x-y}{x_n}\right) \right\} \partial_x^\gamma \left\{ \left[\int_{\mathbb{R}^{n-1}} \eta\left(\frac{x-(z', 0)}{x_n}\right) dz' \right]^{-1} \right\}, \quad (4.9) \end{aligned}$$

then, finally, invoking the chain rule. For the last step, it helps to notice that

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \eta\left(\frac{x - (z', 0)}{x_n}\right) dz' &= \int_{\mathbb{R}^{n-1}} \eta\left(\frac{x' - z'}{x_n}, 1\right) dz' = \\ &= (-x_n)^{n-1} \int_{\mathbb{R}^{n-1}} \eta(w', 1) dw' = c x_n^{n-1}, \end{aligned} \quad (4.10)$$

where $c := (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \eta(w', 1) dw'$ is a real constant. Hence, on the one hand,

$$\begin{aligned} \partial_x^\gamma \left\{ \left[\int_{\mathbb{R}^{n-1}} \eta\left(\frac{x - (z', 0)}{x_n}\right) dz' \right]^{-1} \right\} &= c \partial_x^\gamma (x_n^{1-n}) = \\ &= \begin{cases} c \left(\prod_{j=0}^{|\gamma|-1} (1 - n - j) \right) x_n^{1-n-|\gamma|}, & \text{if } \gamma = (0, \dots, 0, \gamma_n), \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (4.11)$$

On the other hand, we have

$$\begin{aligned} &\partial_x^\beta \left[\eta\left(\frac{x - y}{x_n}\right) \right] = \\ &= \sum_{|\delta| \leq |\beta|} (\partial^\delta \eta)\left(\frac{x - y}{x_n}\right) \frac{P_r^{\beta, \delta}(x_1 - y_1, \dots, x_n - y_n, x_n)}{x_n^{2|\beta|}}, \end{aligned} \quad (4.12)$$

where, generally speaking, $P_r^{\beta, \delta}(t_1, \dots, t_n, t_{n+1})$ is a homogeneous polynomial of degree r in the variables t_1, \dots, t_{n+1} ; that is,

$$P_r^{\beta, \delta}(t) = \sum_{|\gamma|=r} a_\gamma^{\beta, \delta} t^\gamma, \quad t = (t_1, \dots, t_n, t_{n+1}) \in \mathbb{R}^{n+1}, \quad (4.13)$$

where the $a_\gamma^{\beta, \delta}$'s are real-coefficients. Indeed, starting from the observation that, for each $j \in \{1, \dots, n\}$ and for each differentiable function F , there holds

$$\partial_{x_j} \left[F\left(\frac{x - y}{x_n}\right) \right] = \sum_{k=1}^n (\partial_k F)\left(\frac{x - y}{x_n}\right) \frac{\delta_{jk} x_n - (x_k - y_k) \delta_{jn}}{x_n^2}, \quad (4.14)$$

formula (4.12) may be justified by induction on the length of the multi-index $\beta \in \mathbb{N}_0^n$.

In particular, from (4.12) we see that for each $x = (x', x_n) \in \mathbb{R}_+^n$ and $y \in \overline{\mathbb{R}_+^n}$ we have

$$\begin{aligned} \frac{x-y}{x_n} \in \text{supp}(\partial^\delta \eta) &\implies \\ &\implies |x-y| \leq 4x_n \\ &\implies \left| P_{2^{|\beta|-|\beta|}}^{\beta, \delta}(x_1 - y_1, \dots, x_n - y_n, x_n) \right| \leq C_{n, \beta, \delta} x_n^{2^{|\beta|} - |\beta|} \\ &\implies \left| \partial_x^\beta \left[\eta \left(\frac{x-y}{x_n} \right) \right] \right| \leq C x_n^{-|\beta|} \chi_{|x-y| < 4x_n}. \end{aligned} \quad (4.15)$$

Collectively, (4.9), (4.11), and (4.15) imply that the function k satisfies

$$\begin{aligned} |(\partial_x^\alpha k)(x, y)| &\leq C_{n, \alpha} x_n^{1-n-|\alpha|} \chi_{|x-y| < 4x_n}, \\ \forall x = (x', x_n) \in \mathbb{R}_+^n, \forall y \in \overline{\mathbb{R}_+^n}, \forall \alpha \in \mathbb{N}_0^n. \end{aligned} \quad (4.16)$$

As a consequence,

$$|k(x, y)| \leq c_n x_n^{1-n} \chi_{|x-y| < 4x_n}, \quad \forall (x, y) \in \mathbb{R}_+^n \times \overline{\mathbb{R}_+^n} \quad (4.17)$$

and

$$|(\nabla_x k)(x, y)| \leq c_n x_n^{-n} \chi_{|x-y| < 4x_n}, \quad \forall x = (x', x_n) \in \mathbb{R}_+^n, \forall y \in \overline{\mathbb{R}_+^n}. \quad (4.18)$$

Moving on, consider the mapping \mathcal{E} taking functions defined on $\partial\mathbb{R}_+^n \equiv \mathbb{R}^{n-1}$ to functions defined in \mathbb{R}_+^n according to the formula

$$(\mathcal{E}f)(x) := \int_{\mathbb{R}^{n-1}} k(x, (y', 0)) f(y') dy', \quad \forall x \in \mathbb{R}_+^n, \forall f \in \mathcal{C}_c^0(\mathbb{R}^{n-1}). \quad (4.19)$$

Then, for each $f \in \mathcal{C}_c^0(\mathbb{R}^{n-1})$, we may employ (4.17) to conclude that $\mathcal{E}f$ is well-defined. Also, thanks to (4.16), we have that $\mathcal{E}f$ inherits the regularity of k , i.e., $\mathcal{E}f \in \mathcal{C}^\infty(\mathbb{R}_+^n)$.

We claim that for each $p \in (1, +\infty)$ and $a \in (-\frac{1}{p}, 1 - \frac{1}{p})$, there exists $C_{n, p, a} \in (0, +\infty)$ such that for each $f \in \mathcal{C}_c^0(\mathbb{R}^{n-1})$

$$\int_{\mathbb{R}_+^n} |[\nabla(\mathcal{E}f)](x)|^p x_n^{ap} dx \leq C_{n, p, a} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(y') - f(z')|^p}{|y' - z'|^{n-1+sp}} dy' dz', \quad (4.20)$$

where, as usual, $s := 1 - a - 1/p \in (0, 1)$.

To justify (4.20), fix an arbitrary $f \in \mathcal{C}_c^0(\mathbb{R}^{n-1})$ and observe that (4.19) implies that for each fixed $z' \in \mathbb{R}^{n-1}$

$$|[\nabla(\mathcal{E}f)](x)| \leq \int_{\mathbb{R}^{n-1}} |(\nabla_x k)(x, (y', 0))| |f(y') - f(z')| dy', \quad \forall x \in \mathbb{R}_+^n. \quad (4.21)$$

In turn, from (4.21), (4.18), and Hölder's inequality we obtain that for each $x = (x', x_n) \in \mathbb{R}_+^n$ and each $z' \in \mathbb{R}^{n-1}$

$$\begin{aligned} |[\nabla(\mathcal{E}f)](x)| &\leq C \left(x_n^{-n} \int_{|x-(y',0)| < 4x_n} |f(y') - f(z')| dy' \right)^p \\ &\leq C x_n^{-np} \cdot x_n^{(p-1)(n-1)} \int_{|x-(y',0)| < 4x_n} |f(y') - f(z')|^p dy'. \end{aligned} \quad (4.22)$$

At this stage, average the most extreme sides of (4.22) in $z' \in B_{n-1}(x, 4x_n) \subseteq \mathbb{R}^{n-1}$ in order to obtain

$$\begin{aligned} &|[\nabla(\mathcal{E}f)](x)|^p \leq \\ &\leq C x_n^{2-2n-p} \int_{|x-(z',0)| < 4x_n} \int_{|x-(y',0)| < 4x_n} |f(y') - f(z')|^p dy' dz' \end{aligned} \quad (4.23)$$

for each $x = (x', x_n) \in \mathbb{R}_+^n$. Consequently,

$$\begin{aligned} &\int_{\mathbb{R}_+^n} |[\nabla(\mathcal{E}f)](x)|^p x_n^{ap} dx \leq \\ &\leq C \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |f(y') - f(z')|^p \left[\int_{\substack{|x-(z',0)| < 4x_n \\ |x-(y',0)| < 4x_n}} x_n^{ap-p-2n+2} dx \right] dy' dz'. \end{aligned} \quad (4.24)$$

Observe that on the domain of integration of the inner-most integral we have $|x' - z'| < \sqrt{15} x_n$ and $|x' - y'| < \sqrt{15} x_n$, hence also $|y' - z'| < 2\sqrt{15} x_n$ by the triangle inequality. Bearing this in mind and using Fubini's theorem, we may estimate this inner-most integral by writing

$$\begin{aligned} &\int_{\substack{|x-(z',0)| < 4x_n \\ |x-(y',0)| < 4x_n}} x_n^{ap-p-2n+2} dx \leq \\ &\leq \int_{|y'-z'|/(2\sqrt{15})}^{\infty} \left(\int_{|x'-z'| < \sqrt{15} x_n} 1 dx' \right) x_n^{ap-p-2n+2} dx_n \leq \\ &\leq C_n \int_{|y'-z'|/(2\sqrt{15})}^{\infty} x_n^{ap-p-n+1} dx_n = \frac{C_{n,a,p}}{|y' - z'|^{n+p-ap-2}}, \end{aligned} \quad (4.25)$$

where $C_{n,a,p} > 0$ is a finite constant, given that $ap - p - n + 1 < -1$. At this stage, (4.20) follows from (4.24) and (4.25).

Moving on, we claim that for each radius $R \in (0, +\infty)$ there exists a constant $C_{n,p,a,R} \in (0, +\infty)$ with the property that

$$\begin{aligned} \int_{\mathbb{R}_+^n \cap B(0,R)} |(\mathcal{E}f)(x)|^p x_n^{ap} dx &\leq \\ &\leq C_{n,p,a,R} \int_{\mathbb{R}^{n-1}} |f(y')|^p dy', \quad \forall f \in \mathcal{C}_c^0(\mathbb{R}^{n-1}). \end{aligned} \quad (4.26)$$

This estimate follows from a similar argument to that used in the verification of (4.20) (making use of (4.17) in place of (4.18)).

The final property of the operator \mathcal{E} we wish to establish is that for each $f \in \mathcal{C}_c^0(\mathbb{R}^{n-1})$

$$\begin{aligned} \mathcal{E}f \text{ extends continuously to } \overline{\mathbb{R}_+^n} \text{ and} \\ [(\mathcal{E}f)|_{\partial\mathbb{R}_+^n}](x') = f(x'), \quad \forall x' \in \mathbb{R}^{n-1} \equiv \partial\mathbb{R}_+^n. \end{aligned} \quad (4.27)$$

To this end, fix $f \in \mathcal{C}_c^0(\mathbb{R}^{n-1})$ along with some $x'_* \in \mathbb{R}^{n-1}$. Also, let some arbitrary $\varepsilon > 0$ be fixed. Since f is continuous at $(x'_*, 0)$, there exists $\delta > 0$ such that if $y' \in \mathbb{R}^{n-1}$ satisfies $|x'_* - y'| < \delta$ then $|f(x'_*) - f(y')| < \varepsilon$. Then for each $x = (x', x_n) \in \mathbb{R}_+^n$ we may estimate

$$\begin{aligned} |(\mathcal{E}f)(x) - f(x'_*)| &= \left| \int_{\mathbb{R}^{n-1}} k(x, (y', 0)) (f(y') - f(x'_*)) dy' \right| \\ &\leq \int_{\mathbb{R}^{n-1}} |k(x, (y', 0))| |f(y') - f(x'_*)| dy' \\ &\leq C_n \int_{|x' - y'| < \sqrt{15} x_n} |f(y') - f(x'_*)| dy', \end{aligned} \quad (4.28)$$

where the equality is based on (4.8), while for the last inequality we have used (4.17) and that the set $\{y' \in \mathbb{R}^{n-1} : |x - (y', 0)| < 4x_n\}$ is contained in the set $\{y' \in \mathbb{R}^{n-1} : |x' - y'| < \sqrt{15} x_n\}$. Thus,

$$|(\mathcal{E}f)(x) - f(x'_*)| \leq \varepsilon \text{ if } |x' - x'_*| < \delta/2 \text{ and } x_n < \delta/(2\sqrt{15}), \quad (4.29)$$

and the claims in (4.27) readily follow from this. In particular, $\text{Tr } \mathcal{E}f = f$. This completes the discussion in the case when $\Omega = \mathbb{R}_+^n$.

The general situation when Ω is an arbitrary bounded Lipschitz domain may then be reduced to the case just treated via a smooth localization and by locally flattening the boundary via bi-Lipschitz maps (as we have done in the past). Given that $(\mathcal{C}_c^\infty(\mathbb{R}^n))|_{\partial\Omega}$ is dense in $B_s^{p,p}(\partial\Omega)$, the a priori bounds established in the first part of the proof may be used to conclude that all desired properties of the extension operator hold in this degree of generality. \square

In the last part of this section we once again revisit the issue of how weighted Sobolev spaces behave under complex interpolation. Our first result in this regard reads as follows.

Theorem 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$\begin{aligned} & \{\mathring{W}_a^{1,p}(\Omega)\}_{1 < p < \infty, -1/p < a < 1-1/p}, \\ & \{W_a^{-1,p}(\Omega)\}_{1 < p < \infty, -1/p < a < 1-1/p} \end{aligned} \quad (4.30)$$

are complex interpolation scales, in the following precise sense. Suppose that, for $j \in \{0, 1\}$, we have $1 < p_j < \infty$ and $-1/p_j < a_j < 1 - 1/p_j$. Also, fix $\theta \in (0, 1)$ and assume that $p \in (0, \infty)$ and $a \in \mathbb{R}$ are such that $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $a = (1 - \theta)a_0 + \theta a_1$. Then

$$[\mathring{W}_{a_0}^{1,p_0}(\Omega), \mathring{W}_{a_1}^{1,p_1}(\Omega)]_\theta = \mathring{W}_a^{1,p}(\Omega), \quad (4.31)$$

$$[W_{a_0}^{-1,p_0}(\Omega), W_{a_1}^{-1,p_1}(\Omega)]_\theta = W_a^{-1,p}(\Omega). \quad (4.32)$$

In the proof of the above theorem the following abstract interpolation result with constraints is going to be useful. For a proof, see [10, Theorem 14.3, p. 97] (cf. also [8]).

Lemma 4.3. *Let $X_j, Y_j, Z_j, j = 0, 1$, be Banach spaces such that $X_0 \cap X_1$ is dense in both X_0 and X_1 , and similarly for Z_0, Z_1 . Suppose that $Y_j \hookrightarrow Z_j, j = 0, 1$ and there exists a linear operator D such that $D : X_j \rightarrow Z_j$ boundedly for $j = 0, 1$. Define the spaces*

$$X_j(D) := \{u \in X_j : Du \in Y_j\}, \quad j = 0, 1, \quad (4.33)$$

equipped with the graph norm, i.e. $\|u\|_{X_j(D)} := \|u\|_{X_j} + \|Du\|_{Y_j}, j = 0, 1$. Finally, suppose that there exist continuous linear mappings $K : Z_j \rightarrow X_j$ and $R : Z_j \rightarrow Y_j$ with the property $D \circ K = I + R$ on the spaces Z_j for $j = 0, 1$. Then

$$[X_0(D), X_1(D)]_\theta = \left\{ u \in [X_0, X_1]_\theta : Du \in [Y_0, Y_1]_\theta \right\}, \quad \theta \in (0, 1). \quad (4.34)$$

We shall also need the well-known duality formula for the complex method of interpolation (see, for instance, [2]).

Lemma 4.4. *Let X_0, X_1 be a compatible couple of reflexive Banach spaces and let $\theta \in (0, 1)$. Then*

$$([X_0, X_1]_\theta)^* = [X_0^*, X_1^*]_\theta. \quad (4.35)$$

We are prepared to present the

Proof of Theorem 4.2. Formula (4.31) follows from Theorem 4.1 and Lemma 4.3, used with

$$X_j := W_{a_j}^{1,p_j}(\Omega), \quad Y_j := 0, \quad \text{and} \quad Z_j := B_{s_j}^{p_j,p_j}(\partial\Omega) \quad (4.36)$$

(as usual, $s_j := 1 - a_j - 1/p_j$), for $j = 0, 1$, and where

$$D := \text{Tr}, \quad K := \text{Ex}, \quad \text{and} \quad R := 0. \quad (4.37)$$

That $D \circ K = I + R$ on Z_j for $j = 0, 1$ makes the object of (4.2), and (4.34) becomes precisely (4.31), in light of (3.42). Finally, (4.32) is a consequence of (4.31), Lemma 4.4, and Proposition 2.3. \square

5. BOUNDARY PROBLEMS FOR ELLIPTIC SYSTEMS WITH BOUNDED MEASURABLE COEFFICIENTS IN EUCLIDEAN LIPSCHITZ DOMAINS

The goal here is to prove the following sharp well-posedness result.

Theorem 5.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, connected, Lipschitz domain and assume that*

$$\mathbf{A} = (a_{jk}^{\alpha\beta})_{\substack{1 \leq j, k \leq n \\ 1 \leq \alpha, \beta \leq M}}, \quad a_{jk}^{\alpha\beta} \in L^\infty(\Omega), \quad (5.1)$$

is a coefficient tensor satisfying the strong Legendre-Hadamard ellipticity condition

$$\begin{aligned} \operatorname{Re} \left[\sum_{j, k=1}^n \sum_{\alpha, \beta=1}^M a_{jk}^{\alpha\beta}(x) \zeta_j^\alpha \overline{\zeta_k^\beta} \right] &\geq c |\zeta|^2, \\ \forall \zeta = (\zeta_j^\alpha)_{\substack{1 \leq j \leq n \\ 1 \leq \alpha \leq M}} \in \mathbb{C}^{nM}, &\text{ for a.e. } x \in \Omega, \end{aligned} \quad (5.2)$$

for some $c \in (0, \infty)$. Associated with the coefficient tensor \mathbf{A} consider the $M \times M$ second order system in divergence form

$$L\mathbf{u} := \left(\sum_{j=1}^n \partial_j \left(\sum_{k=1}^n \sum_{\beta=1}^M a_{jk}^{\alpha\beta} \partial_k u_\beta \right) \right)_{1 \leq \alpha \leq M}, \quad \mathbf{u} = (u_\beta)_{1 \leq \beta \leq M}. \quad (5.3)$$

Then there exists some $\varepsilon > 0$ such that whenever

$$p \in (2 - \varepsilon, 2 + \varepsilon), \quad a \in (-1/p, 1 - 1/p) \cap (-\varepsilon, \varepsilon), \quad s := 1 - a - 1/p, \quad (5.4)$$

the Poisson boundary value problem with Dirichlet boundary data,

$$\begin{cases} \mathbf{u} \in W_a^{1,p}(\Omega), \\ L\mathbf{u} = \mathbf{f} \in W_a^{-1,p}(\Omega), \\ \operatorname{Tr} \mathbf{u} = \mathbf{g} \in B_s^{p,p}(\partial\Omega), \end{cases} \quad (5.5)$$

is well-posed. That is, assuming p, a, s are as in (5.4), for each $\mathbf{f} \in W_a^{-1,p}(\Omega)$ and $\mathbf{g} \in B_s^{p,p}(\partial\Omega)$ there exists a unique solution \mathbf{u} of (5.5), which also satisfies the estimate

$$\|\mathbf{u}\|_{W_a^{1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{W_a^{-1,p}(\Omega)} + \|\mathbf{g}\|_{B_s^{p,p}(\partial\Omega)} \right), \quad (5.6)$$

where $C \in (0, +\infty)$ is independent of \mathbf{f} and \mathbf{g} .

To set the stage, we first record a useful preliminary result in the proposition below. General abstract stability results of this type have been established in [9].

Proposition 5.2. *Suppose I is a convex Euclidean set and $(X_q)_{q \in I}$, $(Y_q)_{q \in I}$ are two complex interpolation scales of Banach spaces. In addition, assume that T is an operator such that*

$$\begin{aligned} T : X_q &\longrightarrow Y_q \text{ linearly and boundedly for each } q \in I, \text{ and} \\ &\exists q_* \in I \text{ such that } T : X_{q_*} \longrightarrow Y_{q_*} \text{ is an isomorphism.} \end{aligned} \quad (5.7)$$

Then there exists a neighborhood \mathcal{O} of q_ such that $T : X_q \rightarrow Y_q$ is an isomorphism for every $q \in \mathcal{O}$.*

We may now turn our attention to presenting the

Proof of Theorem 5.1. For starters, from the discussion in Section 2 we know that

$$\begin{aligned} W_a^{-1,p}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) : \exists h_0, h_1, \dots, h_n \in L^p(\Omega, \delta^{ap} dx) \right. \\ \left. \text{such that } u = h_0 + \sum_{j=1}^n \partial_j h_j \text{ in } \mathcal{D}'(\Omega) \right\}, \end{aligned} \quad (5.8)$$

and the norm on this space is equivalent to

$$\begin{aligned} \|u\|_{W_a^{-1,p}(\Omega)} = \inf \left\{ \sum_{j=0}^n \|h_j\|_{L^p(\Omega, \delta^{ap} dx)} : h_0, h_1, \dots, h_n \in L^p(\Omega, \delta^{ap} dx) \right. \\ \left. \text{such that } u = h_0 + \sum_{j=1}^n \partial_j h_j \text{ in } \mathcal{D}'(\Omega) \right\}. \end{aligned}$$

Granted these, it follows that

$$\begin{aligned} L : W_a^{-1,p}(\Omega) &\longrightarrow W_a^{-1,p}(\Omega) \text{ linearly and boundedly,} \\ &\text{whenever } p \in (1, \infty) \text{ and } a \in (-1/p, 1 - 1/p). \end{aligned} \quad (5.9)$$

In addition, from the Lax–Milgram Lemma (which, in turn, makes use of the strong ellipticity condition on L) we deduce that

$$L : \mathring{W}^{1,2}(\Omega) \longrightarrow W^{-1,2}(\Omega) \text{ isomorphically.} \quad (5.10)$$

Our next claim is that there exists $\varepsilon > 0$ such that

$$\begin{aligned} L : \mathring{W}_a^{1,p}(\Omega) &\longrightarrow W_a^{-1,p}(\Omega) \text{ isomorphically} \\ &\text{whenever } p \in (2 - \varepsilon, 2 + \varepsilon) \text{ and } a \in (-1/p, 1 - 1/p) \cap (-\varepsilon, \varepsilon). \end{aligned} \quad (5.11)$$

This follows from (5.9), (5.10), and Proposition 5.2.

Having proved (5.11), the final step is to show that, for p , a as above and with $s := 1 - a - 1/p$, the boundary value problem (5.5) is well-posed. Uniqueness is clear from (5.11) and (3.42). For existence, let $\mathbf{f} \in W_a^{-1,p}(\Omega)$ and $\mathbf{g} \in B_s^{p,p}(\partial\Omega)$ be given. From Theorem 4.1, we know that $\mathbf{v} := \text{Ex } \mathbf{g} \in W_a^{1,p}(\Omega)$ satisfies $\text{Tr } \mathbf{v} = \mathbf{g}$. Moreover, since the operator Ex is bounded, we have

$$\|\mathbf{v}\|_{W_a^{1,p}(\Omega)} \leq C \|\mathbf{g}\|_{B_s^{p,p}(\partial\Omega)}, \quad (5.12)$$

where $C \in (0, \infty)$ is independent of \mathbf{g} . Consider the function $\tilde{\mathbf{f}} := \mathbf{f} - L\mathbf{v} \in W_a^{-1,p}(\Omega)$ and note that

$$\|\tilde{\mathbf{f}}\|_{W_a^{-1,p}(\Omega)} \leq C \left(\|\mathbf{f}\|_{W_a^{-1,p}(\Omega)} + \|\mathbf{g}\|_{B_s^p(\partial\Omega)} \right), \quad (5.13)$$

where $C \in (0, \infty)$ is independent of \mathbf{f} and \mathbf{g} . Since $L : \dot{W}_a^{1,p}(\Omega) \rightarrow W_a^{-1,p}(\Omega)$ is an isomorphism and $\tilde{\mathbf{f}} \in W_a^{-1,p}(\Omega)$, it follows that $\mathbf{w} := L^{-1}(\tilde{\mathbf{f}}) \in \dot{W}_a^{1,p}(\Omega)$ and $L\mathbf{w} = \tilde{\mathbf{f}}$. Finally, take $\mathbf{u} := \mathbf{v} + \mathbf{w} \in W_a^{1,p}(\Omega)$ and compute

$$L\mathbf{u} = L\mathbf{v} + \tilde{\mathbf{f}} = L\mathbf{v} + (\mathbf{f} - L\mathbf{v}) = \mathbf{f} \quad (5.14)$$

and

$$\text{Tr } \mathbf{u} = \text{Tr}(\text{Ex } \mathbf{g}) + \text{Tr} (L^{-1}(\tilde{\mathbf{f}})) = \mathbf{g} + \text{Tr } \mathbf{w} = \mathbf{g} + \mathbf{0} = \mathbf{g}. \quad (5.15)$$

This finishes the existence of a function \mathbf{u} satisfying the boundary value problem. \square

Theorem 5.1 is sharp, in the sense that the membership of p to a small neighborhood of 2 is a necessary condition, even when $\Omega \subseteq \mathbb{R}^n$ is a bounded \mathcal{C}^∞ domain, and when $a = 0$ (i.e., in the unweighted case), if the coefficients of the system L are merely bounded and measurable.

When $n \geq 3$, $M = n$, a counterexample may be produced by altering a construction of E. De Giorgi from [5]. Specifically, consider $\Omega := \{x \in \mathbb{R}^n : |x| < 1\}$ and, for each $\gamma \in [0, \frac{n}{2})$ and $\alpha, \beta \in \{1, \dots, n\}$, let $A_{\alpha\beta}$ be the $n \times n$ matrix whose (i, j) -entry is

$$\begin{aligned} a_{ij}^{\alpha\beta}(x) &:= \delta_{\alpha\beta}\delta_{ij} + \\ &+ \frac{\gamma(n-\gamma)(n-2)^2}{(n-2\gamma)^2(n-1)^2} \left[\delta_{i\alpha} + \frac{n}{n-2} \frac{x_i x_\alpha}{|x|^2} \right] \left[\delta_{j\beta} + \frac{n}{n-2} \frac{x_j x_\beta}{|x|^2} \right] \end{aligned} \quad (5.16)$$

for each $x \in \Omega \setminus \{0\}$. Obviously, $a_{ij}^{\alpha\beta} \in L^\infty(\Omega, \mathcal{L}^n)$ and a straightforward calculation shows that

$$\begin{aligned} &\sum_{\alpha, \beta=1}^n \sum_{i, j=1}^n a_{ij}^{\alpha\beta}(x) \zeta_i^\alpha \zeta_j^\beta = \\ &= |\zeta|^2 + \frac{\gamma(n-\gamma)(n-2)^2}{(n-2\gamma)^2(n-1)^2} \left(\sum_{i=1}^n \zeta_i^i + \frac{n}{n-2} \sum_{i, \alpha=1}^n \zeta_i^\alpha \frac{x_i x_\alpha}{|x|^2} \right)^2 \end{aligned} \quad (5.17)$$

for each $\zeta = (\zeta_i^\alpha)_{1 \leq \alpha, i \leq n} \in \mathbb{R}^{n^2}$ and $x \in \Omega \setminus \{0\}$. Given our assumptions on γ , it follows that the strong ellipticity condition holds:

$$\sum_{\alpha, \beta=1}^n \sum_{i, j=1}^n a_{ij}^{\alpha\beta}(x) \zeta_i^\alpha \zeta_j^\beta \geq |\zeta|^2 \quad \mathcal{L}^n\text{-a.e. in } \Omega, \quad (5.18)$$

$$\forall \zeta = (\zeta_i^\alpha)_{1 \leq \alpha, i \leq n} \in \mathbb{R}^{n^2}.$$

Now, the fact that $\gamma < n/2$ ensures that the function

$$u(x) := \frac{x}{|x|^\gamma} - x, \quad \forall x \in \Omega \setminus \{0\}, \quad (5.19)$$

belongs to $W^{1,2}(\Omega)$. Since by design $u|_{\partial\Omega} = 0$, we deduce that actually $u \in \dot{W}^{1,2}(\Omega)$. Furthermore, if

$$f := (f_1, \dots, f_n) \quad \text{with} \quad f_i := - \sum_{\alpha=1}^n \sum_{j=1}^n \partial_\alpha a_{ij}^{\alpha_j} \quad \text{for} \quad 1 \leq i \leq n, \quad (5.20)$$

then clearly

$$f \in \bigcap_{1 < p < \infty} W^{-1,p}(\Omega), \quad (5.21)$$

while a direct computation shows that

$$\sum_{\alpha,\beta=1}^n \partial_\alpha (A_{\alpha\beta}(x) \partial_\beta u) = f \quad \text{in} \quad \mathcal{D}'(\Omega). \quad (5.22)$$

However, on the one hand $u \in W^{1,p}(\Omega)$ if and only if $p < n/\gamma$, while on the other hand $n/\gamma \searrow 2$ as $\gamma \nearrow n/2$. By duality, (note that L is formally self-adjoint), the same type of conclusion holds for $p < 2$.

6. THE SETTING OF WEAKLY LIPSCHITZ DOMAINS

A careful inspection of the arguments in the proof of Theorem 5.1 reveals that we may relax the assumption on the domain Ω , originally assumed to be a Lipschitz domain. Specifically, it suffices to ask that $\Omega \subset \mathbb{R}^n$ is a bounded, open set, with the property that for every $x_0 \in \partial\Omega$ there exist an open neighborhood U of x_0 in \mathbb{R}^n and a mapping $F = (F_1, \dots, F_n) : U \rightarrow \mathbb{R}^n$ with the following properties:

- (i) $F(U)$ is open and $F : U \rightarrow F(U)$ is a bi-Lipschitz map;
- (ii) $\Omega \cap U = \{x \in U : F_n(x) > 0\}$.

In the sequel, we shall refer to such a set Ω as being a *weakly Lipschitz domain*. This is done in order to distinguish the latter from the more familiar category of “strongly” Lipschitz domains considered so far.

Note that if the bi-Lipschitzianity assumption for F is strengthened by demanding that F is a \mathcal{C}^1 -diffeomorphism, then the resulting class becomes precisely the category of bounded \mathcal{C}^1 domains in \mathbb{R}^n . This is easily seen by invoking the standard Implicit Function Theorem for \mathcal{C}^1 functions. However, when dealing with the case when F is only bi-Lipschitz, the nature of the Implicit Function Theorem changes drastically and, as a result, the class of weakly Lipschitz domains is much larger than that of strongly Lipschitz domains. To shed light on this issue, we next discuss some concrete examples. In fact, since the bi-Lipschitz image of a strongly Lipschitz domain is a weakly Lipschitz domain, it suffices to show that the class of strongly Lipschitz domains is not stable under bi-Lipschitz homeomorphisms.

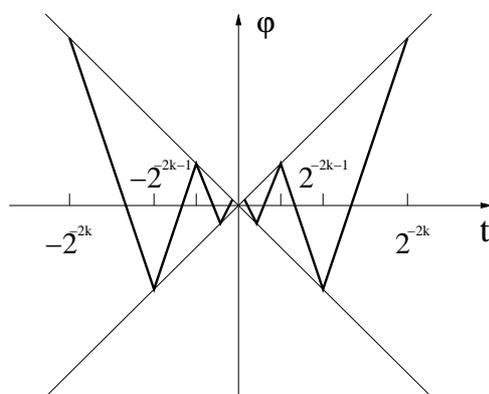
We start with an interesting example from (pp. 7–9 in) [6], where this is attributed to Zerner. Concretely, consider the bi-Lipschitz homeomorphism

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad F(x_1, x_2) := (x_1, \varphi(x_1) + x_2), \quad (6.1)$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is the Lipschitz function

$$\varphi(t) := \begin{cases} 3|t| - \frac{1}{2^{2k-1}} & \text{for } \frac{1}{2^{2k+1}} \leq |t| \leq \frac{1}{2^{2k}}, \\ -3|t| + \frac{1}{2^{2k}} & \text{for } \frac{1}{2^{2k+2}} \leq |t| \leq \frac{1}{2^{2k+1}}. \end{cases} \quad (6.2)$$

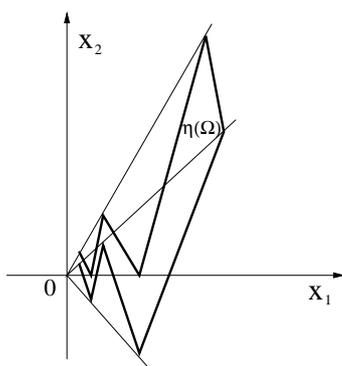
As is also visible from the picture below, the graph of φ is a zigzagged of lines of slopes ± 3 :



If one now considers the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$,

$$\Omega := \left\{ (x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < x_1 \right\}, \quad (6.3)$$

then $F(\Omega)$, depicted below



fails to be a strongly Lipschitz domain, since the cone property is violated at the origin.

In fact, the construction described above can be refined to show that *bi-Lipschitz functions may fail to map even bounded C^∞ planar domains into strongly Lipschitz domains*. Concretely, pick $x_0 \in \Omega$ and let $\varphi : S^1 \rightarrow (0, \infty)$ be the Lipschitz function uniquely determined by the requirement that $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by $G(x) := \varphi((x - x_0)/|x - x_0|)(x - x_0)$ if $x \neq x_0$ and $G(x_0) := 0$, maps $\partial B(x_0, r)$ onto $\partial\Omega$ (for some fixed, sufficiently small $r > 0$). Then $F \circ G$ maps the bounded, C^∞ domain $B(x_0, r)$ onto the domain shown in the picture above. There are many other interesting examples of strongly Lipschitz domains $\Omega \subset \mathbb{R}^n$ and bi-Lipschitz maps $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the property that $F(\Omega)$ fails to be strongly Lipschitz. A large category of such examples can be found within the class of *conical domains*. In order to be more specific, let S^{n-1} stand for the unit sphere in \mathbb{R}^n and denote by S_+^{n-1} its upper hemisphere. Pick a bi-Lipschitz homeomorphism $\psi : S^{n-1} \rightarrow S^{n-1}$ along with an arbitrary Lipschitz function $\varphi : S^{n-1} \rightarrow (0, \infty)$, and set

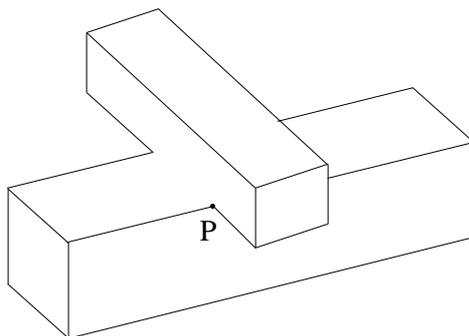
$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad F(r\omega) := r\varphi(\omega)\psi^{-1}(\omega), \quad r \geq 0, \quad \omega \in S^{n-1}, \quad (6.4)$$

$$\Omega := \left\{ r\omega : \omega \in S_+^{n-1}, \quad 0 < r < \varphi(\omega) \right\}. \quad (6.5)$$

Using $|r_1\omega_1 - r_2\omega_2|^2 = |r_1 - r_2|^2 + r_1r_2|\omega_1 - \omega_2|^2$ for every $\omega_1, \omega_2 \in S^{n-1}$, $r_1, r_2 \geq 0$, and the fact that the inverse of (6.4) is $F^{-1}(r\omega) = r\varphi(\omega)^{-1}\psi(\omega)$, it can be easily checked that F above is bi-Lipschitz. However, while $\Omega \subset \mathbb{R}^n$ is clearly a strongly Lipschitz domain in \mathbb{R}^n ,

$$F(\Omega) = \left\{ \rho w : w \in \psi(S_+^{n-1}), \quad 0 < \rho < \varphi(\omega) \right\}, \quad (6.6)$$

may fail to be a strongly Lipschitz domain. In fact, near $0 \in \partial F(\Omega)$, the surface $\partial F(\Omega)$ may fail to be the graph of *any* real-valued function of $n - 1$ variables, in any system of coordinates which is a rigid motion of the standard one (i.e., $\partial F(\Omega)$ is a non-Lipschitz cone). A concrete example, which can be produced using the above recipe, is Maz'ya's so-called *two-brick domain*:



A moment's reflection shows that, indeed, near the point P , the boundary of the above domain is not the graph of any function (as it fails the vertical line test) in any system of coordinates isometric to the original one.

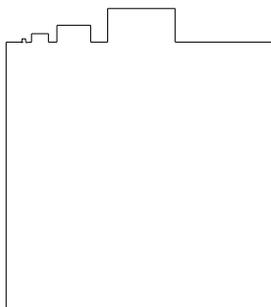
Moreover, images of bounded strongly Lipschitz domains via bi-Lipschitz maps can also develop spiral-like singularities, such as

$$\begin{aligned} F(\Omega) &= \left\{ re^{i(\theta - \ln r)} : 0 < \theta < \pi/4, 0 < r < 1 \right\} \subset \mathbb{R}^2 \equiv \mathbb{C}, \\ \Omega &:= \left\{ re^{i\theta} : 0 < r < 1, 0 < \theta < \pi/4 \right\}, \quad F(re^{i\theta}) := re^{i(\theta - \ln r)}. \end{aligned} \quad (6.7)$$

Another interesting example of the phenomenon described above is as follows. Let

$$\tilde{\Omega} := [(0, 1) \times (-1, 0)] \cup \left[\bigcup_{k=1}^{\infty} (3 \cdot 2^{-k-2}, 5 \cdot 2^{-k-2}) \times [0, 2^{-k-2}] \right] \quad (6.8)$$

be the planar domain in the picture below:



It is not difficult to see that the uniformity of the cone condition is violated in any neighborhood of the origin, so $\tilde{\Omega}$ is not a strongly Lipschitz domain. Nonetheless, on p. 19 of [11], Maz'ya has constructed a bi-Lipschitz map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with the property that $\tilde{\Omega} = F((0, 1) \times (0, 1))$.

In the next section we shall actually take this analysis a step further and indicate that well-posedness results in the spirit of those established so far continue to hold in the setting of Lipschitz manifolds with boundary, which is even more general (as all weakly Lipschitz domains in \mathbb{R}^n fall into the latter category).

7. THE SETTING OF LIPSCHITZ MANIFOLDS WITH BOUNDARY

For the convenience of the reader, here we collect some basic rudiments of analysis on Lipschitz manifolds.

A compact topological manifold with boundary \mathcal{M} of dimension n is a compact, Hausdorff topological space \mathcal{M} with the property that for every $x \in \mathcal{M}$ there exists an open set U in \mathcal{M} , $x \in U$, and a mapping $\phi : U \rightarrow \mathbb{R}^n$ such that $\phi(U)$ is a relatively open subset of $\overline{\mathbb{R}_+^n}$ and $\phi : U \rightarrow \phi(U)$ is a homeomorphism. We shall call (U, ϕ) a *coordinate chart* (about x). An *atlas*

on \mathcal{M} is a finite family $\mathcal{A} = \{U_i, \phi_i\}_{i \in I}$ such that $\mathcal{M} = \bigcup_{i \in I} U_i$ and (U_i, ϕ_i) is a coordinate chart for each $i \in I$.

Define the interior Ω of \mathcal{M} as the collection of points x for which there is a coordinate chart (U, ϕ) about x with the property that $\phi(U)$ is an open subset of \mathbb{R}_+^n . Then set $\partial\Omega := \mathcal{M} \setminus \Omega$ and call it the *boundary of \mathcal{M}* .

A compact topological manifold with boundary \mathcal{M} is called a *compact Lipschitz manifold with boundary* if there exists an atlas (called Lipschitz atlas) $\mathcal{A} = \{U_i, \phi_i\}_{i \in I}$ such that for any $i, j \in I$ the transition map $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$ is by-Lipschitz (with respect to the usual metric in \mathbb{R}^n). Two atlases are called *equivalent* provided their union is an atlas. A *Lipschitz structure* on \mathcal{M} is the equivalence class of a certain Lipschitz atlas, called *structural atlas*. In what follows, given a compact Lipschitz manifold with boundary \mathcal{M} , we shall always assume that a Lipschitz structure on \mathcal{M} has been fixed. Any Lipschitz atlas compatible with this structure will be referred to as a structural atlas.

Given a compact Lipschitz manifold with boundary \mathcal{M} , equipped with a structural atlas $\mathcal{A} = \{U_i, \phi_i\}_{i \in I}$, call a set $S \subseteq \mathcal{M}$ of *zero measure* in \mathcal{M} if $\phi_i(U_i \cap S)$ has measure zero in \mathbb{R}^n with respect to the usual n -dimensional Lebesgue measure for every $(U_i, \phi_i) \in \mathcal{A}$. Accordingly, a property is said to hold *almost everywhere* (a.e.) on \mathcal{M} provided the set of points where it fails has zero measure in \mathcal{M} .

A real-valued function defined a.e. on \mathcal{M} is called *measurable* if it is so in any coordinate chart of a structural atlas. Furthermore, the class $L^p(\mathcal{M})$, $1 \leq p \leq \infty$, of real valued functions L^p -integrable on \mathcal{M} is introduced in a similar fashion.

Next we introduce the *singular set* of \mathcal{M} relative to a structural atlas $\mathcal{A} = \{U_i, \phi_i\}_{i \in I}$ as being

$$\begin{aligned} \text{Sing}(\mathcal{M}; \mathcal{A}) := & \left\{ x \in \mathcal{M} : \text{there exist } i, j \in I \text{ with } x \in U_i \cap U_j \right. \\ & \text{and such that } \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j) \\ & \left. \text{is not differentiable at } \phi_j(x) \right\}. \quad (7.1) \end{aligned}$$

A basic observation is that the singular set of a compact, boundaryless, Lipschitz manifold, relative to any structural atlas, has measure zero. In the sequel, points in $\text{Sing}(\mathcal{M}; \mathcal{A})$ will be called *singular points* (relative to \mathcal{A}), whereas points in $\text{Reg}(\mathcal{M}; \mathcal{A}) := \mathcal{M} \setminus \text{Sing}(\mathcal{M}; \mathcal{A})$ will be referred to as *regular points* (relative to \mathcal{A}).

Definition 7.1. Let $(\mathcal{M}_j, \mathcal{A}_j)$ be two compact Lipschitz manifolds with boundary, $j = 1, 2$. A continuous mapping $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ will be called *differentiable* at $x \in \mathcal{M}_1$ provided the following properties are valid:

- (i) x is a regular point of \mathcal{M}_1 , relative to some structural atlas \mathcal{A}_1 ;
- (ii) $f(x)$ is a regular point of \mathcal{M}_2 relative to some structural atlas \mathcal{A}_2 ;

- (iii) there exist $(U_j, \phi_j) \in \mathcal{A}_j, j = 1, 2$, with $x \in U_1, f(x) \in U_2$, such that the function $\phi_2 \circ f \circ \phi_1^{-1} : \phi_1(U_1 \cap f^{-1}(U_2)) \rightarrow \phi_2(U_2)$ is differentiable at $\phi_1(x)$.

We continue to assume that $(\mathcal{M}_j, \mathcal{A}_j), j = 1, 2$, are two compact Lipschitz manifolds with boundary. A continuous map $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ will be called *Lipschitz* if for any two coordinate charts $(U_j, \phi_j) \in \mathcal{A}_j, j = 1, 2$, the composition $\phi_2 \circ f \circ \phi_1^{-1} : \phi_1(U_1 \cap f^{-1}(U_2)) \rightarrow \phi_2(U_2)$ is a Lipschitz function. Also, call f *bi-Lipschitz*, if f is a homeomorphism and both f and f^{-1} are Lipschitz. Is important to observe that a Lipschitz function maps sets of zero measure into sets of zero measure.

As a consequence of definitions and the celebrated theorem of Rademacher, according to which Lipschitz functions between Euclidean spaces are differentiable almost everywhere, we have the following result.

Proposition 7.2. *Assume that $\mathcal{M}_j, j = 1, 2$, are compact Lipschitz manifolds with boundary and that $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a Lipschitz function. In addition, assume that*

$$f^{-1}(\text{Sing}(\mathcal{M}_2; \mathcal{A}_2)) \text{ has zero measure in } \mathcal{M}_1, \tag{7.2}$$

for any structural atlas \mathcal{A}_2 of \mathcal{M}_2 .

(We note that this condition is automatically verified if f is bi-Lipschitz, or if \mathcal{M}_2 is a \mathcal{C}^1 manifold.) Then f is differentiable almost everywhere in \mathcal{M}_1 .

Moving on, if $x \in \mathcal{M}$, two mappings f, g from a neighborhood of x into \mathbb{R} are called *equivalent at x* (and we denote this by $f \overset{x}{\sim} g$) if there exists V open small neighborhood of x such that $f|_V = g|_V$. Classes of equivalence modulo $\overset{x}{\sim}$ will be called *germs at x* . We shall pay special attention to germs of differentiable functions at a regular point x , relative to a structural atlas \mathcal{A} , which will be denoted by $\text{Diff}_x(\mathcal{M}; \mathcal{A})$. A continuous mapping $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}, \varepsilon > 0$, with $\gamma(0) = x$ and such that there exists $(U, \phi) \in \mathcal{A}$ for which $x \in U$ and $\phi \circ \gamma$ is differentiable at 0, will be called *path* (through x). For such a path γ we define a linear mapping $\frac{d}{d\gamma} : \text{Diff}_x(\mathcal{M}; \mathcal{A}) \rightarrow \mathbb{R}$ called *derivation along γ* (at x) by

$$\frac{d}{d\gamma} ([f]) := \left. \frac{d}{dt} (f \circ \gamma)(t) \right|_{t=0},$$

for any $[f] \in \text{Diff}_x(\mathcal{M}; \mathcal{A})$. Let $\{e_k\}_{1 \leq k \leq n}$ denote the standard orthonormal basis in \mathbb{R}^n . If $(U, \phi) \in \mathcal{A}$ is such that $x \in U$ then, for each $k = 1, 2, \dots, n$, the derivation along $\phi^{-1}(\phi(x) + te_k)$ at $x \in \text{Reg}(\mathcal{M}; \mathcal{A})$ is denoted by $\frac{d}{d\phi_k}$. Note that

$$\frac{d}{d\phi_k} ([f]) = \frac{\partial(f \circ \phi^{-1})}{\partial x_k} (\phi(x)), \quad k = 1, 2, \dots, n. \tag{7.3}$$

Once a structural atlas \mathcal{A} has been fixed, we can define the tangent space at $x \in \mathcal{M}$ to the manifold \mathcal{M} by setting

$$T_x \mathcal{M} := \left\{ \frac{d}{d\gamma} : \gamma \text{ path through } x \right\} \text{ if } x \in \text{Reg}(\mathcal{M}; \mathcal{A}), \quad (7.4)$$

and

$$T_x \mathcal{M} := \{0\} \text{ if } x \in \text{Sing}(\mathcal{M}; \mathcal{A}). \quad (7.5)$$

It is not difficult to check that $T_x \mathcal{M}$ is a vector space and that in fact $\dim(T_x \mathcal{M}) = n$ (i.e., the same as the dimension of \mathcal{M}) at any regular point x , relative to \mathcal{A} . In fact, for $(U, \phi) \in \mathcal{A}$ a basis in $T_x \mathcal{M}$ at any regular point $x \in U$ is given by $\{\frac{d}{d\phi_k}\}_{k=1}^n$. Now, the *tangent bundle* is

$$T\mathcal{M} := \bigsqcup_{x \in \mathcal{M}} T_x \mathcal{M}. \quad (7.6)$$

We wish to emphasize that the tangent bundle $T\mathcal{M}$ depends on the choice of a structural atlas only up to a set of zero measure in \mathcal{M} .

Going further, let $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a continuous function between two compact Lipschitz manifolds with boundary \mathcal{M}_1 and \mathcal{M}_2 which is differentiable almost everywhere. We then define the *gradient* of f as the mapping $\text{Grad } f : T\mathcal{M}_1 \rightarrow T\mathcal{M}_2$ defined almost everywhere in the following way. At almost every differentiability point $x \in \mathcal{M}_1$ of f , $\text{Grad}_{\mathcal{M}} f_x$ is defined as the mapping of $T_x \mathcal{M}_1$ into $T_{f(x)} \mathcal{M}_2$ given by

$$\text{Grad}_{\mathcal{M}} f_x \left(\frac{d}{d\gamma} \right) := \frac{d}{d(f \circ \gamma)}, \quad (7.7)$$

for any path γ through x (note that $f \circ \gamma$ is a path through $f(x)$ for almost every x). Let us also note that if $(U, \phi = (\phi_1, \dots, \phi_n)) \in \mathcal{A}$ then $\text{Grad}_{cM} \phi_j \left(\frac{d}{d\phi_k} \right) = \delta_{jk} \frac{d}{dt}$ for every $1 \leq j, k \leq n$, where we have denoted by $\frac{d}{dt}$ the standard derivation on \mathbb{R} and, as before, δ_{jk} stands for the Kronecker symbol.

Assume next that the compact Lipschitz manifold with boundary \mathcal{M} is *oriented* and equipped with a (*Lipschitz*) *Riemannian metric*. Being oriented is defined essentially as in the smooth case. That is, an orientation has been specified in $T_x \mathcal{M}$ for a.e. $x \in \mathcal{M}$ such that there exists a structural atlas \mathcal{A} which contains only positive coordinate charts. Recall that a chart $(U, \phi) \in \mathcal{A}$ is called *positive* if the ordered n -tuple $(\frac{d}{d\phi_1}, \dots, \frac{d}{d\phi_n})$ is a positively oriented basis of $T_x \mathcal{M}$ for a.e. $x \in U$. Also, by a *Lipschitz Riemannian structure*, we mean that at almost any point $x \in \mathcal{M}$ some inner product $\langle \cdot, \cdot \rangle_x$ has been specified on the tangent space $T_x \mathcal{M}$ with the following properties:

- (i) $\langle \cdot, \cdot \rangle_x$ varies measurably with x , that is, if \mathcal{A} is a structural atlas consisting of positive charts and $(U, \phi) \in \mathcal{A}$, then the functions

$$g_{ij}^U(x) := \left\langle \frac{d}{d\phi_i}, \frac{d}{d\phi_j} \right\rangle_x, \quad x \in U, \quad 1 \leq i, j \leq n, \quad (7.8)$$

are measurable on U ;

- (ii) there exist a structural atlas \mathcal{A} and two finite constants $C_1, C_2 > 0$ such that for any $(U, \phi) \in \mathcal{A}$, for a.e. $x \in U$, and any path γ through x such that $\phi \circ \gamma$ is differentiable at 0, there holds

$$C_1 \|(\phi \circ \gamma)'(0)\|_{\mathbb{R}^n}^2 \leq \left\langle \frac{d}{d\gamma}, \frac{d}{d\gamma} \right\rangle_x \leq C_2 \|(\phi \circ \gamma)'(0)\|_{\mathbb{R}^n}^2 \quad (7.9)$$

(here and elsewhere, $\|\cdot\|_{\mathbb{R}^n}$ refers to the Euclidean norm in \mathbb{R}^n).

This latter condition implies that the matrix $G^U(x) := (g_{ij}^U(x))_{1 \leq i, j \leq n}$ is symmetric, bounded and positive definite in an uniform manner, for a.e. $x \in U$. In fact,

$$C_1 \|v\|_{\mathbb{R}^n}^2 \leq \langle G^U(x)v, v \rangle_{\mathbb{R}^n} \leq C_2 \|v\|_{\mathbb{R}^n}^2, \quad (7.10)$$

for any $v \in \mathbb{R}^n$ and a.e. $x \in U$.

Proposition 7.3. *Any compact Lipschitz manifold with boundary \mathcal{M} has a Lipschitz Riemannian metric.*

Proof. A Lipschitz Riemannian metric on \mathcal{M} can be constructed by locally transferring the Euclidean metric from \mathbb{R}^n in a standard fashion, and then gluing everything together via a Lipschitz partition of unity. \square

The inner product on the tangent space $T_x \mathcal{M}$ induces a natural pointwise inner product $\langle \cdot, \cdot \rangle_{\Lambda^\ell T_x \mathcal{M}}$ on $\Lambda^\ell T_x \mathcal{M}$, the ℓ -th exterior power of the tangent bundle for each $0 \leq \ell \leq n$, at a.e. $x \in \mathcal{M}$. In particular, there exists a unique form, denoted by $dV_{\mathcal{M}}$, of maximal degree, normalized to one (in the norm $|\cdot|_{\Lambda^n T_x \mathcal{M}}$ associated with the above inner product) a.e. on \mathcal{M} and which is positively oriented. We shall refer to this n -form as the volume element on \mathcal{M} . In turn, this gives rise to a Borel regular measure $\mathcal{L}_{\mathcal{M}}$ on \mathcal{M} , uniquely determined by the requirement that if f is a scalar-valued continuous function on \mathcal{M} which is supported on an open subset \mathcal{O} of \mathcal{M} then

$$\int_{\mathcal{O}} f d\mathcal{L}_{\mathcal{M}} = \sum_j \int_{\phi_j(U_j \cap \mathcal{O})} (\phi_j^{-1})^*(\theta_j f dV_{\mathcal{M}}), \quad (7.11)$$

where $\{\theta_j\}_j$ is a Lipschitz partition of unity on \mathcal{M} subordinated to (a finite) open cover $(U_j)_j$ of \mathcal{M} , with the property that $(U_j, \phi_j) \in \mathcal{A}$ for each j .

Proposition 7.4. *Consider a compact, oriented Lipschitz manifold with boundary \mathcal{M} equipped with a Lipschitz Riemannian metric. Also, fix a positive structural atlas \mathcal{A} and denote by $dV_{\mathcal{M}}$ the volume element on \mathcal{M} . Then for every $(U, \phi) \in \mathcal{A}$ one has*

$$\begin{aligned} & (\phi^{-1})^*(dV_{\mathcal{M}}) = \\ & = \left[\det \left(\left\langle \frac{d}{d\phi_i}, \frac{d}{d\phi_j} \right\rangle_{\phi^{-1}(\cdot)} \right)_{i,j} \right]^{1/2} dV_{\mathbb{R}^n} \text{ a.e. on } \phi(U), \end{aligned} \quad (7.12)$$

where $dV_{\mathbb{R}^n} = dx_1 \wedge \cdots \wedge dx_n$ is the volume element in \mathbb{R}^n , and if C_1, C_2 are as in (7.9) then

$$C_1^{n/2} \leq \left[\det \left(\left\langle \frac{d}{d\phi_i}, \frac{d}{d\phi_j} \right\rangle_x \right)_{i,j} \right] \leq C_2^{n/2}, \text{ for a.e. } x \in U. \quad (7.13)$$

Proof. Formula (7.4) is a consequence of definitions and straightforward linear algebra, whereas (7.13) follows from (7.9). \square

Recall that Ω denotes the interior of the compact Lipschitz manifold with boundary \mathcal{M} , and that $\partial\Omega := \mathcal{M} \setminus \Omega$. Fix an atlas $\{(U_i, \phi_i)\}_{i \in I}$ for \mathcal{M} and pick a Lipschitz partition of unity $\{\xi_i\}_{i \in I}$ subordinate to the open cover $\{U_i\}_{i \in I}$ of \mathcal{M} . For $1 < p < \infty$ and $a \in (-1/p, 1 - 1/p)$, we then define the weighted Sobolev space $W_a^{1,p}(\Omega)$ as the collection of all locally integrable functions $u : \Omega \rightarrow \mathbb{C}$ such that

$$\begin{aligned} \|u\|_{W_a^{1,p}(\Omega)} := & \sum_{i \in I} \|(\xi_i u) \circ \phi_i^{-1}\|_{L^p(\mathbb{R}_+^n, x_n^{ap} dx)} + \\ & + \sum_{i \in I} \|\nabla [(\xi_i u) \circ \phi_i^{-1}]\|_{L^p(\mathbb{R}_+^n, x_n^{ap} dx)} < +\infty. \end{aligned} \quad (7.14)$$

Assuming that $1 < p' < \infty$ is such that $1/p + 1/p' = 1$, we also define

$$W_a^{-1,p}(\Omega) := (\mathring{W}_{-a}^{1,p'}(\Omega))^*. \quad (7.15)$$

Moving on, recall that for the range of indices $1 < p < \infty$ and $0 < s < 1$, the membership to the Besov space $B_s^{p,p}(\mathbb{R}^{n-1})$ is defined via the requirement

$$\begin{aligned} \|f\|_{B_s^{p,p}(\mathbb{R}^{n-1})} := & \|f\|_{L^p(\mathbb{R}^{n-1})} + \\ & + \left(\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x') - f(y')|^p}{|x' - y'|^{n-1+sp}} dx' dy' \right)^{1/p} < +\infty. \end{aligned} \quad (7.16)$$

One natural and convenient way of defining Besov spaces $B_s^{p,p}(\partial\Omega)$, for $1 < p < \infty$ and $s \in (0, 1)$, on the boundary $\partial\Omega$ of the Lipschitz manifold \mathcal{M} is to transport the corresponding scale from \mathbb{R}^{n-1} to $\partial\Omega$ via a partition of unity and bi-Lipschitz pull-back in local coordinate charts.

Some of the most useful properties for these weighted Sobolev spaces for us in this paper are collected in the theorem below. We agree to let Lip denote Lipschitz functions and Lip_0 Lipschitz functions with compact support.

Theorem 7.5. *Let Ω denote the interior of the compact Lipschitz manifold with boundary \mathcal{M} , and set $\partial\Omega := \mathcal{M} \setminus \Omega$. Also, assume that*

$$1 < p < \infty, \quad -1/p < a < 1 - 1/p, \quad s := 1 - a - 1/p \in (0, 1). \quad (7.17)$$

Then the following assertions are true.

- (i) When equipped with the norm (7.14), the space $W_a^{1,p}(\Omega)$ becomes complete (hence Banach). Also, $W_a^{1,p}(\Omega)$ is a module over $\text{Lip}(\Omega)$ and

$$\text{Lip}(\Omega) \hookrightarrow W_a^{1,p}(\Omega) \text{ densely.} \quad (7.18)$$

- (ii) The restriction to the boundary operator, $\text{Lip}(\mathcal{M}) \ni u \mapsto u|_{\partial\Omega} \in \text{Lip}(\partial\Omega)$ extends to a well-defined, linear, bounded mapping

$$\text{Tr} : W_a^{1,p}(\Omega) \longrightarrow B_s^{p,p}(\partial\Omega) \quad (7.19)$$

referred to in the sequel as the trace operator. Furthermore, this trace operator has a continuous right inverse, that is, there exists an extension operator

$$\text{Ext} : B_s^{p,p}(\partial\Omega) \longrightarrow W_a^{1,p}(\Omega) \quad (7.20)$$

which is linear and bounded, and such that $\text{Tr} \circ \text{Ext} = \text{I}$, the identity.

- (iii) There holds

$$\text{Lip}_0(\Omega) \hookrightarrow \{u \in W_a^{1,p}(\Omega) : \text{Tr} u = 0\} \text{ densely.} \quad (7.21)$$

- (iv) If we define

$$\mathring{W}_a^{1,p}(\Omega) := \text{the closure of } \text{Lip}_0(\Omega) \text{ in } W_a^{1,p}(\Omega) \quad (7.22)$$

then

$$\mathring{W}_a^{1,p}(\Omega) = \{u \in W_a^{1,p}(\Omega) : \text{Tr} u = 0\}. \quad (7.23)$$

- (v) The spaces $W_a^{1,p}(\Omega)$, $\mathring{W}_a^{1,p}(\Omega)$, and $W_a^{-1,p}(\Omega)$, are all reflexive.
 (vi) Assume that $1 < p' < \infty$ is such that $1/p + 1/p' = 1$. Then every functional $\Lambda \in (W_{-a}^{1,p'}(\Omega, \cdot))^*$ can be described as follows. For each $u \in W_{-a}^{1,p'}(\Omega)$

$$\begin{aligned} \langle \Lambda, u \rangle = & \sum_{i \in I} \left(\int_{\phi_i(U_i)} f_0^i(x) ((\xi_i u) \circ \phi_i^{-1})(x) \sqrt{g(x)} dx + \right. \\ & \left. + \sum_{j=1}^n \int_{\phi_i(U_i)} f_j^i(x) \partial_{x_j} ((\xi_i u) \circ \phi_i^{-1})(x) \sqrt{g(x)} dx \right), \quad (7.24) \end{aligned}$$

where $\{(U_i, \phi_i)\}_{i \in I}$ is a finite atlas for \mathcal{M} , and $\{\xi_i\}_{i \in I} \subset \text{Lip}(\mathcal{M})$ is a partition of unity subordinate to the open cover $\{U_i\}_{i \in I}$ of \mathcal{M} .

Furthermore, for each $i \in I$, the functions f_j^i , $0 \leq j \leq n$, appearing in (7.24) belong to $L^p(\phi_i(U_i), x_n^{\alpha p} dx)$ and the norm $\|\Lambda\|_{(W_{-a}^{1,p'}(\Omega))^*}$ is equivalent to the infimum of the sum of the norms of f_j^i 's over all possible choices of the atlas, local charts, and partitions of unity.

- (vii) *The scales $W_a^{1,p}(\Omega)$, $\mathring{W}_a^{1,p}(\Omega)$, $W_a^{-1,p}(\Omega)$, are stable under complex interpolation. More specifically, if $1 < p_i < \infty$, $-1/p_i < a_i < 1 - 1/p_i$, $i \in \{0, 1\}$, and $\theta \in (0, 1)$, $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $a = (1 - \theta)a_0 + \theta a_1$, then*

$$[W_{a_0}^{1,p_0}(\Omega), W_{a_1}^{1,p_1}(\Omega)]_\theta = W_a^{1,p}(\Omega), \quad (7.25)$$

$$[\mathring{W}_{a_0}^{1,p_0}(\Omega), \mathring{W}_{a_1}^{1,p_1}(\Omega)]_\theta = \mathring{W}_a^{1,p}(\Omega), \quad (7.26)$$

$$[W_{a_0}^{-1,p_0}(\Omega), W_{a_1}^{-1,p_1}(\Omega)]_\theta = W_a^{-1,p}(\Omega), \quad (7.27)$$

where $[\cdot, \cdot]_\theta$ denotes the usual complex interpolation bracket.

Proof. All the claims can then be deduced from their Euclidean counterpart (dealt with in earlier sections), via a standard localization argument and by making bi-Lipschitz changes of coordinates in local coordinate charts. \square

Recall that Ω denotes the interior of \mathcal{M} and that $\partial\Omega := \mathcal{M} \setminus \Omega$. Unraveling definitions to the point that well-known Euclidean results can be invoked, it is not difficult to show that the gradient induces a well-defined and bounded operator

$$\text{Grad}_{\mathcal{M}} : W_a^{1,p}(\Omega) \longrightarrow L^p(\Omega, \delta^{ap} \mathcal{L}_{\mathcal{M}}) \otimes T\mathcal{M} \quad (7.28)$$

whenever $p \in (1, \infty)$ and $a \in (-1/p, 1 - 1/p)$. We denote the (sign) opposite of the adjoint of this operator by $\text{Div}_{\mathcal{M}}$, and refer to it as the divergence operator on the Lipschitz manifold \mathcal{M} . Hence,

$$\text{Div}_{\mathcal{M}} : L^p(\Omega, \delta^{ap} \mathcal{L}_{\mathcal{M}}) \otimes T\mathcal{M} \longrightarrow W_a^{-1,p}(\Omega) \quad (7.29)$$

is a bounded operator if $p \in (1, \infty)$ and $a \in (-1/p, 1 - 1/p)$. Finally, we define the Laplace–Beltrami operator $\Delta_{\mathcal{M}}$ on the Lipschitz manifold \mathcal{M} as the composition

$$\Delta_{\mathcal{M}} := \text{Div}_{\mathcal{M}} \circ \text{Grad}_{\mathcal{M}}. \quad (7.30)$$

Hence, whenever $p \in (1, \infty)$ and $a \in (-1/p, 1 - 1/p)$, this induces a linear and bounded mapping

$$\Delta_{\mathcal{M}} : W_a^{1,p}(\Omega) \longrightarrow W_a^{-1,p}(\Omega). \quad (7.31)$$

Moreover, the adjoint of (7.31) is

$$\Delta_{\mathcal{M}} : W_{-a}^{1,p'}(\Omega) \longrightarrow W_{-a}^{-1,p'}(\Omega), \quad (7.32)$$

where $1/p' + 1/p = 1$, and $\Delta_{\mathcal{M}}$ in (7.31) is an isomorphism when $p = 2$ and $a = 0$.

One final comment pertains to the nature of the Laplace–Beltrami operator $\Delta_{\mathcal{M}}$ in local coordinates. Specifically, for each $(U, \phi) \in \mathcal{A}$, organize the functions introduced in (7.8) as a matrix $G_U := (g_{ij}^U)_{1 \leq i, j \leq n}$ and denote by $(g_U^{jk})_{1 \leq j, k \leq n}$ the inverse of the matrix G_U . Also, set $g_U := \det G_U$ so that, according to Proposition 7.4, the volume element in $dV_{\mathcal{M}}$ has the property that

$$(\phi^{-1})^*(dV_{\mathcal{M}}) = \sqrt{g_U} dx_1 \cdots dx_n \text{ in } \phi(U). \quad (7.33)$$

Then, in the local coordinates associated with the chart (U, ϕ) , the Laplace–Beltrami operator $\Delta_{\mathcal{M}}$ can be described as

$$\Delta_{\mathcal{M}} = \frac{1}{\sqrt{g_U}} \sum_{j,k=1}^n \partial_j (g_U^{j,k} \sqrt{g_U} \partial_k \cdot), \tag{7.34}$$

where, as customary, we have identified $d/d\phi_i$ with ∂_i for each $i \in \{1, \dots, n\}$.

We are now ready to discuss the following sharp well-posedness result in the setting of compact Lipschitz manifolds with boundary.

Theorem 7.6. *Let Ω denote the interior of the compact Lipschitz manifold with boundary \mathcal{M} , and set $\partial\Omega := \mathcal{M} \setminus \Omega$. Then there exists $\varepsilon > 0$ such that whenever*

$$p \in (2 - \varepsilon, 2 + \varepsilon), \quad a \in (-1/p, 1 - 1/p) \cap (-\varepsilon, \varepsilon), \quad s := 1 - a - 1/p, \tag{7.35}$$

the Poisson boundary value problem with Dirichlet boundary data for the Laplace–Beltrami operator

$$\begin{cases} u \in W_a^{1,p}(\Omega), \\ \Delta_{\mathcal{M}} u = f \in W_a^{-1,p}(\Omega), \\ \text{Tr } u = g \in B_s^{p,p}(\partial\Omega) \end{cases} \tag{7.36}$$

is well-posed.

Proof. This follows by arguing as in the proof of Theorem 5.1, making use of the functional analytic theory for weighted Sobolev spaces from Theorem 7.5. □

Theorem 7.6 is, once again, sharp (in that having p near 2 is a necessary condition). This follows from an example given by N. Meyers in [13, Section 5]. Specifically, take

$$\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \tag{7.37}$$

and consider the coefficient matrix given by

$$\begin{aligned} a_{11}(x_1, x_2) &= 1 - (1 - \mu^2)x_2^2(x_1^2 + x_2^2)^{-1}, \\ a_{12}(x_1, x_2) &= A_{21}(x_1, x_2) = (1 - \mu^2)x_1x_2(x_1^2 + x_2^2)^{-1}, \\ a_{22}(x_1, x_2) &= 1 - (1 - \mu^2)x_1^2(x_1^2 + x_2^2)^{-1}, \\ &\quad \forall (x, y) \in \Omega \setminus \{(0, 0)\}, \end{aligned} \tag{7.38}$$

where $\mu \in (0, 1)$ is a fixed parameter. Define the scalar operator $Lu := \sum_{j,k=1,2} \partial_j (a_{jk}(x_1, x_2) \partial_k u)$ in Ω . Note that the a_{jk} ’s belong to $L^\infty(\Omega, \mathcal{L}^2)$ and a direct calculation shows that

$$\sum_{j,k=1,2} a_{jk}(x_1, x_2) \xi_j \xi_k = |\xi|^2 - (1 - \mu^2) \frac{(x_1 \xi_2 - x_2 \xi_1)^2}{|x|^2} \geq \mu^2 |\xi|^2, \tag{7.39}$$

for each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $x = (x_1, x_2) \in \Omega \setminus \{0\}$. Hence, L is elliptic. To proceed, introduce the function

$$v(x_1, x_2) := x_1(x_2^2 + x_2^2)^{(\mu-1)/2} \in L^\infty(\Omega, \mathcal{L}^2) \cap \mathcal{C}^\infty(\bar{\Omega} \setminus \{0\}). \quad (7.40)$$

A straightforward calculation shows that $Lv = 0$ near the origin. Also, fix $\phi \in \mathcal{C}_c^\infty(\Omega)$ so that $\phi \equiv 1$ near the origin, and set $u := \phi v$. It follows that

$$\begin{aligned} u &\in \dot{W}^{1,2}(\Omega), \quad f := Lu \in \mathcal{C}_c^\infty(\Omega), \\ |(\nabla u)(x)| &\approx |x|^{\mu-1} \text{ near } 0 \in \Omega. \end{aligned} \quad (7.41)$$

Consequently,

$$u \in W^{1,p}(\Omega) \iff p < \frac{2}{1-\mu}. \quad (7.42)$$

In particular, the fact that $2/(1-\mu) \searrow 2$ as $\mu \searrow 0$ shows that for each $p > 2$ there exists $\mu \in (0, 1)$ with the property that the operator $L : \dot{W}^{1,p}(\Omega) \rightarrow W^{-1,p}(\Omega)$ fails to be an isomorphism. By duality, (note that L is formally self-adjoint), the same type of conclusion holds for $p < 2$.

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