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INVARIANT DOMAINS AND GLOBAL EXISTENCE FOR REACTION-DIFFUSION SYSTEMS WITH A TRIDIAGONAL MATRIX OF DIFFUSION COEFFICIENTS

Abstract. The aim of this study is to prove the global existence of solutions for reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients and nonhomogeneous boundary conditions. Towards this end, we make use of the appropriate techniques which are based on the invariant domains and on Lyapunov functional methods. The nonlinear reaction term has been supposed to be of polynomial growth. This result is a continuation of that due to Kouachi and Rebiai [13].

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რეზიუმე. ნაშრომის მიზანია დამტკიცდეს დიფუზიის კოეფიციენტების ტრიდიაგონალური მატრიციანი რეაქციულ-დიფუზიური სისტემების გლობალური ამონახსნების არსებობა არაერთგვაროვან სასაზღვრო პირობებში. ამ მიზნით გამოიყენება შესაბამისი ტექნიკა, რომელიც დაფუძნებულია ინვარიანტულ არეებზე და ლიაპუნოვის ფუნქციონალის მეთოდებზე. არაწრფივი რეაქციული წევრის შესახებ გაკეთებულია დაშვება, რომ მისი ზრდის რიგი პოლინომიალურია. ეს შედეგი წარმოდგენს კუაჩისა და რებიაის [13] შედეგის გავრცელებას.

1. Introduction

We consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a_{11}\Delta u - a_{12}\Delta v = f(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - a_{21}\Delta u - a_{22}\Delta v - a_{23}\Delta w = g(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.2}$$

$$\frac{\partial w}{\partial t} - a_{32}\Delta v - a_{33}\Delta w = h(u,v,w) \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.3}$$

with the boundary conditions

$$\lambda u+(1-\lambda)\frac{\partial u}{\partial \eta}=\beta_1, \quad \lambda v+(1-\lambda)\frac{\partial v}{\partial \eta}=\beta_2, \quad \lambda w+(1-\lambda)\frac{\partial w}{\partial \eta}=\beta_3, \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.4}$$

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \text{ in } \Omega, \tag{1.5}$$

where

(i) $0 < \lambda < 1$ and $\beta_i \in \mathbb{R}$, $i = 1,2,3$, for nonhomogeneous Robin boundary conditions.

(ii) $\lambda = \beta_i = 0$, $i = 1,2,3$, for homogeneous Neumann boundary conditions.

(iii) $1 - \lambda = \beta_i = 0$, $i = 1,2,3$, for homogeneous Dirichlet boundary conditions.

Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^N$ with boundary $\partial\Omega$ and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. The diffusion terms a_{ij} ($i,j = 1,2,3$ and $(i,j) \neq (1,3),(3,1)$) are supposed to be positive constants such that

$$a_{12}a_{21}(a_{22} - a_{33}) = a_{23}a_{32}(a_{11} - a_{22})$$

and

$$a_{33}(a_{12} + a_{21})^2 + a_{11}(a_{23} + a_{32})^2 < 4a_{11}a_{22}a_{33}$$

which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

is positive definite. The eigenvalues λ_1, λ_2 and λ_3 ($\lambda_1 < \lambda_2 = a_{22} < \lambda_3$) of A are positive. If we put

$$\underline{a} = \min\{a_{11}, a_{33}\} \text{ and } \overline{a} = \max\{a_{11}, a_{33}\},$$

then the positivity of the a_{ij} implies that

$$\lambda_1 < \underline{a} < \lambda_2 < \overline{a} < \lambda_3.$$

The initial data are assumed to be in the domain

$$\Sigma = \begin{cases} \big\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ \mu_i u_0 + \nu_i w_0 \le v_0, \ i = 1, 2, 3\big\} \\ \qquad \text{if } \ \mu_i\beta_1 + \nu_i\beta_3 \le \beta_2, \ \ i = 1, 2, 3, \\ \big\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ \mu_i u_0 + \nu_i w_0 \le v_0 \le \mu_1 u_0 + \nu_1 w_0, \ i = 2, 3\big\} \\ \qquad \text{if } \ \mu_i\beta_1 + \nu_i\beta_3 \le \beta_2 \le \mu_1\beta_1 + \nu_1\beta_3, \ \ i = 2, 3, \\ \big\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ \mu_i u_0 + \nu_i w_0 \le v_0 \le \mu_2 u_0 + \nu_2 w_0, \ i = 1, 3\big\} \\ \qquad \text{if } \ \mu_i\beta_1 + \nu_i\beta_3 \le \beta_2 \le \mu_2\beta_1 + \nu_2\beta_3, \ \ i = 1, 3, \\ \big\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ \mu_3 u_0 + \nu_3 w_0 \le v_0 \le \mu_i u_0 + \nu_i w_0, \ i = 1, 2\big\} \\ \qquad \text{if } \ \mu_3\beta_1 + \nu_3\beta_3 \le v_0 \le \mu_i\beta_1 + \nu_i\beta_3, \ \ i = 1, 2, \end{cases}$$

where $\mu_1 = a_{21}/(a_{11}-\lambda_1) > 0 > \mu_2 = a_{21}/(a_{11}-\lambda_2) > \mu_3 = a_{21}/(a_{11}-\lambda_3)$, $\nu_1 = a_{23}/(a_{33}-\lambda_1) > \nu_2 = a_{23}/(a_{33}-\lambda_2) > 0 > \nu_3 = a_{23}/(a_{33}-\lambda_3)$, if we assume without loss of generality that $a_{11} < a_{33}$.

Since we use the same methods to treat all the cases, we will tackle only with the first one. We suppose that the functions f, g and h are continuously differentiable, polynomially bounded on Σ,

$$\big(f(r_1, r_2, r_3), g(r_1, r_2, r_3), h(r_1, r_2, r_3)\big) \ \text{ is in } \Sigma \text{ for all } (r_1, r_2, r_3) \text{ in } \partial\Sigma$$

(we say that (f, g, h) points into Σ on $\partial\Sigma$), i.e.,

$$\mu_i f(r_1, r_2, r_3) + \nu_i h(r_1, r_2, r_3) \le g(r_1, r_2, r_3), \tag{1.6}$$

for all r_1, r_2 and r_3 such that $\mu_j r_1 + \nu_j r_3 \le r_2 = \mu_i r_1 + \nu_i r_3$, $j = 1, 2, 3$ $(j \ne i)$, $i = 1, 2, 3$, and for positive constants E and D, we have

$$(Ef + Dg + h)(u, v, w) \le C_1(u + v + w + 1) \tag{1.7}$$

for all (u, v, w) in Σ, where C_1 is a positive constant.

In the two-component case, where $a_{12} = 0$, Kouachi and Youkana [14] generalized the method of Haraux and Youkana [4] with the reaction terms $f(u, v) = -\lambda F(u, v)$ and $g(u, v) = +\mu F(u, v)$ with $F(u, v) \ge 0$, requiring the condition

$$\lim_{s\to+\infty} \Big[\frac{\ln(1 + F(r, s))}{s}\Big] < \alpha^* \ \text{ for any } \ r \ge 0,$$

with

$$\alpha^* = \frac{2a_{11}a_{22}}{n(a_{11} - a_{22})^2\|u_0\|_\infty} \min\Big\{\frac{\lambda}{\mu}, \frac{a_{11} - a_{22}}{a_{21}}\Big\},$$

where the positive diffusion coefficients a_{11}, a_{22} satisfy $a_{11} > a_{22}$ and a_{21}, λ, μ are positive constants. This condition reflects a weak exponential growth of the function F. Kanel and Kirane [6] proved the global existence in the case where $g(u, v) = -f(u, v) = uv^n$ and n is an odd integer, under the embarrassing condition $|a_{12} - a_{21}| < C_p$, where C_p contains a constant from Solonnikov's estimate [19]. Later, in [7] they improved their results to obtain the global existence under the restrictions

H$_1$. $a_{22} < a_{11} + a_{21}$,

H$_2$. $a_{12} < \varepsilon_0 = \dfrac{a_{11}a_{22}(a_{11}+a_{21}-a_{22})}{a_{11}a_{22}+a_{21}(a_{11}+a_{21}-a_{22})}$ if $a_{11} \leq a_{22} < a_{11}+a_{21}$,

H$_3$. $a_{12} < \min\left\{\dfrac{1}{2}(a_{11}+a_{21}), \varepsilon_0\right\}$ if $a_{22} < a_{11}$,

and $|F(v)| \leq C_F(1+|v|^{1-\varepsilon})$, $vF(v) \geq 0$ for all $v \in \mathbb{R}$, where ε and C_F are positive constants with $\varepsilon<1$ and $g(u,v)=-f(u,v)=uF(v)$.

Kouachi [12] has proved the global existence for solutions of two-component reaction-diffusion systems with a general full matrix of diffusion coefficients and nonhomogeneous boundary conditions. Recently, we proved the global existence for solutions of three-component reaction-diffusion systems with a tridiagonal matrix of diffusion coefficients and nonhomogeneous boundary conditions where the positive diffusion coefficients a_{11}, a_{33} are equal (see Kouachi and Rebiai [13]).

The present investigation is a continuation work of that obtained in [13]. In this study we will treat the case where $a_{11} \neq a_{33}$.

We note that the case of strongly coupled systems which are not triangular in the diffusion part is quite more difficult. As a consequence of the blow-up of the solutions found in [17], we can indeed prove that there is the blow-up of the solutions in finite time for such nontriangular systems even though the initial data are regular, the solutions are positive and the nonlinear terms are negative, a structure that ensured the global existence in the diagonal case. For this purpose, we construct the invariant domains in which we can demonstrate that for any initial data in those domains, problem (1.1)–(1.5) is equivalent to the problem for which the global existence follows from the usual techniques based on Lyapunov functionals (see Kirane and Kouachi [8], Kouachi and Youkana [14] and Kouachi [12]).

Many chemical and biological operations are described by means of reaction diffusion systems with a tridiagonal matrix of diffusion coefficients. The components $u(t,x)$, $v(t,x)$ and $w(t,x)$ can be represented either by chemical concentrations or biological population densities (see, e.g., Cussler [1] and [2]). For example, in chemistry, an n-species reaction-diffusion system with cross-diffusion can be described by the following system of partial differential equations

$$\frac{\partial c_i}{\partial t} - \operatorname{div}(\nabla D_{ii}c_i) - \sum_{j\neq i}\operatorname{div}(\nabla D_{ij}c_j) = R_i(c_1,\ldots,c_n), \quad i,j=1,2,\ldots,n,$$

where $R_i(c_1,\ldots,c_n)$ are the reactive terms, D_{ii} are the main-diffusion coefficients and the cross-diffusion term $\operatorname{div}(\nabla D_{ij}c_j)$ links the gradient of species c_j to the flux of species c_i. If $D_{ij} \geq 0$, then the ith species diffuses from larger to smaller concentrations of the jth species, analogous to the case of ordinary self-diffusion. If $D_{ij} < 0$, then the ith species diffuses in the opposite direction, against the gradient ∇c_j.

Throughout this work, we denote by $\|\cdot\|_p$, $p \in [1,+\infty[$ the norm in $L^p(\Omega)$ and $\|\cdot\|_\infty$ the norm in $C(\overline{\Omega})$ or $L^\infty(\Omega)$.

2. The Local Existence and Invariant Domains

The study of local existence and uniqueness of solutions (u,v,w) of (1.1)–(1.5) follows from the basic existence theory for parabolic semilinear equations (see, e.g., [3], [5] and [16]). As a consequence, for any initial data in $C(\overline{\Omega})$ or $L^\infty(\Omega)$ there exists $T^* \in]0,+\infty]$ such that (1.1)–(1.5) has a unique classical solution on $[0,T^*[\times\Omega$. Furthermore, if $T^* < +\infty$, then

$$\lim_{t\uparrow T^*}\left(\|u(t)\|_\infty + \|v(t)\|_\infty + \|w(t)\|_\infty\right) = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty + \|w(t)\|_\infty \le C \text{ for all } t \in [0,T^*[\,,$$

then $T^* = +\infty$.

Since the initial conditions are in Σ, then under the assumptions (1.6), the next proposition says that the classical solution of (1.1)–(1.5) on $[0,T^*[\times\Omega$ remains in Σ for all t in $[0,T^*[$.

Proposition 1. *Suppose that (f,g,h) points into Σ on $\partial\Sigma$. Then for any (u_0,v_0,w_0) in Σ the solution (u,v,w) of the problem (1.1)–(1.5) remains in Σ for all t in $[0,T^*[$.*

Proof. Let $(x_{i1},x_{i2},x_{i3})^t$, $i=1,2,3$, be the eigenvectors of the matrix A^t associate with its eigenvalues λ_i, $i=1,2,3$ ($\lambda_1<\lambda_2<\lambda_3$). Multiplying equations (1.1), (1.2) and (1.3) of the given reaction-diffusion system by x_{i1}, x_{i2} and x_{i3}, respectively, and summing the resulting equations, we get

$$\frac{\partial}{\partial t}z_1 - \lambda_1\Delta z_1 = F_1(z_1,z_2,z_3) \text{ in }]0,T^*[\times\Omega, \tag{2.1}$$

$$\frac{\partial}{\partial t}z_2 - \lambda_2\Delta z_2 = F_2(z_1,z_2,z_3) \text{ in }]0,T^*[\times\Omega, \tag{2.2}$$

$$\frac{\partial}{\partial t}z_3 - \lambda_3\Delta z_3 = F_3(z_1,z_2,z_3) \text{ in }]0,T^*[\times\Omega, \tag{2.3}$$

with the boundary conditions

$$\lambda z_i + (1-\lambda)\frac{\partial z_i}{\partial\eta} = \rho_i, \quad i=1,2,3, \text{ on }]0,T^*[\times\partial\Omega, \tag{2.4}$$

and the initial data

$$z_i(0,x) = z_i^0(x), \quad i=1,2,3, \text{ in } \Omega, \tag{2.5}$$

where

$$z_i = x_{i1}u + x_{i2}v + x_{i3}w, \quad i=1,2,3, \text{ in }]0,T^*[\times\Omega, \tag{2.6}$$
$$\rho_i = x_{i1}\beta_1 + x_{i2}\beta_2 + x_{i3}\beta_3, \quad i=1,2,3,$$

and

$$F_i(z_1,z_2,z_3) = x_{i1}f + x_{i2}g + x_{i3}h, \quad i=1,2,3, \tag{2.7}$$

for all (u,v,w) in Σ.

We note that the condition of the parabolicity of the system (1.1)–(1.3) implies one of (2.1)–(2.3). Since λ_1, λ_2 and λ_3 are the eigenvalues of the matrix A^t, the problem (1.1)–(1.5) is equivalent to the problem (2.1)–(2.5), and to prove that Σ is an invariant domain for the system (1.1)–(1.3) it suffices to prove that the domain

$$\left\{(z_1^0, z_2^0, z_3^0) \in \mathbb{R}^3 : \ z_i^0 \geq 0, \ i = 1, 2, 3\right\} = (\mathbb{R}^+)^3 \tag{2.8}$$

is invariant for the system (2.1)–(2.3) and there exist some constants x_{ij}, $i, j = 1, 2, 3$, such that

$$\Sigma = \left\{(u_0, v_0, w_0) \in \mathbb{R}^3 : \ z_i^0 = x_{i1}u_0 + x_{i2}v_0 + x_{i3}w_0 \geq 0, \ i = 1, 2, 3\right\}. \tag{2.9}$$

Since $(x_{i1}, x_{i2}, x_{i3})^t$ is an eigenvector of the matrix A^t associated to the eigenvalue λ_i, $i = 1, 2, 3$, we have

$$\begin{cases} (a_{11} - \lambda_i)x_{i1} + a_{21}x_{i2} = 0, \\ a_{23}x_{i2} + (a_{33} - \lambda_i)x_{i3} = 0, \end{cases} \qquad i = 1, 2, 3.$$

If we assume, without loss of generality, that $a_{11} < a_{33}$ and choose $x_{12} = x_{22} = x_{32} = 1$, then we have $x_{i1}u_0 + x_{i2}v_0 + x_{i3}w_0 \geq 0, \ i = 1, 2, 3 \Longleftrightarrow \mu_i u_0 + \nu_i w_0 \leq v_0, \ i = 1, 2, 3$. Thus (2.9) is proved and (2.6) can be written as

$$z_i = -\mu_i u + v - \nu_i w, \ \ i = 1, 2, 3. \tag{2.6a}$$

Now, to prove that the domain $(\mathbb{R}^+)^3$ is invariant for the system (2.1)–(2.3), it suffices to show that $F_i(z_1, z_2, z_3) \geq 0$ for all (z_1, z_2, z_3) such that $z_i = 0$ and $z_j \geq 0$, $j = 1, 2, 3$ $(j \neq i)$, $i = 1, 2, 3$, thanks to the invariant domain method (see Smoller [18]). Using the expressions (2.7), we get

$$F_i = -\mu_i f + g - \nu_i h, \ \ i = 1, 2, 3, \tag{2.7a}$$

for all (u, v, w) in Σ. Since from (1.6) we have $F_i(z_1, z_2, z_3) \geq 0$ for all (z_1, z_2, z_3) such that $z_i = 0$ and $z_j \geq 0$, $j = 1, 2, 3$ $(j \neq i)$, $i = 1, 2, 3$, we obtain $z_i(t, x) \geq 0$, $i = 1, 2, 3$, for all $(t, x) \in [0, T^*[\times \Omega$. As a consequence, Σ is an invariant domain for the system (1.1)–(1.3). $\square$

In addition, the system (1.1)–(1.3) with the boundary conditions (1.4) and initial data in Σ is equivalent to the system (2.1)–(2.3) with the boundary conditions (2.4) and positive initial data (2.5).

Once the invariant domains are constructed and since ρ_i, $i = 1, 2, 3$, given by $\rho_i = -\mu_i\beta_1 + \beta_2 - \nu_i\beta_3$, $i = 1, 2, 3$, are positive, we can apply the Lyapunov technique and establish the global existence of unique solutions for (1.1)–(1.5).

3. Global Existence

As the determinant of the linear algebraic system (2.6), with respect to variables u, v and w, is different from zero, to prove the global existence of solutions of the problem (1.1)–(1.5) one needs to prove it for the problem (2.1)–(2.5). To this end, it is well known that (see Henry [5]) it suffices to derive a uniform estimate of $\|F_i(z_1, z_2, z_3)\|_p$, $i = 1, 2, 3$, on $[0, T]$, $T < T^*$, for some $p > N/2$.

Let θ and σ be two positive constants such that

$$\theta > A_{12}, \tag{3.1}$$

$$(\theta^2 - A_{12}^2)(\sigma^2 - A_{23}^2) > (A_{13} - A_{12}A_{23})^2, \tag{3.2}$$

where $A_{ij} = \frac{\lambda_i + \lambda_j}{2\sqrt{\lambda_i \lambda_j}}$, $i, j = 1, 2, 3$ $(i < j)$, and let

$$\theta_q = \theta^{q^2} \ \text{ and } \ \sigma_p = \sigma^{p^2} \ \text{ for } \ q = 0, 1, \ldots, p \ \text{ and } \ p = 0, 1, \ldots, n, \tag{3.3}$$

with n as a positive integer. The main result of this section is

Theorem 1. *Let (z_1, z_2, z_3) be any positive solution of* (2.1)–(2.5) *on* $[0, T^*[\times\Omega$*; let the functional*

$$t \longmapsto L(t) = \int_\Omega H_n\big(z_1(t,x), z_2(t,x), z_3(t,x)\big)\, dx, \tag{3.4}$$

where

$$H_n(z_1, z_2, z_3) = \sum_{p=0}^{n} \sum_{q=0}^{p} C_n^p C_p^q \theta_q \sigma_p z_1^q z_2^{p-q} z_3^{n-p}, \tag{3.5}$$

with n being a positive integer and $C_n^p = \frac{n!}{(n-p)!p!}$.

Then, the functional L is uniformly bounded on $[0, T]$, $T < T^*$.

For the proof of Theorem 1 we need some preparatory Lemmas.

Lemma 1. *Let H_n be the homogeneous polynomial defined by* (3.5)*. Then*

$$\frac{\partial H_n}{\partial z_1} = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_{q+1} \sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-1)-p}, \tag{3.6}$$

$$\frac{\partial H_n}{\partial z_2} = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q \sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-1)-p}, \tag{3.7}$$

$$\frac{\partial H_n}{\partial z_3} = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q \sigma_p z_1^q z_2^{p-q} z_3^{(n-1)-p}. \tag{3.8}$$

Proof. Differentiating H_n with respect to z_1 and using the fact that

$$qC_p^q = pC_{p-1}^{q-1} \ \text{ and } \ pC_n^p = nC_{n-1}^{p-1} \tag{3.9}$$

for $q = 1, 2, \ldots, p$, $p = 1, 2, \ldots, n$, we get

$$\frac{\partial H_n}{\partial z_1} = n \sum_{p=1}^{n} \sum_{q=1}^{p} C_{n-1}^{p-1} C_{p-1}^{q-1} \theta_q \sigma_p z_1^{q-1} z_2^{p-q} z_3^{n-p}.$$

Replacing in the sums the indices $q-1$ by q and $p-1$ by p, we deduce (3.6). For the formula (3.7), differentiating H_n with respect to z_2, taking into account

$$C_p^q = C_p^{p-q}, \ \ q = 0, 1, \ldots, p-1 \ \text{ and } \ p = 1, 2, \ldots, n, \tag{3.10}$$

using (3.9) and replacing the index $p-1$ by p, we get (3.7).

Finally, we have

$$\frac{\partial H_n}{\partial z_3} = \sum_{p=0}^{n-1}\sum_{q=0}^{p}(n-p)C_n^p C_p^q \theta_q \sigma_p z_1^q z_2^{p-q} z_3^{n-p-1}.$$

Since $(n-p)C_n^p = (n-p)C_n^{n-p} = nC_{n-1}^{n-p-1} = nC_{n-1}^p$, we get (3.8). $\square$

Lemma 2. *The second partial derivatives of H_n are given by*

$$\frac{\partial^2 H_n}{\partial z_1^2} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+2}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.11}$$

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+1}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.12}$$

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.13}$$

$$\frac{\partial^2 H_n}{\partial z_2^2} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q}\sigma_{p+2} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.14}$$

$$\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-2)-p}, \tag{3.15}$$

$$\frac{\partial^2 H_n}{\partial z_3^2} = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q}\sigma_{p} z_1^q z_2^{p-q} z_3^{(n-2)-p}. \tag{3.16}$$

Proof. Differentiating $\frac{\partial H_n}{\partial z_1}$ given by (3.6) with respect to z_1, we obtain

$$\frac{\partial^2 H_n}{\partial z_1^2} = n\sum_{p=1}^{n-1}\sum_{q=1}^{p} qC_{n-1}^p C_p^q \theta_{q+1}\sigma_{q+1} z_1^{q-1} z_2^{p-q} z_3^{(n-1)-p}.$$

Using (3.9), we get (3.11).

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_2} = n\sum_{p=1}^{n-1}\sum_{q=0}^{p-1}(p-q)C_{n-1}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q-1} z_3^{(n-1)-p}.$$

Applying (3.10) and then (3.9), we get (3.12).

$$\frac{\partial^2 H_n}{\partial z_1 \partial z_3} = n\sum_{p=0}^{n-2}\sum_{q=0}^{p}((n-1)-p)C_{n-1}^p C_p^q \theta_{q+1}\sigma_{p+1} z_1^q z_2^{p-q} z_3^{(n-2)-p}.$$

Applying successively (3.10), (3.9) and (3.10) for the second time, we deduce (3.13).

$$\frac{\partial^2 H_n}{\partial z_2^2} = n\sum_{p=1}^{n-1}\sum_{q=0}^{p-1}(p-q)C_{n-1}^{p}C_p^q\theta_q\sigma_{p+1}z_1^q z_2^{p-q-1}z_3^{(n-1)-p}.$$

The application of (3.10) and then (3.9) yields (3.14).

$$\frac{\partial^2 H_n}{\partial z_2 \partial z_3} = n\sum_{p=0}^{n-2}\sum_{q=0}^{p}((n-1)-p)C_{n-1}^{p}C_p^q\theta_q\sigma_{p}z_1^q z_2^{p-q}z_3^{(n-2)-p}.$$

Applying (3.10) and then (3.9), we get (3.15). Finally, we get (3.16) by differentiating $\frac{\partial H_n}{\partial z_3}$ with respect to z_3 and applying successively (3.10), (3.9) and (3.10) for the second time. $\square$

Proof of Theorem 1. Differentiating L with respect to t, we find that

$$\begin{aligned} L'(t) &= \int_\Omega \Big(\frac{\partial H_n}{\partial z_1}\frac{\partial z_1}{\partial t} + \frac{\partial H_n}{\partial z_2}\frac{\partial z_2}{\partial t} + \frac{\partial H_n}{\partial z_3}\frac{\partial z_3}{\partial t}\Big)\,dx = \\ &= \int_\Omega \Big(\lambda_1\frac{\partial H_n}{\partial z_1}\Delta z_1 + \lambda_2\frac{\partial H_n}{\partial z_2}\Delta z_2 + \lambda_3\frac{\partial H_n}{\partial z_3}\Delta z_3\Big)\,dx + \\ &\quad + \int_\Omega \Big(\frac{\partial H_n}{\partial z_1}F_1 + \frac{\partial H_n}{\partial z_2}F_2 + \frac{\partial H_n}{\partial z_3}F_3\Big)\,dx =: I + J, \end{aligned}$$

Using Green's formula in I, we get $I = I_1 + I_2$, where

$$I_1 = \int_{\partial\Omega}\Big(\lambda_1\frac{\partial H_n}{\partial z_1}\frac{\partial z_1}{\partial \eta} + \lambda_2\frac{\partial H_n}{\partial z_2}\frac{\partial z_2}{\partial \eta} + \lambda_3\frac{\partial H_n}{\partial z_3}\frac{\partial z_3}{\partial \eta}\Big)\,ds,$$

where ds denotes the $(n-1)$-dimensional surface element, and

$$\begin{aligned} I_2 = -\int_\Omega \Big[&\lambda_1\frac{\partial^2 H_n}{\partial z_1^2}|\nabla z_1|^2 + (\lambda_1+\lambda_2)\frac{\partial^2 H_n}{\partial z_1\partial z_2}\nabla z_1\nabla z_2 + \\ &+ (\lambda_1+\lambda_3)\frac{\partial^2 H_n}{\partial z_1\partial z_3}\nabla z_1\nabla z_3 + \lambda_2\frac{\partial^2 H_n}{\partial z_2^2}|\nabla z_2|^2 + \\ &+ (\lambda_2+\lambda_3)\frac{\partial^2 H_n}{\partial z_2\partial z_3}\nabla z_2\nabla z_3 + \lambda_3\frac{\partial^2 H_n}{\partial z_3^2}|\nabla z_3|^2\Big]\,dx. \end{aligned}$$

We prove that there exists a positive constant C_2 independent of $t \in [0, T^*[$ such that

$$I_1 \le C_2 \ \text{ for all } \ t \in [0, T^*[\,, \tag{3.17}$$

and that

$$I_2 \le 0. \tag{3.18}$$

To see this, we follow the same reasoning as in [11].

(i) If $0 < \lambda < 1$, using the boundary conditions (2.4), we get

$$I_1 = \int_{\partial\Omega}\Big(\lambda_1\frac{\partial H_n}{\partial z_1}(\gamma_1 - \alpha z_1) + \lambda_2\frac{\partial H_n}{\partial z_2}(\gamma_2 - \alpha z_2) + \lambda_3\frac{\partial H_n}{\partial z_3}(\gamma_3 - \alpha z_3)\Big)\,ds,$$

where $\alpha = \frac{\lambda}{1-\lambda}$ and $\gamma_i = \frac{\rho_i}{1-\lambda}$, $i = 1, 2, 3$. Since

$$H(z_1,z_2,z_3)=\lambda_1\frac{\partial H_n}{\partial z_1}(\gamma_1-\alpha z_1)+\lambda_2\frac{\partial H_n}{\partial z_2}(\gamma_2-\alpha z_2)+ \\ +\lambda_3\frac{\partial H_n}{\partial z_3}(\gamma_3-\alpha z_3)=P_{n-1}(z_1,z_2,z_3)-Q_n(z_1,z_2,z_3),$$

where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n, and since the solution is positive, we obtain

$$\limsup_{(|z_1|+|z_2|+|z_3|)\to+\infty} H(z_1,z_2,z_3)=-\infty, \tag{3.19}$$

which proves that H is uniformly bounded on $(\mathbb{R}^+)^3$, and consequently (3.17).

(ii) If $\lambda=0$, then $I_1=0$ on $[0,T^*[$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0,T^*[\times\Omega$ implies $\partial z_1/\partial\eta\le 0$, $\partial z_2/\partial\eta\le 0$ and $\partial z_3/\partial\eta\le 0$ on $[0,T^*[\times\partial\Omega$. Consequently, one again gets (3.17) with $C_2=0$.

We now prove (3.18). Applying Lemma 2, we obtain

$$I_2=-n(n-1)\int_\Omega\sum_{p=0}^{n-2}\sum_{q=0}^{p}C_{n-2}^pC_p^q\big[(B_{pq}z)\cdot z\big]\,dx,$$

where

$$B_{pq}=\begin{pmatrix}\lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1}\\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1}\\ \frac{\lambda_1+\lambda_3}{2}\theta_{q+1}\sigma_{p+1} & \frac{\lambda_2+\lambda_3}{2}\theta_q\sigma_{p+1} & \lambda_3\theta_q\sigma_p\end{pmatrix}$$

for $q=0,1,\dots,p$, $p=0,1,\dots,n-2$ and $z=(\nabla z_1,\nabla z_2,\nabla z_3)^t$.

The quadratic forms (with respect to $\nabla z_1,\nabla z_2$ and ∇z_3) associated with the matrices B_{pq}, $q=0,1,\dots,p$, $p=0,1,\dots,n-2$, are positive, since their main determinants Δ_1, Δ_2 and Δ_3 are positive too, according to the Sylvester criterion. To see this, we have

1) $\Delta_1=\lambda_1\theta_{q+2}\sigma_{p+2}>0$ for $q=0,1,\dots,p$ $p=0,1,\dots,n-2$.

2) $\Delta_2=\begin{vmatrix}\lambda_1\theta_{q+2}\sigma_{p+2} & \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2}\\ \frac{\lambda_1+\lambda_2}{2}\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2}\end{vmatrix}=\lambda_1\lambda_2\theta_{q+1}^2\sigma_{p+2}^2(\theta^2-A_{12}^2),$
for $q=0,1,\dots,p$ and $p=0,1,\dots,n-2$.
Using (3.1), we get $\Delta_2>0$.

$$3)\ \Delta_3=\begin{vmatrix}\lambda_1\theta_{q+2}\sigma_{p+2} & \dfrac{\lambda_1+\lambda_2}{2}\,\theta_{q+1}\sigma_{p+2} & \dfrac{\lambda_1+\lambda_3}{2}\,\theta_{q+1}\sigma_{p+1}\\ \dfrac{\lambda_1+\lambda_2}{2}\,\theta_{q+1}\sigma_{p+2} & \lambda_2\theta_q\sigma_{p+2} & \dfrac{\lambda_2+\lambda_3}{2}\,\theta_q\sigma_{p+1}\\ \dfrac{\lambda_1+\lambda_3}{2}\,\theta_{q+1}\sigma_{p+1} & \dfrac{\lambda_2+\lambda_3}{2}\,\theta_q\sigma_{p+1} & \lambda_3\theta_q\sigma_p\end{vmatrix}=$$

$$=\lambda_1\lambda_2\lambda_3\theta_{q+1}^2\theta_q\sigma_{p+2}\sigma_{p+1}^2\big[(\theta^2-A_{12}^2)(\sigma^2-A_{23}^2)-(A_{13}-A_{12}A_{23})^2\big],$$

for $q=0,1,\dots,p$ and $p=0,1,\dots,n-2$.

Using (3.2), we get $\Delta_3>0$. Consequently we have (3.18).

Substitution of the expressions of the partial derivatives given by Lemma 1 in the second integral yields

$$J=\int_\Omega\Big[n\sum_{p=0}^{n-1}\sum_{q=0}^{p}C_{n-1}^pC_p^qz_1^qz_2^{p-q}z_3^{(n-1)-p}\Big]\times$$
$$\times\,(\theta_{q+1}\sigma_{p+1}F_1+\theta_q\sigma_{p+1}F_2+\theta_q\sigma_pF_3)\,dx.$$

Using the expressions (2.7a), we obtain

$$\theta_{q+1}\sigma_{p+1}F_1+\theta_q\sigma_{p+1}F_2+\theta_q\sigma_pF_3=-(\mu_1\theta_{q+1}\sigma_{p+1}+\mu_2\theta_q\sigma_{p+1}+\mu_3\theta_q\sigma_p)f+$$
$$+(\theta_{q+1}\sigma_{p+1}+\theta_q\sigma_{p+1}+\theta_q\sigma_p)g-(\nu_1\theta_{q+1}\sigma_{p+1}+\nu_2\theta_q\sigma_{p+1}+\nu_3\theta_q\sigma_p)h=$$
$$=-\theta_{q+1}\sigma_{p+1}\Big(\nu_1+\nu_2\frac{\theta_q}{\theta_{q+1}}+\nu_3\,\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}\Big)\times$$
$$\times\left(\frac{\mu_1+\mu_2\,\frac{\theta_q}{\theta_{q+1}}+\mu_3\,\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}}{\nu_1+\nu_2\,\frac{\theta_q}{\theta_{q+1}}+\nu_3\,\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}}\,f-\frac{1+\frac{\theta_q}{\theta_{q+1}}+\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}}{\nu_1+\nu_2\,\frac{\theta_q}{\theta_{q+1}}+\nu_3\,\frac{\theta_q}{\theta_{q+1}}\,\frac{\sigma_p}{\sigma_{p+1}}}\,g+h\right).$$

Since $\frac{\theta_q}{\theta_{q+1}}$ and $\frac{\sigma_p}{\sigma_{p+1}}$ are sufficiently large if we choose θ and σ sufficiently large, by using the condition (1.7) and the relation (2.6a) successively, for an appropriate constant C_3, we get

$$J\le C_3\int_\Omega\Big[\sum_{p=0}^{n-1}\sum_{q=0}^{p}(z_1+z_2+z_3+1)C_{n-1}^pC_p^qz_1^qz_2^{p-q}z_3^{(n-1)-p}\Big]\,dx.$$

To prove that the functional L is uniformly bounded on the interval $[0,T]$, we first write

$$\sum_{p=0}^{n-1}\sum_{q=0}^{p}(z_1+z_2+z_3+1)C_{n-1}^pC_p^qz_1^qz_2^{p-q}z_3^{(n-1)-p}=$$
$$=R_n(z_1,z_2,z_3)+S_{n-1}(z_1,z_2,z_3),$$

where $R_n(z_1,z_2,z_3)$ and $S_{n-1}(z_1,z_2,z_3)$ are two homogeneous polynomials of degrees n and $n-1$, respectively. First, since the polynomials H_n and R_n are of degree n, there exists a positive constant C_4 such that $\int\limits_\Omega R_n(z_1,z_2,z_3)\,dx\le C_4\int\limits_\Omega H_n(z_1,z_2,z_3)\,dx$. Applying Hölder's inequality to the integral $\int\limits_{\Omega} S_{n-1}(z_1, z_2, z_3)\, dx$, one gets

$$\int_{\Omega} S_{n-1}(z_1, z_2, z_3)\, dx \leq (meas\ \Omega)^{\frac{1}{n}} \Big(\int_{\Omega} \big(S_{n-1}(z_1, z_2, z_3)\big)^{\frac{n}{n-1}} dx \Big)^{\frac{n-1}{n}}.$$

Since for all $z_1 \geq 0$ and $z_2, z_3 > 0$,

$$\frac{(S_{n-1}(z_1, z_2, z_3))^{\frac{n}{n-1}}}{H_n(z_1, z_2, z_3)} = \frac{(S_{n-1}(\xi_1, \xi_2, 1))^{\frac{n}{n-1}}}{H_n(\xi_1, \xi_2, 1)},$$

where $\xi_1 = z_1/z_2$, $\xi_2 = z_2/z_3$ and

$$\lim_{\substack{\xi_1 \to +\infty \\ \xi_2 \to +\infty}} \frac{(S_{n-1}(\xi_1, \xi_2, 1))^{\frac{n}{n-1}}}{H_n(\xi_1, \xi_2, 1)} < +\infty,$$

one asserts that there exists a positive constant C_5 such that

$$\frac{(S_{n-1}(z_1, z_2, z_3))^{\frac{n}{n-1}}}{H_n(z_1, z_2, z_3)} \leq C_5 \text{ for all } z_1, z_2, z_3 \geq 0.$$

Due to (3.19), there exist η_i, $i = 1, 2, 3$, such that for all $z_i > \eta_i$ the functional L satisfies the differential inequality $L'(t) \leq C_6 L(t) + C_7 L^{\frac{n-1}{n}}(t)$, which for $Z = L^{\frac{1}{n}}$ can be written as $nZ' \leq C_6 Z + C_7$. A simple integration gives a uniform bound of the functional L on the interval $[0, T]$.

On the other hand, if z_i is in the compact interval $[0, \eta_i]$, then the continuous function $(z_1, z_2, z_3) \longmapsto H_n(z_1, z_2, z_3)$ is bounded. Thus, the functional L is uniformly bounded on $[0, T]$. This completes the proof of Theorem 1. □

Corollary 1. *Suppose that the functions f, g and h are continuously differentiable on Σ, point into Σ on $\partial\Sigma$ and satisfy the condition* (1.7). *Then all uniformly bounded solutions on Ω of* (1.1)–(1.5) *with initial data in Σ are in $L^\infty(0, T; L^p(\Omega))$ for all $p \geq 1$.*

Proof. The proof of this Corollary is an immediate consequence of Theorem 1, the trivial inequality $\int\limits_{\Omega}(z_1+z_2+z_3)^p\, dx \leq L(t)$ on $[0, T^*[$, and (2.6a). □

Proposition 2. *Under the hypothesis of Corollary* 1*, if the functions f, g and h are polynomially bounded on Σ, then all uniformly bounded solutions on Ω of* (1.1)–(1.4) *with the initial data in Σ are global in time.*

Proof. As it has been mentioned above, it suffices to derive a uniform estimate of $\|F_1(z_1, z_2, z_3)\|_p$, $\|F_2(z_1, z_2, z_3)\|_p$ and $\|F_3(z_1, z_2, z_3)\|_p$ on $[0, T]$, $T < T^*$ for some $p > \frac{N}{2}$. Since the reaction terms $f(u, v, w)$, $g(u, v, w)$ and $h(u, v, w)$ are polynomially bounded on Σ, by using the relations (2.6a) and (2.7a) we get that such are $F_1(z_1, z_2, z_3)$, $F_2(z_1, z_2, z_3)$ and $F_3(z_1, z_2, z_3)$, and the proof becomes an immediate consequence of Corollary 1. □

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