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ON OSCILLATION OF SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to the memory of Professor T. Chanturia

Abstract. A new oscillation criterion is proved for second-order linear ordinary differential equations with locally integrable coefficients. It is also shown that a certain generalization of the Hartman–Wintner theorem can be derived from the result obtained.

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1. Introduction

In the present paper we consider the second-order linear differential equation

$$u'' = -p(t)u + g(t)u', \tag{1.1}$$

where $p, g\colon \mathbb{R}_+ \to \mathbb{R}$ are locally integrable functions such that

$$\int\limits_0^{+\infty} \exp\bigg(\int\limits_0^s g(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = +\infty. \tag{1.2}$$

As usual, in the Carathéodory case, a function $u\colon \mathbb{R}_+ \to \mathbb{R}$ is said to be *a solution to equation* (1.1) if it is absolutely continuous together with the first derivative on every compact interval contained in $\mathbb{R}_+$ and satisfies

$$u''(t) = -p(t)u(t) + g(t)u'(t) \quad \text{for a. e. } t \geq 0.$$

Equation (1.1) is said to be *oscillatory* if every solution of this equation has a sequence of zeros tending to infinity.

In [7], the following oscillation criterion is proved for the equation

$$u'' = -p(t)u. \tag{1.3}$$

Theorem 1.1 ([7])**.** *Let the condition*

$$\limsup_{t\to+\infty} \frac{1}{t^\alpha} \int\limits_0^t (t-s)^\alpha p(s)\,\mathrm{d}s = +\infty \tag{1.4}$$

hold for some $\alpha > 1$. *Then equation* (1.3) *is oscillatory.*

It is also mentioned therein that the well-known Wintner criterion (see [10])

$$\lim_{t\to+\infty} \frac{1}{t} \int\limits_0^t \bigg(\int\limits_0^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = +\infty \tag{1.5}$$

follows from this result, because equality (1.5) guarantees the validity of relation (1.4) with $\alpha = 2$. Theorem 1.1 has been then generalized for the second-order equations, e. g., in [8, 9] (see also references therein). For higher-order equations, the integral oscillation criteria have been proved in [2–4].

The aim of the present paper is to establish a new oscillation criterion, which is applicable to the case where the "Kamenev-type" upper limit (1.4) is finite. The main result (namely, Theorem 2.1) and some further remarks are given in Section 2, and the proofs are given in Section 3. Moreover, a certain generalization of the Hartman–Wintner theorem (namely, Corollary 2.1) is derived in Section 2.

2. Main Results

Let

$$\sigma(g)(t) := \exp\left(\int_0^t g(s)\,\mathrm{d}s\right), \quad f(t) := \int_0^t \sigma(g)(s)\,\mathrm{d}s \quad \text{for } t \geq 0. \tag{2.1}$$

For any $\alpha > 1$, $\beta > 0$, and $\lambda < 1$, we put

$$k(t;\alpha,\beta,\lambda) := \frac{1}{f^{\alpha\beta}(t)} \int_0^t \big(f^{\beta}(t) - f^{\beta}(s)\big)^{\alpha} \frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s \quad \text{for } t > 0 \tag{2.2}$$

and

$$c(t;\lambda) := \frac{1-\lambda}{f^{1-\lambda}(t)} \int_0^t \frac{\sigma(g)(s)}{f^{\lambda}(s)} \left(\int_0^s \frac{f^{\lambda}(\xi)p(\xi)}{\sigma(g)(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}s \quad \text{for } t > 0. \tag{2.3}$$

We are now in a position to formulate our main result.

Theorem 2.1. *Let $\alpha > 1$, $\beta > 0$, $\lambda < 1$, condition* (1.2) *hold, and either*

$$\limsup_{t\to+\infty} k(t;\alpha,\beta,\lambda) = +\infty \tag{2.4}$$

or

$$-\infty < \limsup_{t\to+\infty} k(t;\alpha,\beta,\lambda) < +\infty, \tag{2.5}$$

$$\text{the function } c(\cdot\,;\lambda) \text{ does not possess a finite limit as } t \to +\infty. \tag{2.6}$$

Then equation (1.1) *is oscillatory.*

Observe that condition (2.4) with $\beta = 1$, $\lambda = 0$, and $g \equiv 0$ reduces to the Kamenev condition (1.4). Therefore, Theorem 2.1 can be regarded as an extension of Theorem 1.1 to the case where condition (1.4) is violated.

It is well-known that oscillatory properties of equation (1.1) can be also described in terms of lower and upper limits of the function c. We mention, in particular, the following Hartman–Wintner theorem (see A. Wintner [10] and P. Hartman [5,6] for $\lambda = 0$ and $g \equiv 0$).

Theorem 2.2 (Hartman–Wintner)**.** *Let $\lambda < 1$, condition* (1.2) *hold, and either*

$$\lim_{t\to+\infty} c(t;\lambda) = +\infty,$$

or

$$-\infty < \liminf_{t\to+\infty} c(t;\lambda) < \limsup_{t\to+\infty} c(t;\lambda)$$

be satisfied. Then equation (1.1) *is oscillatory.*

It is clear that for the given $\lambda < 1$, the following two cases remain uncovered in the previous theorem:

$$\text{there exists a finite limit } \lim_{t\to+\infty} c(t;\lambda) \tag{2.7}$$

and

$$\liminf_{t\to+\infty} c(t;\lambda) = -\infty. \tag{2.8}$$

The case, where (2.7) holds, is already studied in literature (see, e. g., [1] and references therein), but the authors know that there is still a broad field for further investigation if (2.8) is satisfied. Corollary 2.1 below gives a new oscillation criterion which is applicable also to the case where (2.8) holds.

For any $\lambda < 1$, we put

$$h(t;\lambda) := \frac{2(1-\lambda)}{f^{2(1-\lambda)}(t)} \int_0^t \sigma(g)(s) f^{1-2\lambda}(s) c(s;\lambda)\,\mathrm{d}s \quad \text{for } t > 0.$$

Theorem 2.1 yields

Corollary 2.1. *Let $\lambda < 1$, condition* (1.2) *hold, and either*

$$\limsup_{t\to+\infty} h(t;\lambda) = +\infty$$

or

$$-\infty < \limsup_{t\to+\infty} h(t;\lambda) < +\infty,$$

the function $c(\cdot\,;\lambda)$ does not possess a finite limit as $t \to +\infty$.

Then equation (1.1) *is oscillatory.*

This statement can be regarded as a generalization of Theorem 2.2. Indeed, it is not difficult to verify that if there exists a (finite or infinite) limit $\lim_{t\to+\infty} c(t;\lambda)$, then there exists also a limit $\lim_{t\to+\infty} h(t;\lambda)$ and both limits coincide. Moreover, if

$$\liminf_{t\to+\infty} c(t;\lambda) > -\infty \tag{2.9}$$

then

$$\liminf_{t\to+\infty} h(t;\lambda) > -\infty.$$

Therefore, if the assumptions of Theorem 2.2 are satisfied then the assumptions of Corollary 2.1 hold, as well. Note also that the assumptions of Theorem 2.2 require necessarily the validity of inequality (2.9). The following example shows that in some cases can be applied Corollary 2.1, while condition (2.9) is violated (i. e., (2.8) holds).

Example 2.1. Let $g \equiv 0$ and $p(t) = (2 - t^2)\cos(t) - 4t\sin(t)$ for $t \geq 0$. Then

$$c(t;0) = t\cos(t), \quad h(t;0) = 2\sin(t) + \frac{4}{t}\cos(t) - \frac{4}{t^2}\sin(t) \quad \text{for } t \geq 0,$$

and thus

$$\liminf_{t\to+\infty} c(t;0) = -\infty, \quad \limsup_{t\to+\infty} c(t;0) = +\infty,$$
$$\liminf_{t\to+\infty} h(t;0) = -2, \quad \limsup_{t\to+\infty} h(t;0) = 2.$$

Consequently, Theorem 2.2 with $\lambda = 0$ cannot be applied in this case. However, Corollary 2.1 yields that equation (1.1) is oscillatory.

3. Proofs

In order to prove Theorem 2.1, we need the following two lemmas.

Lemma 3.1. *Let $\alpha > 1$, $\beta > 0$, $\lambda < 1$, condition* (1.2) *hold, and u be a solution to equation* (1.1) *satisfying the relation*

$$u(t) \neq 0 \quad \textit{for } t \geq t_0 \tag{3.1}$$

with $t_0 > 0$. Then

$$\limsup_{t \to +\infty} k(t; \alpha, \beta, \lambda) < +\infty. \tag{3.2}$$

If, in addition, the inequality

$$\limsup_{t \to +\infty} k(t; \alpha, \beta, \lambda) > -\infty \tag{3.3}$$

is satisfied, then

$$\int_{t_0}^{+\infty} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)} \left[f(s)\varrho(s) - \frac{\lambda}{2} \right]^2 \mathrm{d}s < +\infty, \tag{3.4}$$

where

$$\varrho(t) := \frac{u'(t)}{u(t)\sigma(g)(t)} \quad \textit{for } t \geq t_0. \tag{3.5}$$

Proof. In view of (1.1), relation (3.5) yields that

$$\varrho'(t) = -\frac{p(t)}{\sigma(g)(t)} - \sigma(g)(t)\varrho^2(t) \quad \text{for a.e. } t \geq t_0, \tag{3.6}$$

whence we get

$$\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha f^\lambda(s)\varrho'(s)\,\mathrm{d}s =$$

$$= -\int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha \frac{f^\lambda(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s -$$

$$- \int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^\alpha f^\lambda(s)\sigma(g)(s)\varrho^2(s)\,\mathrm{d}s \quad \text{for } t \geq t_0.$$

Integration by parts on the left-hand side of the latter equality results in

$$- \left(f^\beta(t) - f^\beta(t_0)\right)^\alpha f^\lambda(t_0)\varrho(t_0) +$$

$$+ \alpha\beta \int_{t_0}^{t} \left(f^\beta(t) - f^\beta(s)\right)^{\alpha-1} f^{\beta-1}(s) f^\lambda(s)\sigma(g)(s)\varrho(s)\,\mathrm{d}s -$$

$$
\begin{aligned}
&-\lambda\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha}\frac{\sigma(g)(s)}{f^{1-\lambda}(s)}\,\varrho(s)\,\mathrm{d}s=\\
&=-\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s-\\
&-\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha}f^{\lambda}(s)\sigma(g)(s)\varrho^{2}(s)\,\mathrm{d}s\quad\text{for }\ t\geq t_0. \qquad (3.7)
\end{aligned}
$$

We now point out that

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha}f^{\lambda}(s)\sigma(g)(s)\varrho^{2}(s)\,\mathrm{d}s+\\
&+\lambda\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha}\frac{\sigma(g)(s)}{f^{1-\lambda}(s)}\,\varrho(s)\,\mathrm{d}s=\\
&=-\frac{1}{2}\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^{2}\mathrm{d}s+\\
&+\frac{\lambda^{2}}{2}\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\,\mathrm{d}s\quad\text{for }\ t\geq t_0
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha}f^{\lambda}(s)\sigma(g)(s)\varrho^{2}(s)\,\mathrm{d}s-\\
&-\alpha\beta\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-1}f^{\beta-1}(s)f^{\lambda}(s)\sigma(g)(s)\varrho(s)\,\mathrm{d}s=\\
&=-\frac{1}{2}\int\limits_{t_0}^{t}f^{\lambda}(s)\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-2}\times\\
&\times\sigma(g)(s)\left[\left(f^{\beta}(t)-f^{\beta}(s)\right)\varrho(t)+\alpha\beta f^{\beta-1}(s)\right]^{2}\mathrm{d}s+\\
&+\frac{\alpha^{2}\beta^{2}}{2}\int\limits_{t_0}^{t}f^{\lambda}(s)\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-2}f^{2(\beta-1)}(s)\sigma(g)(s)\,\mathrm{d}s\quad\text{for }\ t\geq t_0.
\end{aligned}
$$

Therefore relation (3.7) yields

$$
\begin{aligned}
k(t;\alpha,\beta,\lambda) \leq & -\frac{1}{2}\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s+\\
&+\frac{\lambda^2}{2}\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\,\mathrm{d}s+\\
&+\frac{\alpha^2\beta^2}{2f^{\alpha\beta}(t)}\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-2}f^{2(\beta-1)+\lambda}(s)\sigma(g)(s)\,\mathrm{d}s+\\
&+\int\limits_{0}^{t_0}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s+\\
&+\left(1-\left[\frac{f(t_0)}{f(t)}\right]^{\beta}\right)^{\alpha}f^{\lambda}(t_0)\varrho(t_0) \quad \text{for } t\geq t_0. \qquad (3.8)
\end{aligned}
$$

Since assumption (1.2) and notation (2.1) guarantee that

$$\lim_{t\to+\infty} f(t)=+\infty, \qquad (3.9)$$

it is easy to get

$$\lim_{t\to+\infty}\int\limits_{0}^{t_0}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s=\int\limits_{0}^{t_0}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s \qquad (3.10)$$

and

$$\lim_{t\to+\infty}\left(1-\left[\frac{f(t_0)}{f(t)}\right]^{\beta}\right)^{\alpha}f^{\lambda}(t_0)\varrho(t_0)=f^{\lambda}(t_0)\varrho(t_0). \qquad (3.11)$$

On the other hand, we have

$$
\begin{aligned}
&\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\,\mathrm{d}s\leq\\
&\qquad\leq\int\limits_{t_0}^{t}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\,\mathrm{d}s\leq\frac{1}{(1-\lambda)f^{1-\lambda}(t_0)} \quad \text{for } t\geq t_0 \qquad (3.12)
\end{aligned}
$$

and

$$\frac{1}{f^{\alpha\beta}(t)}\int\limits_{t_0}^{t}\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-2}f^{2(\beta-1)+\lambda}(s)\sigma(g)(s)\,\mathrm{d}s\leq$$

$$\leq \frac{1}{f^{\beta(\alpha-1)}(t)f^{1-\lambda}(t_0)} \int\limits_{t_0}^{t} f^{\beta-1}(s)\left(f^{\beta}(t)-f^{\beta}(s)\right)^{\alpha-2}\sigma(g)(s)\,\mathrm{d}s =$$

$$= \frac{1}{\beta(\alpha-1)f^{1-\lambda}(t_0)}\left(1-\left[\frac{f(t_0)}{f(t)}\right]^{\beta}\right)^{\alpha-1} \leq$$

$$\leq \frac{1}{\beta(\alpha-1)f^{1-\lambda}(t_0)} \quad \text{for } t \geq t_0. \tag{3.13}$$

Consequently, in view of (3.10)–(3.13), relation (3.8) implies that

$$\limsup_{t\to+\infty} k(t;\alpha,\beta,\lambda) \leq \frac{1}{2}\left(\frac{\lambda^2}{1-\lambda}+\frac{\alpha^2\beta}{\alpha-1}\right)\frac{1}{f^{1-\lambda}(t_0)}+ \\ + \int\limits_{0}^{t_0} \frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s + f^{\lambda}(t_0)\varrho(t_0),$$

and thus inequality (3.2) is satisfied.

Assume now that, in addition, relation (3.3) holds. We will show that inequality (3.4) is satisfied. It is obvious that either

$$\int\limits_{t_0}^{+\infty} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s = +\infty, \tag{3.14}$$

or

$$\int\limits_{t_0}^{+\infty} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s < +\infty. \tag{3.15}$$

Suppose that (3.14) holds. For any $\tau \geq a$ we have

$$\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s \geq \\ \geq \int\limits_{t_0}^{\tau}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s \quad \text{for } t \geq \tau$$

and thus

$$\liminf_{t\to+\infty}\int\limits_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s \geq \\ \geq \int\limits_{a}^{\tau}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s \quad \text{for } \tau \geq t_0.$$

The last relation, by virtue of equality (3.14), guarantees that

$$\lim_{t\to+\infty}\int_{t_0}^{t}\left(1-\left[\frac{f(s)}{f(t)}\right]^{\beta}\right)^{\alpha}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\lambda\right]^2\mathrm{d}s=+\infty.$$

Therefore inequality (3.8), together with (3.10)–(3.13), yields

$$\limsup_{t\to+\infty}k(t;\lambda)=-\infty,$$

which contradicts assumption (3.3). The obtained contradiction proves that inequality (3.15) holds. Since the function $\sqrt{\frac{\sigma(g)(\cdot)}{f^{2-\lambda}(\cdot)}}$ is quadratically integrable on $[t_0,+\infty[$, relation (3.4) is fulfilled, as well. □

The next lemma belongs to P. Hartman in the case where $\lambda=0$ and $g\equiv 0$ (see, e. g., [5,6]).

Lemma 3.2. *Let* $\lambda<1$*, condition* (1.2) *hold, and* u *be a solution to equation* (1.1) *satisfying relation* (3.1) *with* $t_0>0$*. Moreover, let condition* (3.4) *be fulfilled, where the function* ϱ *is defined by formula* (3.5)*. Then there exists a finite limit*

$$\lim_{t\to+\infty}c(t;\lambda). \tag{3.16}$$

Proof. In view of (1.1), from relation (3.5) we easily obtain equality (3.6). Multiplying both sides of (3.6) by the expression $f^{\lambda}(t)$ and integrating it by parts from t_0 to t, we arrive at

$$f^{\lambda}(t)\varrho(t)-f^{\lambda}(t_0)\varrho(t_0)-\lambda\int_{t_0}^{t}\frac{\sigma(g)(s)}{f^{1-\lambda}(s)}\varrho(s)\,\mathrm{d}s=$$
$$=-\int_{t_0}^{t}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s-\int_{t_0}^{t}f^{\lambda}(s)\sigma(g)(s)\varrho^2(s)\,\mathrm{d}s\quad\text{for } t\geq t_0,$$

whence we get

$$\frac{1}{f^{1-\lambda}(t)}\left[f(t)\varrho(t)-\frac{\lambda}{2}\right]=\varrho_1-\frac{\lambda(2-\lambda)}{4(1-\lambda)}\,\frac{1}{f^{1-\lambda}(t)}-\int_{0}^{t}\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s+$$
$$+\int_{t}^{+\infty}\frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s)-\frac{\lambda}{2}\right]^2\mathrm{d}s\quad\text{for } t\geq t_0, \tag{3.17}$$

where

$$\varrho_1 := f^{\lambda}(t_0)\varrho(t_0) + \frac{\lambda^2}{4(1-\lambda)f^{1-\lambda}(t_0)} + \\ + \int_0^{t_0} \frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s - \int_{t_0}^{+\infty} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]^2 \mathrm{d}s.$$

We now multiply both sides of equality (3.17) by the expression $f^{-\lambda}(t)\sigma(g)(t)$, integrate them by parts from t_0 to t and thus we get

$$\int_{t_0}^{t} \frac{\sigma(g)(s)}{f(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]\mathrm{d}s = -\int_0^{t} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_0^{s} \frac{f^{\lambda}(\xi)p(\xi)}{\sigma(g)(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}s + \\ + \int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_s^{+\infty} \frac{\sigma(g)(\xi)}{f^{2-\lambda}(\xi)}\left[f(\xi)\varrho(\xi) - \frac{\lambda}{2}\right]^2 \mathrm{d}\xi\right)\mathrm{d}s + \\ + \frac{\varrho_1}{1-\lambda}\, f^{1-\lambda}(t) - \frac{\lambda(2-\lambda)}{4(1-\lambda)}\ln\frac{f(t)}{f(t_0)} + \varrho_3 \quad \text{for } t \geq t_0, \tag{3.18}$$

where

$$\varrho_3 := \int_0^{t_0} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_0^{s} \frac{f^{\lambda}(\xi)p(\xi)}{\sigma(g)(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}s - \frac{\varrho_1}{1-\lambda}\, f^{1-\lambda}(t_0).$$

Since assumption (1.2) and notation (1.3) guarantee that relation (3.9) holds, by using the l'Hospital rule, it is easy to get

$$\lim_{t\to+\infty} \frac{1}{f^{1-\lambda}(t)}\ \ln\frac{f(t)}{f(t_0)}\,\mathrm{d}s = 0 \tag{3.19}$$

and

$$\lim_{t\to+\infty} \frac{1}{f^{1-\lambda}(t)} \int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_s^{+\infty} \frac{\sigma(g)(\xi)}{f^{2-\lambda}(\xi)}\left[f(\xi)\varrho(\xi) - \frac{\lambda}{2}\right]^2 \mathrm{d}\xi\right)\mathrm{d}s = 0. \tag{3.20}$$

On the other hand, by using the Hölder inequality, we obtain

$$\left(\int_{t_0}^{t} \frac{\sigma(g)(s)}{f(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]\mathrm{d}s\right)^2 \leq \\ \leq \int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{\lambda}(s)}\,\mathrm{d}s \int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]^2 \mathrm{d}s \leq \\ \leq \frac{f^{1-\lambda}(t)}{1-\lambda} \int_{t_0}^{t} \frac{\sigma(g)(s)}{f^{2-\lambda}(s)}\left[f(s)\varrho(s) - \frac{\lambda}{2}\right]^2 \mathrm{d}s \quad \text{for } t \geq t_0$$

and thus, by virtue of relations (3.4) and (3.9), we have

$$\lim_{t\to+\infty}\frac{1}{f^{1-\lambda}(t)}\int_{t_0}^{t}\frac{\sigma(g)(s)}{f(s)}\left[f(s)\varrho(s)-\frac{\lambda}{2}\right]\mathrm{d}s=0. \tag{3.21}$$

Consequently, in view of relations (3.9) and (3.19)–(3.21), it follows from equality (3.18) that

$$\lim_{t\to+\infty}c(t;\lambda)=\varrho_1.$$

□

Proof of Theorem 2.1. Assume, to the contrary, that there exists a solution u to equation (1.1) fulfilling relation (3.1) with $t_0>0$.

Then, according to Lemma 3.1, assumption (2.4) of Theorem 2.1 cannot be satisfied and thus assumptions (2.5) and (2.6) hold. By using Lemma 3.1, we obtain the validity of inequality (3.4) in which the function ϱ is defined by formula (3.5). However, Lemma 3.2 then guarantees that there exists a finite limit (3.16) which contradicts assumption (2.6). □

Proof of Corollary 2.1. By direct calculation we can check that

$$k(t;2,1-\lambda,\lambda)=\frac{1}{f^{2(1-\lambda)}(t)}\int_0^t\left(f^{1-\lambda}(t)-f^{1-\lambda}(s)\right)^2\frac{f^{\lambda}(s)p(s)}{\sigma(g)(s)}\,\mathrm{d}s=$$

$$=\frac{2(1-\lambda)}{f^{2(1-\lambda)}(t)}\int_0^t\left(f^{1-\lambda}(t)-f^{1-\lambda}(s)\right)\frac{\sigma(g)(s)}{f^{\lambda}(s)}\left(\int_0^s\frac{f^{\lambda}(\xi)p(\xi)}{\sigma(g)(\xi)}\,\mathrm{d}\xi\right)\mathrm{d}s=$$

$$=\frac{2(1-\lambda)^2}{f^{2(1-\lambda)}(t)}\int_0^t\frac{\sigma(g)(s)}{f^{\lambda}(s)}\left[\int_0^s\frac{\sigma(g)(\xi)}{f^{\lambda}(\xi)}\left(\int_0^{\xi}\frac{f^{\lambda}(\eta)p(\eta)}{\sigma(g)(\eta)}\,\mathrm{d}\eta\right)\mathrm{d}\xi\right]\mathrm{d}s=h(t;\lambda)$$

for $t\geq 0$ and thus the validity of the corollary follows immediately from Theorem 2.1 with $\alpha=2$ and $\beta=1-\lambda$. □

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