Memoirs on Differential Equations and Mathematical Physics
Volume 57, 2012, 109–122

Yūki Naito

REMARKS ON SINGULAR STURM COMPARISON THEOREMS

Cordially dedicated to Professor Takaŝi Kusano on his 80th birthday

Abstract. In a finite or infinite open interval, the linear differential equations of second order with singularities at endpoints are considered. By making use of principal solutions at endpoints of the interval, we obtain sharper forms of the Strum comparison theorem.

2010 Mathematics Subject Classification. 34B24, 34C10.

Key words and phrases. Singular second order linear differential equations, Sturm comparison theorem, principal solutions

რეზიუმე. სასრულ ან უსასრულო შუალედში განხილულია მეორე რიგის წრფივი დიფერენციალური განტოლებები სინგულარობებით საზღვრით წერტილებში. აღნიშნული წერტილების მიმართ მთავარი ამონახსნების გამოყენებით დადგენილია არაგაუმჯობესებადი შტურმის შედარების თეორემა.

1. Introduction

We consider two differential equations

$$(p(t)u')' + q(t)u = 0, \tag{1.1}$$

$$(P(t)v')' + Q(t)v = 0 \tag{1.2}$$

on the intervals (α, ω) with $-\infty \le \alpha < \omega \le \infty$ and $[a, \omega)$ with $a \in (\alpha, \omega)$. Throughout the paper we assume, in (1.1) and (1.2), that $p(t)$, $q(t)$, $P(t)$, and $Q(t)$ are continuous functions on (α, ω), and satisfy

$$p(t) \ge P(t) > 0 \ \text{ and } \ Q(t) \ge q(t) \ \text{ on } \ (\alpha, \omega). \tag{1.3}$$

We consider the Sturm comparison theorems in the case where the continuity of the coefficients of equations is assumed only on (α, ω). The possibility that the interval is unbounded is not excluded. Concerning the Sturm comparison theorems for such singular equations, several results are summarized in Reid [13] and Swanson [14]. In this paper, motivated by the recent works by Chuaqui et. al. [2] and Aharonov and Elias [1], we will show sharper forms of the Strum's comparison theorem by making use of the principal solutions at endpoints of the interval.

Let us recall the definitions of principal and nonprincipal solutions to (1.1). Assume that (1.1) is nonoscillatory at $t = \omega$. It is well known [5, Ch. XI, Theorem 6.4] that (1.1) has a unique (neglecting a constant factor) solution $u_0(t)$ satisfying

$$\int^{\omega} \frac{ds}{p(s)u_0(s)^2} = \infty, \tag{1.4}$$

and any solution $u_1(t)$, linearly independent of $u_0(t)$, satisfies

$$\int^{\omega} \frac{ds}{p(s)u_1(s)^2} < \infty \tag{1.5}$$

and $u_0(t)/u_1(t) \to 0$ as $t \to \omega$. A solution $u_0(t)$ satisfying (1.4) is called a principal solution at $t = \omega$, and a solution $u_1(t)$ satisfying (1.5) is called a nonprincipal solution at $t = \omega$. The principal and nonprincipal solutions of (1.1) at $t = \alpha$ are defined similarly. For further information about the properties of principal and nonprincipal solutions, we refer to Hartman [5, Ch. XI] and Elbert and Kusano [3].

First we consider (1.1) and (1.2) on a half-open interval $[a, \omega)$ with $a \in (\alpha, \omega)$. The Sturm's comparison theorem can be stated usually as follows: (See, e.g., [5, Ch. XI, Theorem 3.1].)

Theorem A. *Let $u(t) \not\equiv 0$ be a solution of (1.1) on $[a, \omega)$, and let $v(t)$ be a solution of (1.2) on $[a, \omega)$. Assume that, for some $n \in \mathbf{N} = \{1, 2, \dots\}$, the solution $u(t)$ has exactly n zeros $t = t_1 < t_2 < \cdots < t_n$ in (a, ω). If either $u(a) = 0$ or*

$$u(a) \ne 0, \ v(a) \ne 0, \ \text{ and } \ \frac{p(a)u'(a)}{u(a)} \ge \frac{P(a)v'(a)}{v(a)},$$

then $v(t)$ has one of the following properties:

(i) *$v(t)$ has at least n zeros in (a, t_n);*

(ii) *$v(t)$ is a constant multiple of $u(t)$ on $[a, t_n]$ and*

$$p(t) \equiv P(t), \quad q(t) \equiv Q(t) \ \text{ on } \ [a, t_n].$$

In the case where $u(t) \neq 0$ on (t_n, ω) in Theorem A, it seems interesting to put a question whether a solution $v(t)$ of (1.2) has at least one zero in (t_n, ω). Our results are the following.

Theorem 1. *Assume that (1.1) is nonoscillatory at $t = \omega$. Let $u_0(t)$ be a principal solution of (1.1) at $t = \omega$, and let $v(t)$ be a solution of (1.2) on $[a, \omega)$. Assume that $u_0(t) > 0$ on (a, ω). If either $u_0(a) = 0$ or*

$$u_0(a) \neq 0, \ v(a) \neq 0, \ \text{ and } \ \frac{p(a)u_0'(a)}{u_0(a)} \geq \frac{P(a)v'(a)}{v(a)}, \tag{1.6}$$

then $v(t)$ has one of the following properties:

(i) *$v(t)$ has at least one zero in (a, ω);*

(ii) *$v(t)$ is a constant multiple of $u_0(t)$ on $[a, \omega)$, and*

$$p(t) \equiv P(t), \ \ q(t) \equiv Q(t) \ \text{ on } \ [a, \omega).$$

Combining Theorems A and 1, we obtain the following

Theorem 2. *Assume that (1.1) is nonoscillatory at $t = \omega$. Let $u_0(t)$ be a principal solution of (1.1) at $t = \omega$, and let $v(t)$ be a solution of (1.2) on $[a, \omega)$. Assume that $u(t)$ has exactly n zeros in (a, ω) for some $n \in \mathbf{N}$. If either $u_0(a) = 0$ or (1.6) holds, then $v(t)$ has one of the following properties:*

(i) *$v(t)$ has at least $n + 1$ zeros in (a, ω);*

(ii) *$v(t)$ is a constant multiple of $u_0(t)$ on $[a, \omega)$ and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on $[a, \omega)$.*

Next, motivated by [1,2,11,12], we consider (1.1) and (1.2) on the interval (α, ω) with $-\infty \leq \alpha < \omega \leq \infty$.

Theorem 3. *Assume that there exists a solution $u_0(t)$ of (1.1) such that $u_0(t)$ has exactly $n - 1$ zeros in (α, ω) for some $n \in \mathbf{N}$ and is principal at both points $t = \alpha$ and $t = \omega$, that is,*

$$\int_{\alpha} \frac{1}{p(t)u_0(t)^2}\, dt = \infty \ \text{ and } \ \int^{\omega} \frac{1}{p(t)u_0(t)^2}\, dt = \infty. \tag{1.7}$$

If $v(t)$ is a solution of (1.2) on (α, ω), then $v(t)$ has one of the following properties:

(i) *$v(t)$ has at least n zeros in (α, ω);*

(ii) *$v(t)$ is a constant multiple of $u_0(t)$ on (α, ω), and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on (α, ω).*

Let us consider some corollaries of Theorem 3. For the case where $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on (α, ω) in Theorem 3, we will obtain the uniqueness of solution of (1.1) with prescribed numbers of zeros in (α, ω).

Corollary 1. *Assume that there exists a solution $u_0(t)$ of* (1.1) *such that $u_0(t)$ has exactly $n-1$ zeros in (α, ω) for some $n \in \mathbf{N}$ and satisfies* (1.7). *Then any solution, linearly independent of u_0, has exactly n zeros in (α, ω), that is, the solution of* (1.1) *with $n-1$ zeros in (α, ω) is unique up to a constant factor.*

In the case where

$$p(t) \not\equiv P(t) \ \text{ or } \ q(t) \not\equiv Q(t) \ \text{ on } \ (\alpha, \omega), \tag{1.8}$$

as a corollary of Theorem 3, we obtain the following

Corollary 2. *Assume that* (1.8) *holds. If there exists a solution $u_0(t)$ of* (1.1) *such that $u_0(t)$ has exactly $n-1$ zeros in (α, ω) for some $n \in \mathbf{N}$ and satisfies* (1.7), *then every solution v of* (1.2) *has at least n zeros in (α, ω).*

Remark 1.

(i) In the case where $u_0(t) > 0$ and $p(t) \equiv P(t) \equiv 1$ on (α, ω), the result in Corollary 2 was shown in [1, Theorem 1 (i)] by a different argument.

(ii) Let us consider the equation with a parameter $\lambda > 0$:

$$(p(t)u')' + \lambda q(t)u = 0 \tag{1.9}$$

on the interval (α, ω). In (1.9) we assume that $q \geq 0$, $q \not\equiv 0$ on (α, ω). For each $n \in \mathbf{N}$, let us denote by λ_n the parameter λ such that (1.9) has a solution u_0 which has exactly $n-1$ zeros in (α, ω) and satisfies (1.7). Corollary 2 implies that λ_n is unique for each $n \in \mathbf{N}$ if it exists. The existence of a sequence $\{\lambda_n\}_{n=1}^{\infty}$ was shown by Kusano and M. Naito [7,8] for the equation (1.9) on (a, ∞) under suitable conditions on p and q. (See also [10].) The extension of the results to the half-linear differential equations was done by [4, 9].

We will show that the condition (1.7) is likewise necessary for the uniqueness of a solution with prescribed numbers of zeros.

Theorem 4. *Assume that* (1.1) *has a solution $u(t)$ which has exactly $n-1$ zeros in (α, ω) with some $n \in \mathbf{N}$, and that any solution, linearly independent of u, has n zeros in (α, ω). Then $u(t)$ is principal at both points $t = \alpha$ and $t = \omega$, that is,* (1.7) *holds with $u_0 = u$.*

Finally, we consider comparison results on the existence of positive solutions of (1.1) and (1.2). Note that, by Corollary 2, if (1.8) holds, and if (1.1) has a positive solution u_0 satisfying (1.7), then (1.2) has no positive solution.

Theorem 5.

(i) *Assume that* (1.8) *holds. If* (1.2) *has a positive solution on* (α, ω), *then* (1.1) *has positive solutions* $u(t)$, $u_0(t)$, $\widetilde{u}_0(t)$ *on* (α, ω) *satisfying*

$$\int_\alpha \frac{1}{p(t)u(t)^2}\,dt < \infty, \quad \int^\omega \frac{1}{p(t)u(t)^2}\,dt < \infty, \tag{1.10}$$

$$\int_\alpha \frac{1}{p(t)u_0(t)^2}\,dt < \infty, \quad \int^\omega \frac{1}{p(t)u_0(t)^2}\,dt = \infty, \tag{1.11}$$

and

$$\int_\alpha \frac{1}{p(t)\widetilde{u}_0(t)^2}\,dt = \infty, \quad \int^\omega \frac{1}{p(t)\widetilde{u}_0(t)^2}\,dt < \infty, \tag{1.12}$$

respectively.

(ii) *Assume that* (1.1) *has a positive solution* $u(t)$ *on* (α, ω) *satisfying*

$$\int_\alpha \frac{1}{p(t)u(t)^2}\,dt < \infty \ \text{ or } \ \int^\omega \frac{1}{p(t)u(t)^2}\,dt < \infty. \tag{1.13}$$

Then there exist continuous functions $P(t)$ *and* $Q(t)$ *satisfying* (1.3) *with* (1.8) *such that* (1.2) *has a positive solution on* (α, ω).

Remark 2. Some concrete examples of Theorem 5 (ii) were constructed by [1].

Theorem 1 is proved by employing Piconne's identity [6] together with some properties of principal solutions. We prove Theorem 3 by combining comparison results for the half-open intervals $(\alpha, a]$ and $[a, \omega)$. Making use of two principal solutions at $t = \alpha$ and $t = \omega$, we obtain Theorems 4 and 5.

2. Proofs of Theorems

To prove Theorem 1, we need the following lemmas.

Lemma 1. *Assume that* $q(t) \le 0$ *on* $[a, \omega)$ *in* (1.1). *Then* (1.1) *is nonoscillatory at* $t = \omega$ *and a principal solution* $u_0(t)$ *of* (1.1) *satisfies* $u_0(t) > 0$ *and* $u_0'(t) \le 0$ *on* $[a, \omega)$.

Lemma 2. *Assume that* (1.1) *is nonoscillatory at* $t = \omega$. *Let* $u_0(t)$ *be a principal solution of* (1.1), *and let* $v(t)$ *be a solution of* (1.2) *satisfying* $v(t) > 0$ *on* $[T, \omega)$ *with some* $T \ge a$. *Then* $u_0(t) > 0$ *on* $[T, \omega)$ *and*

$$\frac{p(t)u_0'(t)}{u_0(t)} \le \frac{P(t)v'(t)}{v(t)} \ \text{ on } \ [T, \omega).$$

Lemmas 1 and 2 are shown in [5, Ch. XI, Corollaries 6.4 and 6.5]. However, for reader's convenience, we give slightly simpler proofs of them.

Proof of Lemma 1. Let $u_i(t)$, $i = 1, 2$, be solutions of (1.1) determined by $u_i(a) = 1$ and $u_i'(a) = i$. It is easy to see that $(p(t)u_i'(t))' \geq 0$ and $u_i(t) > 0$ on $[a, \omega)$, $i = 1, 2$. Since $u_1(t)$ and $u_2(t)$ are linearly independent, either $u_1(t)$ or $u_2(t)$ is a nonprincipal solution. Without loss of generality, we may assume that $u_1(t)$ is a nonprincipal solution. By [5, Ch. XI, Corollary 6.3],

$$u_0(t) = u_1(t) \int_t^\infty \frac{ds}{p(s)u_1(s)^2} \quad \text{for} \quad a \leq t < \omega,$$

is well defined and a principal solution of (1.1). Then we have $u_0(t) > 0$ on $[a, \omega)$. We obtain

$$p(t)u_0'(t) = p(t)u_1'(t) \int_t^\infty \frac{ds}{p(s)u_1(s)^2} - \frac{1}{u_1(t)} \quad \text{for} \quad a \leq t < \omega.$$

Since $p(t)u_1'(t)$ is nondecreasing, we have

$$p(t)u_0'(t) \leq \int_t^\infty \frac{u_1'(s)}{u_1(s)^2}\, ds - \frac{1}{u_1(t)} \quad \text{for} \quad a \leq t < \omega. \tag{2.1}$$

Note here that

$$\int_t^\infty \frac{u_1'(s)}{u_1(s)^2}\, ds - \frac{1}{u_1(t)} =$$

$$= \lim_{\tau \to \infty} \left(\int_t^\tau \frac{u_1'(s)}{u_1(s)^2}\, ds - \frac{1}{u_1(t)} \right) = \lim_{\tau \to \infty} \left(- \frac{1}{u_1(\tau)} \right) \leq 0.$$

Thus, from (2.1), we obtain $u_0'(t) \leq 0$ on $[a, \omega)$. □

Proof of Lemma 2. Let

$$w(t) = \exp\left(\int_T^t \frac{P(s)v'(s)}{p(s)v(s)}\, ds \right) \quad \text{for} \quad T \leq t < \omega.$$

Then $w(t) > 0$ on $[T, \omega)$ and satisfies

$$p(t)w'(t) = \frac{P(t)v'(t)w(t)}{v(t)} \quad \text{for} \quad T \leq t < \omega. \tag{2.2}$$

It follows that

$$(p(t)w')' = (P(t)v')' \frac{w}{v} + P(t)v' \Big(\frac{w}{v} \Big)'.$$

From (2.2) we note that

$$\Big(\frac{w}{v} \Big)' = \frac{vw' - v'w}{v^2} = \frac{w'}{v} - \frac{v'w}{v^2} = \Big(\frac{1}{p(t)} - \frac{1}{P(t)} \Big) \frac{P(t)v'w}{v^2}\,.$$

Thus, w satisfies

$$(p(t)w')' + Q_0(t)w = 0 \ \text{ for } \ T \le t < \omega,$$

where

$$Q_0(t) = Q(t) + \Big(\frac{1}{P(t)} - \frac{1}{p(t)}\Big)\Big(\frac{P(t)v'(t)}{v(t)}\Big)^2 \ \text{ for } \ T \le t < \omega.$$

Let

$$z(t) = \frac{u_0(t)}{w(t)} \ \text{ on } [T, \omega).$$

Since $z(t)$ satisfies

$$p(t)w(t)^2 z'(t) = p(t)u_0'(t)w(t) - p(t)u_0(t)w'(t),$$

we see that

$$(p(t)w(t)^2 z')' + w(t)^2(q(t) - Q_0(t))z = 0 \ \text{ for } \ T \le t < \omega. \tag{2.3}$$

Since $u_0(t)$ is a principal solution, by [5, Ch. XI, Lemma 2.1], we have

$$\int^{\infty} \frac{ds}{p(s)w(s)^2 z(s)^2} = \int^{\infty} \frac{ds}{p(s)u_0(s)^2} = \infty.$$

Thus $z(t)$ is a principal solution of (2.3). Note here that $Q_0(t) \ge Q(t) \ge q(t)$ on $[T, \omega)$. Then, by Lemma 1, we have $z(t) > 0$ and $z'(t) \le 0$ on $[T, \omega)$, which implies $u_0(t) > 0$ on $[T, \omega)$. Then it follows that

$$\frac{u_0'(t)}{u_0(t)} = \frac{w'(t)}{w(t)} + \frac{z'(t)}{z(t)} \le \frac{w'(t)}{w(t)} \ \text{ for } \ T \le t < \omega.$$

From (2.2) we conclude that

$$\frac{p(t)u_0'(t)}{u_0(t)} \le \frac{p(t)w'(t)}{w(t)} = \frac{P(t)v'(t)}{v(t)} \ \text{ for } \ T \le t < \omega. \qquad \square$$

Proof of Theorem 1. Assume that $v(t) > 0$ on (a, ω). By Picone's identity [6], we have

$$\frac{d}{dt}\Big(\frac{u_0}{v}\,(pu_0'v - Pu_0v')\Big) = (Q-q)u_0^2 + (p-P)u_0'^{\,2} + \frac{P(u_0'v - u_0v')^2}{v^2}\,. \tag{2.4}$$

Note that if $u_0(a) = v(a) = 0$, we obtain $\lim\limits_{t\to a} u_0(t)^2/v(t) = 0$ by l'Hospital's rule. Then we have, if $u_0(a) = 0$,

$$\lim_{t\to a} \frac{u_0(t)}{v(t)}\big(p(t)u_0'(t)v(t) - P(t)u_0(t)v'(t)\big) = 0.$$

If (1.6) holds, then

$$\lim_{t\to a} \frac{u_0(t)}{v(t)}\big(p(t)u_0'(t)v(t) - P(t)u_0(t)v(t)'\big) = \\ = u_0(a)^2\Big(\frac{p(a)u_0'(a)}{u_0(a)} - \frac{P(a)v'(a)}{v(a)}\Big) \ge 0.$$

Therefore, integrating (2.4) over $[\tau, t]$ and letting $\tau \to a$, we have

$$u_0(t)^2\Big(\frac{p(t)u_0'(t)}{u_0(t)} - \frac{P(t)v'(t)}{v(t)}\Big) \geq \\ \geq \int_a^t \Big((Q-q)u_0^2 + (p-P){u_0'}^2 + \frac{P(u_0'v - u_0v')^2}{v^2}\Big)\,ds$$

for $a < t < \omega$. From Lemma 2, we have

$$\int_a^t \Big((Q-q)u_0^2 + (p-P){u_0'}^2 + \frac{P(u_0'v - u_0v')^2}{v^2}\Big)\,ds \leq 0 \ \text{ for } \ a < t < \omega,$$

which implies that

$$q(t) \equiv Q(t), \quad p(t) \equiv P(t), \ \text{ and } \ u_0(t)v'(t) \equiv u_0'(t)v(t) \ \text{ on } [a, \omega).$$

Hence, $v(t)$ is a constant multiple of $u_0(t)$ on $[a, \omega)$. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $t = t_1 < t_2 < \cdots < t_n$ be zeros of $u_0(t)$ in (a, ω). We note that $v(t)$ satisfies either (i) or (ii) in Theorem A on $[a, t_n]$.

By applying Theorem 1 on $[t_n, \omega)$, we find that either $v(t)$ has at least one zero in (t_n, ω) or $v(t)$ is a multiple constant of $u_0(t)$ on $[t_n, \omega)$ and $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[t_n, \omega)$. In the former case, $v(t)$ has at least $n+1$ zeros in (a, ω). In the latter case, since $v(t_n) = 0$, we have either $v(t)$ has at least $n+1$ zeros in (a, ω) or $v(t)$ is a multiple constant of $u_0(t)$, $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[a, \omega)$. This completes the proof of Theorem 2. □

In order to prove Theorem 3, we consider (1.1) and (1.2) on the half-open interval of the form $(\alpha, a]$ with $\alpha \geq -\infty$.

Lemma 3. *Assume that* (1.1) *is nonoscillatory at* $t = \alpha$. *Let* $u_0(t)$ *be a principal solution of* (1.1) *at* $t = \alpha$, *and let* $v(t)$ *be a solution of* (1.2) *on* $(\alpha, a]$. *Assume that* $u_0(t) > 0$ *on* (α, a). *If either* $u_0(a) = 0$ *or*

$$u_0(a) \neq 0, \ v(a) \neq 0, \ \text{ and } \ \frac{p(a)u_0'(a)}{u_0(a)} \leq \frac{P(a)v'(a)}{v(a)}\,,$$

then $v(t)$ *has one of the following properties:*

(i) $v(t)$ *has at least one zero in* (α, a);

(ii) $v(t)$ *is a constant multiple of* $u_0(t)$ *on* $(\alpha, a]$, *and* $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ *on* $(\alpha, a]$.

Proof. Put

$$\widetilde{u}_0(t) = u_0(a-t) \ \text{ and } \ \widetilde{v}(t) = v(a-t).$$

Then $\widetilde{u}_0$ and $\widetilde{v}$ satisfy, respectively,

$$\big(\widetilde{p}(t){\widetilde{u}_0}'\big)' + \widetilde{q}(t)\widetilde{u}_0 = 0 \ \text{ and } \ \big(\widetilde{P}(t){\widetilde{v}}'\big)' + \widetilde{Q}(t)\widetilde{v} = 0 \ \text{ on } \ [0, \widetilde{\omega}),$$

where

$$\widetilde{p}(t) = p(a-t),\ \ \widetilde{q}(t) = q(a-t),\ \ \widetilde{P}(t) = P(a-t),$$
$$\widetilde{Q}(t) = Q(a-t),\ \text{ and }\ \widetilde{\omega} = a - \alpha.$$

Furthermore, we have

$$\int^{\widetilde{\omega}} \frac{1}{\widetilde{p}(t)\widetilde{u}_0(t)^2}\,dt = \infty,\ \ \frac{\widetilde{p}(0)\widetilde{u}_0\,'(0)}{\widetilde{u}_0(0)} = -\frac{p(a)u_0'(a)}{u_0(a)}\,,\ \text{ and}$$
$$\frac{\widetilde{P}(0)\widetilde{v}\,'(0)}{\widetilde{v}(0)} = -\frac{P(a)v'(a)}{v(a)}\,.$$

By applying Theorem 1 to $\widetilde{u}_0$ and $\widetilde{v}$ on $[0,\widetilde{\omega})$, we obtain Lemma 3. □

Proof of Theorem 3. First we consider the case $n = 1$. We may assume that $u_0(t) > 0$ on (α,ω). We show that $v(t)$ is a constant multiple of $u_0(t)$ on (α,ω), if $v(t) > 0$ on (α,ω). Assume that $v(t) > 0$ on (α,ω). Take any $t_0 \in (\alpha,\omega)$. First we will verify that

$$\frac{p(t_0)u_0'(t_0)}{u_0(t_0)} = \frac{P(t_0)v'(t_0)}{v(t_0)}\,. \tag{2.5}$$

Assume to the contrary that (2.5) does not hold. If

$$\frac{p(t_0)u_0'(t_0)}{u_0(t_0)} > \frac{P(t_0)v'(t_0)}{v(t_0)}\,, \tag{2.6}$$

then $v(t)$ has at least one zero in (t_0,ω) by applying Theorem 1 with $a = t_0$. This is a contradiction. On the other hand, if the opposite inequality holds in (2.6), then $v(t)$ has at least one zero in (α,t_0) by Lemma 3. This is a contradiction. Thus we obtain (2.5).

By applying Theorem 1 and Lemma 3 with $a = t_0$ again, we conclude that $v(t)$ is a constant multiple of $u_0(t)$ on (α,ω), and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on (α,ω).

Next, we consider the case $n \geq 2$. Let $t = t_1 < t_2 < \cdots < t_{n-1}$ be zeros of $u_0(t)$ in (α,ω). By applying Theorem 2 with $a = t_1$, we have either $v(t)$ has at least $n-1$ zeros in (t_1,ω) or $v(t)$ is a multiple constant of $u_0(t)$ on $[t_1,\omega)$. Thus, $v(t)$ has at least $n-1$ zeros in (α,ω). Therefore, it suffices to show that if $v(t)$ has exactly $n-1$ zeros, then $v(t)$ is a multiple constant of $u_0(t)$ on (α,ω). Assume that $v(t)$ has exactly $n-1$ zeros. First we verify that $v(t_1) = 0$. (Recall that $t = t_1$ is the first zero of $u_0(t)$.) Assume to the contrary that $v(t_1) \neq 0$. By applying Theorem 2 and Lemma 3 with $a = t_1$, we see that $v(t)$ has at least $n-1$ zeros in (t_1,ω) and at least one zero in (α,t_1), respectively. Thus, $v(t)$ has at least n zeros in (α,ω). This is a contradiction. Thus we obtain $v(t_1) = 0$.

By applying Theorem 2 and Lemma 3 with $a = t_1$ again, we conclude that $v(t)$ is a constant multiple of $u_0(t)$ on (α,ω), and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on (α,ω). □

For the proof of Theorem 4, we need the following

Lemma 4. *Assume that* (1.1) *has a solution* $u(t)$ *which has exactly* $n-1$ *zeros in* (α,ω). *Let* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *be principal solutions of* (1.1) *at* $t=\omega$ *and* $t=\alpha$, *respectively. Then* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *have at most* $n-1$ *zeros in* (α,ω).

Proof. First we consider the case where $n=1$, that is, $u(t)$ has no zero in (α,ω). Assume to the contrary that $u_0(t)$ has at least one zero in (α,ω). Let $t_0\in(\alpha,\omega)$ be the largest zero of $u_0(t)$. We may assume that $u_0(t)>0$ on (t_0,ω). By applying Theorem 1 with $a=t_0$, $p(t)\equiv P(t)$, and $q(t)\equiv Q(t)$, we see that $u(t)$ has at least one zero in $[t_0,\omega)$. This is a contradiction. Thus u_0 has no zero on (α,ω). By the similar argument as above, we see that $\widetilde{u}_0$ has no zero on (α,ω). Next, we consider the case where $n\geq 2$, that is $u(t)$ has exactly $n-1$ zeros in (α,ω). Assume to the contrary that $u_0(t)$ has at least n zeros in (α,ω). Let t_{n-1} be the $(n-1)$-th zero of $u(t)$. Note here that zeros of $u(t)$ and $u_0(t)$ do not coincide, since $u(t)$ and $u_0(t)$ are linearly independent. By the Sturm separation theorem, $u_0(t)$ has a zero $t_0\in(t_{n-1},\omega)$. By applying Theorem 1 with $a=t_0$, $p(t)\equiv P(t)$, and $q(t)\equiv Q(t)$, we see that $u(t)$ has at least one zero in (t_{n-1},ω). This is a contradiction. Thus u_0 has at most $n-1$ zeros in (α,ω). By the similar argument as above, we see that $\widetilde{u}_0$ has at most $n-1$ zeros in (α,ω). □

Proof of Theorem 4. Let u_0 and $\widetilde{u}_0$ be principal solutions of (1.1) at $t=\omega$ and $t=\alpha$, respectively. We show that the solution u is a multiple constant of u_0 on (α,ω), and also of $\widetilde{u}_0$ on (α,ω). Assume to the contrary that $u(t)$ and $u_0(t)$ are linearly independent. Then $u_0(t)$ has n zeros in (α,ω). This contradicts Lemma 4. Thus u is a multiple constant of u_0 on (α,ω). Similarly, we see that u is a multiple constant of $\widetilde{u}_0$ on (α,ω). Thus, the solution u is principal at both points $t=\alpha$ and $t=\omega$, and hence (1.7) holds with $u_0=u$. □

To prove Theorem 5, we have the following

Lemma 5. *Assume that there exists a positive solution* $v(t)$ *of* (1.2) *on* (α,ω). (*Then* (1.1) *is nonoscillatory at* $t=\alpha$ *and* $t=\omega$.) *Let* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *be principal solutions of* (1.1) *at* $t=\omega$ *and* $t=\alpha$, *respectively. Then* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *have no zero on* (α,ω). *Furthermore, if* $p(t)\not\equiv P(t)$ *or* $q(t)\not\equiv Q(t)$ *on* (α,ω), *then* $u_0(t)$ *and* $\widetilde{u}_0(t)$ *are linearly independent on* (α,ω).

Proof. Assume to the contrary that $u_0(t)$ has at least one zero in (α,ω). Let $t_0\in(\alpha,\omega)$ be the largest zero of $u_0(t)$. We may assume that $u_0(t)>0$ on (t_0,ω). By applying Theorem 1 with $a=t_0$, we find that any solution of (1.2) has at least one zero in $[t_0,\omega)$. This is a contradiction. Thus u_0 has no zero on (α,ω). By the similar argument, we see that $\widetilde{u}_0$ has no zero on (α,ω).

Assume that $p(t) \not\equiv P(t)$ or $q(t) \not\equiv Q(t)$ on (α, ω). In this case, we will show that $u_0(t)$ and $\widetilde{u}_0(t)$ are linearly independent on (α, ω). Assume to the contrary that $u_0(t)$ is a constant multiple of $\widetilde{u}_0(t)$ on (α, ω). Then $u_0(t)$ is also principal at $t = \alpha$, and hence (1.7) holds. Theorem 3 implies that $v(t)$ is a constant multiple of $u_0(t)$ on (α, ω), and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on (α, ω). This is a contradiction. Thus $u_0(t)$ and $\widetilde{u}_0(t)$ are linearly independent on (α, ω). □

Proof of Theorem 5. (i) Let u_0 and $\widetilde{u}_0$ be principal solutions of (1.1) at $t = \omega$ and $t = \alpha$, respectively. Lemma 5 implies that $u_0(t) > 0$ and $\widetilde{u}_0(t) > 0$ on (α, ω), and that $u_0(t)$ and $\widetilde{u}_0(t)$ are linear independent on (α, ω). Since a principal solution at $t = \alpha$ ($t = \omega$) is unique up to a constant factor, $u_0(t)$ and $\widetilde{u}_0(t)$ are nonprincipal at $t = \alpha$ and $t = \omega$, respectively. Thus we obtain (1.11) and (1.12). Put $u(t) = u_0(t) + \widetilde{u}_0(t)$. Then u is a positive solution of (1.1) on (α, ω), and nonprincipal at both points $t = \alpha$ and $t = \omega$. Thus (1.10) holds.

(ii) Let u_0 and $\widetilde{u}_0$ be principal solutions of (1.1) at $t = \omega$ and $t = \alpha$, respectively. Applying Lemma 5 with $P(t) \equiv p(t)$ and $Q(t) \equiv q(t)$ on (α, ω), we have $u_0(t) > 0$ and $\widetilde{u}_0(t) > 0$ on (α, ω). We show that $u_0(t)$ and $\widetilde{u}_0(t)$ are linearly independent on (α, ω). Assume to the contrary that $u_0(t)$ is a constant multiple of $\widetilde{u}_0(t)$ on (α, ω). Then $u_0(t)$ is also principal at $t = \alpha$, and hence (1.7) holds. Corollary 1 with $n = 1$ implies that any positive solution of (1.1) is a constant multiple of $u_0(t)$ on (α, ω). Since (1.1) has a positive solution u satisfying (1.13), this is a contradiction. Thus $u_0(t)$ and $\widetilde{u}_0(t)$ are linearly independent on (α, ω).

We note here that for any $t \in (\alpha, \omega)$,

$$\frac{p(t)u_0'(t)}{u_0(t)} < \frac{p(t)\widetilde{u}_0\,'(t)}{\widetilde{u}_0(t)}\,. \tag{2.7}$$

In fact, if (2.7) does not hold for some $t = t_0 \in (\alpha, \omega)$, then $\widetilde{u}_0$ has at least one zero in (t_0, ω) by Theorem 1. This is a contradiction. Thus (2.7) holds for any $t \in (\alpha, \omega)$.

For $\lambda \geq 0$, define $P_\lambda(t)$ and $Q_\lambda(t)$ by

$$P_\lambda(t) = \frac{p(t)}{1 + \lambda r(t)} \ \text{ and } \ Q_\lambda(t) = q(t) + \lambda r(t) \ \text{ on } \ (\alpha, \omega),$$

where $r(t)$ is a continuous function on (α, ω) satisfying $r(t) \geq 0$, $r(t) \not\equiv 0$ on (α, ω), and $r(t) \equiv 0$ on $(\alpha, t_1] \cup [t_2, \omega)$ with some $t_1 < t_2$. Let us consider the differential equation

$$(P_\lambda(t)v')' + Q_\lambda(t)v = 0 \ \text{ on } \ (\alpha, \omega). \tag{2.8}$$

Note that

$$P_\lambda(t) \equiv p(t) \ \text{ and } \ Q_\lambda(t) \equiv q(t) \ \text{ on } (\alpha, t_1] \cup [t_2, \omega) \text{ for all } \lambda \geq 0.$$

Then the solutions $u_0(t)$ and $\widetilde{u}_0(t)$ solve (2.8) on each interval $(\alpha, t_1]$ and $[t_2, \omega)$. For $\lambda \geq 0$, define $u_0(t;\lambda)$ and $\widetilde{u}_0(t;\lambda)$ by solutions of (2.8) satisfying

$$u_0(t;\lambda) \equiv u_0(t) \text{ on } [t_2,\omega) \text{ and } \widetilde{u}_0(t;\lambda) \equiv \widetilde{u}_0(t) \text{ on } (\alpha, t_1],$$

respectively. Then $u_0(t;\lambda)$ and $u_0'(t;\lambda)$ depend continuously on $\lambda \geq 0$ uniformly on any compact subinterval of (α,ω). In, particularly, $u_0(t;\lambda) \to u_0(t)$ and $\widetilde{u}_0(t;\lambda) \to \widetilde{u}_0(t)$ as $\lambda \to 0$ uniformly on $[t_1, t_2]$. Since (2.7) holds with $t = t_1$, for $\lambda > 0$ sufficiently small, we have

$$\frac{p(t_1)u_0'(t_1;\lambda)}{u_0(t_1;\lambda)} < \frac{p(t_1)\widetilde{u}_0\,'(t_1;\lambda)}{\widetilde{u}_0(t_1;\lambda)} \text{ and } u_0(t;\lambda) > 0 \text{ on } [t_1,\omega). \tag{2.9}$$

For $\lambda > 0$ satisfying (2.9), we will show that $\widetilde{u}_0(t;\lambda) > 0$ on (t_1,ω). Assume to the contrary that $\widetilde{u}_0(t;\lambda)$ has at least one zero $t_0 \in (t_1,\omega)$. Applying Theorem A with $a = t_0$, $u(t) \equiv \widetilde{u}_0(t;\lambda)$ and $v(t) \equiv u_0(t;\lambda)$, we see that $u_0(t;\lambda)$ has at least one zero in (t_1,ω). This is a contradiction. Thus $\widetilde{u}_0(t;\lambda) > 0$ on (t_1,ω), and hence $\widetilde{u}_0(t;\lambda) > 0$ on (α,ω). Then (1.2) with $P(t) \equiv P_\lambda(t)$ and $Q(t) \equiv Q_\lambda(t)$ has a positive solution on (α,ω). □

References

1. D. Aharonov and U. Elias, Singular Sturm comparison theorems. *J. Math. Anal. Appl.* **371** (2010), No. 2, 759–763.
2. M. Chuaqui, P. Duren, B. Osgood, and D. Stowe, Oscillation of solutions of linear differential equations. *Bull. Aust. Math. Soc.* **79** (2009), No. 1, 161–169.
3. Á. Elbert and T. Kusano, Principal solutions of non-oscillatory half-linear differential equations. *Adv. Math. Sci. Appl.* **8** (1998), No. 2, 745–759.
4. Á. Elbert, T. Kusano, and M. Naito, On the number of zeros of nonoscillatory solutions to second order half-linear differential equations. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **42** (1999), 101–131 (2000).
5. P. Hartman, Ordinary differential equations. *John Wiley & Sons, Inc., New York–London–Sydney*, 1964.
6. E. L. Ince, Ordinary differential equations. *Longmans, Green & Co, London*, 1927.
7. T. Kusano and M. Naito, A singular eigenvalue problem for second order linear ordinary differential equations. *International Symposium on Differential Equations and Mathematical Physics (Tbilisi, 1997). Mem. Differential Equations Math. Phys.* **12** (1997), 122–130.
8. T. Kusano and M. Naito, A singular eigenvalue problem for the Sturm-Liouville equation. (Russian) *Differ. Uravn.* **34** (1998), No. 3, 303–312, 428; English transl.: *Differential Equations* **34** (1998), No. 3, 302–311.
9. T. Kusano and M. Naito, On the number of zeros of nonoscillatory solutions to half-linear ordinary differential equations involving a parameter. *Trans. Amer. Math. Soc.* **354** (2002), No. 12, 4751–4767 (electronic).
10. Y. Naito, Classification of second order linear differential equations and an application to singular elliptic eigenvalue problems. *Bull. Lond. Math. Soc.* (to appear).
11. Z. Nehari, The Schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc.* **55** (1949), 545–551.
12. Z. Nehari, Some criteria of univalence. *Proc. Amer. Math. Soc.* **5** (1954), 700–704.
13. W. T. Reid, Sturmian theory for ordinary differential equations. With a preface by John Burns. *Applied Mathematical Sciences*, 31. *Springer-Verlag, New York–Berlin*, 1980.

14. C. A. Swanson, Comparison and oscillation theory of linear differential equations. *Mathematics in Science and Engineering*, Vol. 48. *Academic Press, New York–London*, 1968.

(Received 22.05.2012)

Author's address:

Department of Mathematics, Ehime University, Matsuyama 790-8577, Japan.

E-mail: ynaito@ehime-u.ac.jp