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VARIATION FORMULAS OF SOLUTION FOR A CONTROLLED DELAY FUNCTIONAL-DIFFERENTIAL EQUATION TAKING INTO ACCOUNT DELAYS PERTURBATIONS AND THE MIXED INITIAL CONDITION

Abstract. Variation formulas of solution are obtained for a nonlinear controlled delay functional-differential equation with respect to perturbations of initial moment, constant delays, initial vector, initial functions and control function. The effects of delay perturbations and the mixed initial condition are discovered in the variation formulas.

რეზიუმე. სამართი დაგვიანებულ არგუმენტიანი ფუნქციონალურ-დიფერენციალური განტოლებისათვის მიღებულია ამონახსნის ვარიაციის ფორმულები საწყისი მომენტის, მუდმივი დაგვიანებების, საწყისი ვექტორის, საწყისი ფუნქციებისა და მართვის ფუნქციის შეშფოთებების მიმართ. ვარიაციის ფორმულებში გამოვლენილია დაგვიანებების შეშფოთებისა და შერეული საწყისი პირობის ეფექტები.

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1. Introduction

In the present paper, variation formulas of solution (variation formulas) are obtained for a nonlinear controlled delay functional-differential equation under perturbations of initial moment, constant delays, initial vector, initial functions and control function. The effects of delays perturbations and the mixed initial condition are discovered in the variation formulas. The mixed initial condition means that at the initial moment, some coordinates of the trajectory do not coincide with the corresponding coordinates of the initial function, whereas the others coincide. The variation formula allows one to construct an approximate solution of the perturbed equation in an analytical form on the one hand, and in the theory of optimal control it plays the basic role in proving the necessary conditions of optimality [1]–[11], on the other. Variation formulas for various classes of functional-differential equations without perturbation of delay are given in [2], [6], [7] and [9]–[13]. Variation formulas for delay functional-differential equations with the continuous and discontinuous initial condition taking into account constant delay perturbation are proved in [14] and [15], respectively. Variation formulas for controlled delay functional-differential equations with the continuous initial condition taking into account constant delay perturbation are proved in [16].

2. Formulation of the Main Results

Let R_x^n be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T denotes transposition; suppose $P \subset R_p^k$, $Z \subset R_z^m$ and $W \subset R_u^r$ are open sets and $O = (P, Z)^T = \{x = (p, z)^T \in R_x^n : \ p \in P, z \in Z\}$, with $k+m = n$. Let the n-dimensional function $f(t, x, p, z, u)$ satisfy the following conditions: for almost all $t \in I = [a, b]$, the function $f(t, \cdot) : O \times P \times Z \times W \to R_x^n$ is continuously differentiable; for any $(x, p, z, u) \in O \times P \times Z \times W$, the functions $f(t, x, p, z, u), f_x(\cdot), f_p(\cdot), f_z(\cdot) f_u(\cdot)$ are measurable on I; for arbitrary compacts $K \subset O$, $U \subset W$ there exists a function $m_{K,U}(\cdot) \in L(I, [0, \infty))$, such that for any $x \in K$, $(p, z)^T \in K$, $u \in U$ and for almost all $t \in I$ the inequality

$$|f(t, x, p, z, u)| + |f_x(\cdot)| + |f_p(\cdot)| + |f_z(\cdot)| + |f_u(\cdot)| \le m_{K,U}(t)$$

is fulfilled.

Let $0 < \tau_1 < \tau_2$, $0 < \sigma_1 < \sigma_2$ be the given numbers and $E_\varphi = E_\varphi(I_1, R_p^k)$ be the space of continuous functions $\varphi : I_1 \to R_p^k$, where $I_1 = [\widehat{\tau}, b], \widehat{\tau} = a - \max\{\tau_2, \sigma_2\}$. Further,

$$\Phi = \big\{\varphi \in E_\varphi : \ \varphi(t) \in P\big\} \ \text{ and } \ G = \big\{g \in E_g = E_g(I_1, R_z^m) : \ g(t) \in Z\big\}$$

are the sets of initial functions. Let E_u be the space of bounded measurable functions $u : I \to R_u^r$ and $\Omega = \{u \in E_u : \ u(t) \in W, \ t \in I, \ \mathrm{cl}\, u(I) \subset W\}$ be a set of control functions, where $u(I) = \{u(t) : \ t \in I\}$ and $\mathrm{cl}\, u(I)$ is the closure of the set $u(I)$.

To any element

$$\mu = (t_0, \tau, \sigma, p_0, \varphi, g, u) \in \Lambda = (a, b) \times (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times P \times \Phi \times G \times \Omega,$$

we assign the controlled delay functional-differential equation

$$\dot{x}(t) = (\dot{p}(t), \dot{z}(t))^T = f\big(t, x(t), p(t-\tau), z(t-\sigma), u(t)\big) \tag{2.1}$$

with a mixed initial condition

$$x(t) = (\varphi(t), g(t))^T, \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = (p_0, g(t_0))^T. \tag{2.2}$$

The condition (2.2) is said to be a mixed initial condition; it consists of two parts: the first part is $p(t) = \varphi(t)$, $t \in [\widehat{\tau}, t_0)$, $p(t_0) = p_0$, the discontinuous part, since generally $p(t_0) \ne \varphi(t_0)$; the second part is $z(t) = g(t)$, $t \in [\widehat{\tau}, t_0]$, the continuous part, since always $z(t_0) = g(t_0)$.

Definition 2.1. Let $\mu = (t_0, \tau, \sigma, p_0, \varphi, g, u) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b)$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies the condition (2.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (2.1) almost everywhere on $[t_0, t_1]$.

Let $\mu_0 = (t_{00}, \tau_0, \sigma_0, p_{00}, \varphi_0, g_0, u_0) \in \Lambda$ be a fixed element. In the space $E_\mu = R^1_{t_0} \times R^1_\tau \times R^1_\sigma \times R^k_p \times E_\varphi \times E_g \times E_u$ we introduce the set of variations

$$V = \Big\{\delta\mu = (\delta t_0, \delta\tau, \delta\sigma, \delta p_0, \delta\varphi, \delta g, \delta u) \in E_\mu - \mu_0 : \ |\delta t_0| \le \alpha,$$

$$|\delta\tau| \le \alpha, \ |\delta\sigma| \le \alpha, \ |\delta p_0| \le \alpha, \ \delta\varphi = \sum_{i=1}^{\nu} \lambda_i \delta\varphi_i,$$

$$\delta g = \sum_{i=1}^{\nu} \lambda_i \delta g_i, \ \delta u = \sum_{i=1}^{\nu} \lambda_i \delta u_i, \ |\lambda_i| \le \alpha, \ i = \overline{1, \nu}\Big\},$$

where $\delta\varphi_i \in E_\varphi - \varphi_0$, $\delta g_i \in E_g - g_0$, $\delta u_i \in E_u - u_0$, $i = \overline{1, \nu}$, are the fixed functions; $\alpha > 0$ is a fixed number.

Let $x_0(t) = (p_0(t), z_0(t))^T$ be the solution corresponding to the element μ_0 and defined on the interval $[\hat{\tau}, t_{10}]$, with $t_{10} < b$. There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon_1] \times V$ we have $\mu_0 + \varepsilon\delta\mu \in \Lambda$. In addition, to this element there corresponds the solution $x(t; \mu_0 + \varepsilon\delta\mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ (see Theorem 5.3 in [17, p. 111]).

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \mu_0 + \varepsilon\delta\mu) - x_0(t), \ \ (t, \varepsilon, \delta\mu) \in [\hat{\tau}, t_{10} + \delta_1] \times [0, \varepsilon_1] \times V.$$

Theorem 2.1. *Let the following conditions hold:*

2.1. $t_{00} + \tau_0 < t_{10}$;

2.2. *the functions $\varphi_0(t)$, $g_0(t)$, $t \in I_1$, are absolutely continuous and $\dot{\varphi}_0(t)$, $\dot{g}_0(t)$ are bounded; there exist compact sets $K_0 \subset O$ and $U_0 \subset W$ containing neighborhoods of sets $(\varphi_0(I_1), g_0(I_1))^T \cup x_0([t_{00}, t_{10}])$ and $\operatorname{cl} u_0(I)$, respectively, such that the function $f(t, x, p, z, u)$, $(t, x) \in I \times K_0$, $(p, z)^T \in K_0$, $u \in U_0$, is bounded;*

2.3. *there exist the limits*

$$\lim_{t \to t_{00}-} \dot{g}_0(t) = \dot{g}_0^-,$$

$$\lim_{w \to w_0} f(w, u_0(t)) = f_0^-, \ \ w \in (t_{00} - \tau_0, t_{00}] \times O \times P \times Z,$$

$$\lim_{(w_1, w_2) \to (w_{01}, w_{02})} \big[f(w_1, u_0(t)) - f(w_2, u_0(t))\big] = f_{01}^-,$$

$$w_1, w_2 \in (t_{00}, t_{00} + \tau_0] \times O \times P \times Z,$$

where

$$
\begin{aligned}
w &= (t, x, p, z),\\
w_0 &= \big(t_{00}, x_{00}, \varphi_0(t_{00}-\tau_0), g_0(t_{00}-\sigma_0)\big),\\
x_{00} &= (p_{00}, g_0(t_{00}))^T,\\
w_{01} &= \big(t_{00}+\tau_0, x_0(t_{00}+\tau_0), p_{00}, z_0(t_{00}+\tau_0-\sigma_0)\big),\\
w_{02} &= \big(t_{00}+\tau_0, x_0(t_{00}+\tau_0), \varphi_0(t_{00}), z_0(t_{00}+\tau_0-\sigma_0)\big).
\end{aligned}
$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1]$ *and* $\delta_2 \in (0, \delta_1]$ *such that*

$$\Delta x(t; \varepsilon\delta\mu) = \varepsilon \delta x(t; \delta\mu) + o(t; \varepsilon\delta\mu) \tag{2.3}$$

for arbitrary

$$(t, \varepsilon, \delta\mu) \in [t_{10}-\delta_2, t_{10}+\delta_2] \times [0, \varepsilon_2] \times \big\{\delta\mu \in V : \delta t_0 \le 0,\ \delta\tau \le 0,\ \delta\sigma \le 0\big\},$$

where

$$
\begin{aligned}
\delta x(t; \delta\mu) = {}& \Big\{Y(t_{00}; t)\big[(\Theta_{k\times 1}, \dot g_0^-)^T - f_0^-\big] - Y(t_{00}+\tau_0; t) f_{01}^-\Big\}\delta t_0 - \\
& - Y(t_{00}+\tau_0; t) f_{01}^- \delta\tau + \beta(t; \varepsilon\delta\mu),
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\beta(t; \varepsilon\delta\mu) = {}& Y(t_{00}; t)(\delta p_0, \delta g(t_{00}))^T - \\
& - \Big\{\int_{t_{00}}^{t} Y(\xi; t) f_p[\xi] \dot p_0(\xi - \tau_0)\, d\xi\Big\}\delta\tau - \\
& - \Big\{\int_{t_{00}}^{t} Y(\xi; t) f_z[\xi] \dot z_0(\xi - \sigma_0)\, d\xi\Big\}\delta\sigma + \\
& + \int_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0; t) f_p[\xi+\tau_0] \delta\varphi(\xi)\, d\xi + \\
& + \int_{t_{00}-\sigma_0}^{t_{00}} Y(\xi+\sigma_0; t) f_z[\xi+\sigma_0] \delta g(\xi)\, d\xi + \\
& + \int_{t_{00}}^{t} Y(\xi; t) f_u[\xi] \delta u(\xi)\, d\xi;
\end{aligned}
\tag{2.5}
$$

$$\lim_{\varepsilon\to 0} \frac{o(t; \varepsilon\delta\mu)}{\varepsilon} = 0$$

uniformly for

$$(t, \delta\mu) \in [t_{10}-\delta_2, t_{10}+\delta_2] \times \big\{\delta\mu \in V :\ \delta t_0 \le 0,\ \delta\tau \le 0,\ \delta\sigma \le 0\big\};$$

$\Theta_{k\times 1}$ is the $k\times 1$ zero matrix, $Y(s;t)$ is the $n\times n$ matrix function satisfying on the interval $[t_{00},t]$ the equation

$$Y_\xi(\xi;t)=-Y(\xi;t)f_x[\xi]-\Big(Y(\xi+\tau_0;t)f_p[\xi+\tau_0],Y(\xi+\sigma_0;t)f_z[\xi+\sigma_0]\Big),$$

and the condition

$$Y(\xi;t)=\begin{cases}H_{n\times n} & \text{for } \xi=t,\\ \Theta_{n\times n} & \text{for } \xi>t.\end{cases}$$

Here, $H_{n\times n}$ is the $n\times n$ identity matrix,

$$f_x[\xi]=f_x\Big(\xi,x_0(\xi),p_0(\xi-\tau_0),z_0(\xi-\sigma_0),u_0(\xi)\Big),\;\; \dot p_0(\xi-\tau_0)=\dot p_0(s)_{|s=\xi-\tau_0},$$

under $\dot p_0(s)$ is assumed derivative of the function $p_0(s)$ on the set $[\widehat{\tau},t_{00})\cup(t_{00},t_{10}+\delta_2]$.

Some comments. The function $\delta x(t;\delta\mu)$ is called the variation of the solution $x_0(t)$ on the interval $[t_{10}-\delta_2,t_{10}+\delta_2]$ and the expression (2.4) is called the variation formula.

c 1) Theorem 2.1 corresponds to the case where the variations at the points t_{00},τ_0,σ_0 are performed simultaneously on the left.

c 2) The addend

$$-\Big\{Y(t_{00}+\tau_0;t)f_{01}^-+\int\limits_{t_{00}}^{t}Y(\xi;t)f_p[\xi]\dot p_0(\xi-\tau_0)\,d\xi\Big\}\delta\tau-$$
$$-\Big\{\int\limits_{t_{00}}^{t}Y(\xi;t)f_z[\xi]\dot z_0(\xi-\sigma_0)\,d\xi\Big\}\delta\sigma$$

is the effect of perturbations of the delays τ_0 and σ_0 (see (2.4) and (2.5)).

c 3) The expression

$$Y(t_{00};t)(\delta p_0,\delta g(t_{00}))^T+$$
$$+\Big\{Y(t_{00};t)\big[(\Theta_{k\times 1},\dot g_0^-)^T-f_0^-\big]-Y(t_{00}+\tau_0;t)f_{01}^-\Big\}\delta t_0$$

is the effect of the mixed initial condition (2.2) under perturbations of initial moment t_{00}, initial vector p_{00} and function $g_0(t)$.

c 4) The expression

$$\int\limits_{t_{00}-\tau_0}^{t_{00}} Y(\xi+\tau_0;t)f_p[\xi+\tau_0]\delta\varphi(\xi)\,d\xi+$$

$$+\int\limits_{t_{00}-\sigma_0}^{t_{00}} Y(\xi+\sigma_0;t)f_z[\xi+\sigma_0]\delta g(\xi)\,d\xi+\int\limits_{t_{00}}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$

in the formula (2.5) is the effect of perturbations of the initial functions $\varphi_0(t)$, $g_0(t)$ and the control function $u_0(t)$.

c 5) The variation formula allows one to obtain an approximate solution of the perturbed functional-differential equation

$$\dot{x}(t)=f\Big(t,x(t),p(t-\tau_0-\varepsilon\delta\tau),z(t-\sigma_0-\varepsilon\delta\sigma),u_0(t)+\varepsilon\delta u(t)\Big)$$

with the perturbed initial condition

$$x(t)=\big(\varphi_0(t)+\varepsilon\delta\varphi(t),g_0(t)+\varepsilon\delta g(t)\big)^T,\ \ t\in[\widehat{\tau},t_{00}+\varepsilon\delta t_0),$$
$$x(t_{00}+\varepsilon\delta t_0)=\big(p_{00}+\varepsilon\delta p_0,g_0(t_{00})+\varepsilon\delta g(t_{00})\big)^T.$$

In fact, for a sufficiently small $\varepsilon\in(0,\varepsilon_2]$ from (2.3) it follows that

$$x(t;\mu_0+\varepsilon\delta\mu)\approx x_0(t)+\varepsilon\delta x(t;\delta\mu).$$

Theorem 2.2. *Let the conditions* 2.1 *and* 2.2 *of Theorem* 2.1 *hold. Moreover, there exist the limits*

$$\lim_{t\to t_{00}+}\dot{g}_0(t)=\dot{g}_0^+,$$
$$\lim_{w\to w_0} f(w,u_0(t))=f_0^+,\ \ w\in[t_{00},t_{10})\times O\times P\times Z,$$
$$\lim_{(w_1,w_2)\to(w_{01},w_{02})}\big[f(w_1,u_0(t))-f_0(w_2,u_0(t))\big]=f_{01}^+,$$
$$w_1,w_2\in[t_{00}+\tau_0,t_{10})\times O\times P\times Z.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ and $\delta_2\in(0,\delta_1]$ such that for arbitrary

$$(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times\{\delta\mu\in V:\ \delta t_0\geq 0,\ \delta\tau\geq 0,\ \delta\sigma\geq 0\}$$

the formula (2.3) *holds, where*

$$\delta x(t;\delta\mu)=\Big\{Y(t_{00};t)\big[(\Theta_{k\times 1},\dot{g}_0^+)^T-f_0^+\big]-Y(t_{00}+\tau_0;t)f_{01}^+\Big\}\delta t_0-$$
$$-Y(t_{00}+\tau_0;t)f_{01}^+\delta\tau+\beta(t;\varepsilon\delta\mu).$$

Theorem 2.2 corresponds to the case where the variations at the points t_{00}, τ_0, σ_0 are performed simultaneously on the right.

Theorem 2.3. *Let the conditions of Theorems* 2.1 *and* 2.2 *hold. Moreover,*

$$(\Theta_{k\times 1},\dot g_0^-)^T-f_0^-=(\Theta_{k\times 1},\dot g_0^+)^T-f_0^+=:\widehat{f}_0,\, f_{01}^-=f_{01}^+=:\widehat{f}_{01}.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1]$ *and* $\delta_2\in(0,\delta_1]$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times[0,\varepsilon_2]\times V$ *the formula* (2.3) *holds, where*

$$\delta x(t;\delta\mu)=\Big\{Y(t_{00};t)\widehat{f}_0-Y(t_{00}+\tau_0;t)\widehat{f}_{01}\Big\}\delta t_0-\\ -Y(t_{00}+\tau_0;t)\widehat{f}_{01}\delta\tau+\beta(t;\varepsilon\delta\mu).$$

Theorem 2.3 corresponds to the case where at the points t_{00}, τ_0, σ_0 the two-sided variations are simultaneously performed. Theorems 2.1–2.3 are proved by the method given in [10]. If $t_{00}+\tau_0>t_{10}$, then Theorems 2.1–2.3 are also valid. In this case the number δ_2 is so small that $t_{00}+\tau_0>t_{10}+\delta_2$, therefore in the variation formulas we have $Y(t_{00}+\tau_0;t)=\Theta_{n\times n}$, $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$. If $t_{00}+\tau_0=t_{10}$, then Theorem 2.1 is valid on the interval $[t_{10},t_{10}+\delta_2]$ and Theorem 2.2 is valid on the interval $[t_{10}-\delta_2,t_{10}]$.

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