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NECESSARY CONDITIONS FOR THE EXISTENCE AND SUFFICIENT CONDITIONS FOR THE NONEXISTENCE OF SOLUTIONS TO A CERTAIN FRACTIONAL TELEGRAPH EQUATION

Abstract. We consider the Cauchy problem for the semi-linear fractional telegraph equation

$$\mathbf{D}_{0|t}^{2\gamma} u + \mathbf{D}_{0|t}^{\gamma} u + (-\Delta)^{\frac{\beta}{2}} u = h(x,t)|u|^p$$

with the given initial data, where $p > 1$, $\frac{1}{2} \le \gamma < 1$ and $0 < \beta < 2$. The Nonexistence results and the necessary conditions for global existence are established.

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რეზიუმე. განხილულია კოშის ამოცანა ნახევრადწრფივი ფრაქციონალური ტელეგრაფის განტოლებისათვის

$$\mathbf{D}_{0|t}^{2\gamma} u + \mathbf{D}_{0|t}^{\gamma} u + (-\Delta)^{\frac{\beta}{2}} u = h(x,t)|u|^p$$

მოცემული საწყისი პირობებით, სადაც $p > 1$, $\frac{1}{2} \le \gamma < 1$ და $0 < \beta < 2$. დადგენილია არარსებობის შედეგები და აუცილებელი პირობები გლობალური არსებობისათვის.

1. Introduction

The telegraph equation has recently been considered by many authors, see for instance [2, 3, 8, 12, 15] and references therein. Cascaval *et al.* [2] discussed the fractional telegraph equations

$$D^{2\beta}u + D^{\beta}u - \Delta u = 0$$

dealing with well-posedness and presenting a study involving asymptotic by using the Riemann–Liouville approach, it has been shown that as t tends to infinity, solutions of the telegraph equations can be approximated by solving the parabolic part. Beghin and Orsingher [15] discussed the time fractional telegraph equations and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations with well-posedness. Chen *et al.* [3] also discussed and derived the solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions.

To focus our motivation, we shall mention below only some results related to Todorova and Yordanov [20] for the Cauchy problem

$$u_{tt} - \Delta u + u_t = |u|^p, \quad u(0) = u_0, \quad u_t(0) = u_1. \tag{1}$$

It has been shown that the damped wave equation has the diffuse structure as $t \to \infty$ (see e.g. [20, 22]). This suggests that problem (1) should have $p_c(n) := 1 + \frac{2}{n}$ as critical exponent which is called the Fujita exponent [5, 7] named after Fujita, in general space dimension. Indeed, Todorova and Yordanov have showed that the critical exponent is exactly $p_c(n)$, that is, if $p > p_c(n)$, then all small initial data solutions of (1) are global, while if $1 < p < p_c(n)$, then all solutions of (1) with initial data having positive average value blow-up in finite time regardless of the smallness of the initial data.

In this paper, we consider the following nonlinear fractional telegraph equation:

$$\begin{cases} \mathbf{D}^{2\gamma}_{0|t}u + \mathbf{D}^{\gamma}_{0|t}u + (-\Delta)^{\frac{\beta}{2}}u = h(x,t)|u|^p & \text{in } \; Q = \mathbb{R}^n \times \mathbb{R}_+ \\ u(0,x) = u_0(x) \;\text{ and }\; u_t(0,x) = u_1(x), & \qquad x \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $\mathbf{D}^{\gamma}_{0|t}$ (resp. $\mathbf{D}^{2\gamma}_{0|t}u$) denotes the so-called fractional time-derivative of power γ (resp. 2γ), $\gamma \in [1/2, 1]$ in the Caputo sense (see [11], [18]), $(-\Delta)^{\frac{\beta}{2}}$ ($\beta \in [0, 2]$) is the $(\beta/2)$-fractional power of the Laplacian $(-\Delta)$ defined by

$$(-\Delta)^{\frac{\beta}{2}}v(x,t) = \mathcal{F}^{-1}(|\xi|^{\beta}\mathcal{F}(v)(\xi))(x,t),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ is its inverse, $h(x,t)$ is the positive function satisfying certain growth condition. We will generalize the results obtained in [20] to the problem (2). The nonexistence results as well as the necessary conditions for local and global existence are obtained.

The difficulties we encounter here arise mainly from the nonlocal nature of the fractional derivative operators; to overcome these difficulties, we present a brief and versatile proof of the equation (2) which is based on the method used by Mitidieri and Pohozaev [14], Pohozaev and Tesei [17], Hakem [6], Berbiche [1], Fino and Karch [4] and Zhang [22]. This method consists in a judicious choice of the test function in the weak formulation of the sought for solution of (2).

This paper is organized as follows: in Section 2, we present some definitions, properties concerning fractional derivative and prove results concerning positivity of solutions; Section 3 contains the proof of the blow-up result; in Section 4, we establish some necessary conditions for local and global existence.

2. Preliminaries

In this section we present some definitions of a fractional derivative and a result concerning the positivity of a solution.

The left-hand fractional derivative and the right-hand fractional derivative in the Riemann–Liouville sense for $\Psi \in L^1(0,T)$, $0<\alpha<1$, are defined as follows:

$$D^\alpha_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dt}\int_0^t \frac{\Psi(\sigma)}{(t-\sigma)^\alpha}\,d\sigma,$$

where the symbol Γ stands for the usual Euler gamma function, and

$$D^\alpha_{t|T}\Psi(t) = -\frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dt}\int_t^T \frac{\Psi(\sigma)}{(\sigma-t)^\alpha}\,d\sigma,$$

respectively.

The Caputo derivative

$$\mathbf{D}^\alpha_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^\alpha}\,d\sigma$$

requires $\Psi' \in L^1(0,T)$. Clearly, we have

$$D^\alpha_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\Psi(0)}{t^\alpha} + \int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^\alpha}\,d\sigma\right]$$

and

$$D^\alpha_{t|T}\Psi(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\Psi(T)}{(T-t)^\alpha} - \int_t^T \frac{\Psi'(\sigma)}{(\sigma-t)^\alpha}\,d\sigma\right]. \tag{3}$$

Therefore, the Caputo derivative is related to the Riemann–Liouville derivative by

$$\mathbf{D}^\alpha_{0|t}\Psi(t) = D^\alpha_{0|t}[\Psi(t)-\Psi(0)] \tag{4}$$

and, in general,

$$\mathbf{D}^{\alpha}_{0|t}\Psi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\Psi^{(n)}(\sigma)}{(t-\sigma)^{2\gamma-n}}\, d\sigma, \quad n = [\alpha]+1, \quad \alpha > 0,$$

we have the formula of integration by parts (see [18, p. 26]),

$$\int_0^T f(t) D^{\alpha}_{0|t} g(t)\, dt = \int_0^T g(t) D^{\alpha}_{t|T} f(t)\, dt, \quad 0 < \alpha < 1.$$

We show the following result:

Proposition 1 (Positivity of solutions)**.** *If* $u_0 \geq 0$, $u_1 = 0$, $f \geq 0$ *and* u *is a solution of the nonhomogeneous problem*

$$\begin{cases} \mathbf{D}^{2\gamma}_{0|t} u + \mathbf{D}^{\gamma}_{0|t} u + (-\Delta)^{\frac{\beta}{2}} u = f(x,t), & (x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \\ u(0,x) = u_0(x) \ \ \textit{and} \ \ u_t(0,x) = 0, & x \in \mathbb{R}^n, \end{cases} \tag{5}$$

then u *is nonnegative.*

Proof. Applying the temporal Laplace and spatial Fourier transforms to (5), we get

$$s^{2\gamma}\widetilde{u}(x,s) - s^{2\gamma-1}u_0(x) + s^{\gamma}\widetilde{u}(x,s) + (-\Delta)^{\beta/2}\widetilde{u}(x,s) = \widetilde{f}(x,s),$$

$$s^{2\gamma}\widehat{\widetilde{u}}(k,s) - s^{2\gamma-1}\hat{u}_0(k) + s^{\gamma}\widehat{\widetilde{u}}(k,s) + |k|^{\beta}\widehat{\widetilde{u}}(k,s) = \widehat{\widetilde{f}}(k,s).$$

Then we derive

$$\widehat{\widetilde{u}}(k,s) = \frac{s^{2\gamma-1} + s^{\gamma-1}}{s^{2\gamma} + s^{\gamma} + |k|^{\beta}}\hat{u}_0(k) + \frac{1}{s^{2\gamma} + s^{\gamma} + |k|^{\beta}}\widehat{\widetilde{f}}(k,s) :=$$

$$:= \widehat{\widetilde{G}}_1(k,s)\hat{u}_0(k) + \widehat{\widetilde{G}}_2(k,s)\widehat{\widetilde{f}}(k,s), \tag{6}$$

where

$$\widehat{\widetilde{G}}_2(k,s) := \frac{1}{s^{2\gamma} + s^{\gamma} + |k|^{\beta}}\,, \tag{7}$$

$$\begin{gathered} \widehat{\widetilde{G}}_1(k,s) := \frac{s^{2\gamma-1} + s^{\gamma-1}}{s^{2\gamma} + s^{\gamma} + |k|^{\beta}} := \widehat{\widetilde{G}}_{1,1}(k,s) + \widehat{\widetilde{G}}_{1,2}, \\ \widehat{\widetilde{G}}_{1,1}(k,s) := \frac{s^{2\gamma-1}}{s^{2\gamma} + s^{\gamma} + |k|^{\beta}}\,, \quad \widehat{\widetilde{G}}_{1,2} := \frac{s^{\gamma-1}}{s^{2\gamma} + s^{\gamma} + |k|^{\beta}}\,. \end{gathered} \tag{8}$$

We invert the Fourier transform in (6) and obtain

$$u(x,t) = \int_{\mathbb{R}^n} G_1(x-y)u_0(y)\, dy + \int_{\mathbb{R}^n}\int_0^t G_2(x-y,\tau)f(x,\tau)\, d\tau\, dy,$$

where $G_1(x,t)$, $G_2(x,t)$ is the corresponding Green's function or the fundamental solution obtained when $u_0(x) = \delta(x)$, $f = 0$ and $u_0(x) = 0$, $f(x,t) = \delta(x)\delta(t)$, respectively, which is characterized by (7), (8).

To express the Green's function, we recall two Laplace transform pairs and one Fourier transform pair,

$$F_1^{(\gamma)}(ct) := t^{-\gamma} M_\gamma(ct^{-\gamma}) \stackrel{\mathcal{L}}{\longleftrightarrow} s^{\gamma-1} e^{-cs^\gamma},$$
$$F_2^{(\gamma)}(ct) := c w_\gamma(ct) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-(s/c)^\gamma},$$

where M_γ denotes the so-called M function (of the Wright type) of order γ, which is defined by

$$M_\mu(z) = \sum_{i=0}^{\infty} \frac{(-z)^i}{i!\Gamma(-\mu i + (1-\mu))}, \quad 0 < \mu < 1.$$

Mainardi, see, for example, [12] has shown that $M_\mu(z)$ is positive for $z > 0$, the other general properties can be found in some references (see e.g. [12, 13, 16]).

w_μ $(0 < \mu < 1)$ denotes the one-sided stable (or Lévy) probability density which can be explicitly expressed by the Fox function [19]

$$w_\mu(t) = \mu^{-1} t^{-2} H_{11}^{10} \left(t^{-1} \middle| \begin{array}{c} (-1, 1) \\ (-1/\mu, 1/\mu) \end{array} \right).$$

It is well known that

$$e^{-\lambda |x|^\beta} \stackrel{\mathcal{F}}{\longrightarrow} p(x, \lambda), \quad 0 < \beta \leq 2,$$

where $p(x, \lambda)$ is the probability density function.

From ([21, pp. 259–263]) we have

$$p(x, \lambda) := \int_0^{+\infty} f_{\lambda, \frac{\beta}{2}}(\tau) T(x, \tau)\, d\tau \quad \text{for} \quad 0 < \beta \leq 2,$$

and

$$p(x, \lambda) = T(x, \lambda) \quad \text{if} \quad \beta = 2,$$

where

$$f_{\lambda, \frac{\beta}{2}}(s) = \int_{\tau - i\infty}^{\tau + i\infty} e^{zs - \lambda z^{\frac{\beta}{2}}}\, dz \geq 0, \quad T(x, \lambda) = \Big(\frac{1}{4\pi\lambda} \Big)^{\frac{n}{2}} e^{-\frac{|x|^2}{4\lambda}}, \quad \tau > 0, \quad \lambda > 0.$$

Then the Fourier–Laplace transform of Green's function G_1 can be rewritten in the integral form

$$\widehat{\widehat{G}}_1(k, s) = (s^{2\gamma-1} + s^{\gamma-1}) \int_0^{+\infty} e^{-v(s^{2\gamma} + s^\gamma + |k|^\beta)}\, dv =$$
$$= \int_0^{+\infty} (s^{2\gamma-1} e^{-vs^{2\gamma}}) e^{-vs^\gamma} e^{-v|k|^\beta}\, dv + \int_0^{+\infty} (s^{\gamma-1} e^{-vs^\gamma}) e^{-vs^{2\gamma}} e^{-v|k|^\beta}\, dv =$$

$$= \int_0^{+\infty} \mathcal{L}\big\{F_1^{(2\gamma)}(vt)\big\}\mathcal{L}\big\{F_2^{(\gamma)}(v^{-1/\gamma}t)\big\}\mathcal{F}\{p(x,v)\}\,dv +$$

$$+ \int_0^{+\infty} \mathcal{L}\big\{F_1^{(\gamma)}(vt)\big\}\mathcal{L}\big\{F_2^{(2\gamma)}(v^{-1/2\gamma}t)\big\}\mathcal{F}\{p(x,v)\}\,dv =$$

$$= \int_0^{+\infty} \mathcal{L}\Big[F_1^{(2\gamma)}(vt) * F_2^{(\gamma)}(v^{-1/\gamma}t)\Big]\mathcal{F}\{p(x,v)\}\,dv +$$

$$+ \int_0^{+\infty} \mathcal{L}\Big[F_1^{(\gamma)}(vt) * F_2^{(2\gamma)}(v^{-1/2\gamma}t)\Big]\mathcal{F}\{p(x,v)\}\,dv.$$

Going back to the space-time domain, we obtain the relation

$$G_1(x,t) = \int_0^{+\infty} F_1^{(2\gamma)}(vt) * F_2^{(\gamma)}(v^{-1/\gamma}t)p(x,v)\,dv +$$

$$+ \int_0^{+\infty} F_1^{(\gamma)}(vt) * F_2^{(2\gamma)}(v^{-1/2\gamma}t)p(x,v)\,dv.$$

By the same technique, we obtain the expression of $G_2(x,t)$

$$\hat{\hat{G}}_2(k,s) = \int_0^{+\infty} e^{-v(s^{2\gamma}+s^{\gamma}+|k|^{\beta})}\,dv = \int_0^{+\infty} e^{-vs^{2\gamma}}e^{-vs^{\gamma}}e^{-v|k|^{\beta}}\,dv =$$

$$= \int_0^{+\infty} \mathcal{L}\left[F_2^{(2\gamma)}(v^{-1/2\gamma}t) * F_2^{(\gamma)}(v^{-1/\gamma}t)\right]\mathcal{F}\{p(x,v)\}\,dv.$$

Going back to the space-time domain, we obtain the relation

$$G_2(x,t) = \int_0^{+\infty} \left[F_2^{(2\gamma)}(v^{-1/2\gamma}t) * F_2^{(\gamma)}(v^{-1/\gamma}t)\right]\{p(x,v)\}\,dv.$$

Thus, by the nonnegativity property of functions $F_1^{(\gamma)}$, $F_2^{(\gamma)}$, $p(x,v)$, we deduce that the solution u is nonnegative. □

3. Blow-up of Solutions

This section is devoted to the blow-up of solutions of the problem (2), where we have assumed that the function h satisfies $h\left(Ry, T^{\beta/\gamma}\tau\right) = R^{\sigma}T^{\rho\beta/\gamma}h\left(y,\tau\right)$ for large R and T, where σ, ρ are some positive constants, under some restrictions on the initial data.

Definition 1. Let $u_0 \geq 0$, $u_0 \in L^1(\mathbb{R}^n)$, $u_1 = 0$. A function $u \in L^p_{loc}(Q_T)$ is a weak solution to (2) defined on $Q_T := \mathbb{R}^n \times [0,T]$, if

$$\int\limits_{Q_T} h\varphi|u|^p \, dx\, dt + \int\limits_{\mathbb{R}^n} u_0 D^{2\gamma-1}_{t|T}\varphi(0)\, dx + \int\limits_{Q_T} u_0 D^{\gamma}_{t|T}\varphi \, dx\, dt =$$
$$= \int\limits_{Q_T} u D^{2\gamma}_{t|T}\varphi \, dx\, dt + \int\limits_{Q_T} u(-\Delta)^{\frac{\beta}{2}}\varphi \, dx\, dt + \int\limits_{Q_T} u D^{\gamma}_{t|T}\varphi \, dx\, dt$$

for any test function $\varphi \in C^{2,1}_{x,t}(Q_T)$ such that

$$\varphi(x,T) = D^{2\gamma-1}_{t|T}\varphi(x,T) = 0.$$

If in the above definition $T = +\infty$, the solution is called global.

We now are in a position to announce our first result.

Theorem 1. *Let $n \geq 1$, $1 < p < \min\left(\rho+1, \frac{1}{1-\gamma}\right)$. Assume that $u_0 \in L^1(\mathbb{R}^n)$, $u_0(x) \geq 0$, and $u_1 = 0$. If*

$$p \leq p_c = 1 + \frac{\gamma(\sigma + \frac{\beta}{\gamma}\rho) + \gamma\beta}{(1-\gamma)\beta + n\gamma}\,,$$

then the problem (2) *admits no global weak positive solutions other than the trivial one.*

Proof. The proof proceeds by contradiction. Suppose that u is a nontrivial nonnegative solution to problem (2) which exists globally in time. For later use, let Φ be a smooth nonincreasing function such that

$$\Phi(z) = \begin{cases} 1 & \text{if } \ z \leq 1, \\ 0 & \text{if } \ z \geq 2, \end{cases}$$

and $0 \leq \Phi \leq 1$. Let

$$\varphi(x,t) := \Phi^l\Big(\frac{t^{2\gamma}}{R^{2\beta}}\Big)\Phi^l\Big(\frac{|x|}{R}\Big) = \varphi_1^l(t)\varphi_2^l(x),$$

where R is a fixed positive number and l is a positive number to be chosen later. Multiplying the equation (2) by $\varphi(x,t)$ and integrating the result on $Q_{TR^{\beta/\gamma}}$, we obtain

$$\int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi|u|^p \, dx\, dt + \int\limits_{\mathbb{R}^n} u_0 D^{2\gamma-1}_{t|TR^{\beta/\gamma}}\varphi(0)\, dx + \int\limits_{Q_{TR^{\beta/\gamma}}} u_0 D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi \, dx\, dt =$$
$$= \int\limits_{Q_{TR^{\beta/\gamma}}} u D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi \, dx\, dt + \int\limits_{Q_{TR^{\beta/\gamma}}} u(-\Delta)^{\frac{\beta}{2}}\varphi \, dx\, dt + \int\limits_{Q_{TR^{\beta/\gamma}}} u D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi \, dx\, dt. \quad (9)$$

Now we estimate the right-hand side of (9). We have

$$\int\limits_{Q_{TR^{\beta/\gamma}}} u(-\Delta)^{\frac{\beta}{2}}\varphi\, dx\, dt = \int\limits_{Q_{TR^{\beta/\gamma}}} (h\Phi^l)^{\frac{1}{p}} u (h\Phi^l)^{-\frac{1}{p}} (-\Delta)^{\frac{\beta}{2}}\Phi^l\, dx\, dt \leq$$

$$\leq l \int\limits_{Q_{TR^{\beta/\gamma}}} (h\Phi^l)^{\frac{1}{p}} u (h\Phi^l)^{-\frac{1}{p}} \Phi^{l-1}(-\Delta)^{\frac{\beta}{2}}\Phi\, dx\, dt,$$

where we have used the Ju's inequality $(-\Delta)^{\beta/2}\xi^l(x) \leq l\xi^{l-1}(x)(-\Delta)^{\beta/2}\xi(x)$ which is satisfied for every $\xi \in C_0^\infty(\mathbb{R}^n)$ (see [10]).

By the ε-Young's inequality, we can estimate

$$\int\limits_{Q_{TR^{\beta/\gamma}}} u(-\Delta)^{\frac{\beta}{2}}\varphi\, dx\, dt \leq \varepsilon l \int\limits_{Q_{TR^{\beta/\gamma}}} h\Phi u^p\, dx\, dt +$$

$$+ C(\varepsilon) \int\limits_{Q_{TR^{\beta/\gamma}}} h^{\frac{-q}{p}} \Phi^{(l-1-\frac{l}{p})q} |(-\Delta)^{\frac{\beta}{2}}\Phi|^q\, dx\, dt =$$

$$= \varepsilon l \int\limits_{Q_{TR^{\beta/\gamma}}} h\Phi u^p\, dx\, dt + C(\varepsilon) \int\limits_{Q_{TR^{\beta/\gamma}}} h^{\frac{-q}{p}} \varphi^{(1-\frac{q}{l})} \left|(-\Delta)^{\frac{\beta}{2}}\varphi^{\frac{1}{l}}\right|^q dx\, dt < \infty, \quad (10)$$

so, we choose $l > q$ to ensure the convergence of the integral in (10).

$$\int\limits_{Q_{TR^{\beta/\gamma}}} u D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi\, dx\, dt \leq$$

$$\leq \varepsilon \int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi u^p\, dx\, dt + C(\varepsilon) \int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q} |D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi|^q\, dx\, dt, \quad (11)$$

and

$$\int\limits_{Q_{TR^{\beta/\gamma}}} u D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi\, dx\, dt \leq$$

$$\leq \varepsilon \int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi u^p\, dx\, dt + C(\varepsilon) \int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q} |D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi|^q\, dx\, dt, \quad (12)$$

where q is the conjugate of p. Gathering up (10), (11) and (12), with ε small enough, we infer that

$$\int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi|u|^p \,dx\,dt + \int\limits_{Q_{TR^{\beta/\gamma}}} u_0 D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi \,dx\,dt \leq$$

$$\leq C \int\limits_{Q_{TR^{\beta/\gamma}}} h^{\frac{-q}{p}}\varphi^{(1-\frac{q}{l})}|(-\Delta)^{\frac{\beta}{2}}\varphi^{\frac{1}{l}}|^q \,dx\,dt +$$

$$+ C \int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q}\left(|D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi|^q + |D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi|^q\right) \,dx\,dt, \quad (13)$$

for some positive constant C independent of R and T. At this stage, let us perform the change of variables $\tau = t/R^{\frac{\beta}{\gamma}}$, $y = \frac{x}{R}$, and $\varphi(x,t) = \psi(y,\tau)$, clearly

$$\tau = t/R^{\frac{\beta}{\gamma}}, \quad x = Ry, \quad dxdt = R^{n+\frac{\beta}{\gamma}}dyd\tau.$$

We have the estimates

$$\int\limits_{Q_{TR^{\beta/\gamma}}} h^{\frac{-q}{p}}\varphi^{(1-\frac{q}{l})}\left|(-\Delta)^{\frac{\beta}{2}}\varphi^{\frac{1}{l}}\right|^q \,dx\,dt =$$

$$= R^{-\beta q+n+\beta/\gamma+(1-q)(\sigma+\frac{\beta}{\gamma}\rho)} \int\limits_{Q_T} h^{1-q}\psi^{(1-\frac{q}{l})}\left|(-\Delta)^{\frac{\beta}{2}}\psi^{\frac{1}{l}}\right|^q \,dy\,d\tau,$$

$$\int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q}\left|D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi\right|^q \,dx\,dt =$$

$$= R^{-\frac{\beta}{\gamma}(2\gamma)q+n+\frac{\beta}{\gamma}+(1-q)(\sigma+\frac{\beta}{\gamma}\rho)} \int\limits_{Q_T} (h\psi)^{1-q}\left|D^{2\gamma}_{\tau|T}\psi\right|^q \,dy\,d\tau,$$

and

$$\int\limits_{Q_{TR^{\beta/\gamma}}} (h\varphi)^{1-q}\left|D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi\right|^q \,dx\,dt =$$

$$= R^{-\beta q+n+\frac{\beta}{\gamma}+(1-q)\left(\sigma+\frac{\beta}{\gamma}\rho\right)} \int\limits_{Q_T} (h\psi)^{1-q}\left|D^{\gamma}_{\tau|T}\psi\right|^q \,dy\,d\tau.$$

It is clear from (3) that $D^{2\gamma-1}_{t|TR^{\beta/\gamma}}\varphi \geq 0$, $D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi \geq 0$. Then we obtain

$$\int\limits_{Q_{TR^{\beta/\gamma}}} h\varphi|u|^p \, dx\, dt \leq$$

$$\leq C(\varepsilon) R^{-\beta q+n+\beta/\gamma+(1-q)(\sigma+\frac{\beta}{\gamma}\rho)} \Bigg[\int\limits_{Q_T} h^{1-q}\psi^{1-\frac{q}{l}} \big|(-\Delta)^{\frac{\beta}{2}}\psi^{\frac{1}{l}}\big|^q \, dy\, d\tau +$$

$$+ \int\limits_{Q_T} (h\psi)^{1-q} \Big(\big|D^{\gamma}_{\tau|T}\psi\big|^q + \big|D^{2\gamma}_{\tau|T}\psi\big|^q \Big) \, dy\, d\tau \Bigg], \quad (14)$$

where C is positive constant independent of R. Now let $R \to +\infty$ in (14). We distinguish two cases. If $p < p_c$ (which is equivalent $-\beta q + n + \beta/\gamma + (1-q)(\sigma + \frac{\beta}{\gamma}\rho) < 0$), then we have

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^+} h|u|^p \, dx\, dt \leq 0.$$

This implies that $u \equiv 0$ a.e. on $\mathbb{R}^n \times \mathbb{R}^+$ since $h(x,t) > 0$ a.e. on $\mathbb{R}^n \times \mathbb{R}^+$. This is a contradiction.

In the case $p = p_c$ (i.e. critical case), from (14) we find that

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^+} h|u|^p \, dx\, dt \leq C. \quad (15)$$

Let us modify the test function φ by introducing a new fixed number S $(1 < S < R)$ such that

$$\varphi(x,t) := \Phi^l\Big(\frac{t^{2\gamma}}{(SR)^{2\beta}}\Big)\Phi^l\Big(\frac{|x|}{R}\Big),$$

we set $x = yR$, $t = (SR)^{\frac{\beta}{\gamma}}\tau$,

$$\Omega_{SR} = \Big\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ : \ |x| \leq 2R, t^{2\gamma} \leq 2(SR)^{2\beta}\Big\},$$
$$\Omega = \Big\{(y,\tau) \in \mathbb{R}^n \times \mathbb{R}^+ : \ |y| \leq 2, \tau^{2\gamma} \leq 2\Big\}.$$

Then we have

$$\int\limits_{\Omega_{SR}} h^{\frac{-q}{p}}\varphi^{(1-\frac{q}{l})}\big|(-\Delta)^{\frac{\beta}{2}}\varphi^{\frac{1}{l}}\big|^q \, dx\, dt =$$

$$= S^{\beta/\gamma+(1-q)\frac{\beta}{\gamma}\rho} \int\limits_{\Omega} h^{1-q}\psi^{1-\frac{q}{l}}\big|(-\Delta)^{\frac{\beta}{2}}\psi^{\frac{1}{l}}\big|^q \, dy\, d\tau,$$

$$\int\limits_{\Omega_{SR}} (h\varphi)^{1-q}\big|D^{\gamma}_{t|TR^{\beta/\gamma}}\varphi\big|^q\,dx\,dt =$$

$$= S^{-\beta q+\beta/\gamma+(1-q)\frac{\beta}{\gamma}\rho}\int\limits_{\Omega}(h\psi)^{1-q}\big|D^{\gamma}_{\tau|T}\psi\big|^q\,dy\,d\tau,$$

and

$$\int\limits_{\Omega_{SR}} (h\varphi)^{1-q}\big|D^{2\gamma}_{t|TR^{\beta/\gamma}}\varphi\big|^q\,dx\,dt =$$

$$= S^{-2\beta q+\beta/\gamma+(1-q)\frac{\beta}{\gamma}\rho}\int\limits_{\Omega}(h\psi)^{1-q}\big|D^{2\gamma}_{t|T}\psi\big|^q\,dy\,d\tau.$$

Combining the above estimates we find

$$\begin{aligned}(1-3\varepsilon)&\int\limits_{\Omega_{SR}} h\varphi u^{p_c}\,dx\,dt \le\\ &\le S^{\frac{\beta}{\gamma}+(1-q)\frac{\beta}{\gamma}\rho}\Bigg(\int\limits_{\Omega} h^{1-q}\psi^{1-\frac{q}{l}}|(-\Delta)^{\frac{\beta}{2}}\psi^{\frac{1}{l}}|^q\,dy\,d\tau\Bigg)+S^{-\beta q+\beta/\gamma+(1-q)\frac{\beta}{\gamma}\rho}\times\\ &\times\Bigg(\int\limits_{\Omega}(h\psi)^{1-q}|D^{\gamma}_{\tau|T}\psi|^q\,dy\,d\tau+\int\limits_{\Omega}(h\psi)^{1-q}|D^{2\gamma}_{\tau|T}\psi|^q\,dy\,d\tau\Bigg).\end{aligned}\tag{16}$$

Now, by taking $\varepsilon=\frac{1}{6}$ and using (15), we obtain via (16), after passing to the limit as $R\to\infty$,

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}_+} hu^p\,dx\,dt \le C\Big(S^{-\beta q+\beta/\gamma+(1-q)\frac{\beta}{\gamma}\rho}+S^{\frac{\beta}{\gamma}+(1-q)\,\frac{\beta}{\gamma}\,\rho}\Big),\tag{17}$$

we notice that the assumption $p<\min(\rho+1,\frac{1}{1-\gamma})$ yields $-\beta q+\beta/\gamma+(1-q)\,\frac{\beta}{\gamma}\,\rho<0$ and $\frac{\beta}{\gamma}+(1-q)\,\frac{\beta}{\gamma}\,\rho<0$, and the left-hand side of (17) is independent of S. Passing to the limit $S\to\infty$, we get immediately

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^+} h|u|^p\,dx\,dt\le 0.$$

Thus $\int\limits_{\mathbb{R}^n\times\mathbb{R}^+} h|u|^p\,dx\,dt=0$, which implies $u\equiv 0$ a.e. and completes the proof. □

Remark 1. When $\beta=2$, $\gamma=1$ and $h=1$, this agrees with Todorova–Yordanov [20].

4. The Necessary Conditions for the Local and Global Existence

In this section we assume that $\inf\limits_{t>0} h(x,t) > 0$, we see that the existence of solutions of the problem (2) depends on the behavior of initial data at infinity.

Theorem 2. *Let u be a local solution to* (2)*, where $T < +\infty$, and $1 < p < \frac{1}{1-\gamma}$. Assume that $u_0 \geq 0$ and $u_1 \geq 0$. Then the following two estimates*

$$\lim_{|x|\to+\infty} \inf \big(\inf_{t>0} h\big)^{q-1} u_0(x) \leq C\big(T^{\gamma(1-q)} + T^{\gamma-2\gamma q}\big),$$
$$\lim_{|x|\to+\infty} \inf \big(\inf_{t>0} h\big)^{q-1} u_1(x) \leq C'\big(T^{2\gamma-1-\gamma q} + T^{2\gamma(1-q)-1}\big)$$

hold for some positive constants C and C'.

Proof. Multiply the equation (2) by $\varphi(x,t)$ and integrating the result on $\Omega_R \times [0,T]$, we get

$$\begin{aligned}\int\limits_{\Omega_R\times[0,T]} h\varphi|u|^p\,dx\,dt + \int\limits_{\Omega_R} u_0 D^{2\gamma-1}_{t|T}\varphi(0)\,dx + \\ + \int\limits_{\Omega_R\times[0,T]} u_0 D^{\gamma}_{t|T}\varphi\,dx\,dt + \int\limits_{\Omega_R\times[0,T]} u_1 D^{2\gamma-1}_{t|T}\varphi\,dx\,dt = \\ = \int\limits_{\Omega_R\times[0,T]} u D^{2\gamma}_{t|T}\varphi\,dx\,dt + \int\limits_{\Omega_R\times[0,T]} u(-\Delta)^{\frac{\beta}{2}}\varphi\,dx\,dt + \int\limits_{\Omega_R\times[0,T]} u D^{\gamma}_{t|T}\varphi\,dx\,dt. \quad (18)\end{aligned}$$

where $\Omega_R := \{x \in \mathbb{R}^n;\ R \leq |x| \leq 2R\}$. Let us consider the function $\Phi \in H^\beta([1,2])$, $\Phi \geq 0$, such that $(-\Delta)^{\beta/2}\Phi = K\Phi$ for some positive constants K. We take

$$\varphi(x,t) := \Phi\Big(\frac{x}{R}\Big)\Big(1 - \frac{t^2}{T^2}\Big)^l,\ \ (x,t) \in \Omega_R \times [0,T],\ \ l > q.$$

Applying the ε-Young's inequality to the right-hand side of (18), one obtains

$$\begin{aligned}\int\limits_{\Omega_R} u_0 D^{2\gamma-1}_{t|T}\varphi(0)\,dx + \int\limits_{\Omega_R\times[0,T]} u_0 D^{\gamma}_{t|T}\varphi\,dx\,dt + \int\limits_{\Omega_R\times[0,T]} u_1 D^{2\gamma-1}_{t|T}\varphi\,dx\,dt \leq \\ \leq C \int\limits_{\Omega_R\times[0,T]} (h\varphi)^{\frac{-q}{p}} \Big(\big|(-\Delta)^{\frac{\beta}{2}}\varphi\big|^q + \big|D^{2\gamma}_{t|T}\varphi\big|^q + \big|D^{\gamma}_{t|T}\varphi\big|^q\Big)\,dx\,dt. \quad (19)\end{aligned}$$

In order to estimate the right-hand side of (19) in terms of T and R, we have

$$\int\limits_{\Omega_R\times[0,T]} (h\varphi)^{1-q}\big|(-\Delta)^{\beta/2}\varphi\big|^q\,dx\,dt = CTR^{-\beta q}\int\limits_{\Omega_R} h^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx,$$

where we have used $(-\Delta)^{\beta/2}\Phi\big(\frac{x}{R}\big) = KR^{-\beta}\Phi(\frac{x}{R})$. An easy computation (using the Euler substitution $y = \frac{s-t}{T-t}$) yields

$$D^{\gamma}_{t|T}\Big(1-\frac{t^2}{T^2}\Big)^{l} = \frac{-T^{2l}}{\Gamma(1-\gamma)}\times$$
$$\times\sum_{k=0}^{l} 2^{l-k}C^l_k M_{lk} t^{l-k-1}(T-t)^{l-k-\gamma}\big[(l-k)T-(2l+1-\gamma)t\big], \quad (20)$$

where $M_{lk} := \Gamma(l+1)\sum\limits_{n=0}^{k} C^k_n \frac{\Gamma(n-\beta+1)}{\Gamma(l-\beta+n+2)}$ and $C^l_k = \frac{l!}{k!(l-k!)}$,

$$D^{2\gamma}_{t|T}(1-\frac{t^2}{T^2})^l = \frac{T^{2l}}{\Gamma(2-2\gamma)}\sum_{k=0}^{l} 2^{l-k}C^l_k M_{lk} t^{l-k-2}(T-t)^{l-k-2\gamma}\times$$
$$\times\Big[(l-k)(l-k-1)T^2-2tT(l-k)(2l-2\gamma+1)+(2l-2\gamma+1)(2l-2\gamma+2)t^2\Big], \quad (21)$$

and

$$\int_0^T D^{\gamma}_{t|T}\Big(1-\frac{t^2}{T^2}\Big)^l dt = \frac{T^{1-\gamma}}{\Gamma(1-\gamma)}\sum_{k=0}^{l} L_{\gamma k}C^l_k, \quad (22)$$

where

$$L_{\gamma k} := \frac{\Gamma(l+1)\Gamma(k+1-\gamma)}{\Gamma(l+k+2-\gamma)}.$$

By (20) and (21), we can see that

$$\Big|D^{\gamma}_{t|T}\Big(1-\frac{t^2}{T^2}\Big)^l\Big| \le \frac{T^{-\gamma}}{\Gamma(1-\gamma)}\sum_{k=0}^{l} 2^{(l-k)}(3l+1-\gamma-k)C^l_k M_{lk} \quad (23)$$

and

$$\Big|D^{2\gamma}_{t|T}\Big(1-\frac{t^2}{T^2}\Big)^l\Big| \le \frac{T^{-2\gamma}}{\Gamma(2-2\gamma)}\times$$
$$\times\sum_{k=0}^{l} 2^{(l-k)}C^l_k M_{lk}\Big[(l-k)(l-k-1)+(2l+1-2\gamma)(4l-2k+2-2\gamma)\Big]. \quad (24)$$

Passing to the new variable $t = T\tau$ and by the relations (22), (23) and (24), we obtain

$$\int_{\Omega_R\times[0,T]} u_1 D^{2\gamma-1}_{t|T}\varphi\, dx\, dt = \frac{C_3}{\Gamma(1-\alpha)}T^{-2\gamma+2}\int_{\Omega_R} u_1(x)\Phi\Big(\frac{x}{R}\Big)dx, \quad (25)$$

$$\int_{\Omega_R\times[0,T]} (h\varphi)^{1-q}\big|D^{\gamma}_{t|T}\varphi\big|^q\, dx\, dt \le CT^{1-\gamma q}\int_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)dx, \quad (26)$$

$$\int_{\Omega_R\times[0,T]} (h\varphi)^{1-q}\big|D^{2\gamma}_{t|T}\varphi\big|^q\, dx\, dt \le CT^{1-2\gamma q}\int_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)dx, \quad (27)$$

and

$$\int\limits_{\Omega_R\times[0,T]} (h\varphi)^{1-q}\big|(-\Delta)^{\frac{\beta}{2}}\varphi\big|^q\,dx\,dt \leq$$
$$\leq CTR^{-\beta q}\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \quad (28)$$

Gathering all the estimates (25)–(28) together with (19), we find

$$T^{1-\gamma}\int\limits_{\Omega_R} u_0(x)\Phi\Big(\frac{x}{R}\Big)\,dx + T^{2-2\gamma}\int\limits_{\Omega_R} u_1(x)\Phi\Big(\frac{x}{R}\Big)\,dx \leq$$
$$\leq C\big(T^{1-\gamma q}+T^{1-2\gamma q}+TR^{-\beta q}\big)\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}(x)\Phi\Big(\frac{x}{R}\Big)\,dx. \quad (29)$$

The estimate (29) and the following estimates

$$\int\limits_{\Omega_R} u_0(x)\Phi\Big(\frac{x}{R}\Big)\,dx \geq$$
$$\geq \inf_{|x|>R}\Big(u_0(x)\big[\inf_{t>0} h(x,t)\big]^{q-1}\Big)\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx,$$
$$\int\limits_{\Omega_R} u_1(x)\Phi\Big(\frac{x}{R}\Big)\,dx \geq$$
$$\geq \inf_{|x|>R}\Big(u_1(xt)\big[\inf_{t>0} h(x,t)\big]^{q-1}\Big)\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx,$$

yield

$$\Big(T^{-\gamma}\inf_{|x|>R}\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)+T^{1-2\gamma}\inf_{|x|>R}\Big(u_1(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\Big)\times$$
$$\times\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \leq$$
$$\leq C\big[T^{-\gamma q}+T^{-2\gamma q}+R^{-\beta q}\big]\int\limits_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \quad (30)$$

Dividing the both sides of (30) by $\int\limits_{\Omega_R}\big[\inf\limits_{t>0} h(x,t)\big]^{1-q}\Phi\big(\frac{x}{R}\big)\,dx > 0$, after passing to the limit $R\to+\infty$, we deduce

$$T^{-\gamma}\lim_{|x|\to+\infty}\inf\Big(u_0(x)\big[\inf_{t>0} h(x,t)\big]^{q-1}\Big)+$$
$$+T^{1-2\gamma}\lim_{|x|\to+\infty}\inf\Big(u_1(x)\big[\inf_{t>0} h(x,t)\big]^{q-1}\Big)\leq C(T^{-\gamma q}+T^{-2\gamma q}).$$

Then we have

$$\lim_{|x|\to+\infty}\inf\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\leq C(T^{\gamma-\gamma q}+T^{\gamma-2\gamma q})$$

and

$$\lim_{|x|\to+\infty}\inf\Big(u_1(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\leq C(T^{2\gamma-1-\gamma q}+T^{2\gamma(1-q)-1}). \qquad \square$$

Corollary 1. *Assume that the problem* (2) *has a nontrivial global solution. Then at least one of the following conditions is satisfied:*

$$\lim_{|x|\to+\infty}\inf\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)=0,$$
$$\lim_{|x|\to+\infty}\inf\Big(u_1(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)=0.$$

Corollary 2. *If one of the conditions*

$$\lim_{|x|\to+\infty}\inf\Big(\big[\inf_{t>0}h(x,t)\big]^{q-1}u_0(x)\Big)=+\infty$$

or

$$\lim_{|x|\to+\infty}\inf\Big(\big[\inf_{t>0}h(x,t)\big]^{q-1}u_1(x)\Big)=+\infty$$

is fulfilled, then the problem (2) *cannot have any local weak solution.*

Theorem 3. *Suppose that the problem* (2) *has a global solution. Then there exist two positive constants K_1 and K_2 such that*

$$\lim_{|x|\to+\infty}\inf\Big(u_0(x)|x|^{\beta(q-1)}\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\leq K_1,$$

and

$$\lim_{|x|\to+\infty}\inf\Big(u_1(x)|x|^{\frac{\beta}{\gamma}(\gamma(q-1)+1-\gamma)}\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\leq K_2.$$

Proof. From the relation (30) we infer that

$$\inf_{|x|>R}\Big(\big[\inf_{t>0}h(x,t)\big]^{q-1}u_0(x)\Big)\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx\leq$$
$$\leq C\big[T^{\gamma-\gamma q}+T^{\gamma-2\gamma q}+T^{\gamma}R^{-\beta q}\big]\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx$$

Then, by taking $T>1$, we have

$$\inf_{|x|>R}\Big(u_0(x)\big[\inf_{t>0}h(x,t)\big]^{q-1}\Big)\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx\leq$$
$$\leq C\big[T^{\gamma-\gamma q}+T^{\gamma}R^{-\beta q}\big]\int\limits_{\Omega_R}\big[\inf_{t>0}h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \qquad (31)$$

Now, taking in (31) $T = R^{\frac{\beta}{\gamma}}$, we find

$$\inf_{|x|>R} \Big(u_0(x) \big[\inf_{t>0} h(x,t) \big]^{q-1} \Big) \int\limits_{\Omega_R} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx \le$$
$$\le C R^{\beta(1-q)} \int\limits_{\Omega_R} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx.$$

The last inequality implies

$$\inf_{|x|>R} \Big(u_0(x) |x|^{\beta(q-1)} \big[\inf_{t>0} h(x,t) \big]^{q-1} \Big) \times$$
$$\times \int\limits_{\Omega_R} |x|^{\beta(1-q)} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx \le$$
$$\le C 2^{2\beta(q-1)} \int\limits_{\Omega_R} |x|^{\beta(1-q)} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx. \qquad (32)$$

After division of both sides of (32) by

$$\int\limits_{\Omega_R} |x|^{\beta(1-q)} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx > 0,$$

we deduce that

$$\inf_{|x|>R} \Big(u_0(x) |x|^{\beta(q-1)} \big[\inf_{t>0} h(x,t) \big]^{q-1} \Big) \le C 2^{2\beta(q-1)}.$$

Finally, we pass to the limit $|x| \to +\infty$.

Similarly, we have

$$\inf_{|x|>R} \Big(u_1(x) \big[\inf_{t>0} h(x,t) \big]^{q-1} \Big) \int\limits_{\Omega_R} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx \le$$
$$\le C \big[T^{2\gamma-1-\gamma q} + T^{2\gamma-1-2\gamma q} + T^{2\gamma-1} R^{-\beta q} \big] \int\limits_{\Omega_R} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx,$$

and, by taking $T > 1$, we get

$$\inf_{|x|>R} \Big(\big[\inf_{t>0} h(x,t) \big]^{q-1} u_1(x) \Big) \int\limits_{\Omega_R} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx \le$$
$$\le C \big[T^{2\gamma-1-\gamma q} + T^{2\gamma-1} R^{-\beta q} \big] \int\limits_{\Omega_R} \big[\inf_{t>0} h(x,t) \big]^{1-q} \Phi\Big(\frac{x}{R} \Big)\, dx.$$

Likewise, $T = R^{\frac{\beta}{\gamma}}$. Therefore, by the substitution, we find

$$\inf_{|x|>R}\Big(\big[\inf_{t>0} h(x,t)\big]^{q-1}u_1(x)\Big)\int_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$
$$\le CR^{\frac{\beta}{\gamma}(2\gamma-1)-\beta q}\int_{\Omega_R}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx.$$

Hence

$$\inf_{|x|>R}\Big(|x|^{\beta q-\frac{\beta}{\gamma}(2\gamma-1)}\big[\inf_{t>0} h(x,t)\big]^{q-1}u_1(x)\Big)\times$$
$$\times\int_{\Omega_R}|x|^{\frac{\beta}{\gamma}(2\gamma-1)-\beta q}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx \le$$
$$\le C2^{2\left(\frac{\beta}{\gamma}(2\gamma-1)-\beta q\right)}\int_{\Omega_R}|x|^{\frac{\beta}{\gamma}(2\gamma-1)-\beta q}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx. \tag{33}$$

Finally, we divide both sides of the resulting relation by the expression

$$\int_{\Omega_R}|x|^{\frac{\beta}{\gamma}(2\gamma-1)-\beta q}\big[\inf_{t>0} h(x,t)\big]^{1-q}\Phi\Big(\frac{x}{R}\Big)\,dx > 0,$$

and pass to the limit as $|x| \to +\infty$. □

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