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UPPER AND LOWER SOLUTIONS METHOD FOR IMPULSIVE DIFFERENTIAL EQUATIONS INVOLVING THE CAPUTO FRACTIONAL DERIVATIVE

Abstract. For impulsive differential equations involving the Caputo fractional derivative, sufficient conditions for the solvability of initial value problem are established using the lower and upper solutions method and Schauder's fixed point theorem.

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1. Introduction

This paper is concerned the existence of solutions for the initial value problems (IVP for short), for impulsive fractional order differential equation

$$\begin{aligned} &{}^cD^\alpha y(t) = f(t, y(t)) \\ &\text{for each } \; t \in J = [0, T], \;\; t \neq t_k, \;\; k = 1, \dots, m, \;\; 0 < \alpha \leq 1, \end{aligned} \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \;\; k = 1, \dots, m, \tag{2}$$

$$y(0) = y_0, \tag{3}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R}$ is a continuous function, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$ and $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y\big|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim\limits_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim\limits_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1, \dots, m$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [12, 16, 17, 19, 25, 26, 28]). There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas *et al.* [21], Kiryakova [22], Lakshmikantham *et al.* [24], Miller and Ross [27], Samko *et al.* [32] and the papers of Agarwal *et al.* [1, 2], Belarbi *et al.* [5, 6], Benchohra *et al.* [7, 8, 10], Diethelm *et al.* [12, 13, 14], Furati and Tatar [15], Kilbas and Marzan [20], Mainardi [25], Podlubny *et al.* [31], and the references therein.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $y(0)$, $y'(0)$, etc. the same requirements of boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville and Caputo types see [18, 30].

Integer order impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Bainov and Simeonov [4], Benchohra *et al.* [9], Lakshmikantham *et al.* [23], and Samoilenko and Perestyuk [33] and the references therein. In [3, 11] Agarwal *et al.* and Benchohra and Slimani have initiated the study of fractional differential equations with impulses.

By means of the concept of upper and lower solutions combined with Schauder's fixed point theorem, we present an existence result for the problem (1)–(3). This paper initiates the application of the upper and lower solution method to impulsive fractional differential equations at fixed moments of impulse.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $[a,b]$ be a compact interval. $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\left\{|y(t)| : \ a \leq t \leq b\right\},$$

and we let $L^1([a,b],\mathbb{R})$ the Banach space of functions $y : [a,b] \longrightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_a^b |y(t)|\,dt.$$

Definition 2.1 ([21, 29])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,h(s)\,ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \rightarrow \delta(t)$ as $\alpha \rightarrow 0$, where δ is the delta function.

Definition 2.2 ([21, 29])**.** For a function h given on the interval $[a,b]$, the αth Riemann–Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}h(s)\,ds.$$

Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([21])**.** For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)\,ds,$$

where $n = [\alpha]+1$.

3. Main Result

Consider the following space

$$\begin{aligned} PC(J,\mathbb{R}) = \Big\{ & y : \ J \rightarrow \mathbb{R} : y \in C((t_k, t_{k+1}],\mathbb{R}), \ k = 0,\ldots,m+1 \\ & \text{and there exist } y(t_k^-) \text{ and } y(t_k^+), \ k = 1,\ldots,m \text{ with } y(t_k^-) = y(t_k)\Big\}. \end{aligned}$$

$C(J,\mathbb{R})$ is a Banach space with norm

$$\|y\|_{PC} = \sup_{t\in J} |y(t)|.$$

Set $J' := [0,T] \setminus \{t_1,\dots,t_m\}$.

Definition 3.1. A function $y \in PC(J,\mathbb{R}) \cap C^1(J',\mathbb{R})$ is said to be a solution of (1)–(3) if satisfies the differential equation ${}^cD^\alpha y(t) = f(t,y(t))$ on J', and conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ \ k = 1,\dots,m,$$

and

$$y(0) = y_0$$

are satisfied.

Definition 3.2. A function $u \in PC(J,\mathbb{R}) \cap C^1(J',\mathbb{R})$ is said to be a lower solution of (1)–(3) if ${}^cD^\alpha u(t) \le f(t,u(t))$ on J', $\Delta u|_{t=t_k} \le I_k(u(t_k^-))$, $k = 1,\dots,m$, and $u(0) \le y_0$. Similarly, a function $v \in PC(J,\mathbb{R}) \cap C^1(J',\mathbb{R})$ is said to be an upper solution of (1)–(3) if ${}^cD^\alpha v(t) \ge f(t,v(t))$ on J', $\Delta v|_{t=t_k} \ge I_k(v(t_k^-))$, $k = 1,\dots,m$, and $v(0) \ge y_0$.

For the existence of solutions for the problem (1)–(3), we need the following auxiliary lemmas:

Lemma 3.3 ([21])**.** *Let $\alpha > 0$. Then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

Lemma 3.4 ([21])**.** *Let $\alpha > 0$. Then*

$$I^{\alpha c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1} + I^\alpha h(t)$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

As a consequence of Lemma 3.3 and Lemma 3.4 we have the following result which is useful in what follows. The proof may be found in [11]. For the completeness we present it.

Lemma 3.5. *Let $0 < \alpha \le 1$ and let $\rho \in PC(J,\mathbb{R})$. A function $y \in PC(J,\mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}\rho(s)\,ds \quad \text{if } t \in [0,t_1], \\ y_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\rho(s)\,ds + \dfrac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1}\rho(s)\,ds + \\ \quad + \displaystyle\sum_{i=1}^k I_i(y(t_i^-)), \quad \text{if } t \in (t_k, t_{k+1}],\ k = 1,\dots,m \end{cases} \tag{4}$$

if and only if $y \in PC(J,\mathbb{R}) \cap C^1(J',\mathbb{R})$ *is a solution of the fractional IVP*

$$^cD^\alpha y(t) = \rho(t) \ \text{ for each } \ t \in J', \tag{5}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{6}$$

$$y(0) = y_0. \tag{7}$$

Proof. Assume that y satisfies (5)–(7). If $t \in [0, t_1]$, then

$$^cD^\alpha y(t) = \rho(t).$$

Lemma 3.4 implies

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \rho(s)\, ds.$$

If $t \in (t_1, t_2]$, then Lemma 3.4 implies

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \rho(s)\, ds = \\ &= \Delta y\big|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \rho(s)\, ds = \\ &= I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} \rho(s)\, ds + \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \rho(s)\, ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 3.4 we get

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} \rho(s)\, ds = \\ &= \Delta y\big|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} \rho(s)\, ds = \\ &= I_2(y(t_2^-)) + I_1(y(t_1^-)) + y_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} \rho(s)\, ds + \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \rho(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} \rho(s)\, ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 3.4 we get (4).

Conversely, assume that y satisfies the impulsive fractional integral equation (4). If $t \in [0, t_1]$, then $y(0) = y_0$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α we get

$$ {}^cD^\alpha y(t) = \rho(t) \quad \text{for each} \quad t \in [0, t_1]. $$

If $t \in [t_k, t_{k+1})$, $k = 1, \ldots, m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$$ {}^cD^\alpha y(t) = \rho(t) \quad \text{for each} \quad t \in [t_k, t_{k+1}). $$

Also, we can easily show that

$$ \Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m. \qquad \square $$

For the study of this problem we first list the following hypotheses:

(H1) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous;

(H2) There exist u and $v \in PC \cap C^1(J', \mathbb{R})$, lower and upper solutions for the problem (1)–(3) such that $u \leq v$.

(H3)

$$ u(t_k^+) \leq \min_{y \in [u(t_k^-), v(t_k^-)]} I_k(y) \leq \max_{y \in [u(t_k^-), v(t_k^-)]} I_k(y) \leq v(t_k^+), \quad k = 1, \ldots, m. $$

Theorem 3.6. *Assume that hypotheses* (H1)–(H3) *hold. Then the problem* (1)–(3) *has at least one solution y such that*

$$ u(t) \leq y(t) \leq v(t) \quad \textit{for all} \quad t \in J. $$

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the following modified problem,

$$ {}^cD^\alpha y(t) = f_1(t, y(t)), \quad t \in J, \quad t \neq t_k, \quad k = 1, \ldots, m, \quad 0 < \alpha \leq 1, \tag{8} $$

$$ \Delta y|_{t=t_k} = I_k(\tau(t_k^-, y(t_k^-)), \quad k = 1, \ldots, m, \tag{9} $$

$$ y(0) = y_0, \tag{10} $$

where

$$ f_1(t, y) = f(t, \tau(t, y)), $$
$$ \tau(t, y) = \max\{u(t), \min(y, v(t))\}. $$

A solution for (8)–(10) is a fixed point of the operator $N : PC(J, \mathbb{R}) \longrightarrow PC(J, \mathbb{R})$ defined by

$$ N(y)(t) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f_1(s, \overline{y}(s))\, ds + $$

$$ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} f_1(s, \overline{y}(s))\, ds + \sum_{0<t_k<t} I_k\big(\tau(t_k^-, y(t_k^-))\big). $$

Note that f_1 is a continuous function and from (H2) there exists $M > 0$ such that

$$|f_1(t, y)| \le M \text{ for each } t \in J \text{ and } y \in \mathbb{R}. \tag{11}$$

Also, by the definition of τ and from (H3) we have

$$u(t_k^+) \le I_k(\tau(t_k, y(t_k))) \le v(t_k^+), \quad k = 1, \dots, m. \tag{12}$$

Set

$$\eta = |y_0| + \frac{M}{\Gamma(\alpha+1)} \sum_{k=1}^{m} (t_k - t_{k-1})^\alpha + \\ + \frac{MT^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^{m} \max\left\{|u(t_k^+)|, |v(t_k^+)|\right\}$$

and consider the subset

$$D = \left\{y \in PC(J, \mathbb{R}) : \|y\|_{PC} \le \eta\right\}.$$

Clearly D is a closed, convex subset of $PC(J, \mathbb{R})$ and N maps D into D. We shall show that N satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in D. Then for each $t \in J$

$$\begin{aligned} &\left|N(y_n)(t) - N(y)(t)\right| \le \\ &\le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \left| f_1(s, \overline{y}_n(s)) - f_1(s, \overline{y}(s)) \right| ds + \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left| f_1(s, \overline{y}_n(s)) - f_1(s, \overline{y}(s)) \right| ds \\ &\quad + \sum_{0<t_k<t} \left| I_k\big(\tau\big(t_k^-, y_n(t_k^-)\big)\big) - I_k\big(\tau\big(t_k^-, y(t_k^-)\big)\big) \right|. \end{aligned}$$

Since f_1, I_k, $k = 1, \dots, m$, and τ are continuous functions, we have

$$\|N(y_n) - N(y)\|_{PC} \to 0 \text{ as } n \to \infty.$$

Step 2: *$N(D)$ is bounded.*

This is clear since $N(D) \subset D$ and D is bounded.

Step 3: *$N(D)$ is equicontinuous.*

Let τ_1, $\tau_2 \in J$, $\tau_1 < \tau_2$, and $y \in D$. Then

$$\left|N(\tau_2) - N(\tau_1)\right| = \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<\tau_2-\tau_1} \int_{t_{k-1}}^{t_k} \left|(t_k - s)^{\alpha-1}\right| \left|f_1(s, \overline{y}(s))\right| ds +$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right|f_1(s,\overline{y}(s))v|\,ds+$$

$$+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(\tau_2-s)^{\alpha-1}\right|\left|f_1(s,\overline{y}(s))\right|ds+\sum_{0<t_k<\tau_2-\tau_1}\left|I_k\big(\tau\big(t_k^-,y(t_k^-)\big)\big)\right|\le$$

$$\le\frac{M}{\Gamma(\alpha+1)}(t_k-t_{k-1})^\alpha+\frac{M}{\Gamma(\alpha)}\int_0^{\tau_1}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right|ds+$$

$$+\frac{M}{\Gamma(\alpha+1)}(\tau_2-\tau_1)^\alpha+\sum_{0<t_k<\tau_2-\tau_1}\left|I_k\big(\tau\big(t_k^-,y(t_k^-)\big)\big)\right|.$$

As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1–3 together with the Arzelá–Ascoli theorem, we can conclude that $N : D \to D$ is continuous and compact. From Schauder's theorem we deduce that N has a fixed point y which is a solution of the problem (8)–(10).

Step 4: *The solution y of* (8)–(10) *satisfies*

$$u(t) \le y(t) \le v(t) \text{ for all } t \in J.$$

Let y be the above solution of (8)–(10). We prove that

$$y(t) \le v(t) \text{ for all } t \in J.$$

Assume that $y-v$ attains a positive maximum on $[t_k^+,t_{k+1}^-]$ at $\bar{t}_k \in [t_k^+,t_{k+1}^-]$ for some $k=0,\dots,m$; that is,

$$(y-v)(\bar{t}_k)=\max\left\{y(t)-v(t):\ t\in[t_k^+,t_{k+1}^-]\right\}>0 \text{ for some } k=0,\dots,m.$$

We distinguish the following cases.

Case 1. If $\bar{t}_k \in (t_k^+,t_{k+1}^-)$, there exists $t_k^* \in (t_k^+,t_{k+1}^-)$ such that

$$y(t_k^*)-v(t_k^*)\le 0, \tag{13}$$

and

$$y(t)-v(t)>0 \text{ for all } t\in(t_k^*,\bar{t}_k]. \tag{14}$$

By the definition of τ one has

$${}^cD^\alpha y(t)=f(t,v(t)) \text{ for all } t\in[t_k^*,\bar{t}_k].$$

An integration on $[t_k^*,t]$ for each $t\in[t_k^*,\bar{t}_k]$ yields

$$y(t)-y(t_k^*)=\frac{1}{\Gamma(\alpha)}\int_{t_k^*}^{t}(t-s)^{\alpha-1}f(s,v(s))\,ds. \tag{15}$$

From (15) and using the fact that v is an upper solution to (1)–(3) we get

$$y(t)-y(t_k^*)\le v(t)-v(t_k^*). \tag{16}$$

Thus from (13), (14) and (16) we obtain the contradiction

$$0 < y(t) - v(t) \leq y(t_k^*) - v(t_k^*) \leq 0 \ \text{ for all } \ t \in [t_k^*, \bar{t}_k].$$

Case 2. If $\bar{t}_k = t_k^+$, $k = 1, \dots, m$, then

$$v(t_k^+) < I_k(\tau(t_k^-, y(t_k^-)) \leq v(t_k^+),$$

which is a contradiction. Thus

$$y(t) \leq v(t) \ \text{ for all } \ t \in [0, T].$$

Analogously, we can prove that

$$y(t) \geq u(t) \ \text{ for all } \ t \in [0, T].$$

This shows that the problem (8)–(10) has a solution in the interval $[u, v]$ which is solution of (1)–(3). □

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