Memoirs on Differential Equations and Mathematical Physics
Volume 56, 2012, 57–71

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SOME PROPERTIES OF THE GENERALIZED EULER BETA FUNCTION

Abstract. A generalization of the Euler beta function for the case of multi-dimensional variable is proposed. In this context ordinary beta function is defined as a function of two-dimensional variable. An analogue of the Euler formula for this new function is proved for arbitrary dimension. There is found out the connection of defined function with multi-dimensional hypergeometric Laurichella's function and the theorem on cancelation of multi-dimensional hypergeometric functions singularities is proved. Such generalizations (among others) may be helpful to construct corresponding physical (string) models including different number of fields, as far the (bosonic) string theory reproduces the Euler beta function (Veneziano amplitude) and its multi-dimensional analogue.

2010 Mathematics Subject Classification. 33B15, 33C65, 81T30.
Key words and phrases. Generalized Euler integrals, multivariable hypergeometric functions, strings theory.

რეზიუმე. შემოთავაზებულია ეილერის ბეტა ფუნქციის განზოგადება მრავალგანზომილებიანი ცვლადისათვის. ამ მიდგომაში, ჩვეულებრივი ბეტა ფუნქცია განიმარტება, როგორც ორგანზომილებიანი ცვლადის ფუნქცია. დამტკიცებულია ეილერის ფორმულის ანალოგი ამ ახალი ფუნქციისათვის ნებისმიერი განზომილების შემთხვევაში. ნაპოვნია კავშირი შემოღებულ ფუნქციასა და ლაურიჩელას ჰიპერგეომეტრიულ ფუნქციას შორის და დამტკიცებულია, რომ მიღებულ გამოსახულებაში მრავალი ცვლადის ჰიპერგეომეტრიული ფუნქციის სინგულარობები კვეცავს ერთმანეთს. შესაძლოა მსგავსი განზოგადებები სასარგებლო აღმოჩნდეს ფიზიკური (სიმების) თეორიის ასაგებად, როცა პროცესებში მონაწილეობს რამდენიმე ველი, რამდენადაც (ბოზონური) სიმის თეორია (ვენეციანოს მოდელი) აღიწერება ეილერის ბეტა ფუნქციით.

In the articles [1], [2] there was proposed and investigated the function

$$B_n(r_0, r_1, \ldots, r_n) = B_n(\mathbf{r}) =$$

$$= \begin{cases} 1 & \text{if } n = 0, \\ \det^{-1}[x_j^{i-1}]_{i,j=\overline{1,n}} \det\left[x_j^{i-1} b_{ij}\right]_{i,j=\overline{1,n}} & \text{if } n \geq 1 \end{cases} \tag{1}$$

$$(0 = x_0 < x_1, x_2 < \cdots < x_n),$$

where

$$b_{ij} = \int\limits_{x_{j-1}/x_j}^{1} u^{i-1}(1-u)^{r_j-1} \prod_{\substack{k=0 \\ k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}^{r_k-1}\Big)\, du \quad (i, j = 1, \ldots, n).$$

The function (1) is a multidimensional generalization of the Euler beta function

$$B(r_0, r_1) = B_1(\mathbf{r}) = \int\limits_0^1 u^{r_0-1}(1-u)^{r_1-1}\, du.$$

A multidimensional analogues of Euler's beta function had been studied by mathematicians such as Selberg, Weil and Deligne among many others (see e.g. [3]–[7]). In [1] we have shown that for any $n \in \mathbb{N}$ and for $r_0 > 0$, $r_j \in \mathbb{N}$ $(j = 1, \ldots, n)$ an analogue of the Euler formula is valid:

$$B_n(\mathbf{r}) = \frac{\prod\limits_{j=0}^{n} \Gamma(r_j)}{\Gamma\Big(\sum\limits_{j=0}^{n} r_j\Big)}, \tag{2}$$

where $\Gamma(t) = \int\limits_0^\infty e^{-u} u^{t-1}\, du$ is the Euler Gamma function.

In [2] there is investigated the case of the dimension $n = 2$. The analogue of the Euler formula has been proved for any complex parameters r_0, r_1, r_2 ($\operatorname{Re} r_j > 0$, $j = 0, 1, 2$) and the complications arising when $n \geq 3$ are shown. The relations between $B_n(\mathbf{r})$ and hypergeometric functions of one and of many variables are shown too. Number of relations for the Gauss hypergeometric function is obtained. The analytic formulae for some new definite integrals of the special functions are obtained as well as for the elementary ones.

In present article we prove (2) for any $n \in \mathbb{N}$ and for $r_j \in \mathbb{C}$ ($\operatorname{Re} r_j > 0$, $j = 0, 1, \ldots, n$). The key of the proof is the known

Carlson theorem ([8])**.** *If the function $f(z)$ of a complex variable z is regular in the semi-plane $\operatorname{Re} z > A$, $A \in \mathbb{R}$, and if the conditions*

(a) $\lim\limits_{|z|\to\infty} |f(z)| \exp(-k|z|) \leq const$, $0 < k < \pi$;

(b) $f(z) = 0$ *for* $z = 0, 1, 2, \ldots,$

are valid, then $f(z) = 0$ *for any* $z \in \mathbb{C}$.

For proving (2) we need the following

Lemma. *On the complex plane of the variable* $r_j \in \mathbb{C}$ $(j = 0, 1, \ldots, n)$ *the function* $B_0(r_0, r_1, \ldots, r_n) = B_n(\mathbf{r})$ *is bounded when* $|r_j| \to \infty$ *if other variables* $r_0, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n$ *are fixed and* $\operatorname{Re} r_j \geq 1$ $(j = 0, 1 \ldots, n)$*:*

$$|B_n(\mathbf{r})| \leq M < \infty. \tag{3}$$

The bound M *does not depend on the variables* $r_0, r_1, \ldots, r_n$ $(\operatorname{Re} r_j \geq 1,$ $j = 0, 1 \ldots, n)$.

Proof. The substitutions

$$u = \widetilde{u}, \;\; j = 1; \quad u = 1 - \widetilde{u}\Big(1 - \frac{x_{j-1}}{x_j}\Big), \;\; j = 2, \ldots, n \;\; (n \geq 2),$$

give to the formula (1) the form

$$B_n(\mathbf{r}) = \frac{\det[x_j^{i-1}\widetilde{b}_{ij}]_{i,j=\overline{1,n}}}{\det[x_j^{i-1}]_{i,j=\overline{1,n}}}\,, \tag{4}$$

where

$$\widetilde{b}_{i1} = \prod_{k=2}^{n}\Big(1 - \frac{x_1}{x_k}\Big)^{1-r_k}\int_0^1 \widetilde{u}^{r_0+i-2}(1-\widetilde{u})^{r_1-1}\prod_{k=2}^{n}\Big(1 - \frac{x_1}{x_k}\,\widetilde{u}\Big)^{r_k-1}\,d\widetilde{u},$$

$$\widetilde{b}_{ij} = \Big(1 - \frac{x_{j-1}}{x_j}\Big)^{r_j}\times$$

$$\times\int_0^1\Big[1 - \Big(1 - \frac{x_{j-1}}{x_j}\Big)\widetilde{u}\Big]^{r_0+i-2}\widetilde{u}^{r_j-1}\prod_{\substack{k=0\\k\neq j}}^{n}\Big(1 - \frac{x_j - x_{j-1}}{x_j - x_k}\Big)^{r_k-1}\,d\widetilde{u},$$

$$i = 1, \ldots, n; \;\; j = 2 \ldots, n.$$

All these integrals converge if the conditions

$$\operatorname{Re} r_j > 0, \;\; j = 0 \ldots, n; \;\; 0 = x_0 < x_1 < x_2 < \cdots < x_n$$

are fulfilled. Note that

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}\prod_{k=1}^{n}(1 - z_k u)^{b_k}\,du =$$

$$= F\Big({a \atop c}; b_1, z_1; \ldots; b_n, z_n\Big) \;\; (\operatorname{Re} c > \operatorname{Re} a > 0),$$

where $F\Big({a \atop c}; b_1, z_1; \ldots; b_n, z_n\Big)$ denotes the multi-variable hypergeometric function – one of the four Lauricella's functions of the arguments $z_1, \ldots, z_n$ (see

[9]), which can be expressed as absolutely convergent power-sum

$$F\Big({a \atop c}; b_1, z_1; \dots; b_n, z_n\Big) \equiv F_D\big(a; b_1, \dots, b_n; c; z_1, \dots, z_n\big) =$$
$$= \sum_{k_1,k_2,\dots,k_n=0}^{\infty} \frac{(a)_{k_1+\dots+k_n}}{(c)_{k_1+\dots+k_n}} \prod_{j=1}^{n} \frac{(b_j)_{k_j} z_j^{k_j}}{k_j!} ,$$

if $|z_j| < 1$, $j = 1, \dots, n$. Here $(a)_k$ denotes the Pochhammer symbol:

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1), \quad k \in \mathbb{N}.$$

Thus the formula (4) expresses the above-defined function (1) via the determinant of Lauricella's multi-variable hypergeometric functions.

Let us rewrite the formula (4) as follows

$$B_n(\mathbf{r}) \det[z_j^{i-1}]_{i,j=\overline{1,n}} = M_1 \det[z_j^{i-1} \tilde{\tilde{b}}_{ij}]_{i,j=\overline{1,n}}, \tag{5}$$

where

$$z_k = \frac{x_k}{x_n}, \quad 0 < z_0 < z_1 < \dots < z_{n-1} < z_n = 1, \tag{6}$$

$$M_1 = \prod_{k=2}^{n} \Big(1 - \frac{z_1}{z_k}\Big)^{1-r_k} \Big(1 - \frac{z_{k-1}}{z_k}\Big)^{r_k} =$$
$$= \prod_{k=2}^{n} \Big(1 - \frac{z_1}{z_k}\Big) \prod_{k=3}^{n} \Big(\frac{z_k - z_{k-1}}{z_k - z_1}\Big)^{r_k} \tag{7}$$

and

$$\tilde{\tilde{b}}_{i1} = \int_0^1 u^{r_0+i-2}(1-u)^{r_1-1} \prod_{k=2}^{n} \Big(1 - u\,\frac{z_1}{z_k}\Big)^{r_k-1} du,$$

$$\tilde{\tilde{b}}_{ij} = \int_0^1 \Big(1 - \frac{z_j - z_{j-1}}{z_j}\Big)^{r_0+i-2} u^{r_j-1} \prod_{\substack{k=1\\k\neq j}}^{n} \Big(1 - \frac{z_j - z_{j-1}}{z_j - z_k}\Big)^{r_k-1} du,$$
$$j = 2 \dots, n.$$

Due to the inequalities (6), the expression (5) can be estimated as follows:

$$\big|B_n(\mathbf{r})\big| \det[z_j^{i-1}]_{i,j=\overline{1,n}} =$$
$$= |M_1| \, \Big|\det[z_j^{i-1}\tilde{\tilde{b}}_{ij}]_{i,j=\overline{1,n}}\Big| \le |M_1| \operatorname{per} \big[z_j^{i-1}|\tilde{\tilde{b}}_{ij}|\big]_{i,j=\overline{1,n}}, \tag{8}$$

where $\operatorname{per}[a_{ij}]_{i,j=\overline{1,n}}$ stands for the permanent of a matrix $[a_{ij}]_{i,j=\overline{1,n}}$ (see e.g. [10]):

$$\operatorname{per}[a_{ij}]_{i,j=\overline{1,n}} = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}, \quad \sigma : i \mapsto \sigma(i).$$

Further estimations give:

$$|\tilde{\tilde{b}}_{i1}| \le \int_0^1 \left| u^{r_0+i-2}(1-u)^{r_1-1} \prod_{k=2}^{n} \Big(1 - u\,\frac{z_1}{z_k}\Big)^{r_k-1} \right| du =$$

$$= \int_0^1 u^{\operatorname{Re} r_0+i-2}(1-u)^{\operatorname{Re} r_1-1} \prod_{k=2}^{n} \Big(1 - u\,\frac{z_1}{z_k}\Big)^{\operatorname{Re} r_k-1} \Bigg| du \equiv$$

$$\equiv \int_0^1 g_{i1}(u) f_1(u)\, du,$$

$$|\tilde{\tilde{b}}_{ij}| \le \tag{9}$$

$$\le \int_0^1 \left| \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j}\Big)^{r_0+i-2} u^{r_j-1} \prod_{\substack{k=1\\k\neq j}}^{n} \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j - z_k}\Big)^{r_k-1} \right| du =$$

$$= \int_0^1 \left| \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j}\Big)^{\operatorname{Re} r_0+i-2} u^{\operatorname{Re} r_j-1} \prod_{\substack{k=1\\k\neq j}}^{n} \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j - z_k}\Big)^{\operatorname{Re} r_k-1} \right| du \equiv$$

$$\equiv \int_0^1 f_{ij}(u) g_j(u)\, du, \quad j = 2, \ldots, n,$$

where we have denoted

$$f_1(u) = \prod_{k=1}^{n} \Big(1 - u\,\frac{z_1}{z_k}\Big)^{\operatorname{Re} r_k-1},$$

$$g_{i1}(u) = u^{\operatorname{Re} r_0+i-2},$$

$$f_{ij}(u) = \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j}\Big)^{\operatorname{Re} r_0+i-2} \prod_{k=1}^{j-1} \Big(1 - u\,\frac{z_j - z_{j-1}}{z_j - z_k}\Big)^{\operatorname{Re} r_k-1},$$

$$g_j(u) = u^{\operatorname{Re} r_j-1} \prod_{k=j+1}^{n} \Big(1 + u\,\frac{z_j - z_{j-1}}{z_k - z_j}\Big)^{\operatorname{Re} r_k-1},$$

$$j = 2, \ldots, n; \quad i = 1, \ldots, n.$$

Let us use the mean value theorem in the integrals (9). As it is known, if the function $f(x)$ is monotonic and $f(x) \ge 0$ when $x \in [a, b]$, and if $g(x)$ is integrable, then the Bonnet formulae are valid (see e.g. [11, II, n°306]):

$$\int_a^b f(u) g(u)\, du = f(a) \int_a^\eta g(u)\, du, \quad a \le \eta \le b \ \text{ if } \ f(x) \ \text{ decreases},$$

$$\int_a^b f(u)g(u)\,du = f(b)\int_\xi^b g(u)\,du, \quad a \le \xi \le b \text{ if } f(x) \text{ increases},$$
$$(x \in [a,b]).$$

It is obvious that if $\operatorname{Re} r_j \ge 1$, $j = 0,1,\ldots,n$, then the factors $f_1(u)$ and $f_{ij}(u)$ in (9) decrease for $u \in [0,1]$ and the factors $g_{i1}(u)$ and $g_j(u)$ increase ($j = 2,\ldots,n$; $i = 1,\ldots,n$). Hence, according to the Bonnet formulae, for $\operatorname{Re} r_j \ge 1$, $j = 0,1,\ldots,n$ (due to this conditions all integrals converge) one gets

$$\begin{aligned}
|\widetilde{\widetilde{b}}_{i1}| &\le f_1(0)\int_0^{\eta_{i1}} g_{i1}(u)\,du = f_1(0)g_{i1}(\eta_{i1})\int_{\xi_{i1}}^{\eta_{i1}} du \le \\
&\le f_1(0)g_{i1}(1)(\eta_{i1}-\xi_{i1}) \le 1, \\
|\widetilde{\widetilde{b}}_{ij}| &\le g_j(1)\int_{\xi_{ij}}^{1} f_{ij}(u)\,du = g_j(1)f_{ij}(\xi_{ij})\int_{\xi_{ij}}^{\eta_{i1}} du \le \\
&\le g_j(1)f_{ij}(0)(\eta_{ij}-\xi_{ij}) \le \prod_{k=j+1}^{n}\Big(\frac{z_k - z_{j-1}}{z_k - z_j}\Big)^{\operatorname{Re} r_k - 1}, \\
&\qquad 0 = z_0 < z_1 < \cdots < z_n, \quad 0 \le \xi_{ij} \le \eta_{ij} \le 1, \quad i,j = 1\ldots,n.
\end{aligned} \tag{10}$$

According to (7), (8) and (10) one has

$$\begin{gathered}
\big|B_n(\mathbf{r})\big| \det[z_j^{i-1}]_{i,j=\overline{1,n}} \le \\
\le |M_1| \operatorname{per}\big[z_j^{i-1}|\widetilde{\widetilde{b}}_{ij}|\big]_{i,j=\overline{1,n}} \le M_2 \operatorname{per}[z_j^{i-1}]_{i,j=\overline{1,n}},
\end{gathered}$$
$$M_2 = \prod_{k=2}^{n}\Big(1-\frac{z_1}{z_k}\Big)\prod_{k=3}^{n}\Big(\frac{z_k - z_{k-1}}{z_k - z_1}\Big)^{\operatorname{Re} r_k}\prod_{j=2}^{n-1}\prod_{k=j+1}^{n}\Big(\frac{z_k - z_{j-1}}{z_k - z_j}\Big)^{\operatorname{Re} r_k - 1}.$$

Because of obvious equalities

$$\begin{aligned}
&\prod_{j=2}^{n-1}\prod_{k=j+1}^{n}\Big(\frac{z_k - z_{j-1}}{z_k - z_j}\Big)^{\operatorname{Re} r_k - 1} = \\
&\qquad = \prod_{k=3}^{n}\Big[\prod_{j=2}^{k-1}\Big(\frac{z_k - z_{j-1}}{z_k - z_j}\Big)\Big]^{\operatorname{Re} r_k - 1} = \prod_{k=3}^{n}\Big(\frac{z_k - z_1}{z_k - z_{k-1}}\Big)^{\operatorname{Re} r_k - 1}
\end{aligned}$$

one obtains

$$M_2 = \prod_{k=2}^{n}\Big(1-\frac{z_1}{z_k}\Big)\prod_{k=3}^{n}\Big(\frac{z_k - z_{k-1}}{z_k - z_1}\Big) = \prod_{k=2}^{n}\Big(1-\frac{z_{k-1}}{z_k}\Big).$$

Inserting these results into the inequality (8), one obtains the estimation:

$$\big|B_n(\mathbf{r})\big| \le \frac{\mathrm{per}[z_j^{i-1}]_{i,j=\overline{1,n}}}{\det[z_j^{i-1}]_{i,j=\overline{1,n}}} \prod_{k=2}^{n} \Big(1 - \frac{z_{k-1}}{z_k}\Big), \quad \mathrm{Re}\, r_j \ge 1, \quad j = 0, 1, \ldots, n.$$

Hence, we have got the estimation (3) with M to be expressed as

$$M = \frac{\mathrm{per}[z_j^{i-1}]_{i,j=\overline{1,n}}}{\det[z_j^{i-1}]_{i,j=\overline{1,n}}} \prod_{k=1}^{n} \Big(1 - \frac{z_{k-1}}{z_k}\Big), \tag{11}$$

which, obviously, does not depend on the variables $r_0, r_1, \ldots, r_n$ ($\mathrm{Re}\, r_j \ge 1$, $j = 0, 1, \ldots, n$). □

Note. The restriction $\mathrm{Re}\, r_j \ge 1$, $j = 0, 1, \ldots, n$, is essential. Let, e.g., $n = 1$. In this case (11) gives $M = 1$ and one gets

$$\big|B_1(\mathbf{r})\big| = \Big| \int_0^1 u^{r_0 - 1}(1 - u)^{r_1 - 1}\, du \Big| = \Big| \frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0 + r_1)} \Big| \le 1 \ \text{ if } \ \mathrm{Re}\, r_0, r_1 \ge 1,$$

while in the opposite case when $\mathrm{Re}\, r_0, r_1 < 1$, e.g. for $r_0 = r_1 = 1/2$ one has $B_1(1/2, 1/2) = \pi > 1$, and the estimation (3) is not fulfilled.

Now we get the following

Statement. *The function*

$$f(r_0, r_1, \ldots, r_n) = B_n(r_0, r_1, \ldots, r_n) - \frac{\prod_{j=0}^{n} \Gamma(r_j)}{\Gamma\big(\sum_{j=0}^{n} r_j\big)} \tag{12}$$

satisfies all conditions of Carlson theorem on the complex plane of each variable r_j if other variables $r_0, r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n$ are fixed and $\mathrm{Re}\, r_j \ge 1$, $j = 0, 1, \ldots, n$.

Proof of the Statement. Due to the fact that the function $\Gamma(z)$ is analytic everywhere on the (open) complex plane except the points $z = 0, -1, -2, \ldots$, the function (12) is analytic if $\mathrm{Re}\, r_j > 0$, $j = 0, 1, \ldots, n$ ($r_j = +\infty$ is a regular point of both summands of (12), $j = 0, 1, \ldots, n$). In [1] we have proved that $f_0(r_0) = f(r_0, r_1, \ldots, r_n) = 0$ if the variable r_0 is real and $r_0 > 0$, $r_j \in \mathbb{N}$, $j = 1, \ldots, n$. Hence, according to the analytic function uniqueness theorem, if $r_j \in \mathbb{N}$, $j = 1, \ldots, n$, then $f_0(r_0) = 0$ everywhere on the complex plane of the variable r_0 except, may be, the points $z = 0, -1, -2, \ldots$. So, the function $f_0(r_0)$ satisfies the Statement.

Let us fix the numbers $r_j \in \mathbb{N}$, $j = 1, \ldots, n$ and $r_0 \in \mathbb{C}$, $\mathrm{Re}\, r_0 > 0$. Under these conditions the function $f_1(r_1) = f(r_0, r_1, \ldots, r_n)$ is analytic on the complex semi-plane $\mathrm{Re}\, r_1 > 0$ and $f_1(r_1) = 0$ for $r_1 = 1, 2, \ldots$.

Let us show that if $\mathrm{Re}\, r_0 \ge 1$, $r_j \in \mathbb{N}$, $j = 2, \ldots, n$, then

$$\lim_{|r_1| \to \infty} |f_1(r_1)| \exp(-k|r_1|) \le const, \quad \mathrm{Re}\, r_1 \ge 1, \quad 0 < k < \pi.$$

Indeed, if $|z| \to \infty$ and $\operatorname{Re} z > 0$, in accordance with the asymptotic behavior of the Euler's Gamma function (see e.g. [12, Eq. 1.18(5)]) for any fixed number $\rho \in \mathbb{C}$ we have

$$\lim_{|z|\to\infty} \frac{\Gamma(z)}{\Gamma(z+\rho)} \exp\big(\rho \ln|z|\big) = \lim_{|z|\to\infty} \frac{\Gamma(z)}{\Gamma(z+\rho)} |z|^{\rho} = 1. \tag{13}$$

Hence, due to the estimation (3) we obtain:

$$0 \le \lim_{|r_1|\to\infty} |f_1(r_1)| \exp(-k|r_1|) =$$

$$= \lim_{|r_1|\to\infty} \exp(-k|r_1|) \left| B_n(\mathbf{r}) - \frac{\Gamma(r_1)}{\Gamma(r_1+\rho_1)} \Gamma(r_0) \prod_{j=2}^{n} \Gamma(r_j) \right| \le$$

$$\le \lim_{|r_1|\to\infty} \exp(-k|r_1|) \left\{ \big|B_n(\mathbf{r})\big| + \left| \frac{\Gamma(r_1)}{\Gamma(r_1+\rho_1)} \Gamma(r_0) \prod_{j=2}^{n} \Gamma(r_j) \right| \right\} \le$$

$$\le M \lim_{|r_1|\to\infty} \exp(-k|r_1|) + \Gamma(r_0) \prod_{j=2}^{n} \Gamma(r_j) \times$$

$$\times \lim_{|r_1|\to\infty} \left\{ \frac{\Gamma(r_1)}{\Gamma(r_1+\rho_1)} \exp\big(\rho_1 \ln|r_1|\big) \right\} \lim_{|r_1|\to\infty} \Big\{ |r_1|^{-\rho_1} \exp(-k|r_1|) \Big\} = 0,$$

where we denote $\rho_1 = r_0 + r_2 + \cdots + r_n$. Therefore we get

$$\lim_{|r_1|\to\infty} |f_1(r_1)| e^{-k|r_1|} = 0$$

for any fixed values of $k > 0$ and ρ_1. In particular, one can choose $0 < k < \pi$, $\operatorname{Re} r_0 \ge 1$, $r_j \in \mathbb{N}$, $j = 2, \ldots, n$.

Thus, if $\operatorname{Re} r_0 \ge 1$, $r_j \in \mathbb{N}$, $j = 2, \ldots, n$, then the function $f_1(r_1) = f(r_0, r_1, \ldots, r_n)$ satisfy all conditions of the Carlson Theorem and therefore $f_1(r_1) = 0$, $\operatorname{Re} r_1 \ge 1$.

Now let us suppose that the Statement is valid for the variables r_j with the indices $j = 0, 1, \ldots, m$, where $1 \le m \le n-1$, i.e. let the function

$$f(r_0, r_1, \ldots, r_{m-1}, r_m, \ldots, r_n) \equiv f_m(r_m)$$

satisfies the conditions:

$$f_m(r_m) = f(r_0, r_1, \ldots, r_{m-1}, r_m, \ldots, r_n) = 0 \ \text{ if } \ r_m = 1, 2, \ldots,$$

$$\lim_{|r_m|\to\infty} |f_m(r_m)| \exp(-k|r_m|) \le \mathit{const}, \ \ 0 < k < \pi \ \text{ if } \ \operatorname{Re} r_m \ge 1$$

$$\big(\operatorname{Re} r_j \ge 1, \ \ j = 0, 1, \ldots, m-1, \ \ r_{m+k} \in \mathbb{N}, \ \ k = 1, \ldots, n-m\big).$$

Therefore, according to the Carlson Theorem one gets

$$f_m(r_m) = f(r_0, r_1, \ldots, r_{m-1}, r_m, \ldots, r_n) = 0 \ \text{ if } \ r_m \in \mathbb{C} \tag{14}$$

$$\big(\operatorname{Re} r_j \ge 1, \ \ j = 0, 1, \ldots, m-1, \ \ r_{m+k} \in \mathbb{N}, \ \ k = 1, \ldots, n-m\big).$$

Thus we have shown that

$$f_{m+1}(r_{m+1}) = f(r_0, r_1, \ldots, r_m, r_{m+1}, \ldots, r_n) = 0$$

if $\operatorname{Re} r_j \geq 1$, $j = 0, 1, \ldots, m$, $r_{m+k} \in \mathbb{N}$, $k = 1, \ldots, n-m$. Besides, due to the estimates (3) and (13)

$$\lim_{|r_{m+1}|\to\infty} \big|f_{m+1}(r_{m+1})\big| \exp(-k|r_{m+1}|) \leq$$
$$\leq \lim_{|r_{m+1}|\to\infty} \big| \exp(-k|r_{m+1}|)\left\{ M + \frac{\Gamma(r_{m+1})}{\Gamma(r_{m+1}+\rho_{m+1})} \prod_{\substack{j=0 \\ j\neq m+1}}^{n} \Gamma(r_j) \right\} = 0$$

if $\operatorname{Re} r_{m+1} \geq 1$, where k and ρ_{m+1} are fixed numbers such that

$$0 < k < \pi, \quad \rho_{m+1} = r_0 + r_1 + \cdots + r_m + r_{m+2} + \cdots + r_n$$
$$\big(\operatorname{Re} r_j \geq 1, \;\; j = 0, 1, \ldots, m, \;\; r_{m+k} \in \mathbb{N}, \;\; k = 2, \ldots, n-m\big).$$

So, the proposition of the Statement is fulfilled according to Full Mathematical Induction Principle. □

The Statement enables one to prove the following

Theorem 1. *For any number $n \in \mathbb{N}$ the function $B_n(\mathbf{r})$ satisfies the formula* (2) *– n-dimensional analogue of the Euler formula – if* $\operatorname{Re} r_j > 0$, $j = 0, 1, \ldots, n$.

Proof. According to the Statement the formula (2) is fulfilled if $\operatorname{Re} r_j \geq 1$, $j = 0, 1, \ldots, n$. Therefore, due to analyticity of the function (12) if $\operatorname{Re} r_j > 0$, $j = 0, 1, \ldots, n$, (2) is fulfilled on the open semi-plane $\operatorname{Re} r_j > 0$ of each variable $r_j \in \mathbb{C}$, $j = 0, 1, \ldots, n$. □

We have shown in [2] that the limits of the function $B_2(\mathbf{r}) = B_2(r_0, r_1, r_2)$ when $x_1 \to x_0 = 0$ ($x_1/x_2 = z \to 0$) and $x_2 \to x_1$ ($x_1/x_2 = z \to 1$) (see the definition (1)) exist and satisfy the relation

$$\frac{\det[x_j^{i-1} b_{ij}]_{i,j=1,2}}{\det[x_j^{i-1}]_{i,j=1,2}}\bigg|_{x_1\to x_0=0} = \frac{\det[x_j^{i-1} b_{ij}]_{i,j=1,2}}{\det[x_j^{i-1}]_{i,j=1,2}}\bigg|_{x_2\to x_1} =$$
$$= B_1(r_0, r_1) B_1(r_0 + r_1, r_2).$$

Therefore for $n = 2$ the formula (2) remains valid even if the only restrictions on the variables $x_0, x_1, \ldots, x_n$ are $x_j \geq x_0 = 0$, $j = 1, \ldots, n$ (instead of the restrictions $0 = x_0 < x_1 < \cdots < x_n$ which we have in (1)). It is easy to show that the same is valid for any $n \geq 3$. Namely, one has the following

Theorem 2. *For any $n, l, k \in \mathbb{N}$, $l \leq k \leq n - l$ and $x_j \geq x_0 = 0$, $\operatorname{Re} r_j > 0$, $j = 1, \cdots, n$, the function $B_n(\mathbf{r})$ satisfies the formula*

$$B_n(r_0, r_1, \ldots, r_n) =$$
$$= B_{n-k}(r_0, r_1, \ldots, r_{l-1}, r_l + \cdots + r_{l+k}, r_{l+k+1}, \ldots, r_n) B_k(r_l, \ldots, r_{l+k}) =$$
$$= \frac{\prod\limits_{j=0}^{n} \Gamma(r_j)}{\Gamma\big(\sum\limits_{j=0}^{n} r_j\big)} . \tag{15}$$

Proof. The theorem is trivial for $n = 0$ and $n = 1$; for $n = 2$, in fact, the formula (15) is proved in [2]. In the case $n \geq 3$ one has to consider separately the cases $x_1 \to x_0 = 0$ and $x_l \to x_{l-1}$, $l \geq 2$.

In the case $x_1 \to x_0 = 0$ from the definition (1) one gets:

$$x_1^{i-1} b_{i1} = x_1^{i-1} \int_0^1 u^{r_0+i-2}(1-u)^{r_1-1} \prod_{k=2}^{n} \Big(\frac{x_1 u - x_k}{x_1 - x_k}\Big)^{r_k-1} du \underset{x_1\to 0}{\longrightarrow}$$

$$\underset{x_1\to 0}{\longrightarrow} \begin{cases} B_1(r_0, r_1), & i = 1, \\ 0 & i \geq 2, \end{cases}$$

$$x_j^{i-1} b_{ij} =$$

$$= x_j^{i-1} \int_{x_{j-1}/x_j}^{1} u^{r_0+i-2}(1-u)^{r_j-1} \prod_{\substack{k=2 \\ k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1} \Big(\frac{x_j u - x_1}{x_j - x_1}\Big)^{r_1-1} du \underset{x_1\to 0}{\longrightarrow}$$

$$\underset{x_1\to 0}{\longrightarrow} x_j^{i-1} \int_{x_{j-1}/x_j}^{1} u^{r_0+r_1+i-3}(1-u)^{r_j-1} \prod_{\substack{k=2 \\ k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1} du, \quad j \geq 2,$$

$$i = 1, \dots, n.$$

Hence, according to (1) and (2),

$$B_n(\mathbf{r})\big|_{x_1\to 0} = B_1(r_0, r_1) \prod_{2\leq k<j\leq n} (x_j - x_k)^{-1} \prod_{2\leq j\leq n} x_j^{-1} \times$$

$$\times \det \left[x_j^{i-1} \int_{x_{j-1}/x_j}^{1} u^{r_0+r_1+i-3}(1-u)^{r_j-1} \prod_{\substack{k=2 \\ k\neq j}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1} du \right]_{i,j=2}^{n} =$$

$$(0 = x_1 < \cdots < x_n)$$

$$= B_1(r_0, r_1) B_{n-1}(r_0 + r_1, r_2, \dots, r_n) =$$

$$= \frac{\Gamma(r_0)\Gamma(r_1)\Gamma(r_0+r_1)\Gamma(r_2)\cdots\Gamma(r_n)}{\Gamma(r_0+r_1)\Gamma(r_0+r_1+r_2+\cdots+r_n)} = \frac{\Gamma(r_0)\Gamma(r_1)\cdots\Gamma(r_n)}{\Gamma(r_0+r_1+\cdots+r_n)}. \quad (16)$$

Similarly, in the case when $x_l \to x_{l-1}$, $l \geq 2$, one obtains:

$$x_{l-1}^{i-1} b_{i\,l-1} = x_{l-1}^{i-1} \Big(1 - \frac{x_{l-1}}{x_l}\Big)^{1-r_l} \times$$

$$\times \int_{x_{l-2}/x_{l-1}}^{1} u^{i-1}(1-u)^{r_{l-1}-1} \Big(1 - \frac{x_{l-1}}{x_l} u\Big)^{r_l-1} \prod_{\substack{k=0 \\ k\neq l-1,l}}^{n} \Big(\frac{x_{l-1}u - x_k}{x_{l-1} - x_k}\Big)^{r_k-1} du,$$

$$x_l^{i-1}b_{il}=x_l^{i-1}\int\limits_{x_{l-1}/x_l}^{1}u^{i-1}\Big(\frac{x_lu-x_{l-1}}{x_l-x_{l-1}}\Big)^{r_{l-1}-1}(1-u)^{r_l-1}\prod_{\substack{k=0\\k\neq l-1,l}}^{n}\Big(\frac{x_lu-x_k}{x_l-x_k}\Big)^{r_k-1}du,$$

$$x_j^{i-1}b_{ij}=x_j^{i-1}\int\limits_{x_{j-1}/x_j}^{1}u^{i-1}(1-u)^{r_j-1}\prod_{\substack{k=0\\k\neq j}}^{n}\Big(\frac{x_ju-x_k}{x_j-x_k}\Big)^{r_k-1}du,$$

$$j=1,\dots,n,\quad j\neq l-1,l$$

$$(i=1,\dots,n).$$

The substitution

$$\frac{x_lu-x_{l-1}}{x_l-x_{l-1}}=1-\widetilde{u},\quad u=1-\Big(1-\frac{x_{l-1}}{x_l}\Big)\widetilde{u},$$

$$x_lu-x_k=x_l-x_k-\widetilde{u}(x_l-x_{l-1})$$

gives to the second of these integrals the form

$$x_l^{i-1}b_{il}=x_l^{i-1}\Big(1-\frac{x_{l-1}}{x_l}\Big)^{r_l}\times$$

$$\times\int\limits_0^1\Big(1-u\,\frac{x_l-x_{l-1}}{x_l}\Big)^{i-1}(1-u)^{r_{l-1}-1}u^{r_l-1}\prod_{\substack{k=0\\k\neq l-1,l}}^{n}\Big(1-u\,\frac{x_l-x_{l-1}}{x_l-x_k}\Big)^{r_k-1}du.$$

Inserting these results in (1), after obvious simplifications we find:

$$B_n(\mathbf{r})\big|_{x_l\to x_{l-1}}=$$

$$=\frac{B_1(r_{l-1},r_l)\det B^{(l)}}{x_{l-1}\det[x_j^{i-1}]_{\substack{i=\overline{1,n-2}\\j=\overline{1,n}\\j\neq l-1,l}}\prod\limits_{1\le k\le l-2}(x_{l-1}-x_k)^2\prod\limits_{l+1\le m\le n}(x_m-x_{l-1})^2}\,,\quad(17)$$

where $B^{(l)}=[B_{ik}^{(l)}]$ is the matrix whose i-th row, $i=1,\dots,n$, has the form

$$\Big[x_1^{i-1}\widetilde{b}_{i1},\dots,x_{l-2}^{i-1}\widetilde{b}_{i\,l-2},x_{l-1}^{i-1}\widetilde{b}_{i\,l-1},x_{l+1}^{i-1}\widetilde{b}_{i\,l+1},\dots,x_n^{i-1}\widetilde{b}_{in}\Big]\qquad(18)$$

and

$$\widetilde{b}_{i\,l-1}=\int\limits_{x_{l-2}/x_{l-1}}^{1}u^{i-1}(1-u)^{r_{l-1}+r_l-2}\prod_{\substack{k=0\\k\neq l-1,l}}^{n}\Big(\frac{x_{l-1}u-x_k}{x_{l-1}-x_k}\Big)^{r_k-1}du,$$

$$\widetilde{b}_{ij}=\int\limits_{x_{j-1}/x_j}^{1}u^{i-1}(1-u)^{r_j-1}\Big(\frac{x_ju-x_{l-1}}{x_j-x_{l-1}}\Big)^{r_{l-1}+r_l-2}\prod_{\substack{k=0\\k\neq j,l-1,l}}^{n}\Big(\frac{x_ju-x_k}{x_j-x_k}\Big)^{r_k-1}du,$$

$$i,j=1,\dots,n,\quad j\neq l-1,l.$$

The last step of our transformations is to multiply the row with number $i-1$ in the determinant of the matrix (18) on x_{l-1} and to extract it from the row with number i. Then in the l-th column of the determinant we obtain the Kroneker symbol δ_{1i},

$$\delta_{1i} = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases}$$

and in the other ones we get:

$$x_{l-1}^{i-1}\widetilde{b}_{i\,l-1} \longrightarrow x_{l-1}^{i-1}\widetilde{b}'_{i\,l-1} =$$

$$= -x_{l-1}^{i-1} \int\limits_{x_{l-2}/x_{l-1}}^{1} u^{i-2}(1-u)^{r_{l-1}+r_l-1} \prod_{\substack{k=0\\k\neq l-1,l}}^{n} \Big(\frac{x_{l-1}u - x_k}{x_{l-1} - x_k}\Big)^{r_k-1} du,$$

$$x_j^{i-1}\widetilde{b}_{ij} \longrightarrow x_j^{i-1}\widetilde{b}'_{ij} = (x_j - x_{l-1})x_j^{i-2}\times \tag{19}$$

$$\times \int\limits_{x_{j-1}/x_j}^{1} u^{i-2}(1-u)^{r_j-1}\Big(\frac{x_j u - x_{l-1}}{x_j - x_{l-1}}\Big)^{r_{l-1}+r_l-1} \prod_{\substack{k=0\\k\neq j,l-1,l}}^{n} \Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1} du,$$

$$i, j = 1, \ldots, n, \quad j \neq l-1, l.$$

Expanding the determinant obtained with respect to the l-th column elements and inserting the result in (17), one gets

$$B_n(\mathbf{r})\big|_{x_l \to x_{l-1}} =$$

$$= \frac{B_1(r_{l-1}, r_l)(-1)^{l-1}}{\det[x_j^{i-1}]_{\substack{i=\overline{1,n-2}\\ j=\overline{1,n}\\ j\neq l-1,l}} \prod\limits_{1\le k\le l-2}(x_{l-1}-x_k) \prod\limits_{l+1\le m\le n}(x_m - x_{l-1})}(-1)^{l+1}\times$$

$$\times \det\Big[x_1^{i-1}\widetilde{b}'_{i1}, \ldots, x_{l-2}^{i-1}\widetilde{b}'_{i\,l-2}, x_{l-1}^{i-1}\widetilde{b}'_{i\,l-1}, x_{l+1}^{i-1}\widetilde{b}'_{i\,l+1}, \ldots, x_n^{i-1}\widetilde{b}'_{in}\Big]_{i=\overline{1,n-1}} =$$

$$= B_1(r_{l-1}, r_l)\, \frac{\det\big[x_1^{i-1}\widetilde{b}'_{i1}, \ldots, x_{l-1}^{i-1}\widetilde{b}'_{i\,l-1}, x_{l+1}^{i-1}\widetilde{b}'_{i\,l+1}, \ldots, x_n^{i-1}\widetilde{b}'_{in}\big]_{i=\overline{1,n-1}}}{\det\big[x_1^{i-1}, \ldots, x_{l-1}^{i-1}, x_{l+1}^{i-1}, \ldots, x_n^{i-1}\big]_{i=\overline{1,n-1}}},$$

where the entries $\widetilde{b}_{ij}$, $i, j = 1, \ldots, n$, are defined in (19). So, we obtain the formula (16) in the case under consideration, too. Now the statement of Theorem 2 follows from (16) according to Full Mathematical Induction Principle. □

Theorem 3. *The integrals' singularities of the formula* (1) *determinant's entries, i.e. the singularities of Lauricella's hypergeometric functions on the complex plane of each variable* $x_j \in \mathbb{C}$, $j = 0, \ldots, n$, *cancel each other in the formula* (1).

Proof. According to Theorem 2, the function (1) does not depend on the variables $x_j \geq 0$ if $\operatorname{Re} r_j > 0$, $j = 0, \ldots, n$. Hence, according to the analytic function uniqueness theorem, the function (1) does not depend on the variables $x_j \in \mathbb{C}$ if $\operatorname{Re} r_j > 0$, $j = 0, \ldots, n$, while the integrals in the formula

(1) – Lauricella's hypergeometric functions – have singularities with respect to variables $x_j \in \mathbb{C}$. □

As far as the (bosonic) string theory [13] reproduces the Euler beta function (Veneziano amplitude) and its multidimensional analogue, it seems to be helpful to take an advantage of the proposed generalization of the Euler beta function when one attempts to construct physical (string) models [14] including a number of fermionic fields, as far as the expression

$$\det\left[x_j^{i-1}\int\limits_{x_{j-1}/x_j}^{1} u^{i-1}(1-u)^{r_j-1}\prod_{\substack{k=0\\k\neq j}}\Big(\frac{x_j u - x_k}{x_j - x_k}\Big)^{r_k-1}du\right]_{i,j=\overline{1,n}} = \\ = \det[x_j^{i-1}]_{i,j=\overline{1,n}}\,\frac{\prod\limits_{j=0}^{n}\Gamma(r_j)}{\Gamma\Big(\sum\limits_{j=0}^{n} r_j\Big)}$$

is skew-symmetryc with respect to the variables $x_j \in \mathbb{C}$.

Acknowledgement

The authors thank gratefully to prof. V. Paatashvili for helpful discussions.

References

1. I. Lomidze, On some generalizations of the vandermonde matrix and their relations with the Euler beta-function. *Georgian Math. J.* **1** (1994), No. 4, 405–417.
2. I. Lomidze, A geralization of the Euler beta function and applications. *Georgian Math. J.* **18** (2011), No. 2, 271–298.
3. G. E. Andrews, R. Askey, and R. Roy, Special functions. Encyclopedia of Mathematics and its Applications, 71. *Cambridge University Press, Cambridge*, 1999.
4. A. Selberg, Remarks on a multiple integral. (Norwegian) *Norsk Mat. Tidsskr.* **26** (1944), 71–78.
5. I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, Generalized Euler integrals and A-hypergeometric functions. *Adv. Math.* **84** (1990), No. 2, 255–271.
6. I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, Hyperdeterminants. *Adv. Math.* **96** (1992), No. 2, 226–263.
7. A. L. Kholodenko, New string amplitudes from old Fermat (hyper)surfaces. *Internat. J. Modern Phys. A* **19** (2004), No. 11, 1655–1703.
8. E. C. Titchmarsh, The theory of functions. 2. ed. *Oxford University Press, Oxford*, 1939.
9. P. Appel and M. J. Kampé de Fériet, Functions hypergeómétriques et hypersphériques. Polynômes d'Hermite. *Gautier-Villars, Paris*, 1926.
10. H. Minc, Permanents. With a foreword by Marvin Marcus. Encyclopedia of Mathematics and its Applications, Vol. 6. Encyclopedia of Mathematics and its Applications, 9999. *Addison-Wesley Publishing Co., Reading, Mass.*, 1978.
11. G. M. Fikhtengolz, Course of differential and integral calculus. II. (Russian) the 7-th edition. *Nauka, Moskow*, 1969.

12. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, Higher transcendental functions. Vols. I, II. Based, in part, on notes left by Harry Bateman. *McGraw-Hill Book Company, Inc., New York–Toronto–London*, 1953.
13. M. B. Green, J. H. Schwarz, and E. Witten, Superstring theory. Vol. 2. Loop amplitudes, anomalies and phenomenology. *Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge*, 1987.
14. N. V. Makhaldiani, Statistical description of extended particle systems. On some modifications of nonlinear n-field model and their exact particle-like solutions in d-dimensions. *Communications of the JINR* E2-97-408, *Dubna*, 1997.

(Received 17.11.2009)

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