## On the Occasion of Givi Khuskivadze's
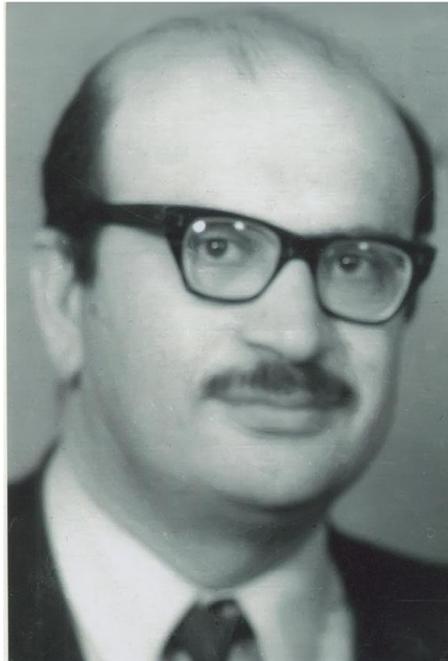## 80th Birthday Anniversary

Givi Khuskivadze, the well-known Georgian mathematician, doctor of sciences in physics and mathematics, has turned 80.

Givi Khuskivadze was born on March 28, 1932. In 1950 he finished with Gold medal the Tbilisi 8th Secondary school and in 1955 he graduated with honour from the Tbilisi State University, faculty of mechanics and mathematics. In 1955–1958 he went through a full post-graduate course at the same University. In 1963 he defended first his candidate's thesis and then, in 1999, he became doctor of phys.-math. sciences. Since 1958, all his life was connected with A. Razmadze Mathematical Institute, where he held posts of junior and then of senior and leading researcher. Just within the precincts of that Institute G. Khuskivadze carried out his remarkable investigations in fundamental problems of the real and complex analysis

which earned him immense authority and respect among specialists engaged in this area.

In G. Khuskivadze's personality were harmoniously combined brilliant talent of a widely and well-educated person, on the one hand, and rare modesty, sensitiveness and honesty, on the other hand. For all these qualities G. Khuskivadze won sincere respect of all his colleagues and friends.

Givi Khuskivadze passed away on December 5, 2011, not having reached only three months to see his 80th anniversary.

Givi Khuskivadze's works deal with various actual problems of the theory of functions. The profound and fine results on the Cauchy type integrals and connected with them singular integrals greatly contributed to further development and elaboration of methods of the theory of the Cauchy type integrals. Of special interest are his works in such important parts of the function theory as the theory of integrals, boundary value problems for analytic and harmonic functions, conformal mappings, etc.

Here we cite an incomplete list of the results obtained by Givi Khuskivadze in a course of his scientific activity.

Since the functions defined by the Cauchy singular integral with density summable in the Lebesgue sense may turn out to be not summable, there naturally arose the question to generalize the notion of the Lebesgue integral in a way that the above-mentioned functions become integrable in that new sense. There were suggested and studied various generalizations of the notion of the Lebesgue integral, the so-called $B$ and $A$ integrals (see, e.g., A. Kolmogorov, E. Titchmarsh, P. Ul'yanov). These integrals, despite the fact that they provide us with a positive solution to the above-posed question, they have certain drawbacks. For example, there exists the $A$-integrable on the interval $[a, b]$ function which is $A$-integrable on neither interval $(\alpha, \beta) \subset [a, b]$.

G. Khuskivadze suggested a simple, free from any exotic drawbacks, easily observable generalization of the Lebesgue integral which he called an $\widetilde{L}$-integral. The basic properties inherent in the $\widetilde{L}$-integrability were revealed, having thus constructed such an extension of the Lebesgue integral which allowed one to get a complete picture of that part of the theory of integral which is connected with the notion of the singular integral in the Cauchy sense. In particular, the representation of the Cauchy type $\widetilde{L}$-integral taken with respect to a closed curve by means of the Cauchy $\widetilde{L}$-integral, is obtained. Besides, the cases are embraced in which the lines of integration are taken from a wide classes of curves, and equalities connecting generalized integrals with the Lebesgue integral are obtained.

The $\widetilde{L}$-integral finds its effective applications. For example, if the Riemann boundary value problem is treated in the assumption that an unknown function is representable by the Cauchy–Lebesgue type integral, then in the framework of the Lebesgue integral, it is impossible to get an acceptable picture of solvability. However, if the statement of the problem is replaced

by the requirement for a solution of the problem to be representable by the Cauchy type $\widetilde{L}$-integral, then the Riemann problem can be solved in a standard way [23], [24].

The notion of the $\widetilde{L}$-integral has been successfully used for investigation of properties of conformal mappings of a unit circle onto a simply-connected domain. (In particular, the criterion for the representability of that integral as an exponential function of the Cauchy type $\widetilde{L}$-integral is established.) [24], [28].

Of great importance is Givi Khuskivadze's contribution to the investigation of problems dealt with the boundedness in Lebesgue spaces of the operator generated by the Cauchy singular integral, and also the properties of the Cauchy type integrals ([1]–[10], [37]). These problems of theoretical and applied interest always attracted his attention. A number of results in this direction have been obtained in collaboration with V. Paatashvili ([11], [12], [18], [19], [22], [31]).

They revealed the necessary and sufficient conditions for the boundedness of the Cauchy singular operator $S_\Gamma$ from $L^p(\Gamma)$ to $L^q(\Gamma)$, $p \geq q$, $p > 1$, when $\Gamma$ is a countable set of concentric circumferences whose sum of lengths is finite ([11], [18], [21]).

Next, it has been proved that for the boundedness of the operator $S_\Gamma$ in the spaces $L^p(\Gamma)$ it is necessary that the condition $\sup\limits_{\zeta \in \Gamma} \ell(\zeta, r) = O(r)$ be fulfilled; $\ell(\zeta, r)$ here is a linear measure of that part of the curve $\Gamma$ which finds itself in a circle with center $\zeta \in \Gamma$, of radius $r$. The fulfilment of this condition is sufficient for curves of a special class which contains the curves $\Gamma_0$ such that $S_{\Gamma_0}$ is bounded in neither class $L^p(\Gamma_0)$, $p > 1$. These results were obtained long before the G. David's work in which he proved that the above-cited condition is necessary and sufficient for the boundedness of the operator $S_\Gamma$ in the spaces $L^p(\Gamma)$, $p > 1$, for an arbitrary rectifiable curve $\Gamma$ [12], [23], [27].

G. Khuskivadze constructed examples illustrating how the geometry of the curve $\Gamma$ affects the boundedness of the operator $S_\Gamma$ in the Lebesgue spaces. He pointed out: (i) a curve $\Gamma_1$ for which $S_{\Gamma_1}$ is bounded from $L^p(\Gamma_1)$ to $L^{p-\varepsilon}(\Gamma)$ for any $p > 1$ and $\varepsilon \in (0, p - \varepsilon)$, but is unbounded in $L^p(\Gamma)$; (ii) a curve $\Gamma_2$ for which even the function $S_{\Gamma_2}(1)$ is summable in neither positive power. It should here be noted that G. Khuskivadze was very skillful in finding "examples in essence" which focused attention on different aspects of the problem under consideration ([10]).

One range of Givi Khuskivadze's works are devoted to the investigation of properties of conformal mappings of simply- connected domains (a portion of these works is carried out jointly with V. Paatashvili).

A new step in studying these properties is application of the methods and results of the theory of the Riemann boundary value problem to the case of a unit circle. This method allowed to obtain new, simple proofs of the well-known theorems on conformal mappings (Lindeloff, Cellogue, Smirnov,

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Warschawskii) and also to establish some important facts dealt with these mappings. Namely, depending on the geometry boundary of the domain $D$, it became possible to reveal those Hardy classes to which the derivative $z'(w)$ of the conformal mapping of a circle onto the domain $D$ belongs, and also those values of numbers $p$ for which the operator $Tf = z'S_\Gamma \frac{f}{z'}$ is bounded in the space $L^p(\Gamma)$ ([13], [14], [20]–[28], [32]).

Of the results mentioned above, a more significant particular case is the representation of $z'(w)$ in the case of arbitrary piecewise smooth curves which generalizes and complements Warschawski's theorem on the behaviour of $z'(w)$ in the case of piecewise Lyapunov curves.

The validity of these investigations is evident owing to a role that conformal mappings play in the analysis. Special attention earns one more of their dignities. They are especially useful in studying the Riemann and Riemann–Hilbert boundary value problems under new assumptions regarding the boundary $D$. (The Riemann's problem, well-known in the case of a circle, made it possible to establish properties of $z'$ for a wide class of domains and then, using these properties, to investigate boundary value problems for these classes of domains.)

Owing to the developments in the investigation of properties of the Cauchy type integrals with density from $L^p(\Gamma; \omega)$, the process G. Khuskivadze took an active part in, it became possible to make progress in studying boundary value problems of the theory of analytic functions, when a solution is required to be represented by the Cauchy type integral with density from $L^p(\Gamma; \omega)$. It also became possible to move forward in investigating the problems under rather general assumptions with regard to $\Gamma$, $\omega$ and to the coefficients in the boundary conditions.

In G. Khuskivadze's works dealt with a simply-connected domain, the Riemann–Hilbert problem is reduced, by using the well-known N. Muskhelishvili method, to the Riemann problem in which the principal coefficient is a product of three functions which depend, respectively, on the curve $\Gamma$, weight $\omega$ and on the coefficients of the initial problem. This allows one to obtain conditions for the solvability under various assumptions regarding the data of the problem and to construct solutions themselves. In the case of doubly-connected domains, the problem is successfully investigated by means of its reduction to an equivalent system of singular integral equations ([15], [17], [23]).

Special attention is focused on the investigation of the Dirichlet problem. Here are revealed such interesting classes of harmonic functions in which the problem in domains with a piecewise smooth boundary (depending on the angle sizes at angular points) may turn out to be uniquely solvable, ambiguously solvable, or unsolvable at all ([16], [23], [33], [35], [36], [38], [40]).

A rather complete picture of the solvability is obtained for Zaremba's mixed boundary value problem, as well, i.e., for that in which we seek for a function, harmonic in the domain $D$, when the value of an unknown function

is given on one part of the boundary and the value of its normal derivative is given on the remaining part of the boundary ([29], [30], [34], [35], [39]).

The results presented above show that Givi Khuskivadze was a person of brilliant talent and subtle mathematical mentality. G. Khuskivadze's remarkable personality will forever remain in the hearts of his friends and colleagues.

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