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**COMPARISON OF ASYMPTOTIC METHOD
WITH EXPLICIT SOLUTIONS
IN RANDOM VIBRATION**

*Authors dedicate this study to the blessed memory
of Academician Nikoloz (Niko) Muskhelishvili
who was rightfully considered
“Mr. Elasticity” of the Twentieth Century*

Abstract. This paper contrasts the exact solutions for the mean-square displacements of a cantilever under random loading with those delivered by asymptotic method. It is shown that probabilistic response evaluated by the latter method may lead to considerable errors: in the case of a beam with viscous damping as much as 100%, and in that of structural damping 188%. A new, closed-form, solution is proposed for a cantilever with Voigt damping. In this case asymptotic method exhibits the maximum error of 312%.

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რეზიუმე. ნაშრომში მოცემული ზუსტი ამოხსნა უპირისპირდება კონსოლის საშუალო კვადრატული გადაადგილების ამოცანის ასიმპტოტურ ამოხსნას შემთხვევითი დატვირთვისათვის, რომლებსაც იძლევა ასიმპტოტური მეთოდი. ნაჩვენებია, რომ ალბათობითი უკუგების შეფასებას ნახსენები მეთოდით მიყვარათ არსებით ცდომილებამდე: ძელისთვის ბლანტი ქრობის შემთხვევაში ცდომილება აღწევს 100%-ს, ხოლო სტრუქტურული ქრობის შემთხვევაში 188%-ს. შემოთავაზებულია ფოიგტის ქრობით კონსოლის ამოცანის ამოხსნა ცხადი სახით. ასეთ შემთხვევაში ასიმპტოტური მეთოდის მაქსიმალური ცდომილება აღწევს 312%-ს.

INTRODUCTION

The normal-mode method is widely used in both deterministic [1] and random [2] vibration analyses. The given and sought functions of both kinds are expanded in terms of the eigenfunctions, whose orthogonality property allows one to derive equations for the coefficients or functions describing the time-wise behavior of the system. The response thus determined is an exact solution of the problem at hand: this straightforward method is however confined to the cases where explicit analytical expressions are available for the eigenfunctions, which is rarely the case.

In the sixties, Bolotin [3] devised a procedure (usually referred to as “dynamic edge effect” method or as “asymptotic” method) which permits approximate evaluation of the eigenvalues and eigenfunctions of a wide range of structures. A concise description of the asymptotic method including a relevant bibliography can be found in a review paper by Elishakoff [4]. The method gained considerable popularity in the East (see, e.g., the monographs, [5] and [6]) and in the West (see, e.g., References [7]–[12]) because of the accurate predictions it yielded for the high frequencies, where the energy methods are quite cumbersome.

The present paper examines the accuracy of the asymptotic approach in random vibration analysis. To this end, we consider a system amenable to closed-form solution, namely a cantilever beam (that is clamped at one end and free at the other) under white-noise “rain-on-the-roof” excitation both in space and in time. This problem was first considered by Houdijk [13], and the solution was reproduced by van Lear and Uhlenbeck [14] who study in addition the effect of gravity under single-term approximation. For detailed derivation of the mean-square tip displacement for a uniform cantilever Bernoulli-Euler beam via the normal-mode method, one can consult Eringen [15] (for final results see also [16]).

BEAM POSSESSING TRANSVERSE VISCOUS-DAMPING

Application of the normal-mode method yields the following formula [17] for the space-time correlation function $R_w(x_1, x_2, \tau)$ of the displacement $w(x, t)$ of a Bernoulli-Euler beam under transverse damping

$$R_w(x_1, x_2, \tau) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \psi_j(x_1) \psi_k(x_2) \int_{-\infty}^{\infty} S_{Q_j Q_k}(\omega) H_j^*(\omega) H_k(\omega) e^{i\omega\tau} d\omega, \quad (1)$$

where x_1 and x_2 are the observation points on the beam axes, τ is the time lag between the observation moments, $\psi_j(x)$ is the beam mode shapes in vacuo, $S_{Q_k Q_j}(\omega)$ are the cross-spectral densities of the generalized forces, $H_j(\omega)$ are the frequency response functions, $\omega =$ frequency, j and k are subscripts denoting the sequential numbers of the eigenvalues, the star denotes

the operation of complex conjugation. For a cantilever beam we have

$$\psi_j(\xi) = \frac{\cosh(m_j \xi) \cos(m_j \xi)}{\cosh(m_j) \cos(m_j)} - \frac{\sinh(m_j \xi) - \sin(m_j \xi)}{\sinh(m_j) + \sin(m_j)}, \quad (2)$$

$$\xi = \frac{x}{L}, \quad m_j^4 = \frac{\omega_j^2 \rho A L^4}{EI}, \quad (3)$$

where L is the length, ξ the dimensionless axial coordinate and m_j satisfies the following transcendental equation

$$\cos(m_j) \cosh(m_j) + 1 = 0. \quad (4)$$

The numerical values of m_j are as follows [18]:

$$\begin{aligned} m_1 = 1.875104, \quad m_2 = 4.694091, \quad m_3 = 7.854757, \\ m_4 = 10.995541, \quad m_5 = 14.137168, \quad m_6 = 17.278759. \end{aligned} \quad (5)$$

For higher values of m_j the following asymptotic formula holds with sufficient accuracy

$$m_j \approx \frac{1}{2} (2j - 1)\pi. \quad (6)$$

The cross-spectral densities are given by the formula

$$S_{Q_j Q_k}(\omega) = \frac{1}{v_j^2 v_k^2} \int_0^L \int_0^L S_q(x_1, x_2, \omega) \psi_k(x_2) dx_1 dx_2, \quad (7)$$

where

$$v_j^2 = \int_0^L \psi_j^2(x) dx = L \quad (j = 1, 2, \dots). \quad (8)$$

For “rain-on-the-roof” excitation the space-time correlation function reads

$$R_q(x_1, x_2, \tau) = \frac{R}{L} \delta(x_1 - x_2) \delta(\tau), \quad (9)$$

where R is a positive constant. The cross-spectral density of the excitation is

$$S_q(x_1, x_2, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_q(x_1, x_2, \tau) e^{-i\omega\tau} d\tau = \frac{R}{2\pi L} \delta(x_1 - x_2), \quad (10)$$

with associated cross-spectral densities given by

$$S_{Q_j Q_k}(\omega) = \frac{1}{v_j^2} \frac{R}{2\pi L} \delta_{jk}, \quad (11)$$

where δ_{jk} is the Kronecker delta. Since $\delta_{jk} = 0$ for non-coincident indices, the double summation in (1) reduces to a single one

$$R_w(x_1, x_2, \tau) = \sum_{j=1}^{\infty} \frac{R}{2\pi L} \frac{1}{v_j^2} \psi_j(x_1) \psi_j(x_2) \int_{-\infty}^{\infty} |H_j(\omega)|^2 e^{i\omega\tau} d\omega. \quad (12)$$

In particular, at the cantilever tip $x = L$ we have

$$R_w(L, L, \tau) = \sum_{j=1}^{\infty} \frac{R}{2\pi L} \frac{1}{v_j^2} \psi_j^2(L) \int_{-\infty}^{\infty} |H_j(\omega)|^2 e^{i\omega\tau} d\omega. \quad (13)$$

It was shown by Lord Rayleigh [18, Eq. 164.9] that

$$v_j^2 = \frac{L}{4} \psi_j^2(L) \quad (14)$$

(see also Prescott [19]). Therefore,

$$R_w(L, L, \tau) = \frac{2R}{\pi L^2} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} |H_j(\omega)|^2 e^{i\omega\tau} d\omega. \quad (15)$$

Now

$$\begin{aligned} \int_{-\infty}^{\infty} |H_j(\omega)|^2 e^{i\omega\tau} d\omega &= \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{(\rho A)^2 [(\omega_j^2 - \omega^2)^2 + 4\zeta_j^2 \omega_j^2 \omega^2]} d\omega = \\ &= \frac{d^2}{(\rho A)^2} e^{-\zeta_j \omega_j |\tau|} \left(\cos \omega_{jd} \tau + \frac{\zeta_j \omega_j}{\omega_{jd}} \sin \omega_{jd} |\tau| \right), \end{aligned} \quad (16)$$

where

$$\zeta_j = \frac{c}{2\rho A \omega_j}, \quad \omega_{jd} = \omega_j (1 - \zeta_j^2)^{1/2}, \quad d^2 = \frac{\pi}{2\eta_j \omega_j^3} \quad (17)$$

and ρ is the mass density, A is the cross sectional area, ω_{jd} = “damped” eigenvalue. For the mean-square tip displacement the formula reads

$$R_w(L, L, 0) = \frac{2R}{c\rho AL^2} \sum_{j=1}^{\infty} \frac{1}{\omega_j^2}, \quad (18)$$

that is, with Eq. (3) taken into account,

$$R_w(L, L, 0) = \frac{2RL^2}{E I c} \sum_{j=1}^{\infty} \frac{1}{m_j^4}, \quad (19)$$

where m_j 's are the roots of Eq. (4) with numerical values as per Eqs. (5) and (6).

According to Lord Rayleigh [18, Eq. 175.3], the sum in Eq. (19) equals

$$\sum_{j=1}^{\infty} \frac{1}{m_j^4} = \frac{1}{12} \quad (20)$$

so that the final expression for the mean-square tip displacement becomes

$$R_w(L, L, 0) = \frac{RL^2}{6EIc}. \quad (21)$$

The equation (21) is due to Houdijk [13] and Eringen [15]. It shows that the mean-square displacement of a cantilever subjected to “rain-on-the-roof” excitation does not depend on the density of the bar material - as

is also the case with a simply-supported beam, see [14] and [16]. Both these results are analogous to that for a single-degree-of-freedom system and ideal white-noise excitation.

It is instructive to predict the mean square displacement, via an asymptotic estimate for the frequency parameter by Eq. (6). Substituting Eq. (6) into (19), we have

$$R_w(L, L, 0) = \frac{32RL^2}{\pi^4 EIc} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^4}. \quad (22)$$

However,

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^4} = \frac{\pi^4}{96} \quad (23)$$

(see Eq. 0.234.5 in [20]). With this value

$$R_w(L, L, 0) = \frac{RL^2}{3EIc}, \quad (24)$$

substituted, we obtain double the exact value according to Eq. (21). This clearly demonstrates the need for caution in straightforward using an asymptotic method for predicting the response.

It is also of interest to look into the error involved in treating an infinite-degree-of-freedom system (a beam) as a single-degree one. In such a case retaining only the first term in Eq. (18) yields

$$R_w(L, L, 0) = \frac{2RL^2}{EIc} \frac{1}{m_1^4} = \frac{2RL^2}{(1.875104)^4} = \frac{RL^2}{6.18EIc}. \quad (25)$$

Comparing the coefficient 6.18 with the exact value 6, we see that the attendant error is less than 3%. We conclude that in this case including only the one accurately calculated frequency results in a much smaller error, than taking an infinite number of frequencies, evaluated asymptotically.

BEAM POSSESSING STRUCTURAL DAMPING

For this case the differential equation reads

$$\tilde{E} \left(\frac{\partial}{\partial t} \right) I \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = q(x, t), \quad (26)$$

where the operator is defined as

$$\tilde{E} \left(\frac{\partial}{\partial t} \right) e^{i\omega t} = E(1 + i\mu)e^{i\omega t}, \quad (27)$$

μ being a structural damping coefficient [17]. For the autocorrelation function with zero time lag we obtain [21]

$$R_w(x_1, x_2, 0) = \frac{R}{2\pi(\rho AL)^2} \sum_{j=1}^{\infty} \varphi_j(x_1) \varphi_j(x_2) \int_{-\infty}^{\infty} \frac{d\omega}{(\omega_j^2 - \omega^2)^2 + \omega_j^2 \mu^2}. \quad (28)$$

For the cantilever tip we have

$$R_w(L, L, 0) = \frac{2R}{(\rho AL)^2} \frac{(1 + \sqrt{1 + \mu^2})^{1/2}}{\mu \sqrt{2(1 + \mu^2)}} \sum_{j=1}^{\infty} \frac{1}{\omega_j^3}. \quad (29)$$

There is no closed-form expression for the sum [22]. Let us first evaluate the appropriate expression for $R_w(L, L, 0)$ through the asymptotic evaluation of ω_j ; from [20, Eq. (1846)] we have

$$\sum_{j=1}^{\infty} \frac{1}{(2j - 1)^{2n}} = \frac{(2^{2n} - 1)\pi^{2n}}{2(2n)!} |B_{2n}|, \quad (30)$$

where B_{2n} is a Bernoulli number with index $2n$. We have $n = 3$, so that

$$|B_6| = \frac{1}{42}, \quad \sum_{j=1}^{\infty} \frac{1}{(2j - 1)^6} = \frac{\pi^6}{960}. \quad (31)$$

Asymptotic evaluation yields then

$$\begin{aligned} R_w(L, L, 0) &= \frac{2^6 \sqrt{2}}{960}, \frac{RL^4}{\sqrt{\rho AEI EI}} \frac{(1 + \sqrt{1 + \mu^2})^{1/2}}{\mu \sqrt{1 + \frac{2}{\mu}}} = \\ &= 0.09428 \frac{RL^4}{\sqrt{\rho AEI EI}} \frac{(1 + \sqrt{1 + \mu^2})^{1/2}}{\mu \sqrt{1 + \frac{2}{\mu}}}. \end{aligned} \quad (32)$$

For comparison with the exact value, we will evaluate Eq. (29) numerically, summing the first six terms exactly, and using an asymptotic expression for the remainder. We thus have

$$\begin{aligned} \sum_{j=1}^6 \frac{1}{\omega_j^3} &= \left(\frac{\rho AL^4}{EI}\right)^{3/2} \sum_{j=1}^6 \frac{1}{m_j^6} = \left(\frac{\rho AL^4}{EI}\right)^{3/2} = \\ &= \left[\frac{1}{1.875104^6} + \frac{1}{4.694091^6} + \frac{1}{7.854757^6} + \right. \\ &\quad \left. + \frac{1}{10.995541^6} + \frac{1}{14.137168^6} + \frac{1}{17.278759^6} \right] = \\ &= 0.02310 \left(\frac{\rho AL^4}{EI}\right)^{3/2}, \end{aligned} \quad (33)$$

$$\sum_{j=7}^{\infty} \frac{1}{\omega_j^6} \left(\frac{\rho AL^4}{EI}\right)^{3/2} \sum_{j=7}^{\infty} \frac{1}{m_j^6} = \left(\frac{\rho AL^4}{EI}\right)^{3/2} \left(\frac{2}{\pi}\right)^6 \sum_{j=7}^{\infty} \frac{1}{(2j - 1)^6}, \quad (34)$$

and

$$\begin{aligned}
\left(\frac{\rho AL^4}{EI}\right)^{3/2} \left(\frac{2}{\pi}\right)^6 \sum_{j=7}^{\infty} \frac{1}{(2j-1)^6} &= \\
&= \left(\frac{\rho AL^4}{EI}\right)^{3/2} \left(\frac{2}{\pi}\right)^6 \left[\sum_{j=1}^{\infty} \frac{1}{(2j-1)^6} - \sum_{j=1}^6 \frac{1}{(2j-1)^6} \right] = \\
&= \left(\frac{\rho AL^4}{EI}\right)^{3/2} \left(\frac{\pi^6}{960} - 1 - \frac{1}{3^6} - \frac{1}{5^6} - \frac{1}{7^6} - \frac{1}{9^6} - \frac{1}{11^6} \right) = \\
&= 4.2871 \times 10^{-6} \left[\frac{\rho AL^4}{EI} \right]^{3/2}. \tag{35}
\end{aligned}$$

Adding up the two subsums (33) and (35) (the latter being practically negligible relative to the former), we finally obtain

$$\sum_{j=1}^{\infty} \frac{1}{\omega_j^3} = 0.023104 \left(\frac{\rho AL^4}{EI} \right)^{3/2}$$

and

$$R_w(L, L, 0)_{exact} = 0.03267 \frac{RL^4}{EI\sqrt{\rho AEI}} \frac{(1 + \sqrt{1 + \mu^2})^{1/2}}{\mu\sqrt{1 + \mu^2}}. \tag{36}$$

Comparison of the formulas (32) and (36) demonstrates that the error incurred by using asymptotic summation is 188%.

If one could treat the beam as a one-degree-of-freedom system, the response would constitute only

$$\begin{aligned}
R_w(L, L, 0) &= \frac{2R}{(\rho AL)^2} \frac{(1 + \sqrt{1 + \mu^2})^{1/2}}{\mu\sqrt{2(1 + \mu^2)}} \frac{1}{\omega_j^3} = \\
&= \frac{\sqrt{2}}{m_1^6} \frac{RL^4}{EI\sqrt{\rho AEI}} \frac{(1 + \sqrt{1 + \mu^2})^{1/2}}{\mu\sqrt{2(1 + \mu^2)}} = \\
&= 0.325359 \frac{RL^4}{EI\sqrt{\rho AEI}} \frac{(1 + \sqrt{1 + \mu^2})^{1/2}}{\mu\sqrt{2(1 + \mu^2)}}
\end{aligned}$$

that is, the difference of less than one percent.

BEAM WITH VOIGT DAMPING

The governing differential equation reads

$$EI \left(1 + \delta \frac{\partial}{\partial t} \right) \frac{\partial^4}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = q(x, t), \tag{37}$$

where δ is the Voigt damping coefficient. For the autocorrelation function of the cantilever tip at zero time-lag we derive (compare with Eq. (15))

$$R_w(L, L, 0) = \frac{2R}{\pi(\rho AL)^2} \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{(\omega_j^2 - \omega^2)^2 + \delta^2 \omega_j^2 \omega^2}. \tag{38}$$

Using Eq. (36) in [21], we arrive at

$$R_w(L, L, 0) = \frac{2R}{\pi(\rho AL)^2} \frac{\pi}{6} \sum_{j=1}^{\infty} \frac{1}{\omega_j^4}. \quad (39)$$

With Eq. (3) in mind, we have

$$R_w(L, L, 0) = \frac{2R}{\pi(\rho AL)^2} \left[\frac{\rho AL^4}{EI} \right] \sum_{j=1}^{\infty} \frac{1}{m_j^8}. \quad (40)$$

According to Lord Rayleigh [18, Eq. (3)] we have

$$\sum_{j=1}^{\infty} \frac{1}{m_j^8} = \frac{1}{12^2} \cdot \frac{33}{35}. \quad (41)$$

Hence, the exact value is

$$R_w(L, L, 0) = \frac{33}{2520} \cdot \frac{RL^6}{\delta(EI)^2} \simeq 0.0131 \cdot \frac{RL^6}{\delta(EI)^2}. \quad (42)$$

The alternate derivation of Eq. (41) is given in the Appendix. Note that the one-term approximation in this case yields a negligible error of 0.066%.

To compare this benchmark solution with an asymptotic one, we will evaluate the approximation

$$\begin{aligned} R_w(L, L, 0) &= \frac{2R}{\delta(\rho AL)^2} \left(\frac{\rho AL^4}{EI} \right)^2 \sum_{j=1}^{\infty} \frac{(2/\pi)^8}{(2j-1)^8} = \\ &= \frac{2^9}{\pi^8} \frac{RL^6}{(EI)^2} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^8}. \end{aligned} \quad (43)$$

In view of Eq. (30) we get

$$B_8 = -\frac{1}{30}, \quad \sum_{j=1}^{\infty} \frac{1}{(2j-1)^8} = \frac{(2^8-1)\pi^8}{2 \cdot 8!} |B_8| = \frac{17\pi^8}{161280}. \quad (44)$$

Hence the asymptotic expression becomes

$$R_w(L, L, 0) = \frac{17}{135} \frac{RL^6}{\delta(EL)^2} \simeq 0.0540 \frac{RL^6}{\delta(EI)^2}. \quad (45)$$

Comparison of Eqs. (42) and (45) reveals the error of 312%. In this case the asymptotic method shows the worst performance.

CONCLUSION

The above examples bring out the errors involved in using asymptotic expression for the eigenvalues in wide-band random vibration analysis. In the case considered, the prediction accuracy of the first eigenvalue by the asymptotic method is very poor: it yields $\pi/2$ instead of 1.875104 for m_1 . Accordingly, extreme care should be exercised in such cases, which are usually

associated with the lower end of the eigenvalues (for example, for plates having some free boundaries, [3]). Therefore, whereas the asymptotic method is convenient, it is advisable to combine it with numerical techniques like the finite-element method, which is especially powerful in the frequency range where the asymptotic method may fail to predict frequencies (and hence the response) with sufficient accuracy.

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The first author had the good fortune to meet academician Niko Muskhelishvili at a scientific conference in 1964 and to have an impressive conversation. The quote attributed to him by Academician Khvedelidze [23] is forever stamped in our memory (in a free translation from the Georgian language): “If you are Riemann, the reader will forgive you a sloppiness, your work will somehow be read and analyzed. But if you are an ordinary mortal mathematician, then, if your manuscript is not well and clearly written, it will have no readers”. We try to impress Muskhelishvili’s insight on our colleagues and students.

This study closes a circle of sorts: the dynamic edge effect method was introduced by Academician Vladimir Vasilievitch Bolotin in 1961 in the Muskhelishvili 60-th Birth Anniversary volume; now, 60 years later, we are revisiting this method with attendant remarkable results.

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APPENDIX: DERIVATION OF EQ. (20) AND EQ. (41)

Using the expression for $\cosh m$ in Eq. (4), as given in [20, Eq. GL.505]

$$\cosh m = \sec m + \sec m \sum_{j=1}^{\infty} (-1)^j \frac{2^{2j} m^{4j}}{(4j)!}, \quad (\text{A.1})$$

we obtain

$$S = \frac{1}{2} (\cos m \cosh m + 1) = 1 + \frac{1}{2} \sum_{j=1}^{\infty} (-1)^j \frac{2^{2j} m^{4j}}{(4j)!}. \quad (\text{A.2})$$

Now, representing S as the following sum

$$S = 1 + \sum_{j=1}^{\infty} S_j, \quad (\text{A.3})$$

we get

$$S_1 = \frac{2^4}{4!} m^4 = -\frac{m^4}{12}, \quad (\text{A.4})$$

$$S_2 = \frac{2^2 m^8}{8!} m^4 = \frac{m^8}{12^2 \cdot 35} \quad (\text{A.5})$$

as they are also given by Rayleigh [18].

Since m_j are the roots of the characteristic equation (4), we have

$$\frac{1}{2} [\cos m \cosh m + 1] = \sum_{j=1}^{\infty} \left(1 - \frac{m^4}{m_j^4}\right) = 1 + \sum_{j=1}^{\infty} a_j m^{4j}. \quad (\text{A.6})$$

The first coefficient in the right-hand side of this equation is

$$a_1 = -\sum_{j=1}^{\infty} \frac{1}{m_j^4}. \quad (\text{A.7})$$

Hence, comparison with Eq. (A.4) reveals that

$$\sum_{j=1}^{\infty} \frac{1}{m_j^4} = \frac{1}{12}. \quad \square \quad (\text{A.8})$$

Now, the coefficient a_2 at m^8 is

$$a_2 = \frac{1}{m_1^4} \sum_{j=2}^{\infty} \frac{1}{m_j^4} + \frac{1}{m_2^4} \sum_{j=3}^{\infty} \frac{1}{m_j^4} + \cdots + \frac{1}{m_n^4} \sum_{j=n+1}^{\infty} \frac{1}{m_j^4} + \cdots = \frac{1}{12^2 \cdot 35}.$$

But

$$\begin{aligned} \frac{1}{m_1^4} \sum_{j=2}^{\infty} \frac{1}{m_j^4} &= \frac{1}{m_1^4} \left[\sum_{j=1}^{\infty} \frac{1}{m_j^4} - \frac{1}{m_1^4} \right] = \frac{1}{12} \frac{1}{m_1^4} - \frac{1}{m_1^8}, \\ M \frac{1}{m_2^4} \sum_{j=3}^{\infty} \frac{1}{m_j^4} &= \frac{1}{m_2^4} \left[\sum_{j=1}^{\infty} \frac{1}{m_j^4} - \frac{1}{m_1^4} - \frac{1}{m_2^4} \right] = \frac{1}{12} \frac{1}{m_1^4} - \frac{1}{m_2^8} - \frac{1}{m_1^4 m_2^4}, \\ \frac{1}{m_3^4} \sum_{j=4}^{\infty} \frac{1}{m_j^4} &= \frac{1}{12} \frac{1}{m_3^4} - \frac{1}{m_3^8} - \frac{1}{m_2^4 m_3^4} - \frac{1}{m_1^4 m_3^4}, \\ a_2 &= \frac{1}{12} \left[\sum_{j=1}^{\infty} \frac{1}{m_j^4} \right] - \sum_{j=1}^{\infty} \frac{1}{m_j^8} - a_2 \end{aligned}$$

and

$$2a_2 = \frac{1}{12} - \sum_{j=1}^{\infty} \frac{1}{m_j^8}.$$

Now, since

$$a_2 = \frac{1}{12^2 \cdot 35}$$

we obtain finally

$$\sum_{j=1}^{\infty} \frac{1}{m_j^8} = \frac{1}{2 \cdot 12^2} - \frac{1}{12^2 \cdot 35} = \frac{33}{35} \cdot \frac{1}{12^2}. \quad \square \quad (\text{A.9})$$

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