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FIXED POINT THEORY FOR MULTIVALUED WEAKLY CONVEX-POWER CONDENSING MAPPINGS WITH APPLICATION TO INTEGRAL INCLUSIONS

Abstract. In this paper we present new fixed point theorems for multivalued maps which are convex-power condensing relative to a measure of weak noncompactness and have weakly sequentially closed graph. These results are then used to investigate the existence of weak solutions to a Volterra integral inclusion with lack of weak compactness. In the last section we discuss convex-power condensing multivalued maps with respect to a measure of noncompactness.

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რეზიუმე. ნაშრომში მოყვანილია ახალი თეორემები უძრავი წერტილის შესახებ მრავალსახა ასახვებისათვის, რომლებიც არიან ამოზნექილ-ხარისხოვანი კუმშვები სუსტი არაკომპაქტურობის ზომის მიმართ და გააჩნიათ სუსტად სეკვენციალურად ჩაკეტილი გრაფიკი. ეს შედეგები შემდეგ გამოყენებულია ვოლტერას ტიპის ინტეგრალური ჩართვების სუსტი ამონახსნების არსებობის საკითხის შესასწავლად სუსტი კომპაქტურობის არარსებობის შემთხვევაში. ბოლო პარაგრაფში განხილულია ამოზნექილ-ხარისხოვანი მრავალსახა ასახვები, რომლებიც არიან კუმშვები არაკომპაქტურობის ზომის მიმართ.

1. Introduction

Since the paper by Szep [32], the theory on the existence of weak solutions to differential equations in Banach spaces has become popular. We quote the contributions of Cramer, Lakshmikantham and Mitchell [6] in 1978 and more recently by Bugajewski [5], Cichon [9], [11], Cichon and Kubiaczyk [10], Mitchell and Smith [23], and O'Regan [24], [25], [26]. Motivated by the paper of Cichon [9], D. O'Regan [30] investigated the existence of weak solutions to the following inclusion which was modelled off a first order differential inclusion [7], [8], [9]

$$x(t) \in x_0 + \int_0^t G(s, x(s))\, ds, \quad t \in [0,T]; \tag{1.1}$$

here $G\colon [0,T] \times E \to 2^E$ and $x_0 \in E$ with E a real reflexive Banach space. The proofs involve a Arino–Gautier–Penot type fixed point theorem for multivalued mappings and the applications depend heavily upon the reflexiveness of the space E. In this paper, we establish existence results for the Volterra integral equation (1.1) in the case where E is nonreflexive. Our approach relies on the concept of convex-power condensing operators with respect to a measure of weak noncompactness. We note that Sun and Zhang [31] introduced the definition of a convex-power condensing operator with respect to the Kuratowski measure of noncompactness for single valued mappings and proved a fixed point theorem which extended the well-known Sadovskii's fixed point theorem and a fixed point theorem in Liu et al. [22]. [35], G. Zhang et al. established some fixed point theorems of Rothe and Altman types about convex-power condensing single valued operators with respect to the Kuratowski measure of noncompactness. These results were applied to a differential equation of first order with integral boundary conditions. In this paper we introduce the concept of a convex-power condensing multivalued operator with respect to a measure of weak noncompactness. We also prove some fixed point principles for this type of operator. Our fixed point results are not only of theoretical interest, but we discuss new applications, namely the existence of solutions to integral inclusions with lack of weak compactness. We illustrate this fact by deriving an existence theory for (1.1) in the case where E is nonreflexive.

For the remainder of this section we gather some notations and preliminary facts. Let X be a Banach space, let $\mathcal{B}(X)$ denote the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Also, let B_r denote the closed ball centered at 0 with radius r.

In our considerations the following definition will play an important role.

Definition 1.1 ([2]). A function $\psi\colon \mathcal{B}(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:

(1) The family $\ker(\psi) = \{M \in \mathcal{B}(X) : \psi(M) = 0\}$ is nonempty and $\ker(\psi)$ is contained in the set of relatively weakly compact sets of X.

(2) $M_1 \subseteq M_2 \Longrightarrow \psi(M_1) \leq \psi(M_2)$.

(3) $\psi(\overline{co}(M)) = \psi(M)$, where $\overline{co}(M)$ is the closed convex hull of M.

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ for $\lambda \in [0,1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n\to\infty} \psi(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker\psi$ described in (1) is said to be the kernel of the measure of weak noncompactness ψ. Note that the intersection set M_∞ from (5) belongs to $\ker\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every n and $\lim_{n\to\infty} \psi(M_n) = 0$. Also, it can be easily verified that the measure ψ satisfies

$$\psi(\overline{M^w}) = \psi(M),$$

where $\overline{M^w}$ is the weak closure of M.

A measure of weak noncompactness ψ is said to be *regular* if

$$\psi(M) = 0 \text{ if and only if } M \text{ is relatively weakly compact.}$$

subadditive if

$$\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2), \tag{1.2}$$

homogeneous if

$$\psi(\lambda M) = |\lambda|\psi(M), \quad \lambda \in \mathbb{R}, \tag{1.3}$$

set additive (or *have the maximum property*) if

$$\psi(M_1 \cup M_2) = \max(\psi(M_1), \psi(M_2)). \tag{1.4}$$

The first important example of a measure of weak noncompactness has been defined by De Blasi [13] as follows:

$$w(M) = \inf\Big\{r > 0 : \text{ there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\Big\},$$

for each $M \in \mathcal{B}(X)$.

Notice that $w(.)$ is regular, homogeneous, subadditive and set additive (see [13]).

The following results are crucial for our purposes. We first state a theorem of Ambrosetti type (see [23, 20] for a proof).

Theorem 1.1. *Let E be a Banach space and let $H \subseteq C([0,T], E)$ be bounded and equicontinuous. Then the map $t \to w(H(t))$ is continuous on $[0,T]$ and*

$$w(H) = \sup_{t\in[0,T]} w(H(t)) = w(H[0,T]),$$

where $H(t) = \{h(t) \colon h \in H\}$ and $H[0,T] = \bigcup_{t\in[0,T]} \{h(t) \colon h \in H\}$.

The following auxiliary result will also be needed.

Lemma 1.1 ([31]). *If $H \subseteq C([0,T],E)$ is equicontinuous and $x_0 \in C([0,T],E)$, then $\overline{co}(H \cup \{x_0\})$ is likewise equicontinuous in $C([0,T],E)$.*

In what follows, we shall recall some classical definitions and results regarding multivalued mappings. Let X and Y be topological spaces. A multivalued map $F\colon X \to 2^Y$ is a point to a set function if for each $x \in X$, $F(x)$ is a nonempty subset of Y. For a subset M of X we write $F(M) = \bigcup_{x\in M} F(x)$ and $F^{-1}(M) = \{x \in X : \ F(x) \cap M \neq \varnothing\}$. The *graph* of F is the set $Gr(F) = \{(x,y) \in X \times Y : \ y \in F(x)\}$. We say that F is *upper semicontinuous* (u.s.c. for short) at $x \in X$ if for every neighborhood V of $F(x)$ there exists a neighborhood U of x with $F(U) \subseteq V$ (equivalently, $F : X \to 2^Y$ is u.s.c. if for any net $\{x_\alpha\}$ in X and any closed set B in Y with $x_\alpha \to x_0 \in X$ and $F(x_\alpha) \cap B \neq \varnothing$ for all α, we have $F(x_0) \cap B \neq \varnothing$). We say that $F\colon X \to 2^Y$ is upper semicontinuous if it is upper semicontinuous at every $x \in X$. The function F is lower semicontinuous (l.s.c.) if the set $F^{-1}(B)$ is open for any open set B in Y . If F is l.s.c. and u.s.c., then F is continuous.
If Y is compact, and the images $F(x)$ are closed, then F is upper semicontinuous if and only if F has a closed graph. In this case, if Y is compact, we find that F is upper semicontinuous if $x_n \to x$, $y_n \to y$, and $y_n \in F(x_n)$, together imply that $y \in F(x)$. When X is a Banach space we say that $F\colon X \to 2^X$ is weakly upper semicontinuous if F is upper semicontinuous in X endowed with the weak topology. Also, $F\colon X \to 2^X$ is said to have weakly sequentially closed graph if the graph of F is sequentially closed w.r.t. the weak topology of X. In Section 4 we present fixed point theorems for multivalued convex-power maps with respect to a measure of noncompactness.
Now, we recall the following extension of the Arino–Gautier–Penot fixed point theorem for multivalued mappings. For a proof we refer the reader to [30, Theorem 2.2.].

Theorem 1.2. *Let X be a metrizable locally convex linear topological space and let C be a weakly compact, convex subset of X. Suppose $F\colon C \to C(C)$ has a weakly sequentially closed graph. Then F has a fixed point; here $C(C)$ denotes the family of nonempty, closed, convex subsets of C.*

In what follows, let X be a Banach space, C a nonempty closed convex subset of X, $F\colon C \to 2^C$ a multivalued mapping and $x_0 \in C$. For any $M \subseteq C$ we set

$$F^{(1,x_0)}(M)=F(M), \quad F^{(n,x_0)}(M)=F\left(\overline{co}\left(F^{(n-1,x_0)}(M) \cup \{x_0\}\right)\right)$$

for $n = 2, 3, \ldots$.

Definition 1.2. Let X be a Banach space, C a nonempty closed convex subset of X and ψ a measure of weak noncompactness on X. Let $F\colon C \to 2^C$ be a bounded multivalued mapping (that is it takes bounded sets into bounded ones) and $x_0 \in C$. We say that F is a ψ- convex-power condensing operator about x_0 and n_0 if for any bounded set $M \subseteq C$ with $\psi(M) > 0$ we have

$$\psi(F^{(n_0,x_0)}(M)) < \psi(M). \tag{1.5}$$

Obviously, $F\colon C \to 2^C$ is ψ-condensing if and only if it is ψ- convex-power condensing operator about x_0 and 1.

2. Fixed Point Theorems for Multivalued Mappings Relative to the Weak Topology

Theorem 2.1. *Let X be a Banach space and ψ be a regular and set additive measure of weak noncompactness on X. Let C be a nonempty closed convex subset of X, $x_0 \in C$ and n_0 be a positive integer. Suppose $F\colon C \to C(C)$ is ψ-convex-power condensing about x_0 and n_0. If F has weakly sequentially closed graph and $F(C)$ is bounded, then F has a fixed point in C.*

Proof. Let

$$\mathcal{F} = \big\{A \subseteq C,\ \overline{co}(A) = A,\ x_0 \in A \text{ and } F(x) \in C(A) \text{ for all } x \in A\big\}.$$

The set $\mathcal{F}$ is nonempty since $C \in \mathcal{F}$. Set $M = \bigcap_{A\in\mathcal{F}} A$. Now we show that for any positive integer n we have

$$\mathcal{P}(n) \qquad M = \overline{co}\left(F^{(n,x_0)}(M) \cup \{x_0\}\right).$$

To see this, we proceed by induction. Clearly, M is a closed convex subset of C and $F(M) \subseteq M$. Thus $M \in \mathcal{F}$. This implies $\overline{co}(F(M) \cup \{x_0\}) \subseteq M$. Hence $F(\overline{co}(F(M) \cup \{x_0\})) \subseteq F(M) \subseteq \overline{co}(F(M) \cup \{x_0\})$. Consequently, $\overline{co}(F(M)\cup\{x_0\}) \in \mathcal{F}$. Hence $M \subseteq \overline{co}(F(M)\cup\{x_0\})$. As a result $\overline{co}(F(M)\cup \{x_0\}) = M$. This shows that $\mathcal{P}(1)$ holds. Let n be fixed and suppose $\mathcal{P}(n)$ holds. This implies $F^{(n+1,x_0)}(M) = F(\overline{co}\left(F^{(n,x_0)}(M) \cup \{x_0\}\right) = F(M)$. Consequently,

$$\overline{co}\left(F^{(n+1,x_0)}(M) \cup \{x_0\}\right) = \overline{co}(F(M) \cup \{x_0\}) = M. \tag{2.1}$$

As a result

$$\overline{co}\left(F^{(n_0,x_0)}(M) \cup \{x_0\}\right) = M. \tag{2.2}$$

Using the properties of the measure of weak noncompactness, we get

$$\psi(M) = \psi\left(\overline{co}\left(F^{(n_0,x_0)}(M) \cup \{x_0\}\right)\right) = \psi(F^{(n_0,x_0)}(M)),$$

which yields that M is weakly compact. Since $F\colon M \to 2^M$ has weakly sequentially closed graph, the result follows from Theorem 1.2. □

As an easy consequence of Theorem 2.1 we obtain the following sharpening of [30, Theorem 2.3].

Corollary 2.1. *Let X be a Banach space and ψ be a regular and set additive measure of weak noncompactness on X. Let C be a nonempty closed convex subset of X. Assume that $F\colon C \to C(C)$ has weakly sequentially closed graph with $F(C)$ bounded. If F is ψ-condensing, i.e. $\psi(F(M)) <$*

$\psi(M)$, whenever M is a bounded non-weakly compact subset of C, the F has a fixed point.

Remark 2.1. Theorem 2.1 is also an extension of its corresponding results in [28], [29].

Lemma 2.1. *Let $F\colon X \to 2^X$ be convex-power condensing about x_0 and n_0 (n_0 is a positive integer) with respect to a regular and set additive measure of weak noncompactness ψ. Let $\widetilde{F}\colon X \to 2^X$ be the operator defined on X by $\widetilde{F}(x) = F(x+x_0)-x_0$. Then, $\widetilde{F}$ is convex-power condensing about 0 and n_0 with respect to ψ. Moreover, F has a fixed point if $\widetilde{F}$ does.*

Proof. Let M be a bounded subset of X with $\psi(M) > 0$. We claim that for all integer $n \geq 1$, we have

$$\widetilde{F}^{(n,0)}(M) \subseteq F^{(n,x_0)}(M+x_0) - x_0. \tag{2.3}$$

To see this, we shall proceed by induction. Clearly,

$$\widetilde{F}^{(1,0)}(M) = \widetilde{F}(M) = F(M+x_0) - x_0 = F^{(1,x_0)}(M+x_0) - x_0. \tag{2.4}$$

Fix an integer $n \geq 1$ and suppose (2.3) holds. Then

$$\widetilde{F}^{(n,0)}(M) \cup \{0\} \subseteq \overline{co}\left(F^{(n,x_0)}(M+x_0) \cup \{x_0\}\right) - x_0. \tag{2.5}$$

Hence

$$\overline{co}\left(\widetilde{F}^{(n,0)}(M) \cup \{0\}\right) \subseteq \overline{co}\left(F^{(n,x_0)}(M+x_0) \cup \{x_0\}\right) - x_0. \tag{2.6}$$

As a result

$$\begin{aligned}\widetilde{F}^{(n+1,0)}(M) &= \widetilde{F}\left(\overline{co}\left(\widetilde{F}^{(n,0)}(M) \cup \{0\}\right)\right) \subseteq \\ &\subseteq \widetilde{F}\left(\overline{co}\left(F^{(n,x_0)}(M+x_0) \cup \{x_0\}\right) - x_0\right) = \\ &= F\left(\overline{co}\left(F^{(n,x_0)}(M+x_0) \cup \{x_0\}\right) - x_0\right) = \\ &= F^{(n+1,x_0)}(M+x_0) - x_0.\end{aligned}$$

This proves our claim. Consequently,

$$\begin{aligned}\psi(\widetilde{F}^{(n_0,0)}(M))) &\leq \psi(F^{(n_0,x_0)}(M+x_0) - x_0) \leq \\ &\leq \psi((F^{(n_0,x_0)}(M+x_0) < \psi(M+x_0) \leq \psi(M).\end{aligned}$$

This proves the first statement. The second statement is straightforward. □

Theorem 2.2. *Let X be a Banach space and let ψ be a regular and set additive measure of weak noncompactness on X. Let Q and C be closed, convex subsets of X with $Q \subseteq C$. In addition, let U be a weakly open subset of Q with $F(\overline{U^w})$ bounded and $x_0 \in U$. Suppose $F\colon X \to 2^X$ is ψ-power-convex condensing map about x_0 and n_0 (n_0 is a positive integer). If F has* *a weakly sequentially closed graph and $F(x) \in C(C)$ for all $x \in \overline{U^w}$, then either*

$$F \text{ has a fixed point,} \tag{2.7}$$

or

$$\text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda F u; \tag{2.8}$$

here $\partial_Q U$ is the weak boundary of U in Q.

Proof. By replacing F, Q, C and U by $\widetilde{F}, Q - x_0, C - x_0$ and $U - x_0$ respectively and using Lemma 2.1, we may assume that $0 \in U$ and F is ψ-power-convex condensing about 0 and n_0. Now suppose (2.8) does not occur and F does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (2.7) occurs). Let

$$M = \left\{ x \in \overline{U^w} : \ x \in \lambda F x \text{ for some } \lambda \in [0,1] \right\}.$$

The set M is nonempty since $0 \in U$. Also, M is weakly sequentially closed. Indeed, let (x_n) be the sequence of M which converges weakly to some $x \in \overline{U^w}$ and let (λ_n) be a sequence of $[0,1]$ satisfying $x_n \in \lambda_n F x_n$. Then for each n there is a $z_n \in F x_n$ with $x_n = \lambda_n z_n$. By passing to a subsequence if necessary, we may assume that (λ_n) converges to some $\lambda \in [0,1]$ and $\lambda_n \neq 0$ for all n. This implies that the sequence (z_n) converges to some $z \in \overline{U^w}$ with $x = \lambda z$. Since F has a weakly sequentially closed graph, then $z \in F(x)$. Hence $x \in \lambda F x$ and therefore $x \in M$. Thus M is weakly sequentially closed. We now claim that M is relatively weakly compact. Suppose $\psi(M) > 0$. Clearly,

$$M \subseteq co(F(M) \cup \{0\}). \tag{2.9}$$

By induction, note for all positive integers n we have

$$M \subseteq co\left(F^{(n,0)}(M) \cup \{0\}\right). \tag{2.10}$$

Indeed, fix an integer $n \geq 1$ and suppose (2.10) holds. Then

$$F(M) \subseteq F\left(\overline{co}\left(F^{(n,0)}(M) \cup \{0\}\right)\right) = F^{(n+1,0)}(M). \tag{2.11}$$

Hence

$$co(F(M) \cup \{0\}) \subseteq co\left(F^{(n+1,0)}(M) \cup \{0\}\right). \tag{2.12}$$

Combining (2.9) and (2.12), we arrive at

$$M \subseteq co\left(F^{(n+1,0)}(M) \cup \{0\}\right).$$

This proves (2.10). In particular, we have

$$M \subseteq co\left(F^{(n_0,0)}(M) \cup \{0\}\right).$$

Thus,

$$\psi(M) \leq \psi\left(co\left(F^{(n_0,0)}(M) \cup \{0\}\right)\right) = \psi(F(M)) < \psi(M), \tag{2.13}$$

which is a contradiction. Hence $\psi(M) = 0$ and therefore $\overline{M^w}$ is weakly compact. This proves our claim. Let now $x \in \overline{M^w}$. Since $\overline{M^w}$ is weakly compact, then there is a sequence (x_n) in M which converges weakly to x. Since M is weakly sequentially closed, we have $x \in M$. Thus $\overline{M^w} = M$. Hence M is weakly closed and therefore weakly compact. From our assumptions we have $M \cap \partial_Q U = \varnothing$. Since X endowed with the weak topology is a locally convex space, then there exists a weakly continuous mapping $\rho\colon \overline{U^w} \to [0,1]$ with $\rho(M) = 1$ and $\rho(\partial_Q U) = 0$ (see [15]). Let

$$T(x) = \begin{cases} \rho(x)F(x), & x \in \overline{U^w}, \\ 0, & x \in X \setminus \overline{U^w}. \end{cases}$$

Clearly, $T\colon X \to 2^X$ has a weakly sequentially closed graph since F does. Moreover, for any $S \subseteq C$ we have

$$T(S) \subseteq co(F(S) \cup \{0\}).$$

This implies that

$$T^{(2,0)}(S) = T\Big(\overline{co}(T(S) \cup \{0\})\Big) \subseteq T\Big(\overline{co}(F(S) \cup \{0\})\Big) \subseteq$$
$$\subseteq \overline{co}\Big(F\Big(\overline{co}(F(S) \cup \{0\}) \cup \{0\}\Big)\Big) = \overline{co}(F^{(2,0)}(S) \cup \{0\}).$$

By induction,

$$T^{(n,0)}(S) = T\left(\overline{co}(T^{(n-1,0)}(S) \cup \{0\})\right) \subseteq T\left(\overline{co}(F^{(n-1,0)}(S) \cup \{0\})\right) \subseteq$$
$$\subseteq \overline{co}\left(F\left(\overline{co}(F^{(n-1,0)}(S) \cup \{0\}) \cup \{0\}\right)\right) = \overline{co}(F^{(n,0)}(S) \cup \{0\}),$$

for each integer $n \geq 1$. Using the properties of the measure of weak noncompactness, we get

$$\psi(T^{(n_0,0)}(S)) \leq \psi(\overline{co}(F^{(n_0,0)}(S) \cup \{0\})) = \psi(F^{(n_0,0)}(S)) < \psi(S), \quad (2.14)$$

if $\psi(S) > 0$. Thus $T\colon X \to 2^X$ has a weakly sequentially closed graph and $T(x) \subseteq C(C)$ for all $x \in C$. Moreover, T is ψ-power-convex condensing about 0 and n_0. By Theorem 2.1 there exists $x \in C$ such that $w \in Tx$. Now $x \in U$ since $0 \in U$. Consequently, $x \in \rho(x)F(x)$ and so $x \in M$. This implies $\rho(x) = 1$ and so $x \in F(x)$. □

Now we present a fixed point theorem of Furi–Pera type for power-convex condensing multivalued mappings with weakly sequentially closed graph.

Theorem 2.3. *Let X be a Banach space and let ψ be a regular and set additive measure of weak noncompactness on X. Let C be a closed convex subset of X and Q a closed convex subset of C with $F(Q)$ bounded and $0 \in Q$. Also, assume $F\colon X \to 2^X$ has a weakly sequentially closed graph and is ψ-power-convex condensing about* 0 *and n_0 (n_0 is a positive integer) and $F(x) \in C(C)$ for all $x \in Q$. In addition, assume that the following conditions are satisfied:*

(i) *there exists a weakly continuous retraction* $r\colon X \to Q$, *with* $r(D) \subseteq \overline{co}(D \cup \{0\})$ *for any bounded subset* D *of* X *and* $r(x) = x$ *for all* $x \in Q$;

(ii) *there exists a* $\delta > 0$ *and a weakly compact set* Q_δ *with* $\Omega_\delta = \{x \in X : d(x, Q) \leq \delta\} \subseteq Q_\delta$; *here* $d(x, y) = \|x - y\|$;

(iii) *for any* $\Omega_\epsilon = \{x \in X : d(x, Q) \leq \epsilon, 0 < \epsilon \leq \delta\}$, *if* $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ *is a sequence in* $Q \times [0, 1]$ *with* $x_j \rightharpoonup x \in \partial_{\Omega_\epsilon} Q$, $\lambda_j \to \lambda$ *and* $x \in \lambda F(x), 0 \leq \lambda < 1$, *then* $\lambda_j F(x_j) \subseteq Q$ *for* j *sufficiently large; here* $\partial_{\Omega_\epsilon} Q$ *is the weak boundary of* Q *relative to* Ω_ϵ.

Then F *has a fixed point in* Q.

Proof. Consider $B = \{x \in X : x \in Fr(x)\}$. We first show that $B \neq \varnothing$. To see this, consider $Fr\colon C \to C(C)$. Clearly Fr has a weakly sequentially closed graph, since F has a weakly sequentially closed graph and r is weakly continuous. Now we show that Fr is ψ-power-convex condensing map about 0 and n_0. To see this, let A be a bounded subset of C and set $A' = \overline{co}(A \cup \{0\})$. Then, using assumption (i) we obtain

$$\begin{aligned}
&(Fr)^{(1,0)}(A) \subseteq F(A'),\\
&(Fr)^{(2,0)}(A) = Fr\left(\overline{co}\left((Fr)^{(1,0)}(A) \cup \{0\}\right)\right) \subseteq \\
&\qquad \subseteq Fr\left(\overline{co}\left(F(A') \cup \{0\}\right)\right) \subseteq F\left(\overline{co}\left(F(A') \cup \{0\}\right)\right) = \\
&\qquad = F^{(2,0)}(A'),
\end{aligned}$$

and by induction,

$$\begin{aligned}
(Fr)^{(n_0,0)}(A) &= Fr\left(\overline{co}\left((Fr)^{(n_0-1,0)}(A) \cup \{0\}\right)\right) \subseteq \\
&\subseteq Fr\left(\overline{co}\left(F^{(n_0-1,0)}(A') \cup \{0\}\right)\right) \subseteq \\
&\subseteq F\left(\overline{co}\left(F^{(n_0-1,0)}(A') \cup \{0\}\right)\right) = \\
&= F^{(n_0,0)}(A').
\end{aligned}$$

Thus

$$\psi\left((Fr)^{(n_0,0)}(A)\right) \leq \psi\left(F^{(n_0,0)}(A')\right) < \psi(A') = \psi(A),$$

whenever $\psi(A) \neq 0$. Invoking Theorem 2.1 we infer that there exists $y \in C$ with $y \in Fr(y)$. Thus $y \in B$ and $B \neq \varnothing$. In addition B is weakly sequentially closed, since Fr has a weakly sequentially closed graph. Moreover, we claim that B is weakly compact. To see this, first notice

$$B \subseteq Fr(B) \subseteq F(B') = F^{(1,0)}(B'),$$

where $B' = \overline{co}(B \cup \{0\})$. Thus

$$B \subseteq Fr(B) \subseteq Fr\left(F(B')\right) \subseteq F\left(\overline{co}\left(F(B') \cup \{0\}\right)\right) = F^{(2,0)}(B'),$$

and by induction

$$B \subseteq Fr(B) \subseteq Fr\left(F^{(n_0-1,0)}(B')\right) \subseteq$$
$$\subseteq F\left(\overline{co}\left(F^{(n_0-1,0)}(B') \cup \{0\}\right)\right) = F^{(n_0,0)}(B'),$$

Now if $\psi(B) \neq 0$, then

$$\psi(B) \leq \psi(F^{(n_0,0)}(B')) < \psi(B') = \psi(B),$$

which is a contradiction. Thus, $\psi(B) = 0$ and so B is relatively weakly compact and therefore $Fr(B)$ is relatively weakly compact, since r is weakly continuous and F has a sequentially closed graph. Now let $x \in \overline{B^w}$. Since $\overline{B^w}$ is weakly compact then there is a sequence (x_n) of elements of B which converges weakly to some x. Since B is weakly sequentially closed then $x \in B$. Thus, $\overline{B^w} = B$. This implies that B is weakly compact. We now show that $B \cap Q \neq \varnothing$. Suppose $B \cap Q = \varnothing$. Then, since B is weakly compact and Q is weakly closed we have from [16] that $d(B,Q) > 0$. Thus there exists ϵ, $0 < \epsilon < \delta$, with $\Omega_\epsilon \cap B = \varnothing$; here $\Omega_\epsilon = \{x \in X :\ d(x,Q) \leq \epsilon\}$. Now Ω_ϵ is closed convex and $\Omega_\epsilon \subseteq Q_\delta$. From our assumptions it follows that Ω_ϵ is weakly compact. Also since X is separable then the weak topology on Ω_ϵ is metrizable [14], [34], let d^* denote the metric. For $i \in \{0, 1 \ldots\}$, let

$$U_i = \left\{x \in \Omega_\epsilon :\ d^*(x,Q) < \frac{\epsilon}{i}\right\}.$$

For each $i \in \{0, 1 \ldots\}$ fixed, U_i is open with respect to d and so U_i is weakly open in Ω_ϵ. Also, $\overline{U_i^w} = \overline{U_i^d} = \left\{x \in \Omega_\epsilon :\ d^*(x,Q) \leq \epsilon/i\right\}$ and $\partial_{\Omega_\epsilon} U_i = \left\{x \in \Omega_\epsilon :\ d^*(x,Q) = \epsilon/i\right\}$. Keeping in mind that $\Omega_\epsilon \cap B = \varnothing$, Theorem 2.2 guarantees that there exists $y_i \in \partial_{\Omega_\epsilon} U_i$ and $\lambda_i \in (0,1)$ with $y_i \in \lambda_i Fr(y_i)$. We now consider $D = \left\{x \in X :\ x \in \lambda Fr(x) \text{ for some } \lambda \in [0,1]\right\}$.

First notice

$$D \subseteq Fr(D) \cup \{0\}.$$

Thus

$$D \subseteq Fr(D) \cup \{0\} \subseteq Fr\Big(\overline{co}\,(Fr(D) \cup \{0\})\Big) \cup \{0\} = (Fr)^{(2,0)} \cup \{0\},$$

and by induction

$$D \subseteq Fr(D) \cup \{0\} \subseteq$$
$$\subseteq Fr\Big(\overline{co}\left((Fr)^{(n_0-1,0)}(D) \cup \{0\}\right)\Big) \cup \{0\} = (Fr)^{(n_0,0)} \cup \{0\},$$

Consequently,

$$\psi(D) \leq \psi\left((Fr)^{(n_0,0)} \cup \{0\}\right) \leq \psi\left((Fr)^{(n_0,0)}\right).$$

Since Fr is ψ-convex-power condensing about 0 and n_0 then $\psi(D) = 0$ and so D is relatively weakly compact.

The same reasoning as above implies that D is weakly compact. Then, up to a subsequence, we may assume that $\lambda_i \to \lambda^* \in [0,1]$ and $y_i \rightharpoonup y^* \in \partial_{\Omega_\epsilon} U_i$.

Since F has a weakly sequentially closed graph then $y^* \in \lambda^* Fr(y^*)$. Notice $\lambda^* Fr(y^*) \nsubseteq Q$ since $y^* \in \partial_{\Omega_\epsilon} U_i$. Thus $\lambda^* \neq 1$ since $B \cap Q = \varnothing$. From assumption (iii) it follows that $\lambda_i Fr(y_i) \subseteq Q$ for j sufficiently large, which is a contradiction. Thus $B \cap Q \neq \varnothing$, so there exists $x \in Q$ with $x \in Fr(x)$, i.e. $x \in Fx$. □

Remark 2.2. In Theorem 2.3, we need $F\colon X \to 2^X$ ψ-convex-power condensing about 0 and n_0. However, the condition $F\colon X \to 2^X$ has weakly sequentially closed graph can be replaced by $F\colon Q \to 2^X$ has weakly sequentially closed graph.

3. Existence Results

In this section we shall discuss the existence of weak solutions to the Volterra integral inclusion

$$x(t) \in x_0 + \int_0^t G(s, x(s))\, ds, \quad t \in [0, T]; \tag{3.1}$$

here $G\colon [0,T] \times E \to C(E)$ and $x_0 \in E$ with E is a real Banach space. The integral in (3.1) is understood to be the Pettis integral and solutions to (3.1) will be sought in $C([0,T], E)$.

This equation will be studied under the following assumptions:

(i) for each continuous function $x\colon [0,T] \to E$ there exists a scalarly measurable function $v\colon [0,T] \to E$ with $v(t) \in G(t, x(t))$ a.e. on $[0,T]$ and v is Pettis integrable on $[0,T]$;

(ii) for any $r > 0$ there exists $\theta_r \in L^1[0,T]$ with $|G(t,u)| \leq \theta_r(t)$ for a.e. $t \in [0,T]$ and all $u \in E$ with $|z| \leq r$; here $|G(t,u)| = \sup\{|w| : w \in G(t,u)\}$;

(iii) there exists $\alpha \in L^1[0,T]$ and $\theta\colon [0,+\infty) \to (0,+\infty)$ a nondecreasing continuous function such that $|G(s,u)| \leq \alpha(s)\theta(|u|)$ for a.e. $s \in [0,t]$, and all $u \in E$, with
$$\int_0^T \alpha(s)\, ds < \int_{|x_0|}^{\infty} \frac{dx}{\theta(x)}\,;$$

(iv) there is a constant $\tau \geq 0$ such that for any bounded subset S of E and for any $t \in [0,T]$ we have
$$w(G([0,t] \times S)) \leq \tau w(S);$$

(v) if (x_n) is a sequence of continuous functions from $[0,T]$ into E which converges weakly to x and if (v_n) is a sequence of Pettis integrable functions from $[0,T]$ into E such that $v_n(s)$ converges weakly to $v(s)$ and $v_n(s) \in G(s, x_n(s))$ for a.e. $s \in [0,T]$, then v is Pettis integrable with $v(s) \in G(s, x(s))$ for a.e. $s \in [0,T]$.

Theorem 3.1. *Let E be a Banach space and suppose* (i)–(iv) *hold. Then* (3.1) *has a solution in* $C([0,T],E)$.

Proof. Define a multivalued operator

$$F\colon C([0,T],E) \to C(C([0,T],E)). \tag{3.2}$$

by letting

$$Fx(t) = \Big\{ x_0 + \int_0^t v(s)\,ds : \ v\colon [0,T] \to E \text{ Pettis integrable with } v(t) \in G(t,x(t)) \text{ a.e. } t \in [0,T] \Big\}. \tag{3.3}$$

We first show that (3.2)–(3.3) make sense. To see this, let $x \in C([0,T],E)$. In view of our assumptions there exists a Pettis integrable $v\colon [0,T] \to E$ with $v(t) \in G(t,x(t))$ for a.e. $t \in [0,T]$. Thus F is well defined. Let $u(t) = x_0 + \int_0^t v(s)\,ds$. To see that $u \in C([0,T],E)$ first notice that there exists $r > 0$ with $|y| = \sup_{[0,T]} |x(t)| \le r$. From assumption (iii) it readily follows that there exists $\theta_r \in L^1[0,T]$ with

$$|G(t,x(t))| \le \theta_r(t) \ \text{ for a.e. } \ t \in [0,T]. \tag{3.4}$$

Let t, $t' \in [0,T]$ with $t < t'$. Without loss of generality assume $u(t) - u(t') \neq 0$. Invoking the Hahn–Banach theorem we deduce that there exists $\phi \in E^*$ (the topological dual of E) with $|\phi| = 1$ and $|u(t) - u(t')| = \phi(u(t) - u(t'))$. Thus

$$|u(t) - u(t')| = \phi\bigg(\int_t^{t'} v(s)\,ds \bigg) \le \int_t^{t'} \theta_r(s)\,ds.$$

Consequently, $u \in C([0,T],E)$. Our next task is to show that F has closed (in $C([0,T],E)$) values (note F has automatically convex values). Let $x \in C([0,T],E)$. Suppose $w_n \in Fx$, $n = 1,2,\dots$. Then there exists Pettis integrable $v_n\colon [0,T] \to E$, $n = 1,2,\dots$ with $v_n(s) \in G(s,x(s))$ a.e. $s \in [0,T]$. Suppose

$$w_n(t) \to x_0 + \int_0^t v(s)\,ds = w(t) \ \text{ in } \ C([0,T],E). \tag{3.5}$$

Fix $t \in (0,T]$ and $\phi \in E^*$. Then $\phi(v_n) \to \phi(v)$ in $L^1[0,t]$ so $\phi(v_n) \to \phi(v)$ in measure. Thus there exists a subsequence S of integers with

$$\phi(v_n(s)) \to \phi(v(s)) \ \text{ for a.e. } \ s \in [0,t] \quad (\text{as } \ n \to \infty \ \text{ in } \ S). \tag{3.6}$$

Now since $v_n(s) \in G(s, x(s))$ for a.e. $s \in [0, t]$ and since the values of G are closed and convex (so weakly closed) we have $v(s) \in G(s, x(s))$ for a.e. $s \in [0, t]$. Thus $w \in Fx$ and so F has closed (in $C([0, T], E)$) values. Now let

$$C = \Big\{ x \in C([0, T], E) : \ |x(t)| \leq b(t) \text{ for } t \in [0, T] \text{ and } |x(t) - x(s)| \leq |b(t) - b(s)| \text{ for } t, s \in [0, T] \Big\},$$

where

$$b(t) = I^{-1}\Big(\int_0^t \alpha(s)\, ds \Big) \ \text{ and } \ I(z) = \int_{|x_0|}^{z} \frac{dx}{\theta(x)}\,.$$

Notice C is a closed, convex, bounded, equicontinuous subset of $C([0, T], E)$ with $0 \in C$. Let F be as defined in (3.2)–(3.3). We claim that $F(C) \subseteq C$. To see this take $u \in F(C)$. Then there exists $y \in C$ with $u \in Fy$ and there exists a Pettis integrable $v\colon [0, T] \to E$ with $u(t) = x_0 + \int_0^t v(s)\, ds$ and $v(t) \in G(t, y(t))$ for a.e. $t \in [0, T]$. Without loss of generality, assume $u(s) \neq 0$ for all $s \in [0, T]$. Then there exists $\phi_s \in E^*$ with $|\phi_s| = 1$ and $\phi_s(u(s)) = |u(s)|$. Consequently, for each fixed $t \in [0, T]$, we have

$$|u(t)| = \phi_t(u(t)) \leq |x_0| + \int_0^t \alpha(s)\theta(|y(s)|)\, ds \leq \leq |x_0| + \int_0^t \alpha(s)\theta(b(s))\, ds = |x_0| + \int_0^t b'(s)\, ds = b(t),$$

since

$$\int_{|x_0|}^{b(s)} \frac{dx}{\theta(x)} = \int_0^s \alpha(x)\, dx.$$

Next suppose $t,\, t' \in [0, T]$ with $t > t'$. Without loss of generality, assume $u(t) - u(t') \neq 0$. Then there exists $\phi \in E^*$ with $|\phi| = 1$ and $\phi(u(t) - u(t')) = |u(t) - u(t')|$. Consequently,

$$|u(t) - u(t')| \leq \int_{t'}^{t} \alpha(s)\theta(|y(s)|)\, ds \leq \leq \int_{t'}^{t} \alpha(s)\theta(|b(s)|)\, ds = \int_{t'}^{t} b'(s)\, ds = b(t) - b(t').$$

Thus, $u \in C$. This proves our claim. Our next task is to show that F has a weakly sequentially closed graph. To see this, let (x_n, y_n) be a sequence in $C \times C$ with $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ and $y_n \in Fx_n$. Then for each $t \in [0, T]$ we have

$$y_n(t) = x_0 + \int_0^t v_n(s)\, ds \tag{3.7}$$

with $v_n\colon [0, T] \to E$, $n = 1, 2, \dots$ Pettis integrable and $v_n(s) \in G(s, x_n(s))$ a.e. $s \in [0, T]$. Recall [23], since C is equicontinuous, that $x_n \rightharpoonup x$ if and only if $x_n(t) \rightharpoonup x(t)$ for each $t \in [0, T]$ and $y_n \rightharpoonup y$ if and only if $y_n(t) \rightharpoonup y(t)$ for each $t \in [0, T]$. Fix $t \in [0, T]$. Since $x_n(s) \rightharpoonup x(s)$ for each $s \in [0, t]$, then $S := \{x_n(s) :\ n \in \mathbb{N}\}$ is a relatively weakly compact subset of E for each $s \in [0, t]$. Using the fact that the De Blasi measure of weak noncompactness is regular we get $w(S) = 0$. From assumption (iv) it follows that $w(G([0, t] \times S) = 0$. Keeping in mind that $v_n(s) \in G(s, x_n(s))$ for a.e. $s \in [0, t]$ we obtain

$$\{v_n(s) :\ n \in \mathbb{N}\} \subseteq G([0, t] \times S)$$

for a.e. $s \in [0, t]$. Hence $w(\{v_n(s) : n \in \mathbb{N}\}) = 0$ for a.e. $s \in [0, t]$. This implies that the set $\{v_n(s) :\ n \in \mathbb{N}\}$ is relatively weakly compact for a.e. $s \in [0, t]$. Hence, by passing to a subsequence if necessary, we may assume that the sequence $v_n(s)$ is weakly convergent in E for a.e. $s \in [0, t]$. Let $v(s)$ be its weak limit. From our assumptions it follows that $v\colon [0, T] \to E$ is Pettis integrable and $v(s) \in G(s, x(s))$ for a. e. $s \in [0, t]$. The Lebesguev Dominated Convergence Theorem for the Pettis integral [18, Corollary 4] implies for each $\phi \in E^*$ that $\phi(y_n(t)) \to \phi\big(x_0 + \int_0^t v(s)\, ds\big)$ i.e. $y_n(t) \rightharpoonup x_0 + \int_0^t v(s) ds$. We can do this for each $t \in [0, T]$. Consequently, $y(t) = x_0 + \int_0^t v(s) ds \in Fx(t)$ for each $t \in [0, T]$, i.e. $y \in Fx$. Now we show that there is an integer n_0 such that F is w-power-convex condensing about 0 and n_0. To see this notice, for each bounded set $H \subseteq C$ and for each $t \in [0, T]$, that

$$F(H)(t) \subseteq x_0 + t\overline{co}\left(G\left([0, t] \times H[0, t]\right)\right). \tag{3.8}$$

Using the properties of the weak measure of noncompactness we get

$$\begin{gathered} w(F^{(1,0)}(H)(t)) = w(F(H)(t)) \leq \\ \leq tw\left(\overline{co}\left(G\left([0, t] \times H[0, t]\right)\right)\right) \leq tw(G([0, t] \times H[0, t]) \leq t\tau w(H[0, t]). \end{gathered}$$

Theorem 1.1 implies (since H is equicontinuous) that

$$w(F^{(1,0)}(H)(t)) \leq t\tau w(H). \tag{3.9}$$

Since $F^{(1,0)}(H)$ is equicontinuous, it follows from Lemma 1.1 that $F^{(2,0)}(H)$ is equicontinuous. Using (3.9) we get

$$w(F^{(2,0)}(H)(t)) =$$

$$= w\Big(\Big\{x_0 + \int_0^t v(s)\,ds \colon v(s) \in G(s, x(s)),\ x \in \overline{co}(F^{(1,0)}(H) \cup \{0\})\Big\}\Big) \leq$$

$$\leq w\Big(\Big\{\int_0^t v(s)\,ds \colon v(s) \in G(s, x(s)),\ x \in \overline{co}(F^{(1,0)}(H) \cup \{0\})\Big\}\Big) =$$

$$= w\Big(\Big\{\int_0^t v(s)\,ds \colon v(s) \in G(s, x(s)),\ x \in V\Big\}\Big),$$

where $V = \overline{co}(F^{(1,0)}(H) \cup \{0\})$. Fix $t \in [0, T]$. We divide the interval $[0, t]$ into m parts $0 = t_0 < t_1 < \cdots < t_m = t$ in such a way that $\Delta t_i = t_i - t_{i-1} = \frac{t}{m}$, $i = 1, \ldots, m$. For each $x \in V$ and for each $v(s) \in G(s, x(s))$ we have

$$\int_0^t v(s)\,ds = \sum_{i=1}^m \int_{t_{i-1}}^{t_i} v(s)\,ds \in \sum_{i=1}^m \Delta t_i \overline{co}\big\{v(s) \colon\ s \in [t_{i-1}, t_i]\big\} \subseteq$$

$$\subseteq \sum_{i=1}^m \Delta t_i \overline{co}\Big(G\big([t_{i-1}, t_i] \times V([t_{i-1}, t_i])\big)\Big).$$

Using again Theorem 1.1 we infer that for each $i = 2, \ldots, m$ there is a $s_i \in [t_{i-1}, t_i]$ such that

$$\sup_{s \in [t_{i-1}, t_i]} w(V(s)) = w(V[t_{i-1}, t_i]) = w(V(s_i)). \tag{3.10}$$

Consequently,

$$w\Big\{\int_0^t v(s)ds \colon x \in V\Big\} \leq \sum_{i=1}^m \Delta t_i w(\overline{co}\Big(G\big([t_{i-1}, t_i] \times V([t_{i-1}, t_i])\big)\Big) \leq$$

$$\leq \tau \sum_{i=1}^m \Delta t_i w\big(\overline{co}(V([t_{i-1}, t_i])\big) \leq \tau \sum_{i=1}^m \Delta t_i w(V((s_i)).$$

On the other hand, if $m \to \infty$ then

$$\sum_{i=1}^m \Delta t_i w(V((s_i)) \longrightarrow \int_0^t w(V(s))\,ds. \tag{3.11}$$

Using the regularity, the set additivity, the convex closure invariance of the De Blasi measure of weak noncompactness together with (3.9) we obtain

$$w(V(s)) = w(F^{(1,0)}(H)(s)) \leq s\tau w(H) \tag{3.12}$$

and therefore

$$\int_0^t w(V(s))\,ds \le s\tau \frac{t^2}{2}\, w(H). \tag{3.13}$$

As a result

$$w(F^{(2,0)}(H)(t)) \le \frac{(\tau t)^2}{2}\, w(H). \tag{3.14}$$

By induction we get

$$w(F^{(n,0)}(H)(t)) \le \frac{(\tau t)^n}{n!}\, w(H). \tag{3.15}$$

Invoking Theorem 1.1 we obtain

$$w(F^{(n,0)}(H)) \le \frac{(\tau T)^n}{n!}\, w(H). \tag{3.16}$$

Since $\lim\limits_{n\to\infty} \frac{(\tau T)^n}{n!} = 0$, then there is a n_0 with $\frac{(\tau T)^{n_0}}{n_0!} < 1$. This implies

$$w(F^{(n_0,0)}(H)) < w(H). \tag{3.17}$$

Consequently, F is w-power-convex condensing about 0 and n_0. The result follows from Theorem 2.1. □

4. Multivalued Convex-Power Maps with Respect to a Measure of Noncompactness

In this section we shall prove some fixed point theorems for multivalued mappings relative to the strong topology on a Banach space. By a measure of noncompactness on a Banach space X we mean a map $\alpha\colon \mathcal{B}(X) \to \mathbb{R}_+$ which satisfies conditions (1)–(5) in Definition 1.1 relative to the strong topology instead of the weak topology. The concept of a measure of noncompactness was initiated by the fundamental papers of Kuratowski [21] and Darbo [12]. Measures of noncompactness play a very important role in nonlinear analysis, namely in the theories of differential and integral equations. Specifically, the so-called Kuratowski measure of noncompactness [21] and Hausdorff (or ball) measure of noncompactness [3] are frequently used. We say that a bounded multivalued mapping $F\colon C \to 2^C$, defined on a nonempty closed convex subset C of X, is a α-convex-power condensing operator about x_0 and n_0 if for any bounded set $M \subseteq C$ with $\alpha(M) > 0$ we have

$$\alpha(F^{(n_0,x_0)}(M)) < \alpha(M). \tag{4.1}$$

Clearly, $F\colon C \to 2^C$ is α-condensing if and only if it is α- convex-power condensing operator about x_0 and 1. We first state the following result:

Theorem 4.1. *Let X be a Banach space and α be a regular and set additive measure of noncompactness on X. Let C be a nonempty closed convex subset of X, $x_0 \in C$ and n_0 be a positive integer. Suppose $F\colon C \to C(C)$ is α-convex-power condensing about x_0 and n_0. If F has a closed graph with $F(C)$ bounded then F has a fixed point in C.*

Proof. Let

$$\mathcal{F} = \Big\{ A \subseteq C,\ \overline{co}(A) = A,\ x_0 \in A \text{ and } F(x) \in C(A) \text{ for all } x \in A \Big\}.$$

The set $\mathcal{F}$ is nonempty since $C \in \mathcal{F}$. Set $M = \bigcap_{A \in \mathcal{F}} A$. The reasoning in Theorem 2.1 shows that for all integer $n \geq 1$ we have:

$$M = \overline{co}\big(F^{(n,x_0)}(M) \cup \{x_0\}\big) \tag{4.2}$$

Using the properties of the measure of noncompactness we get

$$\alpha(M) = \alpha\Big(\overline{co}\big(F^{(n_0,x_0)}(M) \cup \{x_0\}\big)\Big) = \alpha\left(F^{(n_0,x_0)}(M)\right),$$

which yields that M is compact. Since $F\colon M \to 2^M$ has a closed graph then F is upper semi-continuous. The result follows from the Bohnenblust–Karlin fixed point theorem [4]. □

As an easy consequence of Theorem 4.1 we obtain the following result.

Corollary 4.1. *Let X be a Banach space and α be a regular and set additive measure of noncompactness on X. Let C be a nonempty closed convex subset of X. Assume that $F\colon C \to C(C)$ has a closed graph with $F(C)$ bounded. If F is α-condensing, i.e. $\alpha(F(M)) < \alpha(M)$, whenever M is a bounded non-compact subset of C, then F has a fixed point.*

Lemma 4.1. *Let $F\colon X \to 2^X$ be α-convex-power condensing about x_0 and n_0 (n_0 is a positive integer), where α is a regular and set additive measure of noncompactness. Let $\widetilde{F}\colon X \to 2^X$ be the operator defined on X by $\widetilde{F}(x) = F(x + x_0) - x_0$. Then, $\widetilde{F}$ is α-convex-power condensing about 0 and n_0. Moreover, F has a fixed point if $\widetilde{F}$ does.*

Proof. Let M be a bounded subset of X with $\alpha(M) > 0$. The reasoning in Lemma 2.1 yields that for all integer $n \geq 1$, we have

$$\widetilde{F}^{(n,0)}(M) \subseteq F^{(n,x_0)}(M + x_0) - x_0.$$

Hence

$$\alpha\left(\widetilde{F}^{(n_0,0)}(M))\right) \leq \alpha\left(F^{(n_0,x_0)}(M + x_0) - x_0\right) \leq$$
$$\leq \alpha\left((F^{(n_0,x_0)}(M + x_0)\right) < \alpha(M + x_0) \leq \alpha(M).$$

This proves the first statement. The second statement is straightforward. □

Theorem 4.2. *Let X be a Banach space and let α be a regular and set additive measure of noncompactness on X. Let Q and C be closed, convex subsets of X with $Q \subseteq C$. In addition, let U be an open subset of Q with $F(\overline{U})$ bounded and $x_0 \in U$. Suppose $F\colon X \to 2^X$ is α-power-convex condensing map about x_0 and n_0 (n_0 is a positive integer). If F has a closed graph and $F(x) \in C(C)$ for all $x \in \overline{U}$, then either*

$$F \text{ has a fixed point,} \tag{4.3}$$

or

$$\textit{there is a point } u \in \partial_Q U \textit{ and } \lambda \in (0,1) \textit{ with } u \in \lambda F u; \tag{4.4}$$

here $\partial_Q U$ is the boundary of U in Q.

Proof. By replacing F, Q, C and U by $\widetilde{F}, Q - x_0, C - x_0$ and $U - x_0$ respectively and using Lemma 4.1 we may assume that $0 \in U$ and F is α-power-convex condensing about 0 and n_0. Now suppose (4.4) does not occur and F does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (4.3) occurs). Let $M = \{x \in \overline{U} : \ x \in \lambda F x \text{ for some } \lambda \in [0,1]\}$. The set M is nonempty since $0 \in U$. Also M is closed. Indeed let (x_n) be sequence of M which converges to some $x \in \overline{U}$ and let (λ_n) be a sequence of $[0,1]$ satisfying $x_n \in \lambda_n F x_n$. Then for each n there is a $z_n \in F x_n$ with $x_n = \lambda_n z_n$. By passing to a subsequence if necessary, we may assume that (λ_n) converges to some $\lambda \in [0,1]$ and $\lambda_n \neq 0$ for all n. This implies that the sequence (z_n) converges to some $z \in \overline{U}$ with $x = \lambda z$. Since F has a closed graph then $z \in F(x)$. Hence $x \in \lambda F x$ and therefore $x \in M$. Thus M is closed. We now claim that M is relatively compact. Suppose $\alpha(M) > 0$. Clearly,

$$M \subseteq co(F(M) \cup \{0\}).$$

Arguing by induction as in the proof of Theorem 2.2, we can prove that for all integer $n \geq 1$ we have

$$M \subseteq co(F^{(n,0)}(M) \cup \{0\}).$$

This implies

$$\alpha(M) \leq \alpha\Big(co\big(F^{(n_0,0)}(M) \cup \{0\}\big)\Big) = \alpha(F(M)) < \alpha(M), \tag{4.5}$$

which is a contradiction. Hence $\alpha(M) = 0$ and therefore M is compact, since M is closed. From our assumptions we have $M \cap \partial_Q U = \varnothing$. By Urysohn Lemma [15] there exists a continuous mapping $\rho \colon \overline{U} \to [0,1]$ with $\rho(M) = 1$ and $\rho(\partial_Q U) = 0$. Let

$$T(x) = \begin{cases} \rho(x) F(x), & x \in \overline{U}, \\ 0, & x \in X \setminus \overline{U}. \end{cases}$$

Clearly, $T \colon X \to 2^X$ has a closed graph since F does. Moreover, for any $S \subseteq C$ we have

$$T(S) \subseteq co(F(S) \cup \{0\}).$$

This implies that

$$\begin{aligned} T^{(2,0)}(S) &= T\left(\overline{co}(T(S) \cup \{0\})\right) \subseteq T\left(\overline{co}(F(S) \cup \{0\})\right) \subseteq \\ &\subseteq \overline{co}\Big(F(\overline{co}(F(S) \cup \{0\}) \cup \{0\})\Big) = \overline{co}(F^{(2,0)}(S) \cup \{0\}). \end{aligned}$$

By induction

$$T^{(n,0)}(S) = T\left(\overline{co}(T^{(n-1,0)}(S) \cup \{0\})\right) \subseteq T\left(\overline{co}(F^{(n-1,0)}(S) \cup \{0\})\right) \subseteq$$
$$\subseteq \overline{co}\left(F(\overline{co}(F^{(n-1,0)}(S) \cup \{0\}) \cup \{0\})\right) = \overline{co}(F^{(n,0)}(S) \cup \{0\}),$$

for each integer $n \geq 1$. Using the properties of the measure of noncompactness we get

$$\alpha(T^{(n_0,0)}(S)) \leq \alpha\Big(\overline{co}(F^{(n_0,0)}(S) \cup \{0\})\Big) = \alpha(F^{(n_0,0)}(S)) < \alpha(S), \quad (4.6)$$

if $\alpha(S) > 0$. Thus $T\colon X \to 2^X$ has a closed graph and $T(x) \subseteq C(C)$ for all $x \in C$. Moreover, T is α-power-convex condensing about 0 and n_0. By Theorem 4.1 there exists $x \in C$ such that $x \in Tx$. Now $x \in U$ since $0 \in U$. Consequently, $x \in \rho(x)F(x)$ and so $x \in M$. This implies $\rho(x) = 1$ and so $x \in F(x)$. □

Theorem 4.3. *Let X be a Banach space and α a regular set additive measure of noncompactness on X. Let Q be a closed convex subset of X with $0 \in Q$ and n_0 a positive integer. Assume $F\colon X \to 2^X$ has a sequentially closed graph with $F(Q)$ bounded and $F(x) \in C(X)$ for all $x \in Q$. Also assume F is α-convex-power condensing about 0 and n_0 and*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\} \text{ is a sequence in } \partial Q \times [0,1] \\ \text{converging to } (x, \lambda) \text{ with } x \in \lambda F(x) \text{ and } 0 < \lambda < 1, \\ \text{then } \lambda_j F(x_j) \subseteq Q \text{ for } j \text{ sufficiently large} \end{cases} \quad (4.7)$$

holding. Also suppose the following condition holds:

$$\begin{cases} \text{there exists a continuous retraction } r\colon X \to Q \\ \text{with } r(z) \in \partial Q \text{ for } z \in X \setminus Q \text{ and } r(D) \subseteq co(D \cup \{0\}) \\ \text{for any bounded subset } D \text{ of } X. \end{cases} \quad (4.8)$$

Then, F has a fixed point.

Proof. Let $r\colon X \to Q$ be as described in (4.8). Consider $B = \{x \in X :\ x = Fr(x)\}$.

We first show that $B \neq \varnothing$. To see this, consider $Fr\colon X \to C(X)$. Clearly Fr has a sequentially closed graph, since F has a sequentially closed graph and r is continuous. Now we show that Fr is α-power-convex condensing map about 0 and n_0. To see this, let A be a bounded subset of X and set $A' = \overline{co}(A \cup \{0\})$. Then, using (4.8) we obtain

$$(Fr)^{(1,0)}(A) \subseteq F(A'),$$
$$(Fr)^{(2,0)}(A) = Fr\left(\overline{co}\left((Fr)^{(1,0)}(A) \cup \{0\}\right)\right) \subseteq$$
$$\subseteq Fr\left(\overline{co}\left(F(A') \cup \{0\}\right)\right) \subseteq F\left(\overline{co}\left(F(A') \cup \{0\}\right)\right) =$$
$$= F^{(2,0)}(A'),$$

and by induction,

$$(Fr)^{(n_0,0)}(A) = Fr\left(\overline{co}\left((Fr)^{(n_0-1,0)}(A)\cup\{0\}\right)\right) \subseteq$$
$$\subseteq Fr\left(\overline{co}\left(F^{(n_0-1,0)}(A')\cup\{0\}\right)\right) \subseteq$$
$$\subseteq F\left(\overline{co}\left(F^{(n_0-1,0)}(A')\cup\{0\}\right)\right) =$$
$$= F^{(n_0,0)}(A').$$

Thus

$$\alpha\left((Fr)^{(n_0,0)}(A)\right) \leq \alpha\left(F^{(n_0,0)}(A')\right) < \alpha(A') = \alpha(A),$$

whenever $\alpha(A) \neq 0$. Invoking Theorem 4.1 we infer that there exists $y \in X$ with $y \in Fr(y)$. Thus $y \in B$ and $B \neq \varnothing$. In addition B is closed, since Fr has a sequentially closed graph. Moreover, we claim that B is compact. To see this, first notice

$$B \subseteq Fr(B) \subseteq F(B') = F^{(1,0)}(B'),$$

where $B' = \overline{co}(B \cup \{0\})$. Thus

$$B \subseteq Fr(B) \subseteq Fr\left(F(B')\right) \subseteq F\left(\overline{co}\left(F(B')\cup\{0\}\right)\right) = F^{(2,0)}(B'),$$

and by induction

$$B \subseteq Fr(B) \subseteq Fr\left(F^{(n_0-1,0)}(B')\right) \subseteq$$
$$\subseteq F\left(\overline{co}\left(F^{(n_0-1,0)}(B')\cup\{0\}\right)\right) = F^{(n_0,0)}(B'),$$

Now if $\alpha(B) \neq 0$, then

$$\alpha(B) \leq \alpha(F^{(n_0,0)}(B')) < \alpha(B') = \alpha(B),$$

which is a contradiction. Thus, $\alpha(B) = 0$ and so B is relatively compact. Consequently, $B = \overline{B}$ is compact. We now show that $B \cap Q \neq \varnothing$. To do this, we argue by contradiction. Suppose $B \cap Q = \varnothing$. Then since B is compact and Q is closed there exists $\delta > 0$ with $dist(B, Q) > \delta$. Choose $N \in \{1, 2, \ldots\}$ such that $N\delta > 1$. Define

$$U_i = \left\{x \in X :\ d(x, Q) < 1/i\right\} \ \text{ for }\ i \in \{N, N+1, \ldots\};$$

here $d(x, Q) = \inf\{\|x - y\| \ :\ y \in Q\}$. Fix $i \in \{N, N+1, \ldots\}$. Since $dist(B, Q) > \delta$ then $B \cap \overline{U_i} = \varnothing$. Applying Theorem 4.2 to $Fr\colon \overline{U_i} \to C(X)$ we may deduce that there exists $(y_i, \lambda_i) \in \partial U_i \times (0, 1)$ with $y_i = \lambda_i Fr(y_i)$. Notice in particular since $y_i \in \partial U_i \times (0, 1)$ that

$$\lambda_i Fr(y_i) \notin Q \ \text{ for }\ i \in \{N, N+1, \ldots\}. \tag{4.9}$$

We now consider

$$D = \left\{x \in X :\ x = \lambda Fr(x) \text{ for some } \lambda \in [0, 1]\right\}.$$

Clearly D is closed since F has a sequentially closed graph and r is continuous. Now we claim that D is compact. To see this, first notice

$$D \subseteq Fr(D) \cup \{0\}.$$

Thus

$$D \subseteq Fr(D) \cup \{0\} \subseteq Fr\Big(\overline{co}\,(Fr(D) \cup \{0\})\Big) \cup \{0\} = (Fr)^{(2,0)} \cup \{0\},$$

and by induction

$$D \subseteq Fr(D) \cup \{0\} \subseteq Fr\left(\overline{co}\left((Fr)^{(n_0-1,0)}(D) \cup \{0\}\right)\right) \cup \{0\} = (Fr)^{(n_0,0)} \cup \{0\},$$

Consequently,

$$\alpha(D) \leq \alpha\left((Fr)^{(n_0,0)} \cup \{0\}\right) \leq \alpha\left((Fr)^{(n_0,0)}\right).$$

Since Fr is α-convex-power condensing about 0 and n_0 then $\alpha(D) = 0$ and so D is relatively weakly compact. Consequently, $D = \overline{D}$ is compact. Then, up to a subsequence, we may assume that $\lambda_i \to \lambda^* \in [0,1]$ and $y_i \to y^* \in \partial U_i$. Hence $\lambda_i Fr(y_i) \to \lambda^* Fr(y^*)$ and therefore $y^* = \lambda^* Fr(y^*)$. Notice $\lambda^* Fr(y^*) \notin Q$ since $y^* \in \partial U_i$. Thus $\lambda^* \neq 1$ since $B \cap Q = \varnothing$. From assumption (4.7) it follows that $\lambda_i Fr(y_i) \in Q$ for j sufficiently large, which is a contradiction. Thus $B \cap Q \neq \varnothing$, so there exists $x \in Q$ with $x = Fr(x)$, i.e. $x = Fx$. □

Remark 4.1. If $0 \in int(Q)$ then we can choose $r\colon X \to Q$ in the statement of Theorem 4.3 as

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for } \; x \in X;$$

here μ is the Minkowski functional [33] defined by

$$\mu(x) = \inf\big\{\lambda > 0 : \; x \in \lambda Q\big\},$$

for all $x \in X$. Clearly r is continuous, $r(X) \subseteq Q$ and $r(x) = x$ for all $x \in Q$. Also, for any subset A of X we have $r(A) \subseteq co(A \cup \{0\})$.

Remark 4.2. In Theorem 4.3, we need $F\colon X \to 2^X$ α-convex-power condensing about 0 and n_0. However, the condition $F\colon X \to 2^X$ has sequentially closed graph can be replaced by $F\colon Q \to 2^X$ has sequentially closed graph.

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