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THREE-DIMENSIONAL BOUNDARY VALUE PROBLEMS OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY

Abstract. The purpose of this paper is to consider three-dimensional version of Aifantis' equations of statics of the theory of consolidation with double porosity and to study the uniqueness and existence of solutions of basic boundary value problems (BVPs). In this work we intend to extend the potential method and the theory of integral equation to BVPs of the theory of consolidation with double porosity. Using these equations, the potential method and generalized Green's formulas, we prove the existence and uniqueness theorems of solutions for the first and second BVPs for bounded and unbounded domains. For Aifantis' equation of statics we construct one particular solution and we reduce the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic body.

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Key words and phrases. Porous media, double porosity, consolidation, fundamental solution.

რეზიუმე. ნაშრომის მიზანია აიფანტისის ორგვარი ფოროვნობის კონსოლიდაციის თეორიის სტატიკის განტოლებების განხილვა სამი განზომილების შემთხვევაში და ძირითადი სასაზღვრო ამოცანებისათვის ამონახსნების არსებობისა და ერთადერთობის საკითხების შესწავლა. ამ ამოცანების შესასწავლად ნაშრომში გამოყენებულია პოტენციალთა მეთოდი და ინტეგრალური განტოლებები ორგვარი ფოროვნობის კონსოლიდაციის თეორიისათვის. პირველი და მეორე სასაზღვრო ამოცანებისათვის ფრედჰოლმის ინტეგრალური განტოლებებისა და გრინის ფორმულების გამოყენებით დამტკიცებულია მათი ამონახსნების არსებობა და ერთადერთობა, როგორც სასრული, ისე უსასრულო არეებისათვის. აიფანტისის ორგვარი ფოროვნობის კონსოლიდაციის თეორიის სტატიკის განტოლებებისათვის ერთი კერძო ამონახსნის აგებით სტატიკის ამოცანების ამოხსნა მიყვანილია იზოტროპული დრეკადი ტანის სტატიკის ძირითადი ამოცანების ამოხსნაზე.

Introduction

In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example, in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or secondary porosity. When fluid flow and deformation processes occur simultaneously, three coupled partial differential equations can be derived [1], [2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

A theory of consolidation with double porosity has been proposed by Aifantis. The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]–[3], where analytical solutions of the relevant equations are also given. In part I of a series of papers on the subject, R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. They also solved several representative boundary value problems. In part II of that series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity, while in part III M. Y. Khaled, D. E. Beskos and E. C. Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1]–[3] and the references therein). The basic results and the historical information on the theory of porous media were summarized by R. de. Boer in [4]. The fundamental solution in the theory of consolidation with double porosity is given in [5].

In this work we prove the existence and uniqueness theorems of solutions of basic BVPs of the theory of consolidation with double porosity for bounded and unbounded domains. For the proof of all theorems we used the method given in [6].

1. Formulation of Boundary Value Problems and Uniqueness Theorems

The basic equations of statics of the theory of consolidation with double porosity are given by the partial differential equations in the form ([1], [2])

$$A(\partial x)u = \operatorname{grad}(\beta_1 p_1 + \beta_2 p_2), \tag{1.1}$$

$$(m_1\Delta - k)p_1 + kp_2 = 0, \quad kp_1 + (m_2\Delta - k)p_2 = 0, \tag{1.2}$$

$$A(\partial x)u = \mu\Delta u + (\lambda+\mu)\operatorname{grad}\operatorname{div} u, \tag{1.3}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores, $m_j = \frac{k_j}{\mu^*}$, $j = 1, 2$. The constant λ is the Lamé modulus, μ is the shear modulus and the constants β_1 and β_2 measure the change of porosities due to an applied volumetric strain. The constant μ^* denotes the viscosity of the pore fluid and the constant k measures the transfer of fluid from the secondary pores to the primary pores. All quantities λ, μ, β_j, k $(j=1,2)$ and μ^* are positive constants; $\triangle=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$ is the Laplace operator.

Let $D^+(D^-)$ be a bounded (an unbounded) three-dimensional domain surrounded by the surface S. $\overline{D}^+=D^+\cup S$; $D^-=E_2\setminus\overline{D}^+$. Suppose that $S\in C^{1,\alpha}$, $0<\alpha\leq 1$.

First of all, we introduce the definition of a regular vector-function.

Definition 1. A vector-function $U(U_1,U_2,U_3,U_4,U_5)=(u_1,u_2,u_3,p_1,p_2)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second order derivatives in $D^+(D^-)$, and U and its first order derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i.e., $U\in C^2(D^+)\cap C^1(\overline{D^+})$ $(U\in C^2(D^-)\cap C^1(\overline{D^-}))$. Note that for the infinite domain D^- the vector $U(x)$ additionally satisfies the following conditions at infinity:

$$U_k(x)=O(|x|^{-1}),\quad \frac{\partial U_k}{\partial x_j}=O(|x|^{-2}),$$
$$|x|^2=x_1^2+x_2^2+x_3^2,\ \ k=1,2,\dots,5,\ \ j=1,2,3. \tag{1.4}$$

For the equations (1.1)–(1.2) we pose the following boundary value problems:

Find a regular vector U satisfying in $D^+(D^-)$ the equations (1.1)–(1.2), and on the boundary S one of the following conditions is given:

Problem 1. The displacement vector and the fluid pressures are given on S :

$$u^{\pm}(z)=f(z)^{\pm},\quad p_1^{\pm}(z)=f_4^{\pm}(z),\quad p_2^{\pm}(z)=f_5^{\pm}(z),\ \ z\in S.$$

Problem 2. The stress vector and the normal derivatives of the pressure functions $\frac{\partial p_j}{\partial n}$ are given on S :

$$(Pu(z))^{\pm}=f(z)^{\pm},\quad \Big(\frac{\partial p_1(z)}{\partial n}\Big)^{\pm}=f_4^{\pm}(z),\quad \Big(\frac{\partial p_2(z)}{\partial n}\Big)^{\pm}=f_5^{\pm}(z),\ \ z\in S.$$

Problem 3.

$$u^{\pm}(z)=f(z)^{\pm},\quad \Big(\frac{\partial p_1(z)}{\partial n}\Big)^{\pm}=f_4^{\pm}(z),\quad \Big(\frac{\partial p_2(z)}{\partial n}\Big)^{\pm}=f_5^{\pm}(z),\ \ z\in S.$$

Problem 4.

$$(Pu(z))^{\pm}=f(z)^{\pm},\quad p_1^{\pm}(z)=f_4^{\pm}(z),\quad p_2^{\pm}(z)=f_5^{\pm}(z),\ \ z\in S,$$

where $(\cdot)^{\pm}$ denote the limiting values on S from $D^{\pm}$ and $f=(f_1,f_2,f_3)$, f_4, f_5 are given functions. $Pu(x)$ is the stress vector which acts on an element of the surface with the exterior to D^+ unit normal vector $n(x)=(n_1(x),n_2(x),n_3(x))$ at the point $x\in S$,

$$P(\partial x,n)u=T(\partial x,n)u-n(\beta_1p_1+\beta_2p_2), \tag{1.5}$$

where [6]

$$
\begin{aligned}
&T(\partial x,n)=\| T_{kj}(\partial x,n)\|_{3\times 3},\\
&T_{kj}(\partial x,n)=\mu\delta_{kj}\,\frac{\partial}{\partial n}+\lambda n_k\,\frac{\partial}{\partial x_j}+\mu n_j\,\frac{\partial}{\partial x_k}\,,\quad k,j,=1,2,3,\\
&\qquad\frac{\partial}{\partial n}=n_1\,\frac{\partial}{\partial x_1}+n_2\,\frac{\partial}{\partial x_2}+n_3\,\frac{\partial}{\partial x_3}\,.
\end{aligned}
\tag{1.6}
$$

Now we introduce the generalized stress vector. Denoting the generalized stress vector by $\overset{\kappa}{\mathrm{P}}(\partial x,n)u$, we have

$$
\overset{\kappa}{\mathrm{P}}(\partial x,n)u=\overset{\kappa}{\mathrm{T}}(\partial x,n)u-n(\beta_1p_1+\beta_2p_2),
$$

where κ is an arbitrary positive constant and

$$
\overset{\kappa}{\mathrm{T}}(\partial x,n)u=(2\mu-\kappa)\,\frac{\partial u}{\partial n}+(\lambda+\kappa)n\operatorname{div}u+(\kappa-\mu)n\times\operatorname{rot}u,\tag{1.7}
$$

with $a\times b$ denoting the cross product of two vectors a and b. Further, let us introduce the generalized stress tensor, $\|\sigma_{kj}(\partial x,n)\|_{3\times 3}$: [6]

$$
\begin{aligned}
&\sigma_{jj}=(\lambda+\kappa)\operatorname{div}u+(2\mu-\kappa)\,\frac{\partial u_j}{\partial x_j}\,,\quad j=1,2,3,\\
&\sigma_{12}=\mu\,\frac{\partial u_2}{\partial x_1}+(\mu-\kappa)\,\frac{\partial u_1}{\partial x_2},\sigma_{21}=\mu\,\frac{\partial u_1}{\partial x_2}+(\mu-\kappa)\,\frac{\partial u_2}{\partial x_1}\,,\\
&\sigma_{13}=\mu\,\frac{\partial u_3}{\partial x_1}+(\mu-\kappa)\,\frac{\partial u_1}{\partial x_3},\sigma_{31}=\mu\,\frac{\partial u_1}{\partial x_3}+(\mu-\kappa)\,\frac{\partial u_3}{\partial x_1}\,,\\
&\sigma_{23}=\mu\,\frac{\partial u_3}{\partial x_2}+(\mu-\kappa)\,\frac{\partial u_2}{\partial x_3},\quad\sigma_{32}=\mu\,\frac{\partial u_2}{\partial x_3}+(\mu-\kappa)\,\frac{\partial u_3}{\partial x_2}\,.
\end{aligned}
\tag{1.8}
$$

If $\kappa=0$, from (1.7) we have $\overset{\kappa}{\mathrm{T}}(\partial x,n)u=T(\partial x,n)u$. We set $\overset{\kappa}{\mathrm{T}}(\partial x,n)u=N(\partial x,n)u$ for $\kappa=\frac{2\lambda+3\mu}{\lambda+3\mu}$.

Generalized Green's formulas. Let us write the generalized Green's formulas for the domains D^+ and D^-. Let u be a regular solution of the equation (1.1) in D^+. Multiply first equation of (1.1) by u. Integrate the result over D^+ and apply the integration by parts formula to obtain

$$
\int\limits_{D^+}\Big[\overset{\kappa}{\mathrm{E}}(u,u)-(\beta_1p_1+\beta_2p_2)\operatorname{div}u\Big]\,dx=\int\limits_S u\;\overset{\kappa}{\mathrm{P}}(\partial x,n)u\,dS,\tag{1.9}
$$

where

$$
\begin{aligned}
\overset{\kappa}{\mathrm{E}}(u,u)=\;&\frac{3\lambda+2\mu-\kappa}{2}\,(\operatorname{div})^2+\frac{\kappa}{2}\,(\operatorname{rot}u)^2+\frac{2\mu-\kappa}{4}\sum_{k\neq j}\Big(\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\Big)^2+\\
&+\frac{2\mu-\kappa}{6}\Big[\Big(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\Big)^2+\Big(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_3}{\partial x_3}\Big)^2+\Big(\frac{\partial u_2}{\partial x_2}-\frac{\partial u_3}{\partial x_3}\Big)^2\Big].
\end{aligned}
$$

If the vector u satisfies the conditions (1.4), Green's formula for the domain D^- takes the form

$$\int\limits_{D^-}\Big[\overset{\kappa}{\mathrm{E}}(u,u)-(\beta_1p_1+\beta_2p_2)\operatorname{div}u\Big]\,dx=-\int\limits_S u\,\overset{\kappa}{\mathrm{P}}(\partial x,n)u\,dS. \tag{1.10}$$

For the positive definiteness of the potential energy, the inequalities

$$3\lambda+2\mu-\kappa>0,\ \ \kappa>0,\ \ \kappa<2\mu$$

are necessary and sufficient.

Analogously we obtain Green's formula for p_j, $j=1,2$,

$$\begin{aligned}
&\int\limits_{D^+}\Big[m_1(\operatorname{grad}p_1)^2+m_2(\operatorname{grad}p_2)^2+k(p_1-p_2)^2\Big]\,dx=\\
&\qquad=\int\limits_S\Big[m_1p_1\,\frac{\partial p_1}{\partial n}+m_2p_2\,\frac{\partial p_2}{\partial n}\Big]\,dS,\\
&\int\limits_{D^-}\Big[m_1(\operatorname{grad}p_1)^2+m_2(\operatorname{grad}p_2)^2+k(p_1-p_2)^2\Big]\,dx=\\
&\qquad=-\int\limits_S\Big[m_1p_1\,\frac{\partial p_1}{\partial n}+m_2p_2\,\frac{\partial p_2}{\partial n}\Big]\,dS.
\end{aligned} \tag{1.11}$$

Note that if $\beta_1p_1+\beta_2p_2=const$, in view of the equality $\int\limits_{D^+}\operatorname{div}u\,dx=\int\limits_S nu\,dS$ from (1.9) we get

$$\int\limits_{D^+}\overset{\kappa}{\mathrm{E}}(u,u)\,dx=\int\limits_S u\,\overset{\kappa}{\mathrm{T}}(\partial x,n)u\,dS. \tag{1.12}$$

Uniqueness theorems. In this subsection we prove the uniqueness theorems of solutions to the above formulated problems. Let the above formulated problems have two regular solutions $U^{(1)}$ and $U^{(2)}$, where $U^{(k)}=(u_1^{(k)},u_2^{(k)},u_3^{(k)},p_1^{(k)},p_2^{(k)})$, $k=1,2$. We put

$$U=U^{(1)}-U^{(2)}.$$

Evidently, the vector U satisfies the equations (1.1)–(1.2) and the homogeneous boundary conditions

1. $u^{\pm}(z)=0$, $p_j^{\pm}(z)=0$, $j=1,2$, $z\in S$,
2. $(P(\partial z,n)u(z))^{\pm}=0$, $\Big(\dfrac{\partial p_j(z)}{\partial n}\Big)^{\pm}=0$, $j=3,4$, $z\in S$,
3. $u^{\pm}(z)=0$, $\Big(\dfrac{\partial p_j(z)}{\partial n}\Big)^{\pm}=0$, $j=1,2$, $z\in S$,
4. $\big(P(\partial z,n)u\big)^{\pm}(z)=0$, $p_j^{\pm}(z)=0$, $j=1,2$, $z\in S$.

Now we prove the following theorems.

Theorem 1. *The first boundary value problem has at most one regular solution in the bounded domain D^+.*

Proof. Evidently, the vector U satisfies (1.1)–(1.2) and the boundary condition $U^+ = 0$ on S. Note that if U is a regular solution of (1.1)–(1.2), we have Green's formulas (1.9), (1.11). Taking into account the fact that the potential energy $\overset{\kappa}{\mathrm{E}}(u,u)$ is positive definite, we conclude that $U = C$, $x \in D^+$, where $C = const$. Since $U^+ = 0$, we have $C = 0$ and $U(x) = 0$, $x \in D^+$. □

Theorem 2. *The first boundary value problem has at most one regular solution in the infinite domain D^-.*

Proof. The vectors $U^{(1)}$ and $U^{(2)}$ in the domain D^- must satisfy the condition (1.4). In this case the formulas (1.11) are valid and $U(x) = C$, $x \in D^-$, where C is again a constant vector. But U on the boundary satisfies the condition $U^- = 0$, which implies that $C = 0$ and $U(x) = 0$, $x \in D^-$. □

Analogously can be proved the following theorems.

Theorem 3. *A regular solution of the second boundary value problem is not unique in the domain D^+. Two regular solutions may differ by a vector (u, p_1, p_2), where $u(x) = a + b\times x + c(\beta_1+\beta_2)x$, $p_j(x) = c$, $j = 1, 2$, $x \in D^+$, with a and b constant vectors, and c be an arbitrary constant.*

Theorem 4. *Two regular solutions of the boundary value problem $(III)^+$ may differ by the vector (u, p_1, p_2), where $u = 0$ and $p_j = c$, $j = 1, 2$, with c be an arbitrary constant.*

Theorem 5. *Two regular solutions of the boundary value problem $(IV)^+$ may differ by the vector (u, p_1, p_2), where u is a rigid displacement and $p_j = 0$, $j = 1, 2$.*

Theorem 6. *The boundary value problems $(II)^-$, $(III)^-$, $(IV)^-$ have at most one regular solution in the domain D^-.*

Note that from the equation (1.2) one may define the functions $p_j(x)$, $j = 1, 2$. Further we assume that p_j is known, when $x \in D^+$ or $x \in D^-$. Substitute $\beta_1 p_1 + \beta_2 p_2$ in (1.1) and search a particular solution of the following equation

$$\mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u = \mathrm{grad}(\beta_1 p_1 + \beta_2 p_2).$$

We put

$$u_0 = -\frac{1}{4\pi}\int\limits_D \Gamma(x-y)\,\mathrm{grad}(\beta_1 p_1 + \beta_2 p_2)\,dx, \tag{1.13}$$

where

$$\Gamma(x-y) =$$
$$= \frac{1}{4\mu(\lambda+2\mu)}\left\|\frac{(\lambda+3\mu)\delta_{kj}}{r} + \frac{(\lambda+\mu)(x_k-y_k)(x_j-y_j)}{r^3}\right\|_{3\times 3}, \quad r = |x-y|.$$

Substituting the volume potential u_0 into (1.1), we obtain (see [6])

$$\mu\Delta u_0 + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u_0 = \mathrm{grad}(\beta_1 p_1 + \beta_2 p_2). \tag{1.14}$$

Thus we have proved that $u_0(x)$ is a particular solution of the equation (1.1). In (1.13) D denotes either D^+ or D^-, $\mathrm{grad}(\beta_1 p_1 + \beta_2 p_2)$ is a continuous vector in D^+ along with its first order derivatives. When $D = D^-$, the vector $\mathrm{grad}(\beta_1 p_1 + \beta_2 p_2)$ has to satisfy the following decay condition at infinity

$$\mathrm{grad}(\beta_1 p_1 + \beta_2 p_2) = O(|x|^{-2-\alpha}), \quad \alpha > 0.$$

Thus the general solution of the equation (1.1) is representable in the form $u = V + u_0$, where

$$A(\partial x)V = \mu\Delta V + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,V = 0. \tag{1.15}$$

This equation is the equation of an isotropic elastic body. Thus we have reduced the solution of basic BVPs of the theory of consolidation with double porosity to the solution of the basic BVPs for the equation of an isotropic elastic body.

First of all we will construct a fundamental matrix of solutions for the equation (1.2). We look for p_j in the form

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} m_2\Delta - k & -k \\ -k & m_1\Delta - k \end{pmatrix}\psi, \tag{1.16}$$

where the vector $\psi(x)$ is the fundamental solution of the scalar equation

$$\Delta(\Delta - \lambda_0^2)\psi = 0, \;\; \lambda_0^2 = \frac{k}{m_1} + \frac{k}{m_2}\,, \;\; \psi = \frac{e^{-\lambda_0 r} - 1}{\lambda_0^2 r}\,.$$

From (1.16) it follows that the fundamental matrix of solutions of the equation (1.2) is the following matrix

$$M(x-y) =$$
$$= \begin{pmatrix} m_2\,\dfrac{e^{-\lambda_0 r}}{r} - \dfrac{k}{\lambda_0^2}\,\dfrac{e^{-\lambda_0 r}-1}{r} & -\dfrac{k}{\lambda_0^2}\,\dfrac{e^{-\lambda_0 r}-1}{r} \\ -\dfrac{k}{\lambda_0^2}\,\dfrac{e^{-\lambda_0 r}-1}{r} & m_1\,\dfrac{e^{-\lambda_0 r}}{r} - \dfrac{k}{\lambda_0^2}\,\dfrac{e^{-\lambda_0 r}-1}{r} \end{pmatrix}. \tag{1.17}$$

The following theorem is valid:

Theorem 7. *Each column of the matrix $M(x-y)$ is a solution to the equation (1.2) with respect to x for $x \neq y$.*

2. Integral Equations of BVPs

A solution of the first boundary value problem ($p_1^\pm = f_4^\pm$, $p_2^\pm = f_5^\pm$, $V^\pm = F^\pm$) in the domains $D^\pm$ for the systems (1.2), (1.15) will be sought in the form of the double layer potential

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial n(y)} M(x-y)\varphi(y)\, d_y S, \tag{2.1}$$

$$V(x) = \frac{1}{\pi} \int\limits_S \big[N(\partial y, n)\Gamma(x-y)\big]^T g(y)\, d_y S, \tag{2.2}$$

where $S \in C^{1,\alpha}$, $\varphi \in C^{0,\beta}$, $g \in C^{0,\beta}$, $0 < \beta < \alpha \le 1$, $M(x-y)$ is given by (1.17),

$$\big[N(\partial y, n)\Gamma(x-y)\big]^T_{kj} =$$

$$= \frac{\partial}{\partial n} \frac{\delta_{kj}}{r} + \sum_{k=1}^{3} M_{kj}(\partial y, n)\Big[\frac{(\lambda+\mu)(x_k-y_k)(x_j-y_j)}{(\lambda+3\mu)r^3}\Big],$$

$$M_{kj} = n_j \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial x_j}\,.$$

Then for determining the unknown vectors φ and g we obtain the following system of Fredholm integral equations of the second kind on S

$$\pm(\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix} \varphi(z) + \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial n} M(z-y)\varphi(y)\, d_y S = \begin{pmatrix} f_4^{\pm}(z) \\ f_5^{\pm}(z) \end{pmatrix}, \tag{2.3}$$

$$\mp g(z) + \frac{1}{\pi} \int\limits_S \big[T(\partial y, n)\Gamma(y-z)\big] g(y)\, d_y S = F^{\pm}(z). \tag{2.4}$$

If a solution of the first BVP is sought in the form

$$V(x) = \frac{1}{\pi} \int\limits_S \big[T(\partial y, n)\Gamma(x-y)\big]^T g(y)\, d_y S, \tag{2.5}$$

for determining of the unknown vector g we obtain the following singular integral equation of the second kind

$$\mp g(z) + \frac{1}{\pi} \int\limits_S \big[T(\partial y, n)\Gamma(y-z)\big] g(y)\, d_y S = F^{\pm}(z). \tag{2.6}$$

A solution of the second boundary value problem $((\frac{\partial p_1}{\partial n})^{\pm} = f_4^{\pm}$, $(\frac{\partial p_2}{\partial n})^{\pm} = f_5^{\pm}$, $(T(\partial x, n)V)^{\pm} = \Phi^{\pm})$ in the domains $D^{\pm}$ for the systems (1.2)–(1.15) will be sought in terms of the single layer potential

$$\begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix} = \frac{1}{2\pi} \int\limits_S M(x-y)\varphi(y)\, d_y S, \tag{2.7}$$

$$V(x) = \frac{1}{\pi} \int\limits_S \Gamma(x-y)h(y)\, d_y S, \tag{2.8}$$

Then for determining the unknown vectors φ and g we obtain the following system of Fredholm integral equations of the second kind

$$\mp\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\varphi(z)+\frac{1}{2\pi}\int\limits_S \frac{\partial}{\partial n(z)}\, M(z-y)\varphi(y)\, d_yS=\begin{pmatrix} f_4^{\pm}(z) \\ f_5^{\pm}(z) \end{pmatrix}, \qquad (2.9)$$

$$\pm h(z)+\frac{1}{\pi}\int\limits_S T(\partial z,n)\Gamma(z-y)h(y)\, d_yS=\Phi^{\pm}(z), \qquad (2.10)$$

where [6]

$$T(\partial y,n)\Gamma(x-y)=$$

$$=\left\|\frac{\partial}{\partial n}\,\frac{\delta_{kj}}{r}+\sum_{k=1}^{3}M_{kj}(\partial y,n)\Big[\frac{2\delta_{kj}}{(\lambda+2\mu)r}+\frac{2(\lambda+\mu)(x_k-y_k)(x_j-y_j)}{(\lambda+2\mu)r^3}\Big]\right\|_{3\times 3}.$$

3. Analysis of the Basic BVPs in the Domains D^+ and D^-

Problem $(I)^+$. First let us prove the existence of solution of the first boundary value problem in the domain D^+. Consider the equation (2.3)

$$-\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\varphi+\frac{1}{2\pi}\int\limits_S \frac{\partial}{\partial n}\, M(z-y)\varphi(y)\, d_yS=\begin{pmatrix} f_4^{+}(z) \\ f_5^{+}(z) \end{pmatrix}, \qquad (3.1)$$

Let us prove that the equation (3.1) is solvable for any continuous right-hand side. To this end, consider the associated to (3.1) homogeneous equation

$$-\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}\psi(z)+\frac{1}{2\pi}\int\limits_S \frac{\partial}{\partial n}\, M(z-y)\psi(y)\, d_yS=0, \qquad (3.2)$$

and prove that it has only the trivial solution. Assume the contrary and denote by ψ_0 a nontrivial solution of (3.2). The equation (3.2) corresponds to the boundary conditions

$$\Big(\frac{\partial p_1}{\partial \nu}\Big)^{-}=0, \quad \Big(\frac{\partial p_2}{\partial \nu}\Big)^{-}=0,$$

whence we have $\int\limits_S \psi_k\, ds=0$, $k=4,5$.

Now taking into account the continuity of the simple layer potential and using the uniqueness theorem for the solution of the first boundary value problem, we will have $p_k(x)=c$, $x\in D^-$.

Note that

$$\Big(\frac{\partial p_1}{\partial \nu}\Big)^{-}-\Big(\frac{\partial p_1}{\partial \nu}\Big)^{+}=2m_2\psi_4=0, \quad \Big(\frac{\partial p_2}{\partial \nu}\Big)^{-}-\Big(\frac{\partial p_2}{\partial \nu}\Big)^{+}=2m_1\psi_5=0,$$

hence the equation (3.2) has only the trivial solution. This implies that the associated to (3.2) homogeneous equation also has only the trivial solution, and the equation (3.1) is solvable for any continuous right-hand side.

For the regularity of the double layer potential in the domain D^+ it is sufficient to assume that $S \in C^{2,\beta}$ $(0 < \beta < 1)$ and $f_k \in C^{1,\alpha}(S)$ $(0 < \alpha < \beta)$, $k = 4, 5$.

Problem $(I)^-$**.** Now consider the first boundary value problem in the domain D^-. Consider the equation (2.3)

$$\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix} \varphi + \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial n} M(z-y)\varphi(y)\, d_y S = \begin{pmatrix} f_4^-(z) \\ f_5^-(z) \end{pmatrix}. \tag{3.3}$$

Prove that the equation (3.3) is solvable for any continuous right-hand side. We consider the associated to (3.3) homogeneous equation

$$\begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix} \varphi + \frac{1}{2\pi} \int\limits_S \frac{\partial}{\partial \nu} M(z-y)\varphi(y)\, d_y S = 0. \tag{3.4}$$

Let us prove that (3.4) has only the trivial solution. Suppose that it has a nonzero solution $\varphi(z)$. From (3.4) by integration we obtain

$$\int\limits_S \varphi\, dS = 0.$$

In this case the equation (3.4) corresponds to the boundary condition

$$\left(\frac{\partial p_k}{\partial \nu}\right)^+ = 0.$$

We find that $p_k = c$, $x \in D^+$, where c is a constant vector.

Taking into account the equation $\int\limits_S \varphi\, ds = 0$ and the fact that the single layer potential is continuous while passing through the boundary, and using Green's formula for $\kappa = \kappa_n$, we obtain $p_k = 0$, $x \in D^-$. Since

$$\left(\frac{\partial p_1}{\partial \nu}\right)^- - \left(\frac{\partial p_1}{\partial \nu}\right)^+ = 2m_2\varphi_4 = 0, \quad \left(\frac{\partial p_2}{\partial \nu}\right)^- - \left(\frac{\partial p_2}{\partial \nu}\right)^+ = 2m_1\varphi_5 = 0,$$

we have $\varphi(x) = 0$.

Thus we conclude that the associated to (3.4) homogeneous equation has only the trivial solution, and the equation (3.3) is solvable for any continuous right-hand side.

To prove the regularity of the potential (2.1) in the domain D^-, it is sufficient to assume that $S \in C^{2,\beta}(0 < \beta < 1)$ and $f_k \in C^{1,\alpha}(S)$ $(0 < \alpha < \beta)$, $k = 4, 5$.

4. Problems $(1)^+$ and $(2)^-$

Consider the equations (2.4), (2.10)

$$-g(z) + \frac{1}{\pi} \int\limits_S \left[T(\partial y, n)\Gamma(y-z)\right]^T g(y)\, d_y S = F^+(z), \tag{4.1}$$

$$-h(z) + \frac{1}{\pi}\int\limits_S T(\partial z,\nu)\Gamma(y-z)h(y)\,d_yS = \Phi^-(z), \tag{4.2}$$

where $F^+ \in C^{1,\beta}(S)$, $\Phi^- \in C^{1,\beta}(S)$, $0<\alpha<\beta$ are given vectors on the boundary.

Let us prove that the homogeneous equation corresponding to (4.2) has only the trivial solution. Assume that it has a nontrivial solution denoted by $h_0(z)$. Compose the simple layer potential

$$V(x) = \frac{1}{4\pi}\int\limits_S \Gamma(y-x)h_0(y)\,dS. \tag{4.3}$$

It is obvious, that $[T(\partial z,n)V(z)]^- = 0$, $\int\limits_S h_0(y)\,ds = 0$. $V \in C^{0,\beta}(D^-)$ and satisfies the conditions (1.4). This implies that $V(z)=0$, $z\in D^-$, whence $V^+ = V^- = 0$. Now taking into account the continuity of the simple layer potential and using the uniqueness theorem for the solution of the first boundary value problem, we will have $V(x)=0$, $x\in D^+$. Thus $V(x)$ vanishes on the whole space and therefore $h_0(x)=0$. Due to the fredholm theorem we conclude that the nonhomogeneous equation is solvable for an arbitrary Hölder continuous vector Φ^-.

Finally, from the solvability of the equations (4.1) and (4.2) it follows that the solutions of BVPs $(1)^+$ and $(2)^-$ are representable in the form of second kind double and single-layer potentials, respectively. On the basis of the general theory, the following theorems are valid.

Theorem 8. *If $S\in C^{2,\beta}(S)$ and $F^+\in C^{1,\beta}$, then the BVP $(1)^+$ has unique solution. Moreover, this solution is given in the form of the double-layer potential* (2.5)*, where g is a solution of the equation* (4.1).

Theorem 9. *If $S\in C^{2,\beta}(S)$ and $\Phi^-\in C^{1,\beta}$, then the BVP $(2)^-$ has unique solution satisfying the conditions* (1.4) *in the neighborhood of infinity. Moreover, this solution is given in the form of the single-layer potential* (2.8)*, where h is a solution of the equation* (4.2).

5. Problems $(1)^-$ and $(2)^+$

Consider the first external BVP (when on S it is given $V^- = F^-$). A solution of the equation (1.15) is sought in the form

$$V(x,g) = \frac{1}{2\pi}\int\limits_S \big[N(\partial y,n)\Gamma(z-x)\big]^* g(y)\,d_yS + \frac{1}{2}\,\Gamma(x)\alpha, \tag{5.1}$$

where

$$\alpha = \frac{1}{2\pi}\int\limits_S \big[N(\partial y,n)\Gamma(y)\big]^* g(y)\,d_yS.$$

The origin is assumed to lie in the domain D^+. Taking into account the boundary behavior of the potential $V(x)$ and the boundary condition, to define the unknown vector g from (5.1) we obtain the Fredholm integral equation of the second kind

$$g(z) + \frac{1}{2\pi} \int\limits_S \big[N(\partial y, n)\Gamma(y-z)\big]^* g(y)\, d_y S + \frac{1}{2}\,\Gamma(z)\alpha = F^-(z). \tag{5.2}$$

The conjugate equation is

$$h(z) + \frac{1}{2\pi} \int\limits_S \Big[N(\partial y, n)\Gamma(y-z) + \frac{1}{2}\, N(\partial z, n)\Gamma(z)\alpha\Big] h(y)\, d_y S = \Phi^+(z). \tag{5.3}$$

Let us show that the equation (5.3) is always solvable. For this it is sufficient to show that the homogeneous equation corresponding to (5.3) has only the trivial solution. Denote the homogeneous equation by $(5.3)_0$ and assume that it has a solution h_0 different from zero.

From $(5.3)_0$ we get

$$\frac{1}{2\pi} \int\limits_S \Gamma(y)h(y)\, d_y S = 0. \tag{5.4}$$

and the equation $(5.3)_0$ obtain the form

$$h(z) + \frac{1}{2\pi} \int\limits_S N(\partial y, n)\Gamma(y-z)h(y)\, d_y S = 0. \tag{5.5}$$

Construct now the potential

$$V(x) = \frac{1}{2\pi} \int\limits_S \Gamma(x-y)h_0(y)\, d_y S.$$

Here $N(V)^+ = 0$ and $V(0) = 0$. From this we get $V(x) = 0$, $x \in D^-$. Since $h_0(x) = 0$. Thus our assumption is not valid. The equation (5.2) has a solution for an arbitrary right-hand side. Note that a solution of the equation (5.2) exists if $S \in C^{2,\beta}(S)$, $F^-(z) \in C^{1,\beta}(S)$, $0 < \beta < \alpha \leq 1$.

Consider the second BVP. The solution of the equation (1.15) is sought in the form (when on S it is given $(TV)^+ = \Phi^+$)

$$V(x) = \frac{1}{2\pi} \int\limits_S \Gamma(y-z)h(y)\, d_y S - \frac{1}{2}\,\Gamma(x)A - \frac{1}{2}\,\Gamma_0(x)B, \tag{5.6}$$

where A and B are defined as follows:

$$A = \frac{1}{2\pi} \int\limits_S \Gamma(y)h(y)\, d_y S, \quad B = \frac{1}{2\pi} \int\limits_S \Gamma_0(y)h(y)\, d_y S, \tag{5.7}$$

$$\Gamma_0 = \operatorname{rot}_x \Gamma(x-y)_{x=0}.$$

To define $h(z)$, we have the integral equation

$$h(z) + \frac{1}{2\pi} \int_S T(\partial y, n)\Gamma(y-z)h(y)\, d_y S -$$
$$\frac{1}{4}\, T(\partial z, n)\Gamma(z)A - \frac{1}{4}\, T(\partial z, n)\Gamma_0(z)B = \Phi^+(z), \quad (5.8)$$

Let us now show that the integral equation (5.8) is always solvable. Let $h(y) \neq 0$. From (5.8) we have

$$A = \int_S \Phi^+(z)\, dS, \tag{5.9}$$

$$B = \frac{1}{2\pi} \int_S r(y) \times \Phi^+\, dS, \tag{5.10}$$

where $r(y) = (y_1, y_2, y_3)$. If $\Phi^+ = 0$, then $A = 0$, $B = 0$, $(Tu)^+ = 0$, $u = a + [b, r]$. If the principal vector A and the principal moment B are equal to zero, we have $u = 0$, $h = 0$. Thus (5.8) is solvable for any right-hand side.

Consider the conjugate equation

$$g(z) + \frac{1}{2\pi} \int_S T(\partial y, n)\Gamma(y-z)^* g(y)\, d_y S - \frac{1}{2}\, \Gamma(z)\alpha - \frac{1}{4}\, \Gamma_0(z)\beta =$$
$$= F^-(z), \tag{5.11}$$

where

$$\alpha = \frac{1}{2\pi} \int_S \big[T(\partial y, n)\Gamma(y)\big]^* g(y)\, d_y S,$$
$$\beta = \frac{1}{2\pi} \int_S \big[T(\partial y, n)\Gamma_0(y)\big]^* g(y)\, d_y S.$$

The equation (5.11) is always solvable if $F^- \in C^{1,\alpha}(S)$, $S \in C^{1,\alpha}(S)$, $0 < \beta < \alpha \leq 1$.

If the solution of BVP $(1)^-$ is sought in the form

$$V(x) = \frac{1}{2\pi} \int_S \big[T(\partial y, n)\Gamma(y-z)\big]^* g(y)\, dS - \frac{1}{2}\, \Gamma(x)\alpha - \frac{1}{4}\, \Gamma_0(x)\beta, \tag{5.12}$$

then to define the unknown vector g we obtain the integral equation (5.11).

Therefore we formulate the final result.

Theorem 10. *The problem $(1)^-$ is solvable for an arbitrary vector $F^- \in C^{1,\beta}(S)$ for $S \in C^{2,\alpha}(S)$, and the solution is represented by the formula (5.12).*

Theorem 11. *The problem* $(2)^+$ *is solvable for the vector* $\Phi^+ \in C^{0,\beta}(S)$, *only if the principal vector and the principal moment of external stresses are equal to zero,* $A = 0$ *and* $B = 0$. *The solution is represented by the formula* (5.6). *The solution is defined to within rigid displacement.*

The existence theorems for the third and fourth BVPs will be proved analogously.

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