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ON THE INITIAL VALUE PROBLEM FOR TWO-DIMENSIONAL LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. The presented work deals with the question on the existence and uniqueness of a solution of the initial value problem for two-dimensional systems of linear functional differential equations.

Unimprovable efficient conditions sufficient for the unique solvability of the problem considered are established. The question on the existence of a constant-sign solution is also studied in detail. In other words, theorems on systems of linear functional differential inequalities (maximum principles) are discussed, which play a crucial role not only in studies of solvability of linear and non-linear problems but also for other topics related to the theory of boundary value problems (e. g., oscillation theory, asymptotic theory, etc.).

The general results are applied to special cases of functional differential systems, namely, to systems of differential equations with arguments deviations and integro-differential systems, in which case further results are derived; the criteria obtained contain results well-know for ordinary differential systems.

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Key words and phrases. Two-dimensional linear functional differential system, initial value problem, unique solvability, system of differential inequalities.

რეზიუმე. ნაშრომი ეხება წრფივი ფუნქციონალურ დიფერენციალური განტოლებების სისტემებისათვის საწყისი ამოცანის ამონახსნის არსებობისა და ერთადერთობის საკითხს.

დადგენილია განხილული ამოცანის ცალსახად ამოხსნადობის ეფექტური საკმარისი პირობები. აგრეთვე დეტალურადაა განხილული ნიშანმუდმივი ამონახსნის არსებობის საკითხი. სხვა სიტყვებით რომ ვთქვათ, განხილულია თეორემები წრფივ ფუნქციონალურ დიფერენციალურ უტოლობათა სისტემების შესახებ (მაქსიმუმის პრინციპები), რომლებიც გადამწყვეტ როლს თამაშობენ არა მხოლოდ წრფივი და არაწრფივი ამოცანების ამოხსნადობის, არამედ სასაზღვრო ამოცანათა თეორიასთან დაკავშირებული სხვა საკითხების (მაგ., რხევადობის თეორია, ასიმპტოტური თეორია და ა. შ.) შესწავლაშიც.

ზოგადი შედეგები გამოყენებულია ფუნქციონალურ დიფერენციალური სისტემების კერძო შემთხვევებისათვის, კერძოდ, გადახრილარგუმენტიან დიფერენციალურ განტოლებათა და ინტეგრო-დიფერენციალური სისტემებისათვის, და მათთვის მიღებულია დამატებითი შედეგები; მიღებული კრიტერიუმები მოიცავს ჩვეულებრივი დიფერენციალური სისტემებისათვის კარგად ცნობილ შედეგებს.

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Preface

The presented work is devoted to the study of certain qualitative properties of functional differential systems. The author's first results, concerned with the question on the solvability of a two-point boundary value problem for the first-order scalar functional differential equations (see [26–31, 78]), were published in a comprehensive form in the monograph [32]. The techniques used therein had later been generalized and modified for the case of higher dimensions and, as a result, some efficient conditions for the solvability of the initial value problem for functional differential systems were obtained. The present work collects material from the papers [73–77]. The results established in [73–75] are reformulated for the two-dimensional case. Some of the statements given in [76, 77] are also incorporated into the text.

The main part of the work is Section 6, where the question on the unique solvability of the initial value problem for two-dimensional systems of linear functional differential equations is studied. Since the results stated therein are proved using the techniques of differential inequalities, theorems on systems of functional differential inequalities are investigated in Sections 4 and 5.

For the sake of convenience, each section is organized as follows. At first, all the main results are formulated and discussed. These results are then applied to a special case of functional differential systems, namely, to systems of differential equations with arguments deviations, in which case further results are obtained. Then, necessary auxiliary lemmas and the detailed proofs of all the statements formulated above are presented. Finally, we construct several counterexamples showing that some of the results obtained are unimprovable in a certain sense.

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1. Basic Notation and Definitions

(1) $\mathbb{N}$ is the set of all natural numbers.
(2) $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.
(3) For any $x \in \mathbb{R}$, we put

$$[x]_+ = \frac{1}{2}(|x| + x), \quad [x]_- = \frac{1}{2}(|x| - x).$$

(4) $\mathbb{R}^2$ is the space of two-dimensional columns $x = (x_i)_{i=1}^2$ with the elements $x_1, x_2 \in \mathbb{R}$ and the norm

$$\|x\| = |x_1| + |x_2|.$$

(5) $\mathbb{R}^2_+ = \big\{(x_i)_{i=1}^2 \in \mathbb{R}^2 : \ x_1 \geq 0, \ x_2 \geq 0\big\}$.
(6) $x \cdot y$ denotes the scalar product of the vectors $x, y \in \mathbb{R}^2$.

(7) $\mathbb{R}^{2\times 2}$ is the space of 2×2-matrices $X = (x_{ik})_{i,k=1}^2$ with the elements $x_{ik} \in \mathbb{R}$ $(i,k = 1,2)$ and the norm
$$\|X\| = \sum_{i,k=1}^{2} |x_{ik}|.$$
(8) X^{-1} denotes the inverse matrix to $X \in \mathbb{R}^{2\times 2}$.
(9) X^T stands for the transposed matrix to an $n \times m$-matrix X.
(10) For $x = (x_i)_{i=1}^2, y = (y_i)_{i=1}^2 \in \mathbb{R}^2$ and $X = (x_{ik})_{i,k=1}^2$, $Y = (y_{ik})_{i,k=1}^2 \in \mathbb{R}^{2\times 2}$, we put
$$\begin{aligned} x \le y &\text{ if and only if } x_i \le y_i \text{ for } i = 1,2,\\ x < y &\text{ if and only if } x_i < y_i \text{ for } i = 1,2,\\ X \le Y &\text{ if and only if } x_{ik} \le y_{ik} \text{ for } i,k = 1,2. \end{aligned}$$
We write $x \ge 0$ and $x > 0$ instead of $x \ge (0,0)^T$ and $x > (0,0)^T$, respectively.
(11) If $x = (x_i)_{i=1}^2 \in \mathbb{R}^2$, then we denote $|x| = \big(|x_i|\big)_{i=1}^2$, $[x]_+ = \big([x_i]_+\big)_{i=1}^2$, $[x]_- = \big([x_i]_-\big)_{i=1}^2$, $\operatorname{sgn}(x) = \big(|\operatorname{sgn} x_i|\big)_{i=1}^2$, and
$$\operatorname{Sgn}(x) = \begin{pmatrix} \operatorname{sgn} x_1 & 0 \\ 0 & \operatorname{sgn} x_2 \end{pmatrix}.$$
(12) Having $x_1, x_2 \in \mathbb{R}$, we put
$$\operatorname{diag}(x_1, x_2) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}.$$
(13) $C([a,b];\mathbb{R}^2)$ denotes the Banach space of continuous vector functions $u\colon [a,b] \to \mathbb{R}^2$ equipped with the norm
$$\|u\|_C = \max\big\{\|u(t)\| : \ t \in [a,b]\big\}.$$
(14) $C_a([a,b];D)$, where $D \subseteq \mathbb{R}^2$, is the set of the continuous vector functions $u\colon [a,b] \to D$ such that $u(a) = 0$.
(15) $AC([a,b];\mathbb{R}^2)$ is the set of absolutely continuous vector functions $u\colon [a,b] \to \mathbb{R}^2$.
(16) $C([a,b];\mathbb{R})$ denotes the Banach space of continuous scalar functions $z\colon [a,b] \to \mathbb{R}$ equipped with the norm
$$\|z\|_C = \max\big\{|z(t)| : \ t \in [a,b]\big\}.$$
(17) $C([a,b];\mathbb{R}_+) = \big\{z \in C([a,b];\mathbb{R}) : \ z(t) \ge 0 \text{ for } t \in [a,b]\big\}$.
(18) $C_a([a,b];\mathbb{R})$ is the set of the continuous functions $u\colon [a,b] \to \mathbb{R}$ such that $u(a) = 0$.
(19) $C_{loc}([a,b[\,;\mathbb{R})$ is the set of continuous functions $z\colon [a,b[\, \to \mathbb{R}$.
(20) $AC([a,b];\mathbb{R})$ stands for the set of absolutely continuous scalar functions $z\colon [a,b] \to \mathbb{R}$.
(21) $AC_{loc}([a,b[\,;\mathbb{R})$ is the set of the functions $z\colon [a,b[\, \to \mathbb{R}$ such that $z|_{[a,\beta]} \in AC([a,\beta];\mathbb{R})$ for every $\beta \in \,]a,b[\,$.

(22) $L([a,b];\mathbb{R}^2)$ is the Banach space of Lebesgue integrable vector functions $q\colon [a,b] \to \mathbb{R}^2$ equipped with the norm

$$\|q\|_L = \int_a^b \|q(s)\| \, \mathrm{d}s.$$

(23) $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable scalar functions $f\colon [a,b] \to \mathbb{R}$ equipped with the norm

$$\|f\|_L = \int_a^b |f(s)| \, \mathrm{d}s.$$

(24) $L([a,b];\mathbb{R}_+) = \big\{ f \in L([a,b];\mathbb{R}) : f(t) \geq 0 \text{ for a.e. } t \in [a,b] \big\}$.

(25) $\mathcal{L}^2_{ab}$ denotes the set of linear bounded operators $\ell\colon C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$.

(26) $\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

(27) If $\ell\colon C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$ is a linear operator, then for any $i \in \{1,2\}$ and $u \in C([a,b];\mathbb{R}^2)$ $\ell_i(u)$ denotes the ith component of the vector function $\ell(u)$. Moreover, for any $i,k \in \{1,2\}$ and $z \in C([a,b];\mathbb{R})$, we put

$$\ell_{ik}(z) = \ell_i(z_k), \quad \text{where } z_k = \begin{cases} (z,0)^T & \text{if } k = 1, \\ (0,z)^T & \text{if } k = 2. \end{cases}$$

The linear operators $\ell_{ik}\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ $(i,k = 1,2)$ are said to be *components of the operator* ℓ. Obviously, for any $u = (u_1,u_2)^T \in C([a,b];\mathbb{R}^2)$, we have

$$\ell(u) = \big(\ell_1(u), \ell_2(u)\big)^T \quad \text{and} \quad \ell_i(u) = \sum_{k=1}^{2} \ell_{ik}(u_k) \quad \text{for } i = 1,2.$$

(28) Given a linear operator $\ell\colon C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$, we put

$$P_\ell(t) = \big(\ell_{ik}(1)(t)\big)_{i,k=1}^{2} \quad \text{for a.e. } t \in [a,b].$$

It is clear that $P_\ell\colon [a,b] \to \mathbb{R}^{2\times 2}$ is an integrable matrix function.

Definition 1.1. An operator $\ell \in \mathcal{L}^2_{ab}$ (resp., $\ell \in \mathcal{L}_{ab}$) is said to be *strongly bounded* if there exists a function $\eta \in L([a,b];\mathbb{R}_+)$ such that

$$\|\ell(u)(t)\| \leq \eta(t)\|u\|_C \quad \text{for a.e. } t \in [a,b] \text{ and all } u \in C([a,b];\mathbb{R}^2)$$
$$\Big(\text{resp., } |\ell(z)(t)| \leq \eta(t)\|z\|_C \quad \text{for a.e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R})\Big).$$

Example 1.2. Let the operator $\ell\colon C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$ be defined by the relation

$$\ell(v)(t) = \begin{pmatrix} p_{11}(t)v_1\big(\tau_{11}(t)\big) + p_{12}(t)v_2\big(\tau_{12}(t)\big) \\ p_{21}(t)v_1\big(\tau_{21}(t)\big) + p_{22}(t)v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$

$$\text{for a. e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2), \quad (1.1)$$

where $p_{ik} \in L([a,b];\mathbb{R})$ and $\tau_{ik}\colon [a,b] \to [a,b]$ are measurable functions $(i,k = 1,2)$. Then it is clear that ℓ is strongly bounded and

$$\ell_{ik}(z)(t) = p_{ik}(t)z\big(\tau_{ik}(t)\big) \text{ for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}).$$

Definition 1.3. An operator $\ell \in \mathcal{L}^2_{ab}$ (resp., $\ell \in \mathcal{L}_{ab}$) is said to be *positive* if the relation

$$\ell(u)(t) \geq 0 \text{ for a. e. } t \in [a,b]$$

holds for every function $u \in C([a,b];\mathbb{R}^2)$ (resp., $u \in C([a,b];\mathbb{R})$) satisfying the condition

$$u(t) \geq 0 \text{ for } t \in [a,b].$$

We denote the set of positive operators by $\mathcal{P}^2_{ab}$ (resp., $\mathcal{P}_{ab}$).

We say that an operator $\ell \in \mathcal{L}^2_{ab}$ (resp., $\ell \in \mathcal{L}_{ab}$) is *negative* if $-\ell \in \mathcal{P}^2_{ab}$ (resp., $-\ell \in \mathcal{P}_{ab}$). An operator ℓ is called *monotone* if it is either positive or negative.

Remark 1.4. It is clear that every monotone operator is strongly bounded.

Remark 1.5. It is not difficult to verify that $\ell \in \mathcal{P}^2_{ab}$ if and only if

$$\ell_{ik} \in \mathcal{P}_{ab} \text{ for } i,k = 1,2.$$

In particular, the operator ℓ given by the formula (1.1) is positive if and only if

$$p_{ik}(t) \geq 0 \text{ for a. e. } t \in [a,b], \quad i,k = 1,2.$$

Definition 1.6. A linear operator $\ell\colon C([a,b];\mathbb{R}^n) \to L([a,b];\mathbb{R}^n)$, where $n \in \{1,2\}$, is said to be an a-Volterra operator if, for every $b_0 \in]a,b]$ and $v \in C([a,b];\mathbb{R}^n)$ such that $v(t) = 0$ holds for $t \in [a,b_0]$, we have $\ell(v)(t) = 0$ for a. e. $t \in [a,b_0]$.

Remark 1.7. Clearly, $\ell \in \mathcal{L}^2_{ab}$ is an a-Volterra operator if and only if all its components ℓ_{ik} $(i,k = 1,2)$ are a-Volterra operators.

In particular, the operator ℓ given by the formula (1.1) is an a-Volterra operator if and only if the condition

$$|p_{ik}(t)|(\tau_{ik}(t) - t) \leq 0 \text{ for a. e. } t \in [a,b], \quad i,k = 1,2$$

is satisfied.

Definition 1.8. Let $\ell\colon C([a,b];\mathbb{R}^n) \to L([a,b];\mathbb{R}^n)$, where $n \in \{1,2\}$, be an arbitrary operator and $b_0 \in]a,b]$. The operator $\ell^{ab_0}\colon C([a,b_0];\mathbb{R}^n) \to L([a,b_0];\mathbb{R}^n)$ defined by the equality

$$\ell^{ab_0}(z)(t) = \ell\left(z_{b_0}\right)(t) \text{ for a. e. } t \in [a,b_0] \text{ and all } z \in C([a,b_0];\mathbb{R}^n),$$

where

$$z_{b_0}(t) = \begin{cases} z(t) & \text{for } t \in [a, b_0[, \\ z(b_0) & \text{for } t \in [b_0, b], \end{cases}$$

is called the *restriction of the operator* ℓ *to the space* $C([a, b_0]; \mathbb{R}^n)$.

If $b_0 < b_1 \le b$ and $z \in C([a, b_1]; \mathbb{R}^n)$, then we write $\ell^{ab_0}(z)$ instead of $\ell^{ab_0}\big(z|_{[a,b_0]}\big)$.

Remark 1.9. If $\ell\colon C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ is an a-Volterra operator, then it is clear that for every $b_0 \in]a, b[$ and $z \in C([a, b]; \mathbb{R})$ the condition

$$\ell^{ab_0}(z)(t) = \ell(z)(t) \ \text{ for a. e. } \ t \in [a, b_0]$$

is satisfied.

2. Motivation and Illustrative Example

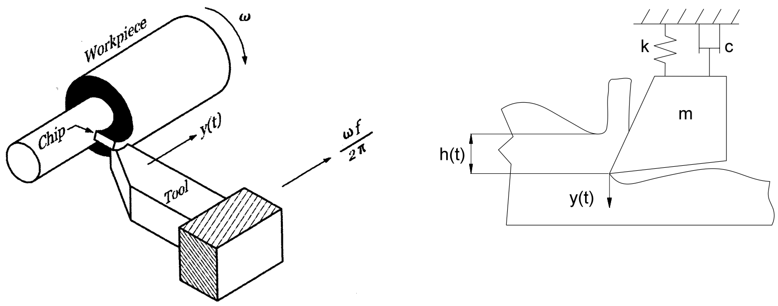
It is well-known that differential equations appear in mathematical models of various phenomena in physics, economy, biology, engineering, and other fields of science. In many cases, these equations can be written in the form of the ordinary differential system

$$x' = f(t, x), \tag{2.1}$$

where $f\colon [0, +\infty[\, \times \mathbb{R}^n \to \mathbb{R}^n$ is a certain vector function (in general nonlinear). The system (2.1) characterizes the evolution of the state variables $x \in \mathbb{R}^n$ in time. The basic assumption that is made on the process when models of this kind are used is that the evolution of the system at the moment t is completely determined by the value of the state variables at the same moment. In other words, for any $t_0 \in [0, +\infty[$, in order to find the value of $x(t)$ of the state variables at the moment $t \ge t_0$, it is sufficient to know the initial value $x(t_0)$.

In practice, however, many phenomena cannot be satisfactorily modelled by ordinary differential systems. Indeed, for many processes, the evolution of the state variables x at the moment t depends not only on the current value $x(t)$ but also on their past or future values. Consequently, when constructing mathematical models for such processes, we obtain differential systems with deviating arguments or more general functional differential systems. Many illustrative examples of such models can be found in the literature (see, e. g., [18, 35, 35, 48] and references therein).

In order to explain how the delay terms in differential systems may arise, let us consider the modelling of regenerative effects in metal cutting in a lathe. The situation under examination is described as follows. A cylindrical workpiece rotates with constant angular velocity ω and the lathe carriage moves along the axis of the workpiece with constant linear velocity $\frac{\omega f}{2\pi}$, where f denotes the feed rate in length per revolution (see Fig. 2.1(a), which is taken from [18]). In such a way, the tool removes a chip whose steady thickness is equal to the feed rate f. Because of some external perturbations, the tool starts a damped oscillation $y(t)$ relative to the lathe



(a) Geometry of turning (b) One-degree-of-freedom model

Fig. 2.1.

carriage and the surface of the workpiece becomes wavy. After a round of the workpiece, the chip thickness will vary because of relative oscillation of the tool. The cutting force is commensurable, among other quantities, to the instantaneous chip thickness and thus depends not only on the steady chip thickness and the actual relative displacement of the tool, but also on the delayed values of relative displacement of the tool. It should be noted that the value of the delay is equal to the time-period τ of one revolution of the workpiece.

The following one-degree-of-freedom model of the cutting process shown on Fig. 2.1(b) goes back to 1950s (see, e. g., [1,16]). The equation of motion takes the form

$$my'' + cy' + ky = -\Delta F_y(h, \vartheta),$$

where m, c, and k denote the inertia, damping, and stiffness characteristics of the tool, F_y is the component of the cutting-force in the direction of oscillations, h is the instantaneous chip thickness, and ϑ stands for the vector of the other input quantities (cutting velocity, chip width, material characteristics, etc.). The instantaneous chip thickness $h(t)$ at the moment t is often written as a deviation from the steady chip thickness f, i. e.,

$$h(t) = f + y(t) - y(t - \tau),$$

where τ is the time-period of one revolution of the workpiece (i. e., $\tau = \frac{2\pi}{\omega}$). Consequently, the equation of motion can be rewritten in the form

$$y''(t) + 2\zeta\omega_n y'(t) + \omega_n^2 y(t) = -\frac{1}{m}\Big(F_y\big(f + y(t) - y(t-\tau), \vartheta(t)\big) - F_y(f, \vartheta_0)\Big),$$

where ω_n is the natural angular frequency of the non-damped free oscillator, ζ is the relative damping factor, and ϑ_0 is the steady value of the input quantities. Hence, we have obtained *a second-order non-linear delay differential equation.*

Assuming that the vector of the input quantities ϑ is constant in time, after linearization of the cutting-force F_y at the steady chip thickness f, the linearized equation of motion becomes (see, e. g., [80])

$$y''(t) + 2\zeta\omega_n y'(t) + \Big(\omega_n^2 + \frac{k_1}{m}\Big) y(t) - \frac{k_1}{m}\, y(t-\tau) = 0, \tag{2.2}$$

where $k_1 = \frac{\partial F_y(h,\vartheta_0)}{\partial h}\Big|_{h=f}$, which is *a second-order linear delay differential equation.*

Another type of linearization of the cutting-force in the equation of motion is presented in [79]. Assuming that p describes the shape of the stationary stress distribution along the active face of the tool in the time-domain $[-\sigma, 0]$, where σ is the time-period needed for the chip to slip along the active face of the tool, the linearized equation has the form

$$y''(t) + 2\zeta\omega_n y'(t) + \omega_n^2 y(t) + \frac{k_2}{m} \int\limits_{t-\sigma}^{t} p(s-t)\big(y(s) - y(s-\tau)\big)\,\mathrm{d}s = 0, \tag{2.3}$$

where k_2 is a suitable constant, which is *a second-order linear integro-differential equation with a delayed argument.*

Now let us go back to the equation (2.2). We will consider this equation on the interval $[0, T]$, where $T > 0$ is large enough. Because of the delayed term $y(t-\tau)$, to define correctly the initial value problem for (2.2), it is not sufficient to prescribe the initial values $y(0)$ and $y'(0)$; we must prescribe values of the function y on the interval $[-\tau, 0]$. Consequently, the initial conditions subjected to the equation (2.2) can be written in the form

$$y(t) = \varphi(t) \;\text{ for }\; t \in [-\tau, 0], \quad y'(0) = d, \tag{2.4}$$

where $d \in \mathbb{R}$ and $\varphi\colon [-\tau, 0] \to \mathbb{R}$ is a suitable initial function. From the application point of view, the problem on the stability of the equation (2.2) is very interesting. But the first very important question reads as follows: *Does there exist any solution of the equation* (2.2) *satisfying the condition* (2.4)? It can be easily shown by the method of steps that, for any initial function $\varphi \in C([-\tau, 0]; \mathbb{R})$ and an arbitrary $d \in \mathbb{R}$, there exists a unique function $y \in C([-\tau, T]; \mathbb{R})$, possessing the continuous on $[0, T]$ second-order derivative, such that the initial conditions (2.4) hold and the equality (2.2) is satisfied for every $t \in [0, T]$.

The equation (2.2) can be reduced to the form in which the deviation maps the interval $[0, T]$ into itself. Indeed, if we put

$$g(t) = \begin{cases} 0 & \text{for } \; t \in [0, \tau[\,, \\ \dfrac{k_1}{m} & \text{for } \; t \in [\tau, T], \end{cases} \qquad q_\varphi(t) = \begin{cases} \dfrac{k_1}{m}\,\varphi(t-\tau) & \text{for } \; t \in [0, \tau[\,, \\ 0 & \text{for } \; t \in [\tau, T], \end{cases}$$

and

$$\widehat{\tau}(t) = \begin{cases} 0 & \text{for } \; t \in [0, \tau[\,, \\ t - \tau & \text{for } \; t \in [\tau, T], \end{cases} \tag{2.5}$$

then the problem (2.2), (2.4) takes the form

$$y''(t) + 2\zeta\omega_n y'(t) + \Big(\omega_n^2 + \frac{k_1}{m}\Big)y(t) - g(t)y\big(\widehat{\tau}(t)\big) = q_\varphi(t), \tag{2.6}$$

$$y(0) = \varphi(0), \quad y'(0) = d \tag{2.7}$$

in which $\widehat{\tau}\colon [0,T] \to [0,T]$.

Equations of the type (2.6) are often studied in the phase space ("displacement–velocity"). To do this, one puts $u = (y, y')^T$ and rewrites the equation (2.6) as follows:

$$u'(t) = -\begin{pmatrix} 0 & -1 \\ \omega_n^2 + \dfrac{k_1}{m} & 2\zeta\omega_n \end{pmatrix} u(t) + \begin{pmatrix} 0 & 0 \\ g(t) & 0 \end{pmatrix} u\big(\widehat{\tau}(t)\big) + \begin{pmatrix} 0 \\ q_\varphi(t) \end{pmatrix}. \tag{2.8}$$

We thus obtain *a two-dimensional differential system with a delayed argument*, which is a particular case of the system (6.20) considered in Section 6.1.3 below. Moreover, the initial condition (2.7) takes the form

$$u(a) = \begin{pmatrix} \varphi(0) \\ d \end{pmatrix}. \tag{2.9}$$

As it has been noted above, for any function $\varphi \in C([-\tau,0];\mathbb{R})$ and an arbitrary $d \in \mathbb{R}$ there exists a unique vector function $u \in C([0,T];\mathbb{R}^2)$, continuously differentiable on $[0,T]$, such that the initial condition (2.9) holds and the equality (2.8) is satisfied for every $t \in [0,T]$. In other words, for every φ and d the initial value problem (2.8), (2.9) has a unique solution without any additional assumption. However, if the system (2.8) is not a delay system, i. e., $\widehat{\tau}(t) > t$ for some $t \in [0,T]$, then certain additional assumptions are necessary to ensure the unique solvability of the problem (2.8), (2.9). Some such additional conditions can be found in Section 6.1.3 and one of them reads as

$$\Big(\omega_n^2 + 2\,\frac{k_1}{m} + 2\zeta\omega_n + 1\Big)\Big(\widehat{\tau}(t) - t\Big) \le \frac{1}{\mathrm{e}} \quad \text{for} \quad t \in [0,T]$$

and means that the deviation $\widehat{\tau}(t) - t$ is "small enough".

In a similar way we can rewrite the equation (2.3) in the form of *the two-dimensional integro-differential system with a delayed argument*

$$u'(t) = -G_1 u(t) - \int\limits_{\widehat{\sigma}(t)}^{t} G_2(s-t)u(s)\,\mathrm{d}s + {}$$
$$+ \int\limits_{\widehat{\sigma}(t)}^{t} \chi(s)G_2(s-t)u\big(\widehat{\tau}(s)\big)\,\mathrm{d}s + \begin{pmatrix} 0 \\ q_\psi(t) \end{pmatrix}, \tag{2.10}$$

where

$$G_1 = \begin{pmatrix} 0 & -1 \\ \omega_n^2 & 2\zeta\omega_n \end{pmatrix}, \quad G_2(\xi) = \begin{pmatrix} 0 & 0 \\ \dfrac{k_2}{m}\,p(\xi) & 0 \end{pmatrix} \quad \text{for} \quad \xi \in [-\sigma, 0],$$

$$\chi(t) = \begin{cases} 0 & \text{for } t \in [0, \sigma[\,, \\ 1 & \text{for } t \in [\sigma, T], \end{cases} \quad \widehat{\sigma}(t) = \begin{cases} 0 & \text{for } t \in [0, \sigma[\,, \\ t - \sigma & \text{for } t \in [\sigma, T], \end{cases}$$

$$q_\psi(t) = \frac{k_2}{m} \int\limits_{t-\sigma}^{\widehat{\sigma}(t)} p(s-t)\big(\psi(s-\tau) - \psi(s)\big)\,\mathrm{d}s + \\ + \frac{k_2}{m} \int\limits_{\widehat{\sigma}(t)}^{t} (1 - \chi(s))p(s-t)\psi(s-\tau)\,\mathrm{d}s \quad \text{for } t \in [0, T],$$

the function $\widehat{\tau}$ is defined by the relation (2.5), and $\psi \in C([-\tau - \sigma, 0]; \mathbb{R})$ is the initial function appearing in the initial conditions

$$y(t) = \psi(t) \quad \text{for } t \in [-\tau - \sigma, 0], \quad y'(0) = d$$

subjected to the second-order equation (2.3).

Finally, we note that the systems (2.8) and (2.10) have many common properties irrespective of their representations, namely, the so-called Fredholm property of the linear boundary value problems, continuous dependence of solutions on the initial conditions and parameters, etc. They both are particular cases of the *linear functional differential system* in the general form (3.1) in which the operator ℓ and the vector function q are defined by the formulas

$$\ell(v)(t) = -\begin{pmatrix} 0 & -1 \\ \omega_n^2 + \frac{k_1}{m} & 2\zeta\omega_n \end{pmatrix} v(t) + \begin{pmatrix} 0 & 0 \\ g(t) & 0 \end{pmatrix} v\big(\widehat{\tau}(t)\big) \\ \text{for } t \in [a, b], \quad v \in C([a, b]; \mathbb{R}^2)$$

and $q = (0, q_\varphi)^T$, and

$$\ell(v)(t) = -G_1 u(t) - \int\limits_{\widehat{\sigma}(t)}^{t} G_2(s-t)u(s)\,\mathrm{d}s + \int\limits_{\widehat{\sigma}(t)}^{t} \chi(s)G_2(s-t)u\big(\widehat{\tau}(s)\big)\,\mathrm{d}s \\ \text{for } t \in [a, b], \quad v \in C([a, b]; \mathbb{R}^2)$$

and $q = (0, q_\psi)^T$, respectively.

Below we investigate functional differential systems in the general form (3.1). Its particular cases are also considered in order to illustrate the applicability of the main results.

3. Statement of the Problem

On the interval $[a, b]$, we consider the Cauchy problem for the two-dimensional linear differential system

$$u'(t) = \ell(u)(t) + q(t), \tag{3.1}$$

$$u(a) = c, \tag{3.2}$$

where $\ell \in \mathcal{L}^2_{ab}$, $q \in L([a,b];\mathbb{R}^2)$, and $c \in \mathbb{R}^2$. As usual in the Carathéodory case, by a solution to this problem we understand a vector function $u \in AC([a,b];\mathbb{R}^2)$ satisfying the equality (3.1) almost everywhere on the interval $[a,b]$ and verifying the initial condition (3.2). Using the components ℓ_{ik} $(i,k=1,2)$ of the operator ℓ (see the item 27 in Section 1), the system (3.1) can be rewritten to the form

$$\begin{aligned} u_1'(t) &= \ell_{11}(u_1)(t) + \ell_{12}(u_2)(t) + q_1(t), \\ u_2'(t) &= \ell_{21}(u_2)(t) + \ell_{22}(u_2)(t) + q_2(t). \end{aligned}$$

Along with the problem (3.1), (3.2) we consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \quad u(a) = 0. \tag{3.3}$$

It is well-known that the linear problem (3.1), (3.2) has the so-called Fredholm property (see [33]; for the case where the operator ℓ is strongly bounded, see also [1, 16]). More precisely, the following statement holds.

Proposition 3.1. *The problem* (3.1), (3.2) *is uniquely solvable for any* $q \in L([a,b];\mathbb{R}^2)$ *and* $c \in \mathbb{R}^2$ *if and only if the homogeneous problem* (3.3) *has only the trivial solution.*

The differential system with argument deviations

$$\begin{aligned} u_1'(t) &= p_{11}(t)u_1\big(\tau_{11}(t)\big) + p_{12}(t)u_2\big(\tau_{12}(t)\big) + q_1(t), \\ u_2'(t) &= p_{21}(t)u_1\big(\tau_{21}(t)\big) + p_{22}(t)u_2\big(\tau_{22}(t)\big) + q_2(t) \end{aligned} \tag{3.1$'$}$$

is investigated in more detail. Here we suppose that $p_{ik}, q_k \in L([a,b];\mathbb{R})$ and $\tau_{ik}\colon [a,b] \to [a,b]$ are measurable functions $(i,k=1,2)$. It is clear that the system (3.1$'$) is a particular case of (3.1) in which the operator ℓ is defined by the formula (1.1).

4. Theorems on Differential Inequalities

It is well-known that theorems on differential inequalities play an important role not only in the theory of boundary value problems, but also in many topics related to the theory of differential equations (asymptotic theory, oscillation theory, etc.). Therefore, the question on the validity of theorems on differential inequalities is studied by many authors (see, e. g., [2,9,10,15,17,21,24,28,29,32,36,37,39,46,47,55,63,65,66,69,71,74,81,82,84]). Although for ordinary differential equations and their systems the question indicated is studied in detail (see, e. g., [9, 10, 14, 36, 37, 47, 82, 84] and references therein), for functional differential systems, and even for rather simple systems (3.1$'$), there is still a broad field for further investigation.

Consider the initial value problem for the system of ordinary differential equations

$$u' = P(t)u + q(t), \quad u(a) = c \tag{4.1}$$

with an integrable matrix function $P\colon [a,b] \to \mathbb{R}^{2\times 2}$. The following proposition on systems of ordinary differential inequalities is well-known (see, e. g., [42]).

Proposition 4.1. *Let the matrix function* $P = (p_{ik})_{i,k=1}^2$ *satisfy*

$$p_{12}(t) \geq 0, \quad p_{21}(t) \geq 0 \ \textit{ for a. e. } \ t \in [a,b].$$

Then every absolutely continuous vector function $x\colon [a,b] \to \mathbb{R}^2$ *such that*

$$x'(t) \leq P(t)x(t) + q(t) \ \textit{ for a. e. } \ t \in [a,b], \quad x(a) \leq c$$

satisfies the condition

$$x(t) \leq u(t) \ \textit{ for } \ t \in [a,b], \tag{4.2}$$

where u is a solution to the problem (4.1).

Below we will give sufficient conditions under which an analogous result is true also for the problem (3.1), (3.2). In other words, we will establish efficient conditions for the operator ℓ which guaranteeing that a certain maximum principle holds for the functional differential system (3.1).

We first introduce

Definition 4.2. Let $n \in \{1,2\}$. We say that a linear bounded operator $\ell\colon C([a,b];\mathbb{R}^n) \to L([a,b];\mathbb{R}^n)$ belongs to the set $\mathcal{S}_{ab}^n(a)$ if for an arbitrary function $u \in AC([a,b];\mathbb{R}^n)$ such that

$$u'(t) \geq \ell(u)(t) \ \text{ for a. e. } \ t \in [a,b], \tag{4.3}$$

$$u(a) \geq 0, \tag{4.4}$$

the relation

$$u(t) \geq 0 \ \text{ for } \ t \in [a,b] \tag{4.5}$$

is satisfied.

If $\ell \in \mathcal{S}_{ab}^2(a)$, then we say that the *theorem on differential inequalities* holds for the system (3.1). Note also that, following [24], we write $\mathcal{S}_{ab}(a)$ instead of $\mathcal{S}_{ab}^1(a)$.

From Definition 4.2 it immediately follows

Proposition 4.3. *Let $n \in \{1,2\}$. The following three statements are equivalent:*

(1) $\ell \in \mathcal{S}_{ab}^n(a)$*;*
(2) *The problem* (3.1), (3.2) *has a unique solution u for arbitrary $q \in L([a,b];\mathbb{R}^n)$ and $c \in \mathbb{R}^n$. Moreover, the solution u satisfies the condition* (4.5) *provided that*

$$q(t) \geq 0 \ \textit{ for a. e. } \ t \in [a,b], \quad c \geq 0;$$

(3) *The operator $K_\ell\colon AC([a,b];\mathbb{R}^n) \to L([a,b];\mathbb{R}^n)$ defined by the formula*

$$K_\ell(v)(t) = v'(t) - \ell(v)(t) \ \textit{ for a. e. } \ t \in [a,b] \ \textit{ and all } \ v \in AC([a,b];\mathbb{R}^n)$$

is inverse positive[1] *in the set* $B = \{v \in AC([a,b];\mathbb{R}^n) : \; v(a) \geq 0\}$.

Remark 4.4. The following analogue of Proposition 4.1 for the problem (3.1), (3.2) is true.

Let $\ell \in \mathcal{S}^2_{ab}(a)$. *Then for an arbitrary vector function* $x \in AC([a,b];\mathbb{R}^2)$ *satisfying the conditions*

$$x'(t) \leq \ell(x)(t) + q(t) \;\; \text{for a. e.} \;\; t \in [a,b], \quad x(a) \leq c,$$

the inequality (4.2) *holds, where* u *is a solution to the problem* (3.1), (3.2).

In what follows, we establish efficient conditions under which the theorem on differential inequalities holds for the system (3.1). Analogous results for the first and second-order scalar functional differential equations are given in [24] and [55], respectively. As mentioned above, these results play very important role not only in the theory of boundary value problems, but also in many topics related to the theory of differential equations. In particular, results of this section will be used in further sections for the study of the solvability of the Cauchy problem for linear functional differential systems.

In Section 4.1, main results are formulated, their proofs being postponed till Section 4.4. Differential systems with argument deviations are studied in more detail in Section 4.2, in which case further results are obtained. In Section 4.5, counterexamples are constructed verifying that the results obtained are unimprovable in a certain sense.

4.1. **Main results.** A certain "characteristic" structure of the set $\mathcal{S}^2_{ab}(a)$ is described in the following theorem.

Theorem 4.5. *Let* $\ell = \ell^+ - \ell^-$, *where* $\ell^+, \ell^- \in \mathcal{P}^2_{ab}$ *are such that*

$$\ell^+ \in \mathcal{S}^2_{ab}(a), \quad -\ell^- \in \mathcal{S}^2_{ab}(a). \tag{4.6}$$

Then $\ell \in \mathcal{S}^2_{ab}(a)$.

Remark 4.6. The assumption (4.6) in Theorem 4.5 can be replaced neither by the assumption

$$(1-\varepsilon)\ell^+ \in \mathcal{S}^2_{ab}(a), \quad -\ell^- \in \mathcal{S}^2_{ab}(a)$$

nor by the assumption

$$\ell^+ \in \mathcal{S}^2_{ab}(a), \quad -(1-\varepsilon)\ell^- \in \mathcal{S}^2_{ab}(a),$$

no matter how small $\varepsilon > 0$ is (see Examples 4.48 and 4.49).

In order to apply Theorem 4.5, we should find some conditions sufficient for the inclusion $\ell \in \mathcal{S}^2_{ab}(a)$ both if the operator ℓ is positive and negative. The conditions indicated are given in Sections 4.1.1 and 4.1.2.

Below we will show (see Theorem 4.21) that if $\ell \in \mathcal{S}^2_{ab}(a)$ is a negative operator, then the components ℓ_{12} and ℓ_{21} of the operator ℓ are necessarily

[1]The notion of an *inverse positive* operator is used by A. Cabada, P. Torres, and others (see, e. g., [8, 82]).

zero operators. Consequently, using Theorem 4.5 and the results of Sections 4.1.1 and 4.1.2, we can derive several efficient conditions sufficient for the inclusion $\ell \in \mathcal{S}^2_{ab}(a)$ if the operator ℓ satisfies

$$\ell_{ii} = \ell^+_{ii} - \ell^-_{ii} \ \textit{ with } \ \ell^+_{ii}, \ell^-_{ii} \in \mathcal{P}_{ab} \ \ (i = 1, 2) \ \text{ and } \ \ell_{12}, \ell_{21} \in \mathcal{P}_{ab}\,.$$

Now we give a rather simple assertion.

Proposition 4.7. *Let $\ell \in \mathcal{L}^2_{ab}$ be such that either*

$$\ell_{i3-i} = 0,^2 \qquad \ell_{3-ii} \in \mathcal{P}_{ab}$$

holds for some $i \in \{1, 2\}$. Then $\ell \in \mathcal{S}^2_{ab}(a)$ if and only if $\ell_{11} \in \mathcal{S}_{ab}(a)$ and $\ell_{22} \in \mathcal{S}_{ab}(a)$.

Recall that the efficient conditions sufficient for the validity of the inclusion $\ell \in \mathcal{S}_{ab}(a)$ are established in the paper [24].

4.1.1. *The case $\ell \in \mathcal{P}^2_{ab}$.* A sufficient and necessary condition for a positive operator ℓ is stated in the next theorem.

Theorem 4.8. *Let $\ell \in \mathcal{P}^2_{ab}$. Then $\ell \in \mathcal{S}^2_{ab}(a)$ if and only if there exists $\gamma \in AC([a, b]; \mathbb{R}^2)$ satisfying*

$$\gamma(t) > 0 \ \textit{ for } \ t \in [a, b], \tag{4.7}$$

$$\gamma'(t) \geq \ell(\gamma)(t) \ \textit{ for a. e. } \ t \in [a, b]. \tag{4.8}$$

By a suitable choice of the function γ in Theorem 4.8, we obtain the following corollary.

Corollary 4.9. *Let $\ell \in \mathcal{P}^2_{ab}$ and let there exist numbers $m, k \in \mathbb{N}$ and $\alpha \in [0, 1[$ such that $m > k$ and*

$$\varrho^m(t) \leq \alpha\varrho^k(t) \ \textit{ for } \ t \in [a, b], \tag{4.9}$$

where $\varrho^1 \in \mathbb{R}^2$ is such that

$$\varrho^1 > 0, \tag{4.10}$$

$$\varrho^{i+1}(t) = \varphi(\varrho^i)(t) \ \textit{ for } \ t \in [a, b], \ \ i \in \mathbb{N}, \tag{4.11}$$

and

$$\varphi(v)(t) = \int_a^t \ell(v)(s)\,\mathrm{d}s \ \textit{ for } \ t \in [a, b], \ \ v \in C([a, b]; \mathbb{R}^2). \tag{4.12}$$

Then $\ell \in \mathcal{S}^2_{ab}(a)$.

Remark 4.10. The assumption $\alpha \in [0, 1[$ in Corollary 4.9 cannot be replaced by the assumption $\alpha \in [0, 1]$ (see Example 4.50).

From the last corollary we get

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[2]The symbol 0 stands here for the zero operator.

Corollary 4.11. *Let $\ell \in \mathcal{P}_{ab}^2$ and there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that*

$$\max\left\{ \frac{1}{\delta_i} \sum_{k=1}^{2} \delta_k \int_a^b \ell_{ik}(1)(s)\,\mathrm{d}s : \ i = 1,2 \right\} < 1. \tag{4.13}$$

Then $\ell \in \mathcal{S}_{ab}^2(a)$.

Remark 4.12. Example 4.50 shows that, in general, the strict inequality (4.13) in Corollary 4.11 cannot be replaced by the nonstrict one. However, in the case where the equality

$$\max\left\{ \frac{1}{\delta_i} \sum_{k=1}^{2} \delta_k \int_a^b \ell_{ik}(1)(s)\,\mathrm{d}s : \ i = 1,2 \right\} = 1 \tag{4.14}$$

is satisfied with some $\delta_1 > 0$ and $\delta_2 > 0$, the inclusion $\ell \in \mathcal{S}_{ab}^2(a)$ is still true under additional assumptions. Some such additional conditions are presented in the next proposition.

Proposition 4.13. *Let $\ell \in \mathcal{P}_{ab}^2$ and there exist numbers $\delta_1 > 0$, $\delta_2 > 0$, and $i \in \{1,2\}$ such that*

$$\frac{1}{\delta_i} \sum_{k=1}^{2} \delta_k \int_a^b \ell_{ik}(1)(s)\,\mathrm{d}s < 1, \tag{4.15}$$

$$\frac{1}{\delta_{3-i}} \sum_{k=1}^{2} \delta_k \int_a^b \ell_{3-ik}(1)(s)\,\mathrm{d}s = 1, \tag{4.16}$$

and

$$\ell_{3-ii}(1) \not\equiv 0. \tag{4.17}$$

Then $\ell \in \mathcal{S}_{ab}^2(a)$.

The last proposition cannot be applied to the case where

$$\frac{1}{\delta_i} \sum_{k=1}^{2} \delta_k \int_a^b \ell_{ik}(1)(s)\,\mathrm{d}s = 1 \ \text{ for } \ i = 1,2.$$

Nevertheless, the following more general statement can be used in the case indicated.

Proposition 4.14. *Let $\ell \in \mathcal{P}_{ab}^2$ and there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that the equality (4.14) is fulfilled. Then $\ell \in \mathcal{S}_{ab}^2(a)$ if and only if the homogeneous problem (3.3) has only the trivial solution.*

The next corollary of Theorem 4.8 contains another type of conditions sufficient for the validity of the inclusion $\ell \in \mathcal{S}_{ab}^2(a)$.

Corollary 4.15. *Let $\ell \in \mathcal{P}_{ab}^2$ and $Y\colon [a,b] \to \mathbb{R}^{2\times 2}$ be a fundamental matrix of the ordinary differential system*

$$x' = \widetilde{P}(t)x, \tag{4.18}$$

where the matrix function $\widetilde{P} = (\widetilde{p}_{ik})_{i,k=1}^2\colon [a,b] \to \mathbb{R}^{2\times 2}$ is defined by the formulas

$$\widetilde{p}_{11} \equiv 0, \quad \widetilde{p}_{22} \equiv 0,$$

$$\widetilde{p}_{i3-i}(t) = \ell_{i3-i}(1)(t)\, \mathrm{e}^{\int_a^t [\ell_{3-i3-i}(1)(s) - \ell_{ii}(1)(s)]\,\mathrm{d}s} \quad \text{for a. e. } t \in [a,b], \ \ i = 1,2. \tag{4.19}$$

Let, moreover, there exist an operator $\bar{\ell} \in \mathcal{P}_{ab}^2$ such that the inequality

$$\ell\big(\varphi(v)\big)(t) - P_\ell(t)\varphi(v)(t) \le \bar{\ell}(v)(t) \ \ \text{for a. e. } t \in [a,b]\,^3 \tag{4.20}$$

holds on the set $C_a([a,b];\mathbb{R}_+^2)$ and

$$BY(b)\int_a^b Y^{-1}(s)\widetilde{q}(s)\,\mathrm{d}s < \mathbf{1}, \tag{4.21}$$

where the operator φ is defined by the relation (4.12), $\mathbf{1} = (1,1)^T$, *the vector function $\widetilde{q} = (\widetilde{q}_1, \widetilde{q}_2)^T \in L([a,b];\mathbb{R}^2)$ is given by the formula*

$$\widetilde{q}_i(t) = \bar{\ell}_i\big(\mathbf{1}\big)(t)\ \mathrm{e}^{-\int_a^t \ell_{ii}(1)(s)\,\mathrm{d}s} \quad \text{for a. e. } t \in [a,b], \ \ i = 1,2, \tag{4.22}$$

and

$$B = \operatorname{diag}\left(\mathrm{e}^{\int_a^b \ell_{11}(1)(s)\,\mathrm{d}s}, \mathrm{e}^{\int_a^b \ell_{22}(1)(s)\,\mathrm{d}s}\right).$$

Then $\ell \in \mathcal{S}_{ab}^2(a)$.

In Corollaries 4.16 and 4.18 efficient conditions are given under which the fundamental matrix Y of the system (4.18) satisfies the condition (4.21).

Corollary 4.16. *Let $\ell \in \mathcal{P}_{ab}^2$ and there exist an operator $\bar{\ell} \in \mathcal{P}_{ab}^2$ such that the inequality* (4.20) *holds on the set $C_a([a,b];\mathbb{R}_+^2)$ and*

$$\mathrm{e}^{\max\left\{\int_a^b \ell_{11}(1)(s)\,\mathrm{d}s, \int_a^b \ell_{22}(1)(s)\,\mathrm{d}s\right\}} \int_a^b h(s)\,\mathrm{e}^{\int_s^b p(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s < 1, \tag{4.23}$$

where the operator φ is defined by the relation (4.12),

$$p(t) = \max\big\{\widetilde{p}_{12}(t), \widetilde{p}_{21}(t)\big\} \ \ \text{for a. e. } t \in [a,b], \tag{4.24}$$

$$h(t) = \max\big\{\widetilde{q}_1(t), \widetilde{q}_2(t)\big\} \quad \text{for a. e. } t \in [a,b], \tag{4.25}$$

and the functions $\widetilde{p}_{12}$, $\widetilde{p}_{21}$ and $\widetilde{q}_1$, $\widetilde{q}_2$ are given by the formulas (4.19) *and* (4.22), *respectively. Then $\ell \in \mathcal{S}_{ab}^2(a)$.*

[3]For the definition of the matrix function P_ℓ, see the item 28 in Section 1.

Remark 4.17. The strict inequality (4.23) in Corollary 4.16 cannot be replaced by the nonstrict one (see Example 4.50).

Corollary 4.18. *Let $\ell \in \mathcal{P}^2_{ab}$ and there exist an operator $\bar{\ell} \in \mathcal{P}^2_{ab}$ such that the inequality* (4.20) *is satisfied on the set $C_a([a,b];\mathbb{R}^2_+)$. Let, moreover, the inequality*

$$\max\left\{\lambda_1 \,\mathrm{e}^{\int_a^b \ell_{11}(1)(s)\,\mathrm{d}s}, \lambda_2 \,\mathrm{e}^{\int_a^b \ell_{22}(1)(s)\,\mathrm{d}s}\right\} < 1 \tag{4.26}$$

hold, where

$$\lambda_i = \int_a^b \cosh\left(\int_s^b p(\xi)\,\mathrm{d}\xi\right)\widetilde{q}_i(s)\,\mathrm{d}s + \\ + \int_a^b \sinh\left(\int_s^b p(\xi)\,\mathrm{d}\xi\right)\widetilde{q}_{3-i}(s)\,\mathrm{d}s \;\; \textit{for} \;\; i = 1,2, \tag{4.27}$$

$$p(t) = \max\left\{\widetilde{p}_{12}(t), \widetilde{p}_{21}(t)\right\} \;\; \textit{for a. e.} \;\; t \in [a,b], \tag{4.28}$$

and the functions $\widetilde{p}_{12}$, $\widetilde{p}_{21}$ and $\widetilde{q}_1$, $\widetilde{q}_2$ are defined by the relations (4.19) *and* (4.22), *respectively. Then $\ell \in \mathcal{S}^2_{ab}(a)$.*

Remark 4.19. Example 4.50 shows that the strict inequality (4.26) in Corollary 4.18 cannot be replaced by the nonstrict one.

The next proposition also follows from Corollary 4.15.

Proposition 4.20. *Let $\ell \in \mathcal{P}^2_{ab}$ be an a-Volterra operator. Then ℓ belongs to the set $\mathcal{S}^2_{ab}(a)$.*

4.1.2. *The case $-\ell \in \mathcal{P}^2_{ab}$.* In the case of a negative operator ℓ we have also sufficient and necessary condition for the validity of the inclusion $\ell \in \mathcal{S}^2_{ab}(a)$, which requires that the system considered consists of two independent scalar equations such that a theorem on scalar differential inequalities holds for each of them.

Theorem 4.21. *Let $-\ell \in \mathcal{P}^2_{ab}$. Then $\ell \in \mathcal{S}^2_{ab}(a)$ if and only if*

$$\ell_{11} \in \mathcal{S}_{ab}(a), \quad \ell_{22} \in \mathcal{S}_{ab}(a) \tag{4.29}$$

and

$$\ell_{12} = 0, \quad \ell_{21} = 0.^4 \tag{4.30}$$

Using the results stated in [20, 24], we can immediately formulate the following corollary of Theorem 4.21.

[4] The symbol 0 stands here for the zero operator.

Corollary 4.22. *Let the operator ℓ be defined by the formula*

$$\ell(v)(t)=\begin{pmatrix}\ell_1(v_1)(t)\\ \ell_2(v_2)(t)\end{pmatrix} \quad \textit{for a. e. } t\in[a,b] \textit{ and all } v=(v_1,v_2)^T\in C([a,b];\mathbb{R}^2),$$

where $\ell_1,\ell_2 \in \mathcal{L}_{ab}$ are negative a-Volterra operators. Let, moreover, for every $i \in \{1,2\}$, at least one of the following conditions be satisfied:

(a) *there exists an absolutely continuous function $\gamma\colon [a,b] \to \mathbb{R}$ such that*

$$\gamma(t) > 0 \quad \textit{for } t \in [a,b[\,,$$
$$\gamma'(t) \le \ell_i(\gamma)(t) \quad \textit{for a. e. } t \in [a,b];$$

(b) *the inequality*

$$\int_a^b |\ell_i(1)(s)|\, \mathrm{d}s \le 1$$

holds;

(c) *the condition*

$$\int_a^b \big|\widetilde{\ell}_i(1)(s)\big| \,\mathrm{e}^{\int_a^s |\ell_i(1)(\xi)|\,\mathrm{d}\xi}\, \mathrm{d}s \le 1$$

is fulfilled, where

$$\widetilde{\ell}_i(z)(t) = \ell_i\big(\theta_i(z)\big)(t) - \ell_i(1)(t)\theta_i(z)(t)$$
$$\textit{for a. e. } t \in [a,b] \textit{ and all } z \in C([a,b];\mathbb{R}),$$

$$\theta_i(z)(t) = \int_a^t \ell_i\big(\omega_i(z)\big)(s)\,\mathrm{d}s \quad \textit{for a. e. } t \in [a,b] \textit{ and all } z \in C([a,b];\mathbb{R}),$$

and

$$\omega_i(z)(t) = z(t)\,\mathrm{e}^{\int_a^t \ell_i(1)(s)\,\mathrm{d}s} \quad \textit{for a. e. } t \in [a,b] \textit{ and all } z \in C([a,b];\mathbb{R}).$$

Then $\ell \in \mathcal{S}^2_{ab}(a)$.

Remark 4.23. The assumption on the operators ℓ_1 and ℓ_2 to be a-Volterra ones is necessary in Corollary 4.22 (see [6, Thm. 2]).

4.2. **Systems with argument deviations.** In this part, we establish some corollaries of the results stated in the previous section for the differential system with argument deviations (3.1′). More precisely, efficient conditions are found for the validity of the inclusion $\ell \in \mathcal{S}^2_{ab}(a)$ whenever the operator

ℓ is defined by one of the formulas

$$\ell(v)(t) = \begin{pmatrix} p_{11}(t)v_1\big(\tau_{11}(t)\big) + p_{12}(t)v_2\big(\tau_{12}(t)\big) \\ p_{21}(t)v_1\big(\tau_{21}(t)\big) + p_{22}(t)v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$

$$\text{for a.e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2) \tag{4.31}$$

and

$$\ell(v)(t) = -\begin{pmatrix} g_1(t)v_1\big(\mu_1(t)\big) \\ g_2(t)v_2\big(\mu_2(t)\big) \end{pmatrix}$$

$$\text{for a.e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2). \tag{4.32}$$

Here we suppose that $p_{ik}, g_k \in L([a,b];\mathbb{R}_+)$ and $\tau_{ik}, \mu_k \colon [a,b] \to [a,b]$ are measurable functions $(i,k = 1,2)$. Throughout this section, the following notation is used:

$$\tau_{11}^* = \text{ess sup}\big\{\tau_{11}(t) : \ t \in [a,b]\big\}, \quad \tau_{22}^* = \text{ess sup}\big\{\tau_{22}(t) : \ t \in [a,b]\big\},$$
$$\tau^* = \max\Big\{\text{ess sup}\big\{\tau_{ik}(t) : \ t \in [a,b]\big\} : \ i,k = 1,2\Big\}. \tag{4.33}$$

Theorem 4.24. *Let there exist numbers $\delta_1 > 0$, $\delta_2 > 0$, and $\alpha \in [0,1[$ such that*

$$\sum_{j=1}^{2} \int_a^t p_{ij}(s)\bigg(\sum_{k=1}^{2} \delta_k \int_a^{\tau_{ij}(s)} p_{jk}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \leq$$
$$\leq \alpha \sum_{k=1}^{2} \delta_k \int_a^t p_{ik}(s)\,\mathrm{d}s \ \ \textit{for} \ \ t \in [a,b], \ \ i = 1,2. \tag{4.34}$$

Then the operator ℓ defined by the formula (4.31) *belongs to the set $\mathcal{S}_{ab}^2(a)$.*

Remark 4.25. Example 4.50 shows that the assumption $\alpha \in [0,1[$ in Theorem 4.24 cannot be replaced by the assumption $\alpha \in [0,1]$.

The following corollary follows immediately from Theorem 4.24.

Corollary 4.26. *Let there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that the inequality*

$$\max\bigg\{\frac{1}{\delta_i}\sum_{k=1}^{2} \delta_k \int_a^{\tau^*} p_{ik}(s)\,\mathrm{d}s : \ i = 1,2\bigg\} < 1 \tag{4.35}$$

is satisfied. Then the operator ℓ defined by the formula (4.31) *belongs to the set $\mathcal{S}_{ab}^2(a)$.*

Remark 4.27. Example 4.50 shows that, in general, the strict inequality (4.35) in Corollary 4.26 cannot be replaced by the nonstrict one. However, in the case where the equality

$$\max\left\{\frac{1}{\delta_i}\sum_{k=1}^{2}\delta_k\int_a^{\tau^*}p_{ik}(s)\,\mathrm{d}s:\ i=1,2\right\}=1$$

is satisfied with some $\delta_1>0$ and $\delta_2>0$, the operator ℓ defined by the formula (4.31) still belongs to the set $\mathcal{S}^2_{ab}(a)$ under additional assumptions. Such additional conditions are presented in the next two theorems.

Theorem 4.28. *Let $i\in\{1,2\}$ and there exist $\delta_1>0$ and $\delta_2>0$ such that*

$$\delta_1\int_a^{\tau^*}p_{i1}(s)\,\mathrm{d}s+\delta_2\int_a^{\tau^*}p_{i2}(s)\,\mathrm{d}s=\delta_i \tag{4.36}$$

and

$$\delta_1\int_a^{\tau^*}p_{3-i1}(s)\,\mathrm{d}s+\delta_2\int_a^{\tau^*}p_{3-i2}(s)\,\mathrm{d}s<\delta_{3-i}\,. \tag{4.37}$$

Then the following assertions are true:

(a) *If the condition*

$$\int_a^{\tau^*_{ii}}p_{ii}(s)\,\mathrm{d}s<1 \tag{4.38}$$

is satisfied, then the operator ℓ defined by the formula (4.31) belongs to the set $\mathcal{S}^2_{ab}(a)$.

(b) *Let*

$$\int_a^{\tau^*_{ii}}p_{ii}(s)\,\mathrm{d}s=1. \tag{4.39}$$

Then the operator ℓ defined by the formula (4.31) belongs to the set $\mathcal{S}^2_{ab}(a)$ if and only if

$$\int_a^{\tau^*_{ii}}p_{ii}(s)\left(\int_a^{\tau_{ii}(s)}p_{ii}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s<1. \tag{4.40}$$

Theorem 4.29. *Let there exist $\delta_1>0$ and $\delta_2>0$ such that the relation (4.36) is satisfied for $i=1,2$. Then the following assertions are true:*

(a) *Let*

$$\int_a^{\tau^*}p_{12}(s)\,\mathrm{d}s\int_a^{\tau^*}p_{21}(s)\,\mathrm{d}s=0 \tag{4.41}$$

and the condition (4.38) hold for $i=1,2$. Then the operator ℓ defined by the formula (4.31) belongs to the set $\mathcal{S}^2_{ab}(a)$.

(b) *Let the condition* (4.41) *hold and there exist* $i \in \{1,2\}$ *such that the condition* (4.39) *is fulfilled. Then the operator* ℓ *defined by the formula* (4.31) *belongs to the set* $\mathcal{S}^2_{ab}(a)$ *if and only if the condition* (4.40) *is satisfied for every* $i \in \{1,2\}$ *verifying the condition* (4.39).

(c) *Let*

$$\int\limits_a^{\tau^*} p_{12}(s)\,\mathrm{d}s \int\limits_a^{\tau^*} p_{21}(s)\,\mathrm{d}s \neq 0. \tag{4.42}$$

Then the operator ℓ *defined by the formula* (4.31) *belongs to the set* $\mathcal{S}^2_{ab}(a)$ *if and only if there exists* $i \in \{1,2\}$ *such that the inequality*

$$\sum_{j=1}^{2} \int\limits_a^{\tau^*} p_{ij}(s) \bigg(\sum_{k=1}^{2} \delta_k \int\limits_a^{\tau_{ij}(s)} p_{jk}(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s < \delta_i \tag{4.43}$$

holds.

The following theorem can be regarded as a supplement of Corollary 4.26 and Theorems 4.28 and 4.29 for the case where neither of the statements indicated can be applied.

Theorem 4.30. *Let* $p_{ik} \not\equiv 0$ *on the interval* $[a,\tau^*]$ *for some* $i,k \in \{1,2\}$ *and let the condition*

$$\operatorname{ess\,sup} \bigg\{ \int\limits_t^{\tau_{ik}(t)} p(s)\,\mathrm{d}s :\ t \in [a,b] \bigg\} < \eta^* \ \ \textit{for}\ \ i,k = 1,2 \tag{4.44}$$

hold, where

$$\eta^* = \sup \left\{ \frac{1}{x} \ln \left(x + \frac{x}{\exp\big(x \int_a^{\tau^*} p(s)\,\mathrm{d}s\big) - 1} \right) :\ x > 0 \right\} \tag{4.45}$$

and

$$p(t) = \max\big\{p_{11}(t) + p_{12}(t), p_{21}(t) + p_{22}(t)\big\} \ \ \textit{for a. e.}\ \ t \in [a,b]. \tag{4.46}$$

Then the operator ℓ *defined by the formula* (4.31) *belongs to the set* $\mathcal{S}^2_{ab}(a)$.

The previous theorem yields

Corollary 4.31. *Let the condition*

$$\int\limits_t^{\tau_{ik}(t)} p(s)\,\mathrm{d}s \leq \frac{1}{\mathrm{e}} \ \ \textit{for a. e.}\ \ t \in [a,b],\ \ i,k = 1,2$$

hold, where the function p *is given by the relation* (4.46)*. Then the operator* ℓ *defined by the formula* (4.31) *belongs to the set* $\mathcal{S}^2_{ab}(a)$.

The next theorem follows from Corollary 4.15.

Theorem 4.32. *Let $Y\colon [a,b] \to \mathbb{R}^{2\times 2}$ be a fundamental matrix of the ordinary differential system* (4.18)*, where $\widetilde{P} = (\widetilde{p}_{ik})_{i,k=1}^2\colon [a,b] \to \mathbb{R}^{2\times 2}$ is defined by the relations*

$$\begin{gathered}\widetilde{p}_{11} \equiv 0, \quad \widetilde{p}_{22} \equiv 0, \\ \widetilde{p}_{i3-i}(t) = p_{i3-i}(t)\,\mathrm{e}^{\int_a^t [p_{3-i3-i}(s) - p_{ii}(s)]\,\mathrm{d}s} \quad \text{for a. e. } t \in [a,b], \quad i = 1,2.\end{gathered} \tag{4.47}$$

Let, moreover, the inequality (4.21) *hold, where the vector function $\widetilde{q} = (\widetilde{q}_i)_{i=1}^2 \in L([a,b];\mathbb{R}^2)$ is defined by the formula*

$$\widetilde{q}_i(t) = \sum_{k=1}^{2} p_{ik}(t)\,\omega_{ik}(t) \left(\sum_{j=1}^{2} \int_t^{\tau_{ik}(t)} p_{kj}(s)\,\mathrm{d}s \right) \mathrm{e}^{-\int_a^t p_{ii}(\eta)\,\mathrm{d}\eta} \quad \text{for a. e. } t \in [a,b], \quad i = 1,2, \tag{4.48}$$

$$\omega_{ik}(t) = \frac{1}{2}\big(1 + \operatorname{sgn}(\tau_{ik}(t) - t)\big) \quad \text{for a. e. } t \in [a,b], \quad i,k = 1,2, \tag{4.49}$$

and

$$B = \operatorname{diag}\left(\mathrm{e}^{\int_a^b p_{11}(s)\,\mathrm{d}s}, \mathrm{e}^{\int_a^b p_{22}(s)\,\mathrm{d}s} \right). \tag{4.50}$$

Then the operator ℓ defined by the formula (4.31) *belongs to the set $\mathcal{S}_{ab}^2(a)$.*

In the following two corollaries, efficient conditions are presented under which the fundamental matrix Y of the system (4.18) satisfies the condition (4.21) in the case where the matrix function $\widetilde{P}$ is defined by the relations (4.47).

Corollary 4.33. *Let*

$$\mathrm{e}^{\max\left\{ \int_a^b p_{11}(s)\,\mathrm{d}s, \int_a^b p_{22}(s)\,\mathrm{d}s \right\}} \int_a^b h(s)\,\mathrm{e}^{\int_s^b p(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s < 1, \tag{4.51}$$

where the functions p and h are given by the equalities (4.24) *and* (4.25)*, respectively, and the functions $\widetilde{p}_{12}$, $\widetilde{p}_{21}$ and $\widetilde{q}_1$, $\widetilde{q}_2$ are defined by the relations* (4.47)–(4.49)*. Then the operator ℓ defined by the formula* (4.31) *belongs to the set $\mathcal{S}_{ab}^2(a)$.*

Remark 4.34. The strict inequality (4.51) in the last corollary cannot be replaced by the nonstrict one (see Example 4.50).

Corollary 4.35. *Let*

$$\max\left\{ \lambda_1\,\mathrm{e}^{\int_a^b p_{11}(s)\,\mathrm{d}s}, \lambda_2\,\mathrm{e}^{\int_a^b p_{22}(s)\,\mathrm{d}s} \right\} < 1, \tag{4.52}$$

where the numbers λ_1, λ_2 are given by the equalities (4.27)*,* (4.28) *and the functions $\widetilde{p}_{12}$, $\widetilde{p}_{21}$ and $\widetilde{q}_1$, $\widetilde{q}_2$ are defined by the relations* (4.47)–(4.49)*. Then the operator ℓ defined by the formula* (4.31) *belongs to the set $\mathcal{S}_{ab}^2(a)$.*

Remark 4.36. Example 4.50 shows that the strict inequality (4.52) in Corollary 4.35 cannot be replaced by the nonstrict one.

Theorem 4.8 also yields the following proposition which is a particular case of Theorem 3.2(a) stated in [69].

Proposition 4.37. *Let the functions p_{ik} $(i,k=1,2)$ be essentially bounded and there exist numbers $\delta_1>0$, $\delta_2>0$ such that the inequality*

$$\max\left\{\frac{1}{\delta_i}\ \operatorname{ess\,sup}\left\{\sum_{k=1}^{2}\delta_k p_{ik}(t)\big(\tau_{ik}(t)-a\big):\ t\in[a,b]\right\}:\ i=1,2\right\}<1$$

holds. Then the operator ℓ defined by the formula (4.31) *belongs to the set $\mathcal{S}^2_{ab}(a)$.*

From Corollary 4.22 and the results stated in [24] we obtain

Theorem 4.38. *Let*

$$g_i(t)\big(\mu_i(t)-t\big)\le 0\ \ \text{for a. e.}\ \ t\in[a,b],\quad i=1,2$$

and, for every $i\in\{1,2\}$, at least one of the following conditions be fulfilled:

(a) *the inequality*

$$\int_a^b g_i(s)\,\mathrm{d}s\le 1$$

holds;

(b) *the inequality*

$$\int_a^b g_i(s)\int_{\mu_i(s)}^{s} g_i(\xi)\exp\left(\int_{\mu_i(\xi)}^{s} g_i(\eta)\,\mathrm{d}\eta\right)\mathrm{d}\xi\,\mathrm{d}s\le 1$$

is satisfied;

(c) *the inequality*

$$\int_{\mu_i(t)}^{t} g_i(s)\,\mathrm{d}s\le\frac{1}{\mathrm{e}}\ \ \text{for a. e.}\ \ t\in[a,b]$$

holds.

Then the operator ℓ defined by the formula (4.32) *belongs to the set $\mathcal{S}^2_{ab}(a)$.*

Remark 4.39. Using Theorem 4.5 and combining the results stated above, we can immediately derive several conditions sufficient for the validity of the inclusion $\ell\in\mathcal{S}^2_{ab}(a)$ if the operator ℓ is defined by the formula

$$\ell(v)(t)=\begin{pmatrix}p_{11}(t)v_1\big(\tau_{11}(t)\big)-g_1(t)v_1\big(\mu_1(t)\big)+p_{12}(t)v_2\big(\tau_{12}(t)\big)\\ p_{21}(t)v_1\big(\tau_{21}(t)\big)+p_{22}(t)v_2\big(\tau_{22}(t)\big)-g_2(t)v_2\big(\mu_2(t)\big)\end{pmatrix}$$

$$\text{for a. e. }\ t\in[a,b]\ \text{ and all }\ v=(v_1,v_2)^T\in C([a,b];\mathbb{R}^2).$$

where $p_{ik}, g_k \in L([a,b];\mathbb{R}_+)$ and $\tau_{ik}, \mu_k\colon [a,b] \to [a,b]$ are measurable functions $(i,k=1,2)$. However, we do not formulate them here in detail.

4.3. **Auxiliary lemmas.** In this part, we give several lemmas that we will need in the proofs of the results stated in Sections 4.1 and 4.2.

Lemma 4.40. *Let $\ell \in \mathcal{P}^2_{ab}$. Then $\ell \in \mathcal{S}^2_{ab}(a)$ if and only if there is no nonzero non-negative vector function $v \in AC([a,b];\mathbb{R}^2)$ possessing the properties*

$$v(a) = 0, \tag{4.53}$$

$$v'(t) \le \ell(v)(t) \quad \text{for a. e. } t \in [a,b]. \tag{4.54}$$

Proof. If $\ell \in \mathcal{S}^2_{ab}(a)$, then it is clear that every non-negative vector function $v \in AC([a,b];\mathbb{R}^2)$ satisfying the conditions (4.53) and (4.54) is identically equal to zero.

Conversely, let there be no nonzero vector function $v = (v_1, v_2)^T \in AC([a,b];\mathbb{R}^2)$ possessing the properties (4.53) and (4.54), and let $u = (u_1,u_2)^T \in AC([a,b];\mathbb{R}^2)$ be such that the conditions (4.3) and (4.4) are satisfied. We will show that the vector function u is non-negative. Indeed, put

$$v(t) = [u(t)]_- \quad \text{for } t \in [a,b].$$

According to the inequality (4.4), it is clear that the vector function v is non-negative and satisfies the condition (4.53). Moreover, by virtue of the assumption $\ell \in \mathcal{P}^2_{ab}$ and Remark 1.5, the inclusion $\ell_{jk} \in \mathcal{P}_{ab}$ holds for $j,k=1,2$. Therefore, using the inequality (4.3) and Lemma 6.17 below, we get

$$\begin{aligned}
v_i'(t) &= \frac{1}{2}\, u_i'(t)\big(\operatorname{sgn} u_i(t) - 1\big) \le \frac{1}{2}\, \ell_i(u)(t)\big(\operatorname{sgn} u_i(t) - 1\big) = \\
&= \frac{1}{2}\Big(\sum_{k=1}^{2} \ell_{ik}(u_k)(t) \operatorname{sgn} u_i(t) - \ell_i(u)(t)\Big) \le \\
&\le \frac{1}{2}\Big(\sum_{k=1}^{2} \ell_{ik}(|u_k|)(t) - \sum_{k=1}^{2} \ell_{ik}(u_k)(t)\Big) = \\
&= \sum_{k=1}^{2} \ell_{ik}([u_k]_-)(t) = \sum_{k=1}^{2} \ell_{ik}(v_k)(t) = \\
&= \ell_i(v)(t) \quad \text{for a. e. } t \in [a,b], \quad i=1,2.
\end{aligned}$$

We have proved that the vector function v satisfies the conditions (4.53) and (4.54), whence we get $v \equiv 0$. However, it means that the condition (4.5) is fulfilled and thus $\ell \in \mathcal{S}^2_{ab}(a)$. $\square$

Lemma 4.41. *Let $p \in L([a,b];\mathbb{R})$, $\widetilde{q} = (\widetilde{q}_1, \widetilde{q}_2)^T \in L([a,b];\mathbb{R}^2)$, and let $v = (v_1,v_2)^T$ be a solution to the problem*

$$v' = A(t)v + \widetilde{q}(t), \quad v(a) = 0, \tag{4.55}$$

where

$$A(t) = \begin{pmatrix} 0 & p(t) \\ p(t) & 0 \end{pmatrix} \quad \textit{for a. e. } t \in [a,b]. \tag{4.56}$$

Then

$$v_i(t) = \int_a^t \cosh\Bigg(\int_s^t p(\xi)\,\mathrm{d}\xi\Bigg)\widetilde{q}_i(s)\,\mathrm{d}s + \\ + \int_a^t \sinh\Bigg(\int_s^t p(\xi)\,\mathrm{d}\xi\Bigg)\widetilde{q}_{3-i}(s)\,\mathrm{d}s \quad \textit{for } t \in [a,b], \quad i = 1,2. \tag{4.57}$$

Proof. It is easy to see that, for an arbitrary $s \in [a,b]$, we have

$$A(t)\Bigg(\int_s^t A(s)\,\mathrm{d}s\Bigg) = \Bigg(\int_s^t A(s)\,\mathrm{d}s\Bigg)A(t) \quad \text{for } t \in [a,b],$$

and thus the solution v to the problem (4.55) has the form

$$v(t) = \int_a^t \mathrm{e}^{\int_s^t A(\xi)\,\mathrm{d}\xi}\,\widetilde{q}(s)\,\mathrm{d}s \quad \text{for } t \in [a,b] \tag{4.58}$$

(see, e. g., [42, Thm. 1.3]). It can be verified by direct calculation that

$$e^{\int_s^t A(\xi)\,\mathrm{d}\xi} = \sum_{k=0}^{+\infty} \frac{1}{(2k)!}\Bigg(\int_s^t p(\xi)\,\mathrm{d}\xi\Bigg)^{2k} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \\ + \sum_{k=0}^{+\infty} \frac{1}{(2k+1)!}\Bigg(\int_s^t p(\xi)\,\mathrm{d}\xi\Bigg)^{2k+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } a \le s \le t \le b,$$

whence we get

$$\mathrm{e}^{\int_s^t A(\xi)\,\mathrm{d}\xi} = \\ = \begin{pmatrix} \cosh\Bigg(\int_s^t p(\xi)\,\mathrm{d}\xi\Bigg) & \sinh\Bigg(\int_s^t p(\xi)\,\mathrm{d}\xi\Bigg) \\ \sinh\Bigg(\int_s^t p(\xi)\,\mathrm{d}\xi\Bigg) & \cosh\Bigg(\int_s^t p(\xi)\,\mathrm{d}\xi\Bigg) \end{pmatrix} \quad \text{for } a \le s \le t \le b. \tag{4.59}$$

Therefore, the relations (4.58) and (4.3) yield the desired representation (4.57) of the solution v. $\square$

Lemma 4.42. *Let $h \in \mathcal{P}_{ab}$ be an a-Volterra operator. Then for an arbitrary non-decreasing function $z \in C([a,b];\mathbb{R})$ the inequality*

$$h(z)(t) \le h(1)(t)z(t) \quad \textit{for a. e. } t \in [a,b] \tag{4.60}$$

is satisfied.

Proof. Let $z \in C([a,b];\mathbb{R})$ be a non-decreasing function. It is clear that for any $t \in [a,b]$ we have

$$h(z)(s) \leq h(1)(s)z(t) \ \text{ for a. e. } \ s \in [a,t]. \tag{4.61}$$

Moreover, there exists a set $E \subseteq \,]a,b]$ such that $\operatorname{mes} E = b - a$[5] and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t h(z)(s)\,\mathrm{d}s = h(z)(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t h(1)(s)\,\mathrm{d}s = h(1)(t) \ \text{ for } \ t \in E. \tag{4.62}$$

Let $t \in E$ be arbitrary but fixed. Then, using the condition (4.61), we get

$$\frac{1}{\delta}\int_{t-\delta}^t h(z)(s)\,\mathrm{d}s \leq z(t)\frac{1}{\delta}\int_{t-\delta}^t h(1)(s)\,\mathrm{d}s \ \text{ for } \ 0 < \delta \leq t - a.$$

Passing to the limit as $\delta \to 0+$ in the last inequality and taking the relations (4.62) into account, we obtain

$$h(z)(t) \leq z(t)h(1)(t)$$

and thus the condition (4.60) is satisfied, because the point $t \in E$ was chosen arbitrarily. □

Lemma 4.43. *Let the operator ℓ be defined by the formula* (4.31) *in which $p_{ik} \in L([a,b];\mathbb{R}_+)$ and $\tau_{ik} \colon [a,b] \to [a,b]$ are measurable functions $(i,k = 1,2)$ and let the number τ^* be given by the relation* (4.33)*. Then $\ell \in \mathcal{S}^2_{ab}(a)$ if and only if $\ell^{a\tau^*} \in \mathcal{S}^2_{a\tau^*}(a)$.*[6]

Proof. According to the relations (4.31) and (4.33), it is clear that the operator ℓ has the following property:

$$\left.\begin{aligned} w \in C([a,b];\mathbb{R}^2),\\ w(t) = 0 \ \text{ for } \ t \in [a,\tau^*] \end{aligned}\right\} \quad \Longrightarrow \quad \ell(w)(t) = 0 \ \text{ for a. e. } \ t \in [a,b]. \tag{4.63}$$

Let $\ell \in \mathcal{S}^2_{ab}(a)$ and $u \in AC([a,\tau^*];\mathbb{R}^2)$ be a vector function satisfying the conditions

$$u'(t) \geq \ell^{a\tau^*}(u)(t) \ \text{ for a. e. } \ t \in [a,\tau^*], \quad u(a) \geq 0. \tag{4.64}$$

We will show that the function u is non-negative on the interval $[a,\tau^*]$. Put

$$v(t) = \begin{cases} u(t) & \text{for; } \ t \in [a,\tau^*[\,, \\ u(\tau^*) + \displaystyle\int_{\tau^*}^t \ell(u_{\tau^*})(s)\,\mathrm{d}s & \text{for } \ t \in [\tau^*,b], \end{cases}$$

---

[5] $\operatorname{mes} E$ stands for the Lebesgue measure of the set E.

[6] $\ell^{a\tau^*}$ denotes the restriction of the operator ℓ to the space $C([a,\tau^*];\mathbb{R}^2)$ (see Definition 1.8).

where the function $u_{\tau^*} \in C([a,b];\mathbb{R}^2)$ is defined by the formula

$$u_{\tau^*}(t) = \begin{cases} u(t) & \text{for } t \in [a,\tau^*[\,, \\ u(\tau^*) & \text{for } t \in [\tau^*,b]. \end{cases} \tag{4.65}$$

Using the relations (4.64), the property (4.63) and Definition 1.8, we easily get

$$v'(t) \geq \ell(u_{\tau^*})(t) = \ell(v)(t) \text{ for a.e. } t \in [a,b], \quad v(a) \geq 0.$$

However, the inclusion $\ell \in \mathcal{S}^2_{ab}(a)$ guarantees that the function v is non-negative on the interval $[a,b]$. Consequently, the inequality $u(t) \geq 0$ holds for $t \in [a,\tau^*]$ and thus $\ell^{a\tau^*} \in \mathcal{S}^2_{a\tau^*}(a)$.

Conversely, let $\ell^{a\tau^*} \in \mathcal{S}^2_{a\tau^*}(a)$ and $u \in AC([a,b];\mathbb{R}^2)$ be a vector function satisfying the conditions (4.3) and (4.4). We will show that the function u is non-negative on the interval $[a,b]$. Let the function u_{τ^*} be defined by the formula (4.65) and $v \equiv u|_{[a,\tau^*]}$. Taking the relation (4.3) and the property (4.63) into account, we get

$$v'(t) = u'(t) \geq \ell(u)(t) = \ell(u_{\tau^*})(t) = \ell^{a\tau^*}(v)(t) \text{ for a.e. } t \in [a,\tau^*].$$

Moreover, the inequality $v(a) \geq 0$ follows immediately from the relation (4.4). Therefore, the inclusion $\ell^{a\tau^*} \in \mathcal{S}^2_{a\tau^*}(a)$ guarantees that the function v is non-negative on the interval $[a,\tau^*]$. Consequently, we have

$$u(t) \geq 0 \text{ for } t \in [a,\tau^*].$$

Using the latter inequality and the property (4.63), it is easy to verify that

$$u'(t) \geq \ell(u)(t) = \ell(u_{\tau^*})(t) \geq 0 \text{ for a.e. } t \in [a,b],$$

because the functions p_{ik} $(i,k=1,2)$ are non-negative. This means that the vector function u is non-decreasing on the interval $[a,b]$ and thus, in view of the condition (4.4), we obtain $u(t) \geq 0$ for $t \in [a,b]$. Consequently, the inclusion $\ell \in \mathcal{S}^2_{ab}(a)$ holds. □

Lemma 4.44. *Let $p_{ik} \in L([a,b];\mathbb{R}_+)$ $(i,k=1,2)$ and numbers $\delta_1 > 0$, $\delta_2 > 0$ be such that the relation* (4.36) *is satisfied for $i=1,2$. Let, moreover, $(u_1,u_2)^T$ be a solution to the homogeneous problem*

$$u_i'(t) = p_{i1}(t)u_1\big(\tau_{i1}(t)\big) + p_{i2}(t)u_2\big(\tau_{i2}(t)\big) \quad \big(t \in [a,\tau^*], \quad i=1,2\big) \tag{4.66}$$

$$u_1(a) = 0, \quad u_2(a) = 0. \tag{4.67}$$

Then both functions u_1 and u_2 do not change their signs on the interval $[a,\tau^*]$. If, in addition,

$$\int_a^{\tau^*} p_{12}(s)\,\mathrm{d}s + \int_a^{\tau^*} p_{21}(s)\,\mathrm{d}s > 0, \tag{4.68}$$

then the relation

$$u_1(t)u_2(t) \geq 0 \text{ for } t \in [a,\tau^*] \tag{4.69}$$

is satisfied.

Proof. For $i = 1, 2$, we put

$$M_i = \max\big\{u_i(t) : \ t \in [a, \tau^*]\big\}, \quad m_i = -\min\big\{u_i(t) : \ t \in [a, \tau^*]\big\}. \tag{4.70}$$

Choose $t_i, T_i \in [a, \tau^*]$ $(i = 1, 2)$ such that

$$u_i(T_i) = M_i\,, \quad u_i(t_i) = -m_i \ \text{ for } \ i = 1, 2 \tag{4.71}$$

and denote

$$\widetilde{p}_{ik} = \int_a^{\tau^*} p_{ik}(s)\,\mathrm{d}s \ \text{ for } \ i, k = 1, 2. \tag{4.72}$$

First suppose that both functions u_1 and u_2 change their signs on $[a, \tau^*]$. Then we have

$$M_i > 0, \quad m_i > 0 \tag{4.73$_i$}$$

for $i = 1, 2$. We can assume without loss of generality that $T_1 < t_1$. The integration of the equality (4.66) with $i = 1$ from T_1 to t_1, in view of the conditions (4.70)–(4.72), yields

$$\begin{aligned} M_1 + m_1 &= -\int_{T_1}^{t_1} p_{11}(s) u_1\big(\tau_{11}(s)\big)\,\mathrm{d}s - \int_{T_1}^{t_1} p_{12}(s) u_2\big(\tau_{12}(s)\big)\,\mathrm{d}s \le \\ &\le m_1 \int_{T_1}^{t_1} p_{11}(s)\,\mathrm{d}s + m_2 \int_{T_1}^{t_1} p_{12}(s)\,\mathrm{d}s \le m_1 \widetilde{p}_{11} + m_2 \widetilde{p}_{12}\,. \end{aligned} \tag{4.74}$$

It is clear that either $T_2 < t_2$ or $T_2 > t_2$ is satisfied.

Case 1*: $T_2 < t_2$ holds.* The integration of the equality (4.66) with $i = 2$ from T_2 to t_2, on account of the conditions (4.70)–(4.72), implies

$$\begin{aligned} M_2 + m_2 &= -\int_{T_2}^{t_2} p_{21}(s) u_1\big(\tau_{21}(s)\big)\,\mathrm{d}s - \int_{T_2}^{t_2} p_{22}(s) u_2\big(\tau_{22}(s)\big)\,\mathrm{d}s \le \\ &\le m_1 \int_{T_2}^{t_2} p_{21}(s)\,\mathrm{d}s + m_2 \int_{T_2}^{t_2} p_{22}(s)\,\mathrm{d}s \le m_1 \widetilde{p}_{21} + m_2 \widetilde{p}_{22}\,. \end{aligned} \tag{4.75}$$

If $\delta_1 m_2 \le \delta_2 m_1$, then from the relations (4.74) and the equality (4.36) with $i = 1$ we get

$$M_1 + m_1 \le m_1 \widetilde{p}_{11} + \frac{\delta_2}{\delta_1}\, m_1 \widetilde{p}_{12} = m_1, \tag{4.76}$$

which contradicts the first inequality in (4.73_1).

If $\delta_1 m_2 > \delta_2 m_1$, then the relations (4.75) and the equality (4.36) with $i = 2$ result in

$$M_2 + m_2 \le \frac{\delta_1}{\delta_2}\, m_2 \widetilde{p}_{21} + m_2 \widetilde{p}_{22} = m_2,$$

which contradicts the first inequality in (4.73_2).

Case 2*:* $T_2 > t_2$ *holds.* The integrations of the equality (4.66) with $i = 2$ from a to t_2 and from t_2 to T_2, on account of the conditions (4.67) and (4.70)–(4.72), yield

$$m_2 = -\int_a^{t_2} p_{21}(s)u_1\big(\tau_{21}(s)\big)\,\mathrm{d}s - \int_a^{t_2} p_{22}(s)u_2\big(\tau_{22}(s)\big)\,\mathrm{d}s \leq$$

$$\leq m_1 \int_a^{t_2} p_{21}(s)\,\mathrm{d}s + m_2 \int_a^{t_2} p_{22}(s)\,\mathrm{d}s \leq m_1\widetilde{p}_{21} + m_2\widetilde{p}_{22} \quad (4.77)$$

and

$$M_2 + m_2 = \int_{t_2}^{T_2} p_{21}(s)u_1\big(\tau_{21}(s)\big)\,\mathrm{d}s + \int_{t_2}^{T_2} p_{22}(s)u_2\big(\tau_{22}(s)\big)\,\mathrm{d}s \leq$$

$$\leq M_1 \int_{t_2}^{T_2} p_{21}(s)\,\mathrm{d}s + M_2 \int_{t_2}^{T_2} p_{22}(s)\,\mathrm{d}s \leq M_1\widetilde{p}_{21} + M_2\widetilde{p}_{22}\,. \quad (4.78)$$

If $\delta_1 m_2 \leq \delta_2 m_1$, then from the condition (4.74) and the equality (4.36) with $i = 1$ we get the relation (4.76), which contradicts the first inequality in (4.73_1).

If $\delta_1 m_2 > \delta_2 m_1$ and $\widetilde{p}_{21} > 0$, then the relations (4.77) and the equality (4.36) with $i = 2$ imply

$$m_2 < \frac{\delta_1}{\delta_2}\, m_2\widetilde{p}_{21} + m_2\widetilde{p}_{22} = m_2\,,$$

which is a contradiction.

If $\delta_1 m_2 > \delta_2 m_1$ and $\widetilde{p}_{21} = 0$, then the relations (4.78) and the equality (4.36) with $i = 2$ result in

$$M_2 + m_2 \leq M_2\widetilde{p}_{22} = M_2\,, \quad (4.79)$$

which contradicts the second inequality in (4.73_2).

The contradictions obtained prove that at least one of the functions u_1 and u_2 does not change its sign on $[a, \tau^*]$. We can assume without loss of generality that

$$u_1(t) \geq 0 \ \text{ for } \ t \in [a, \tau^*]. \quad (4.80)$$

Suppose that, on the contrary, u_2 changes its sign. Then the inequalities (4.73_2) are satisfied and either the relation $T_2 < t_2$ or $T_2 > t_2$ is true.

Case 1*:* $T_2 < t_2$ *holds.* The integration of the equality (4.66) with $i = 2$ from T_2 to t_2, in view of the equality (4.36) with $i = 2$ and the relations

(4.70)–(4.72) and (4.80), implies

$$M_2 + m_2 = -\int_{T_2}^{t_2} p_{21}(s)u_1\big(\tau_{21}(s)\big)\,\mathrm{d}s - \int_{T_2}^{t_2} p_{22}(s)u_2\big(\tau_{22}(s)\big)\,\mathrm{d}s \leq$$

$$\leq m_2 \int_{T_2}^{t_2} p_{22}(s)\,\mathrm{d}s \leq m_2,$$

which contradicts the first inequality in (4.73_2).

Case 2: $T_2 > t_2$ holds. The integrations of the equality (4.66) with $i = 2$ from t_2 to T_2 and from a to t_2, with respect to the conditions (4.70)–(4.72) and (4.80), result in the relations (4.78) and

$$m_2 = -\int_{a}^{t_2} p_{21}(s)u_1\big(\tau_{21}(s)\big)\,\mathrm{d}s - \int_{a}^{t_2} p_{22}(s)u_2\big(\tau_{22}(s)\big)\,\mathrm{d}s \leq$$

$$\leq m_2 \int_{a}^{t_2} p_{22}(s)\,\mathrm{d}s \leq m_2\widetilde{p}_{22}. \tag{4.81}$$

If $\widetilde{p}_{21} = 0$, then from the condition (4.78) and the equality (4.36) with $i = 2$ we get the relation (4.79), which contradicts the second inequality in (4.73_2).

If $\widetilde{p}_{21} > 0$, then the equality (4.36) with $i = 2$ guarantees that $\widetilde{p}_{22} < 1$. Consequently, the relation (4.81) implies $m_2 \leq 0$, which contradicts the second inequality in (4.73_2).

We have proved that both functions u_1 and u_2 do not change their signs on $[a, \tau^*]$. Let, in addition, the relation (4.68) hold. We will show that the inequality (4.69) is satisfied. We can assume without loss of generality that $\widetilde{p}_{12} > 0$ and the relation (4.80) is fulfilled. Suppose that, on the contrary, the condition (4.69) does not hold. Then

$$u_2(t) \leq 0 \quad \text{for} \quad t \in [a, \tau^*] \tag{4.82}$$

and

$$M_1 > 0. \tag{4.83}$$

It is clear that the equality (4.36) with $i = 1$ implies

$$\widetilde{p}_{11} < 1. \tag{4.84}$$

The integration of the equality (4.66) with $i = 1$ from a to T_1, in view of the conditions (4.67), (4.70)–(4.72), and (4.82), results in

$$M_1 = \int_a^{T_1} p_{11}(s)u_1\big(\tau_{11}(s)\big)\,\mathrm{d}s + \int_a^{T_1} p_{12}(s)u_2\big(\tau_{12}(s)\big)\,\mathrm{d}s \leq \\ \leq M_1 \int_a^{T_1} p_{11}(s)\,\mathrm{d}s \leq M_1 \widetilde{p}_{11}.$$

Using the condition (4.84) in the last relation we get $M_1 \leq 0$, which contradicts the inequality (4.83). The contradiction obtained proves that the desired relation (4.69) holds provided that the inequality (4.68) is satisfied. □

Lemma 4.45 ([24, Rem. 1.1])**.** *Let $h \in \mathcal{P}_{ab}$. If the condition*

$$\int_a^b h(1)(s)\,\mathrm{d}s < 1$$

is satisfied, then $h \in \mathcal{S}_{ab}(a)$.

If the equality

$$\int_a^b h(1)(s)\,\mathrm{d}s = 1$$

holds, then the operator h belongs to the set $\mathcal{S}_{ab}(a)$ if and only if the homogeneous problem

$$z'(t) = h(z)(t), \quad z(a) = 0 \tag{4.85}$$

has only the trivial solution.[7]

Lemma 4.46. *Let the operator h be defined by the formula*

$$h(z)(t) = f(t)z\big(\zeta(t)\big) \text{ for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),$$

where $f \in L([a,b];\mathbb{R}_+)$ and $\zeta\colon [a,b] \to [a,b]$ is a measurable function. Put

$$\zeta^* = \operatorname{ess\,sup}\big\{\zeta(t) :\ t \in [a,b]\big\}. \tag{4.86}$$

Then the following assertions are true:

(a) *If the inequality*

$$\int_a^{\zeta^*} f(s)\,\mathrm{d}s < 1 \tag{4.87}$$

is satisfied, then the operator h belongs to the set $\mathcal{S}_{ab}(a)$.

---

[7] Under a solution to the problem (4.85) is understood an absolutely continuous function $z\colon [a,b] \to \mathbb{R}$ satisfying the initial condition $z(a) = 0$ and the differential equality in (4.85) almost everywhere on the interval $[a,b]$.

(b) *Let*

$$\int_a^{\zeta^*} f(s)\,\mathrm{d}s = 1. \tag{4.88}$$

Then the operator h belongs to the set $\mathcal{S}_{ab}(a)$ if and only if

$$\int_a^{\zeta^*} f(s)\left(\int_a^{\zeta(s)} f(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s < 1. \tag{4.89}$$

The results of the last lemma are partly contained in [24]. For the sake of completeness, we prove it here in detail.

Proof. According to the notation (4.86), the restriction $h^{a\zeta^*}$ of the operator h to the space $C([a,\zeta^*];\mathbb{R})$[8] is defined by the formula

$$h^{a\zeta^*}(z)(t) = f(t)z\big(\zeta(t)\big) \ \text{ for a.e. } t\in[a,\zeta^*] \ \text{ and all } \ z\in C([a,\zeta^*];\mathbb{R}).$$

Since, moreover,

$$f(t)\geq 0 \ \text{ for a.e. } t\in[a,b], \tag{4.90}$$

in a similar way as in the proof of Lemma 4.43 it can be shown that $h\in\mathcal{S}_{ab}(a)$ if and only if $h^{a\zeta^*}\in\mathcal{S}_{a\zeta^*}(a)$.

Case (a). Let the condition (4.87) be satisfied. By virtue of Lemma 4.45, we find $h^{a\zeta^*}\in\mathcal{S}_{a\zeta^*}(a)$ and thus $h\in\mathcal{S}_{ab}(a)$.

Case (b). Let the condition (4.88) be fulfilled. According to Lemma 4.45, the operator $h^{a\zeta^*}$ belongs to the set $\mathcal{S}_{a\zeta^*}(a)$ if and only if the homogeneous problem

$$z'(t) = f(t)z\big(\zeta(t)\big) \quad \big(t\in[a,\zeta^*]\big), \tag{4.91}$$

$$z(a) = 0 \tag{4.92}$$

has only the trivial solution[9]. Consequently, to prove the item (b) of the lemma it is sufficient to show that the homogeneous problem (4.91), (4.92) has only the trivial solution if and only if the condition (4.89) is satisfied.

Let z be a solution to the problem (4.91), (4.92). Put

$$M = \max\big\{z(t):\ t\in[a,\zeta^*]\big\}, \quad m = \min\big\{z(t):\ t\in[a,\zeta^*]\big\} \tag{4.93}$$

and choose $t_M, t_m\in[a,\zeta^*]$ such that

$$z(t_M) = M, \quad z(t_m) = m. \tag{4.94}$$

It is clear that the relations (4.92) and (4.93) imply

$$M\geq 0. \tag{4.95}$$

---

[8] See Definition 1.8.

[9] Under a solution to the problem (4.91), (4.92) is understood an absolutely continuous function $z\colon[a,\zeta^*]\to\mathbb{R}$ satisfying the equality (4.91) almost everywhere on the interval $[a,\zeta^*]$ and verifying also the initial condition (4.92).

We can assume without loss of generality that $t_m \le t_M$. The integration of the equality (4.91) from t_m to t_M, in view of the conditions (4.88), (4.90), and (4.93)–(4.95), yields

$$M - m = \int\limits_{t_m}^{t_M} f(s) z\big(\zeta(s)\big)\,\mathrm{d}s \le M \int\limits_{t_m}^{t_M} f(s)\,\mathrm{d}s \le M.$$

Hence we get $m \ge 0$, i. e.,

$$z(t) \ge 0 \ \text{ for } \ t \in [a, \zeta^*]. \tag{4.96}$$

From the relations (4.90), (4.91), and (4.96) we obtain

$$z(t) \le z(\zeta^*) \ \text{ for } \ t \in [a, \zeta^*]. \tag{4.97}$$

Now we put

$$k(t) = \int\limits_a^t f(s)\,\mathrm{d}s \ \text{ for } \ t \in [a, \zeta^*]. \tag{4.98}$$

The integration of the equality (4.91) from t to ζ^*, on account of the conditions (4.90) and (4.97), yields

$$z(\zeta^*) - z(t) = \int\limits_t^{\zeta^*} f(s) z\big(\zeta(s)\big)\,\mathrm{d}s \le z(\zeta^*) \int\limits_t^{\zeta^*} f(s)\,\mathrm{d}s \ \text{ for } \ t \in [a, \zeta^*].$$

Using the conditions (4.88), (4.98) and the last relation, we get

$$z(\zeta^*) k(t) = z(\zeta^*) \bigg(1 - \int\limits_t^{\zeta^*} f(s)\,\mathrm{d}s\bigg) \le z(t) \ \text{ for } \ t \in [a, \zeta^*]. \tag{4.99}$$

On the other hand, the integration of the equality (4.91) from a to t, on account of the conditions (4.90), (4.92), (4.97), and (4.98), results in

$$z(t) = \int\limits_a^t f(s) z\big(\zeta(s)\big)\,\mathrm{d}s \le z(\zeta^*) \int\limits_a^t f(s)\,\mathrm{d}s = z(\zeta^*) k(t) \ \text{ for } \ t \in [a, \zeta^*].$$

Now from the last relation and the condition (4.99) we obtain that

$$z(t) = z(\zeta^*) k(t) \ \text{ for } \ t \in [a, \zeta^*]. \tag{4.100}$$

Finally, the integration of the equality (4.91) from a to ζ^*, with respect to the conditions (4.92) and (4.100), implies

$$z(\zeta^*) = \int\limits_a^{\zeta^*} f(s) z\big(\zeta(s)\big)\,\mathrm{d}s = z(\zeta^*) \int\limits_a^{\zeta^*} f(s) k\big(\zeta(s)\big)\,\mathrm{d}s,$$

whence we get

$$z(\zeta^*)\Big[1-\int\limits_a^{\zeta^*} f(s)\Big(\int\limits_a^{\zeta(s)} f(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s\Big]=0. \tag{4.101}$$

We have proved that every solution z to the problem (4.91), (4.92) admits the representation (4.100), where $z(\zeta^*)$ satisfies the condition (4.101). Consequently, if the inequality (4.89) holds, then the homogeneous problem (4.91), (4.92) has only the trivial solution.

It remains to show that if the condition (4.89) is not satisfied, i. e.,

$$\int\limits_a^{\zeta^*} f(s)k\big(\zeta(s)\big)\,\mathrm{d}s=1, \tag{4.102}$$

then the homogeneous problem (4.91), (4.92) has a nontrivial solution. Indeed, in view of the conditions (4.88) and (4.90), the notation (4.98) yields that

$$k(t)\le k(\zeta^*)=1 \quad\text{for}\quad t\in[a,\zeta^*].$$

Therefore, using the conditions (4.88), (4.90), and (4.102), it is easy to verify that

$$0\le\int\limits_a^{t} f(s)\Big[1-k\big(\zeta(s)\big)\Big]\,\mathrm{d}s\le\int\limits_a^{\zeta^*} f(s)\Big[1-k\big(\zeta(s)\big)\Big]\,\mathrm{d}s=$$
$$=1-\int\limits_a^{\zeta^*} f(s)k\big(\zeta(s)\big)\,\mathrm{d}s=0 \quad\text{for}\quad t\in[a,\zeta^*],$$

whence we get

$$k(t)=\int\limits_a^{t} f(s)k\big(\zeta(s)\big)\,\mathrm{d}s \quad\text{for}\quad t\in[a,\zeta^*].$$

Consequently, k is a nontrivial solution to the problem (4.91), (4.92). □

Lemma 4.47 ([24, Thm. 1.10])**.** *Let the operator h be defined by the formula*

$$h(z)(t)=-f(t)z\big(\zeta(t)\big) \quad\textit{for a. e.}\quad t\in[a,b] \quad\textit{and all}\quad z\in C([a,b];\mathbb{R}),$$

where $f\in L([a,b];\mathbb{R}_+)$ and $\zeta\colon[a,b]\to[a,b]$ is a measurable function such that

$$f(t)\big(\zeta(t)-t\big)\le 0 \quad\textit{for a. e.}\quad t\in[a,b].$$

Let, moreover, either

$$\int\limits_a^{b} f(s)\,\mathrm{d}s\le 1,$$

or

$$\int_a^b f(s) \int_{\zeta(s)}^s f(\xi) \exp\Bigg(\int_{\zeta(\xi)}^s f(\eta)\, \mathrm{d}\eta \Bigg) \mathrm{d}\xi\, \mathrm{d}s \le 1,$$

or

$$\int_{\zeta(t)}^t f(s)\, \mathrm{d}s \le \frac{1}{\mathrm{e}} \ \textit{ for a. e. } \ t \in [a,b].$$

Then the operator h belongs to the set $\mathcal{S}_{ab}(a)$.

4.4. **Proofs.** Now we prove the results stated in Sections 4.1 and 4.2.

Proof of Theorem 4.5. Let $u \in AC([a,b];\mathbb{R}^2)$ be a vector function satisfying the conditions (4.3) and (4.4). We will show that the function u is non-negative. According to the inclusion $-\ell^- \in \mathcal{S}^2_{ab}(a)$ and Proposition 4.3, the problem

$$w'(t) = -\ell^-(w)(t) - \ell^+\big([u]_-\big)(t), \tag{4.103}$$

$$w(a) = 0 \tag{4.104}$$

has a unique solution w and

$$w(t) \le 0 \ \text{ for } \ t \in [a,b]. \tag{4.105}$$

Using the conditions (4.3), (4.103) and the assumption $\ell^+ \in \mathcal{P}^2_{ab}$, we get

$$\begin{aligned}(u-w)'(t) \ge -\ell^-(u-w)(t) + \ell^+([u]_+)(t) \ge \\ \ge -\ell^-(u-w)(t) \ \text{ for a. e. } \ t \in [a,b].\end{aligned} \tag{4.106}$$

Since the inclusion $-\ell^- \in \mathcal{S}^2_{ab}(a)$ holds, the relations (4.4), (4.104), and (4.106) result in

$$u(t) \ge w(t) \ \text{ for } \ t \in [a,b]. \tag{4.107}$$

In view of the relation (4.105), it follows from the inequality (4.107) that

$$-[u(t)]_- \ge w(t) \ \text{ for } \ t \in [a,b]. \tag{4.108}$$

Finally, by virtue of the inequalities (4.105), (4.108), and the assumptions $\ell^+, \ell^- \in \mathcal{P}^2_{ab}$, from the equality (4.103) we get

$$w'(t) \ge \ell^+(w)(t) \ \text{ for a. e. } \ t \in [a,b].$$

Hence, on account of the initial condition (4.104), the inclusion $\ell^+ \in \mathcal{S}^2_{ab}(a)$ guarantees that

$$w(t) \ge 0 \ \text{ for } \ t \in [a,b],$$

which, together with the inequality (4.107), implies the validity of the condition (4.5). Consequently, $\ell \in \mathcal{S}^2_{ab}(a)$. □

Proof of Proposition 4.7. We can assume without loss of generality that $i = 1$.

First suppose that

$$\ell_{11} \in \mathcal{S}_{ab}(a) \ \text{ and } \ \ell_{22} \in \mathcal{S}_{ab}(a).$$

Let $u = (u_1, u_2)^T \in AC([a,b];\mathbb{R}^2)$ be a vector function satisfying the conditions (4.3) and (4.4). We will show that the function u is non-negative. In view of the assumption $\ell_{12} = 0$, it follows from (4.3) and (4.4) that

$$u_1'(t) \geq \ell_{11}(u_1)(t) \ \text{ for a.e. } \ t \in [a,b], \quad u_1(a) \geq 0,$$

and thus the assumption $\ell_{11} \in \mathcal{S}_{ab}(a)$ implies

$$u_1(t) \geq 0 \ \text{ for } \ t \in [a,b]. \tag{4.109}$$

Taking the assumption $\ell_{21} \in \mathcal{P}_{ab}$ into account and using the inequality (4.109) in the relation (4.3), we get

$$u_2'(t) \geq \ell_{21}(u_1)(t) + \ell_{22}(u_2)(t) \geq \ell_{22}(u_2)(t) \ \text{ for a.e. } \ t \in [a,b].$$

Hence, the inclusion $\ell_{22} \in \mathcal{S}_{ab}^2(a)$ yields that

$$u_2(t) \geq 0 \ \text{ for } \ t \in [a,b],$$

which, together with the inequality (4.109), guarantees the validity of the condition (4.5). Consequently, we have proved that $\ell \in \mathcal{S}_{ab}^2(a)$.

Now suppose that $\ell \in \mathcal{S}_{ab}^2(a)$. We first show that $\ell_{22} \in \mathcal{S}_{ab}(a)$. Indeed, let $z \in AC([a,b];\mathbb{R})$ be a function such that

$$z'(t) \geq \ell_{22}(z)(t) \ \text{ for a.e. } \ t \in [a,b], \quad z(a) \geq 0.$$

Put

$$u(t) = \begin{pmatrix} 0 \\ z(t) \end{pmatrix} \ \text{ for } \ t \in [a,b].$$

It is clear that the vector function u is absolutely continuous and satisfies the conditions (4.3) and (4.4). Therefore, the assumption $\ell \in \mathcal{S}_{ab}^2(a)$ yields that the vector function u is non-negative. Consequently,

$$z(t) \geq 0 \ \text{ for } \ t \in [a,b]$$

and thus $\ell_{22} \in \mathcal{S}_{ab}(a)$.

It remains to show that also $\ell_{11} \in \mathcal{S}_{ab}(a)$. Let $y \in AC([a,b];\mathbb{R})$ be a function satisfying the conditions

$$y'(t) \geq \ell_{11}(y)(t) \ \text{ for a.e. } \ t \in [a,b], \ \ y(a) \geq 0.$$

We will show that

$$y(t) \geq 0 \ \text{ for } \ t \in [a,b]. \tag{4.110}$$

According to Proposition 4.3 and the above-proved inclusion $\ell_{22} \in \mathcal{S}_{ab}(a)$, the problem

$$v'(t) = \ell_{22}(v)(t) + \ell_{21}(y)(t), \quad v(a) = 0$$

has a unique solution v. Put

$$u(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} \ \text{ for } \ t \in [a,b].$$

It is clear that the vector function u is absolutely continuous and satisfies the conditions (4.3) and (4.4). Therefore, the assumption $\ell \in \mathcal{S}^2_{ab}(a)$ yields that the vector function u is non-negative. Consequently, the condition (4.110) holds and thus $\ell_{11} \in \mathcal{S}_{ab}(a)$. □

Proof of Theorem 4.8. First suppose that there exists $\gamma \in AC([a,b];\mathbb{R}^2)$ satisfying the inequalities (4.7) and (4.8). Let $u \in AC([a,b];\mathbb{R}^2)$ be such that the conditions (4.3) and (4.4) hold. We will show that the vector function u is non-negative. Put

$$\mathcal{A} = \Big\{\lambda \in \mathbb{R}_+ : \ \lambda\gamma(t) + u(t) \geq 0 \text{ for } t \in [a,b]\Big\}. \tag{4.111}$$

Since the function γ is positive, we have $\mathcal{A} \neq \varnothing$. Setting

$$\lambda_0 = \inf \mathcal{A}, \tag{4.112}$$

we put

$$w(t) = \lambda_0\gamma(t) + u(t) \ \text{ for } \ t \in [a,b]. \tag{4.113}$$

It is clear that $\lambda_0 \geq 0$, $w \in AC([a,b];\mathbb{R}^2)$, and

$$w(t) \geq 0 \ \text{ for } \ t \in [a,b]. \tag{4.114}$$

Therefore, by virtue of the assumption $\ell \in \mathcal{P}^2_{ab}$, we get

$$w'(t) \geq \ell(w)(t) \geq 0 \ \text{ for a. e. } \ t \in [a,b]. \tag{4.115}$$

Assume that

$$\lambda_0 > 0. \tag{4.116}$$

Then it follows from the relations (4.4), (4.7) and (4.116) that $w(a) > 0$. Hence, using the inequality (4.115), we get

$$w(t) > 0 \ \text{ for } \ t \in [a,b].$$

Consequently, there exists $\varepsilon \in \,]0, \lambda_0]$ such that

$$w(t) \geq \varepsilon\gamma(t) \ \text{ for } \ t \in [a,b],$$

i. e.,

$$(\lambda_0 - \varepsilon)\gamma(t) + u(t) \geq 0 \ \text{ for } \ t \in [a,b].$$

Hence, by virtue of the notation (4.111), we get $\lambda_0 - \varepsilon \in \mathcal{A}$, which contradicts the relation (4.112).

The contradiction obtained proves that $\lambda_0 = 0$. Consequently, the relations (4.113) and (4.114) yield the validity of the condition (4.5) and thus $\ell \in \mathcal{S}^2_{ab}(a)$.

Now suppose that $\ell \in \mathcal{S}^2_{ab}(a)$. Then, according to Proposition 4.3, the problem

$$\gamma'(t) = \ell(\gamma)(t), \quad \gamma(a) = (1,1)^T \tag{4.117}$$

has a unique solution γ and

$$\gamma(t) \geq 0 \ \text{ for } \ t \in [a,b].$$

Hence, by virtue of the assumption $\ell \in \mathcal{P}^2_{ab}$, the first equation in (4.117) implies that

$$\gamma'(t) = \ell(\gamma)(t) \geq 0 \quad \text{for a. e. } t \in [a,b].$$

Therefore, in view of the initial condition in (4.117), we get

$$\gamma(t) \geq \gamma(a) > 0 \quad \text{for } t \in [a,b].$$

Consequently, the function $\gamma \in AC([a,b];\mathbb{R}^2)$ satisfies the conditions (4.7) and (4.8). □

Proof of Corollary 4.9. Put

$$\gamma(t) = (1-\alpha)\sum_{j=1}^{k} \varrho^j(t) + \sum_{j=k+1}^{m} \varrho^j(t) \quad \text{for } t \in [a,b].$$

It is clear that $\gamma \in AC([a,b];\mathbb{R}^2)$. According to the relations (4.10)–(4.12) and the assumption $\ell \in \mathcal{P}^2_{ab}$, it is not difficult to verify that

$$\gamma(t) \geq (1-\alpha)\varrho^1 > 0 \quad \text{for } t \in [a,b],$$

i. e., the condition (4.7) is satisfied. Moreover, we have

$$\begin{aligned} \gamma'(t) = (1-\alpha)\sum_{j=1}^{k-1} \ell\big(\varrho^j\big)(t) + \sum_{j=k}^{m-1} \ell\big(\varrho^j\big)(t) = \\ = \ell(\gamma)(t) + \ell\big(\alpha\varrho^k - \varrho^m\big)(t) \quad \text{for a. e. } t \in [a,b]. \end{aligned}$$

Therefore, in view of the inequality (4.9) and the assumption $\ell \in \mathcal{P}^2_{ab}$, the condition (4.8) holds. Consequently, using Theorem 4.8, we get $\ell \in \mathcal{S}^2_{ab}(a)$. □

Proof of Corollary 4.11. The validity of the corollary follows immediately from Corollary 4.9 with $k=1$, $m=2$, $\varrho^1 = (\delta_1,\delta_2)^T$, and

$$\alpha = \max\left\{ \frac{1}{\delta_i}\sum_{k=1}^{2} \delta_k \int_a^b \ell_{ik}(1)(s)\,\mathrm{d}s : \ i = 1,2 \right\}.$$

□

Proof of Proposition 4.13. Put $\varrho^1 = (\delta_1,\delta_2)^T$. We first show that

$$\int_a^b \ell_j\big(\varphi(\varrho^1)\big)(s)\,\mathrm{d}s < \delta_j \quad \text{for } j = 1,2, \tag{4.118}$$

where the operator φ is given by the formula (4.12). Indeed, according to the assumption $\ell \in \mathcal{P}^2_{ab}$ and Remark 1.5, the inclusion $\ell_{jk} \in \mathcal{P}_{ab}$ holds for

$j, k = 1, 2$. Therefore, we get

$$\int_a^b \ell_j\big(\varphi(\varrho^1)\big)(s)\,\mathrm{d}s = \sum_{k=1}^{2} \int_a^b \ell_{jk}\big(\varphi_k(\varrho^1)\big)(s)\,\mathrm{d}s \leq$$
$$\leq \sum_{k=1}^{2} \int_a^b \ell_k(\varrho^1)(\eta)\,\mathrm{d}\eta \int_a^b \ell_{jk}(1)(s)\,\mathrm{d}s =$$
$$= \sum_{k=1}^{2} \int_a^b \ell_{jk}(1)(s)\,\mathrm{d}s \sum_{m=1}^{2} \int_a^b \ell_{km}(\delta_m)(\eta)\,\mathrm{d}\eta =$$
$$= \sum_{k=1}^{2} \int_a^b \ell_{jk}(1)(s)\,\mathrm{d}s \sum_{m=1}^{2} \int_a^b \delta_m \ell_{km}(1)(\eta)\,\mathrm{d}\eta \ \text{ for } \ j = 1, 2. \tag{4.119}$$

Then, by virtue of the assumptions (4.15) and (4.16), we obtain

$$\sum_{k=1}^{2} \int_a^b \ell_{ik}(1)(s)\,\mathrm{d}s \sum_{m=1}^{2} \int_a^b \delta_m \ell_{km}(1)(\eta)\,\mathrm{d}\eta \leq \sum_{k=1}^{2} \delta_k \int_a^b \ell_{ik}(1)(s)\,\mathrm{d}s < \delta_i\,.$$

On the other hand, according to the condition (4.17), the relation

$$\int_a^b \ell_{3-ii}(1)(s)\,\mathrm{d}s > 0$$

holds. Therefore, using the assumptions (4.15) and (4.16), we get

$$\sum_{k=1}^{2} \int_a^b \ell_{3-ik}(1)(s)\,\mathrm{d}s \sum_{m=1}^{2} \delta_m \int_a^b \ell_{km}(1)(\eta)\,\mathrm{d}\eta <$$
$$< \delta_i \int_a^b \ell_{3-ii}(1)(s)\,\mathrm{d}s + \delta_{3-i} \int_a^b \ell_{3-i3-i}(1)(s)\,\mathrm{d}s =$$
$$= \sum_{k=1}^{2} \delta_k \int_a^b \ell_{3-ik}(1)(s)\,\mathrm{d}s = \delta_{3-i}\,.$$

Consequently, the relation (4.119) yields the validity of the condition (4.118). If we put

$$\alpha = \max\left\{ \frac{1}{\delta_j} \int_a^b \ell_j\big(\varphi(\varrho^1)\big)(s)\,\mathrm{d}s : \ j = 1, 2 \right\}, \tag{4.120}$$

it is clear $\alpha \in [0, 1[$ and

$$\varrho^3(b) \leq \alpha(\delta_1, \delta_2)^T = \alpha\varrho^1,$$

where the function ϱ^3 is defined by the formula (4.11). Consequently, the assumptions of Corollary 4.9 are satisfied with $k = 1$, $m = 3$, and α given by the relation (4.120). $\square$

Proof of Proposition 4.14. First suppose that the homogeneous problem (3.3) has only the trivial solution. We will show that $\ell \in \mathcal{S}^2_{ab}(a)$. According to Proposition 3.1, the problem (4.117) has a unique solution $\gamma = (\gamma_1, \gamma_2)^T$. Put

$$\gamma^* = \max\Big\{ -\frac{\gamma_i(t)}{\delta_i} : \ t \in [a,b], \ i = 1,2 \Big\}.$$

Then there exist $t_0 \in [a,b]$ and $i_0 \in \{1,2\}$ such that

$$-\gamma_{i_0}(t_0) = \gamma^* \delta_{i_0}. \tag{4.121}$$

It is clear that

$$-\gamma_i(t) \le \gamma^* \delta_i \ \text{ for } \ t \in [a,b], \quad i = 1,2. \tag{4.122}$$

Note also that, by virtue of the assumption $\ell \in \mathcal{P}^2_{ab}$ and Remark 1.5, the inclusion $\ell_{jk} \in \mathcal{P}_{ab}$ holds for $j,k = 1,2$.

Assume that

$$\gamma^* \ge 0. \tag{4.123}$$

Then it follows from the relations (4.117) that

$$1 - \gamma_i(t_0) = -\int_a^{t_0} \ell_i(\gamma)(s)\,\mathrm{d}s \ \text{ for } \ i = 1,2.$$

Therefore, in view of the conditions (4.14) and (4.121)–(4.123), we get

$$\begin{aligned}
1 + \gamma^*\delta_{i_0} &= -\int_a^{t_0} \ell_{i_0}(\gamma)(s)\,\mathrm{d}s = -\int_a^{t_0} \sum_{k=1}^{2} \ell_{i_0k}(\gamma_k)(s)\,\mathrm{d}s = \\
&= \sum_{k=1}^{2} \int_a^{t_0} \ell_{i_0k}(-\gamma_k)(s)\,\mathrm{d}s \le \sum_{k=1}^{2} \gamma^*\delta_k \int_a^{t_0} \ell_{i_0k}(1)(s)\,\mathrm{d}s = \\
&= \gamma^* \sum_{k=1}^{2} \delta_k \int_a^{t_0} \ell_{i_0k}(1)(s)\,\mathrm{d}s \le \gamma^*\delta_{i_0},
\end{aligned}$$

which is impossible.

The contradiction obtained proves that $\gamma^* < 0$. Hence the relation (4.122) implies the validity of the inequality

$$\gamma(t) \ge -\gamma^*(\delta_1, \delta_2)^T > 0 \ \text{ for } \ t \in [a,b]$$

and thus Theorem 4.8 guarantees that $\ell \in \mathcal{S}^2_{ab}(a)$.

The converse implication follows immediately from Proposition 4.3. $\square$

Proof of Corollary 4.15. According to Lemma 4.40, to prove the corollary it is sufficient to show that there is no nonzero non-negative vector function $v \in AC([a,b];\mathbb{R}^2)$ satisfying the conditions (4.53) and (4.54).

We first show that

$$\widetilde{\ell} \in \mathcal{S}^2_{ab}(a), \tag{4.124}$$

where the operator $\widetilde{\ell} \in \mathcal{P}^2_{ab}$ is defined by the formula

$$\widetilde{\ell}(w)(t) = P_\ell(t)w(t) + \bar{\ell}(w)(t) \text{ for a.e. } t \in [a,b] \text{ and all } w \in C([a,b];\mathbb{R}^2).^{10}$$

According to the inequality (4.21), there exists a two-dimensional vector $\varepsilon > 0$ such that

$$B\left(Y(b)Y^{-1}(a)\varepsilon + Y(b)\int\limits_a^b Y^{-1}(s)\widetilde{q}(s)\,\mathrm{d}s\right) \leq \mathbf{1}, \tag{4.125}$$

where $\mathbf{1} = (1,1)^T$. Put

$$z(t) = Y(t)Y^{-1}(a)\varepsilon + Y(t)\int\limits_a^t Y^{-1}(s)\widetilde{q}(s)\,\mathrm{d}s \text{ for } t \in [a,b].$$

It is clear that z is a solution to the Cauchy problem

$$z' = \widetilde{P}(t)z + \widetilde{q}(t), \quad z(a) = \varepsilon, \tag{4.126}$$

because Y is a fundamental matrix of the system (4.18). In view of the relations (4.19) and (4.22), we get

$$\widetilde{P}(t) \geq \Theta, \quad \widetilde{q}(t) \geq 0 \text{ for a.e. } t \in [a,b], \tag{4.127}$$

where Θ denotes the zero matrix, and thus

$$0 < z(t) \leq z(b) \text{ for } t \in [a,b]. \tag{4.128}$$

Put

$$\gamma_i(t) = z_i(t)\,\mathrm{e}^{\int_a^t \ell_{ii}(1)(s)\,\mathrm{d}s} \quad \text{for } t \in [a,b], \quad i = 1,2. \tag{4.129}$$

It is not difficult to verify that, on account of the conditions (4.19), (4.22), (4.126) and (4.129), $\gamma = (\gamma_1,\gamma_2)^T \in AC([a,b];\mathbb{R}^2)$ is a solution to the system

$$\gamma' = P_\ell(t)\gamma + \bar{\ell}(\mathbf{1})(t). \tag{4.130}$$

Moreover, the conditions (4.125) and (4.128) imply

$$0 < \gamma(t) \leq Bz(b) \leq \mathbf{1} \text{ for } t \in [a,b].$$

Therefore, the vector function γ satisfies the condition (4.7) and, in view of the assumption $\bar{\ell} \in \mathcal{P}^2_{ab}$, from the equality (4.130) we get

$$\gamma'(t) \geq P_\ell(t)\gamma(t) + \bar{\ell}(\gamma)(t) = \widetilde{\ell}(\gamma)(t) \text{ for a.e. } t \in [a,b].$$

Consequently, by virtue of Theorem 4.8, the inclusion (4.124) is fulfilled.

---

[9]For the definition of the matrix function P_ℓ, see the item 28 in Section 1.

Let $v \in AC([a,b];\mathbb{R}^2)$ be a non-negative vector function satisfying the conditions (4.53) and (4.54). We will show that $v \equiv 0$. Put

$$u(t) = \varphi(v)(t) \ \text{ for } \ t \in [a,b], \tag{4.131}$$

where the operator φ is defined by the formula (4.12). Obviously, the conditions (4.53), (4.54) and (4.131) yield

$$u'(t) = \ell(v)(t) \ \text{ for a. e. } \ t \in [a,b], \quad u(a) = 0,$$

and

$$0 \le v(t) \le \int\limits_a^t \ell(v)(s)\,\mathrm{d}s = \varphi(v)(t) = u(t) \ \text{ for } \ t \in [a,b]. \tag{4.132}$$

On the other hand, since the operator ℓ is positive, we have

$$\begin{aligned} u'(t) = \ell(v)(t) \le \ell(u)(t) = \\ = P_\ell(t)u(t) + \ell\big(\varphi(v)\big)(t) - P_\ell(t)\varphi(v)(t) \ \text{ for a. e. } \ t \in [a,b]. \end{aligned}$$

Now, by virtue of the conditions (4.20), (4.53), and (4.132), the last relation yields

$$u'(t) \le P_\ell(t)u(t) + \bar{\ell}(v)(t) \ \text{ for a. e. } \ t \in [a,b].$$

Hence, in view of the inequalities (4.132) and the assumption $\bar{\ell} \in \mathcal{P}_{ab}^2$, we get

$$u'(t) \le P_\ell(t)u(t) + \bar{\ell}(u)(t) = \widetilde{\ell}(u)(t) \ \text{ for a. e. } \ t \in [a,b].$$

Consequently, the inclusion (4.124) yields

$$u(t) \le 0 \ \text{ for } \ t \in [a,b],$$

and thus it follows from the condition (4.132) that $v \equiv 0$. However, this means that there exists no nonzero non-negative vector function $v \in AC([a,b];\mathbb{R}^2)$ satisfying the conditions (4.53) and (4.54). $\square$

Proof of Corollary 4.16. We will show that all the assumptions of Corollary 4.15 are satisfied. To do this, it is sufficient to show that the inequality (4.23) yields the validity of the condition (4.21). Since Y is a fundamental matrix of the system (4.18), the condition (4.21) is fulfilled if and only if the solution $x = (x_1, x_2)^T$ to the Cauchy problem

$$x' = \widetilde{P}(t)x + \widetilde{q}(t), \quad x(a) = 0 \tag{4.133}$$

satisfies the condition

$$x_i(b)\,\mathrm{e}^{\int\limits_a^b \ell_{ii}(1)(s)\,\mathrm{d}s} < 1 \ \text{ for } \ i = 1,2. \tag{4.134}$$

Put

$$v_i(t) = \int\limits_a^t h(s)\,\mathrm{e}^{\int\limits_s^t p(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s \ \text{ for } \ t \in [a,b], \quad i = 1,2. \tag{4.135}$$

It is clear that
$$v_i(t) \geq 0 \ \text{ for } \ t \in [a,b], \quad i = 1,2,$$
because $\widetilde{P}$ and $\widetilde{q}$ satisfy the relations (4.127). Therefore, from the equality (4.135) we get

$$v_i'(t) = p(t)v_i(t) + h(t) \geq \widetilde{p}_{i3-i}(t)v_i(t) + \widetilde{q}_i(t) = \\ = \sum_{k=1}^{2} \widetilde{p}_{ik}(t)v_k(t) + \widetilde{q}_i(t) \ \text{ for a. e. } \ t \in [a,b], \quad i = 1,2.$$

But this means that the vector function $v = (v_1, v_2)^T$ satisfies the initial condition $v(a) = 0$ and the differential inequality
$$v'(t) \geq \widetilde{P}(t)v(t) + \widetilde{q}(t) \ \text{ for a. e. } \ t \in [a,b].$$
According to Proposition 4.1, we get
$$x(t) \leq v(t) \ \text{ for } \ t \in [a,b],$$
where x is the unique solution to the problem (4.133). Consequently, by virtue of the relations (4.23) and (4.135), the solution $x = (x_1, x_2)^T$ to the problem (4.133) fulfills the condition (4.134), and thus the inequality (4.21) holds. Hence, the assumptions of Corollary 4.15 are satisfied. □

Proof of Corollary 4.18. We will show that all the assumptions of Corollary 4.15 are satisfied. To do this, it is sufficient to show that the inequality (4.26) yields the validity of the condition (4.21). Since Y is a fundamental matrix of the system (4.18), the condition (4.21) is fulfilled if and only if the solution $x = (x_1, x_2)^T$ to the problem (4.133) satisfies the condition (4.134).

Let $x = (x_1, x_2)^T$ be the unique solution to the problem (4.133). The vector function x is non-negative, because the relations (4.127) hold. Therefore, the equation in (4.133) yields that
$$x'(t) \leq A(t)x(t) + \widetilde{q}(t) \ \text{ for a. e. } \ t \in [a,b],$$
where the matrix function A is given by the formula (4.56). According to Proposition 4.1, we get
$$x(t) \leq v(t) \ \text{ for } \ t \in [a,b],$$
where v is a solution to the problem (4.55). Consequently, by virtue of the condition (4.26) and Lemma 4.41, the functions x_1 and x_2 satisfy the condition (4.134) and thus the inequality (4.21) holds. Hence, the assumptions of Corollary 4.15 are satisfied. □

Proof of Proposition 4.20. According to Remarks 1.5 and 1.7, the components ℓ_{ik} $(i, k = 1,2)$ of the operator ℓ are positive a-Volterra operators. Using Lemma 4.42, it is not difficult to verify that, for any $v \in C([a,b]; \mathbb{R}_+^2)$, the inequality
$$\ell_{ik}\big(\varphi_k(v)\big)(t) \leq \ell_{ik}(1)(t)\varphi_k(v)(t) \ \text{ for a. e. } \ t \in [a,b], \quad i,k = 1,2,$$

holds, where the operator φ is defined by the formula (4.12) and φ_k denotes its kth component.

Consequently, the assumptions of Corollary 4.15 are satisfied with $\bar{\ell} = 0$.[11] $\square$

Proof of Theorem 4.21. First suppose that the conditions (4.29) and (4.30) hold. Let $u = (u_1, u_2)^T \in AC([a,b];\mathbb{R}^2)$ be a vector function satisfying the conditions (4.3) and (4.4). We will show that the function u is non-negative. According to the assumption (4.30), we have

$$u_i'(t) \geq \ell_{ii}(u_i)(t) \ \text{ for a. e. } \ t \in [a,b], \quad u_i(a) \geq 0$$

for $i = 1, 2$ and thus, by virtue of the inclusions (4.29), we get

$$u_i(t) \geq 0 \ \text{ for } \ t \in [a,b], \ \ i = 1, 2.$$

Consequently, the operator ℓ belongs to the set $\mathcal{S}^2_{ab}(a)$.

Now suppose that $\ell \in \mathcal{S}^2_{ab}(a)$. According to the assumption $-\ell \in \mathcal{P}_{ab}$ and Remark 1.5, we have

$$-\ell_{ik} \in \mathcal{P}_{ab} \ \text{ for } \ i, k = 1, 2. \tag{4.136}$$

We will show that $\ell_{11} \in \mathcal{S}_{ab}(a)$ (the validity of the inclusion $\ell_{22} \in \mathcal{S}_{ab}(a)$ can be proved analogously). Let $u_1 \in AC([a,b];\mathbb{R})$ be a function satisfying the relations

$$u_1'(t) \geq \ell_{11}(u_1)(t) \ \text{ for a. e. } \ t \in [a,b], \quad u_1(a) \geq 0. \tag{4.137}$$

Put

$$u_2(t) = \int_a^t |\ell_{21}(u_1)(s)| \, \mathrm{d}s \ \text{ for } \ t \in [a,b]. \tag{4.138}$$

Then it is clear that

$$u_2'(t) = |\ell_{21}(u_1)(t)| \geq \ell_{21}(u_1)(t) \ \text{ for a. e. } \ t \in [a,b]. \tag{4.139}$$

Moreover, the relation (4.138) guarantees that

$$u_2(t) \geq 0 \ \text{ for } \ t \in [a,b].$$

From the inequalities (4.137) and (4.139), in view of the conditions (4.136) we get

$$u_i'(t) \geq \ell_{i1}(u_1)(t) \geq \sum_{k=1}^{2} \ell_{ik}(u_k)(t) = \ell_i(u)(t) \ \text{ for a. e. } \ t \in [a,b], \ \ i = 1, 2,$$

where $u = (u_1, u_2)^T$. Consequently, the vector function u satisfies the conditions (4.3) and (4.4) which, together with the assumption $\ell \in \mathcal{S}^2_{ab}(a)$, guarantees the validity of the condition (4.5). Hence,

$$u_1(t) \geq 0 \ \text{ for } \ t \in [a,b],$$

and thus $\ell_{11} \in \mathcal{S}_{ab}(a)$.

[11] The symbol 0 stands here for the zero operator.

It remains to show that $\ell_{21} = 0$[12] (the equality $\ell_{12} = 0$ can be proved analogously). We have proved above that $\ell_{11} \in \mathcal{S}_{ab}(a)$ and thus, by virtue of the inclusion $-\ell_{11} \in \mathcal{P}_{ab}$ and [6, Thm. 2], the operator ℓ_{11} is an a-Volterra one. Define the operators $\widetilde{\ell}$ and $\bar{\ell}$ by the formulas

$$\widetilde{\ell}(v)(t)=\begin{pmatrix}\ell_{12}(v_2)(t)\\ \ell_2(v)(t)\end{pmatrix} \quad \text{for a.e. } t\in[a,b] \text{ and all } v=(v_1,v_2)^T\in C([a,b];\mathbb{R}^2)$$

and

$$\bar{\ell}(v)(t) = \begin{pmatrix}-\ell_{11}(v_1)(t)\\ 0\end{pmatrix}$$
$$\text{for a.e. } t \in [a,b] \text{ and all } v = (v_1,v_2)^T \in C([a,b];\mathbb{R}^2).$$

It is clear that $\bar{\ell} \in \mathcal{P}^2_{ab}$ and $\widetilde{\ell} = \bar{\ell} + \ell$. Since ℓ_{11} is an a-Volterra operator, the operator $\bar{\ell}$ is also an a-Volterra one (see Remark 1.7). Hence, using Proposition 4.20, we get $\bar{\ell} \in \mathcal{S}^2_{ab}(a)$ and thus, by virtue of Theorem 4.5 (with $\ell^+ = \bar{\ell}$ and $\ell^- = -\ell$), we obtain that $\widetilde{\ell} \in \mathcal{S}^2_{ab}(a)$. Consequently, the problem

$$u'(t) = \widetilde{\ell}(u)(t), \tag{4.140}$$

$$u(a) = (1,0)^T \tag{4.141}$$

has a unique solution $u = (u_1,u_2)^T$ and this solution is non-negative (see Proposition 4.3). Therefore, in view of the inclusions (4.136), the second equation in the system (4.140) implies

$$u_2'(t) = \widetilde{\ell}_2(u)(t) = \ell_2(u)(t) = \sum_{k=1}^{2} \ell_{2k}(u_k)(t) \le 0 \quad \text{for a.e. } t \in [a,b],$$

which, together with the conditions (4.5) and (4.141), yields that $u_2 \equiv 0$.

On the other hand, from the first equation in the system (4.140) we get

$$u_1'(t) = \widetilde{\ell}_1(u)(t) = \ell_{12}(u_2)(t) = 0 \quad \text{for a.e. } t \in [a,b],$$

and thus $u_1 \equiv 1$. Finally, the second equation in the system (4.140) implies

$$0 = u_2'(t) = \widetilde{\ell}_2(u)(t) = \ell_2(u)(t) = \ell_{21}(1)(t) \quad \text{for a.e. } t \in [a,b],$$

i.e., $\ell_{21}(1) \equiv 0$. However, this means that $\ell_{21} = 0$, because the operator ℓ_{21} is negative. $\square$

Proof of Corollary 4.22. Each of the conditions (a)–(c) of the corollary guarantees the validity of the inclusion $\ell_i \in \mathcal{S}_{ab}(a)$ (see Theorems 1.2 and 1.3, and Corollary 1.2 established in the paper [24]). $\square$

Proof of Theorem 4.24. Let the operator ℓ be defined by the formula (4.31). It is clear that $\ell \in \mathcal{P}^2_{ab}$. Moreover, according to the condition (4.34), we have

$$\varrho^3(t) \le \alpha\varrho^2(t) \quad \text{for } t \in [a,b],$$

[12]The symbol 0 stands here for the zero operator.

where the functions ϱ^2, ϱ^3 are given by the formulas (4.11), (4.12), and

$$\varrho^1 = (\delta_1, \delta_2)^T.$$

Therefore, the assumptions of Corollary 4.9 are satisfied with $k = 2$ and $m = 3$. □

Proof of Corollary 4.26. The validity of the corollary follows immediately from Theorem 4.24 with

$$\alpha = \max\left\{ \frac{1}{\delta_i} \sum_{k=1}^{2} \delta_k \int_a^{\tau^*} p_{ik}(s)\,\mathrm{d}s : \; i = 1,2 \right\}. \qquad \square$$

Proof of Theorem 4.28. We can assume without loss of generality that $i = 1$. Let the operator ℓ be defined by the formula (4.31). It is clear that $\ell \in \mathcal{P}^2_{ab}$. Lemma 4.43 guarantees that

$$\ell \in \mathcal{S}^2_{ab}(a) \iff \ell^{a\tau^*} \in \mathcal{S}^2_{a\tau^*}(a). \tag{4.142}$$

According to the notation (4.33), the restriction $\ell^{a\tau^*}$ of the operator ℓ to the space $C([a,\tau^*];\mathbb{R})$ is given by the formula

$$\ell^{a\tau^*}(v)(t) = \begin{pmatrix} p_{11}(t)v_1\big(\tau_{11}(t)\big) + p_{12}(t)v_2\big(\tau_{12}(t)\big) \\ p_{21}(t)v_1\big(\tau_{21}(t)\big) + p_{22}(t)v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$
$$\text{for a.e. } t \in [a,\tau^*] \text{ and all } v = (v_1,v_2)^T \in C([a,\tau^*];\mathbb{R}^2). \tag{4.143}$$

Moreover, in view of Example 1.2, the components $\ell^{a\tau^*}_{ik}$ $(i,k = 1,2)$ of the operator $\ell^{a\tau^*}$ are defined by the relations

$$\ell^{a\tau^*}_{ik}(z)(t) = p_{ik}(t)z\big(\tau_{ik}(t)\big)$$
$$\text{for a.e. } t \in [a,\tau^*] \text{ and all } z \in C([a,\tau^*];\mathbb{R}), \;\; i,k = 1,2. \tag{4.144}$$

It is clear that $\ell^{a\tau^*}_{ik} \in \mathcal{P}_{ab}$ for $i,k = 1,2$.

We first note that the condition (4.37) implies

$$\int_a^{\tau^*_{22}} p_{22}(s)\,\mathrm{d}s < 1.$$

Hence, Lemma 4.46(a) guarantees that

$$\ell^{a\tau^*}_{22} \in \mathcal{S}_{a\tau^*}(a). \tag{4.145}$$

Case (a). If $p_{12} \not\equiv 0$ on the interval $[a,\tau^*]$, then, in view of the implication (4.142), the assertion of the theorem follows from Proposition 4.13. Therefore, suppose that $p_{12} \equiv 0$. Then, by virtue of the inclusion (4.145), Proposition 4.7 and the implication (4.142), it is sufficient to show that $\ell^{a\tau^*}_{11} \in \mathcal{S}_{a\tau^*}(a)$. However, using the condition (4.38) and Lemma 4.46(a), we see that the inclusion desired holds.

Case (b). According to the conditions (4.36) and (4.39), we find that $p_{12} \equiv 0$ on the interval $[a, \tau^*]$. By virtue of the inclusion (4.145), Proposition 4.7 and the implication (4.142), the operator ℓ belongs to the set $\mathcal{S}^2_{ab}(a)$ if and only if $\ell_{11}^{a\tau^*} \in \mathcal{S}_{a\tau^*}(a)$. However, in view of the condition (4.39) and Lemma 4.46(b), the inclusion $\ell_{11}^{a\tau^*} \in \mathcal{S}_{a\tau^*}(a)$ holds if and only if the condition (4.40) is satisfied. $\square$

Proof of Theorem 4.29. Let the operator ℓ be defined by the formula (4.31). It is clear that $\ell \in \mathcal{P}^2_{ab}$. Lemma 4.43 guarantees that

$$\ell \in \mathcal{S}^2_{ab}(a) \iff \ell^{a\tau^*} \in \mathcal{S}^2_{a\tau^*}(a).$$

According to the notation (4.33), the restriction $\ell^{a\tau^*}$ of the operator ℓ to the space $C([a, \tau^*]; \mathbb{R}^2)$ is given by the formula (4.143). Moreover, in view of Example 1.2, the components $\ell_{ik}^{a\tau^*}$ $(i, k = 1, 2)$ of the operator $\ell^{a\tau^*}$ are defined by the relations (4.144). It is clear that $\ell_{ik}^{a\tau^*} \in \mathcal{P}_{ab}$ for $i, k = 1, 2$.

Case (a). Using the condition (4.38) and Lemma 4.46(a), we get

$$\ell_{11}^{a\tau^*} \in \mathcal{S}_{a\tau^*}(a), \quad \ell_{22}^{a\tau^*} \in \mathcal{S}_{a\tau^*}(a).$$

Hence, by virtue of the condition (4.41) and Proposition 4.7, it is clear that $\ell^{a\tau^*} \in \mathcal{S}^2_{a\tau^*}(a)$ and thus $\ell \in \mathcal{S}^2_{ab}(a)$.

Case (b). It is easy to see from the equality (4.36) that for $i = 1, 2$ either the inequality (4.38) or the equality (4.39) is satisfied. Therefore, in view of the condition (4.41), the assertion of the theorem follows immediately from the implication (4.142), Proposition 4.7 and Lemma 4.46.

Case (c). Since the equality (4.36) is satisfied for $i = 1, 2$, by virtue of Proposition 4.14 the operator $\ell^{a\tau^*}$ belongs to the set $\mathcal{S}^2_{a\tau^*}(a)$ if and only if the homogeneous problem (4.66), (4.67) has only the trivial solution. Consequently, to prove the item (c) of the theorem it is sufficient to show that the homogeneous problem (4.66), (4.67) has only the trivial solution if and only if there exists $i \in \{1, 2\}$ such that the inequality (4.43) is satisfied.

Let $u = (u_1, u_2)^T$ be a solution to the problem (4.66), (4.67). According to the condition (4.42) and Lemma 4.44, we can assume without loss of generality that

$$u_i(t) \geq 0 \ \text{ for } \ t \in [a, \tau^*], \quad i = 1, 2.$$

Therefore from the system (4.66) we get

$$u_i(t) \leq u_i(\tau^*) \ \text{ for } \ t \in [a, \tau^*], \quad i = 1, 2. \tag{4.146}$$

Put

$$u_i^* = \frac{1}{\delta_i}\, u_i(\tau^*) \ \text{ for } \ i = 1, 2, \tag{4.147}$$

$$f_i(t) = \sum_{k=1}^{2} \delta_k \int_a^t p_{ik}(s)\, \mathrm{d}s \ \text{ for } \ t \in [a, \tau^*], \quad i = 1, 2. \tag{4.148}$$

The integration of the system (4.66) from t to τ^*, on account of the inequalities (4.146), implies

$$u_i(\tau^*) - u_i(t) = \int_t^{\tau^*} p_{i1}(s)u_1\big(\tau_{i1}(s)\big)\,\mathrm{d}s + \int_t^{\tau^*} p_{i2}(s)u_2\big(\tau_{i2}(s)\big)\,\mathrm{d}s \leq$$
$$\leq u_1(\tau^*)\int_t^{\tau^*} p_{i1}(s)\,\mathrm{d}s + u_2(\tau^*)\int_t^{\tau^*} p_{i2}(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,\tau^*], \quad i=1,2.$$

Using the notation (4.147), we get

$$\delta_i u_i^* + \sum_{k=1}^{2} \delta_k u_k^* \int_a^t p_{ik}(s)\,\mathrm{d}s \leq$$
$$\leq u_i(t) + \sum_{k=1}^{2} \delta_k u_k^* \int_a^{\tau^*} p_{ik}(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,\tau^*], \quad i=1,2. \tag{4.149}$$

On the other hand, the integration of the system (4.66) from a to t, in view of the conditions (4.67), (4.146) and (4.147), yields that

$$u_i(t) = \int_a^t p_{i1}(s)u_1\big(\tau_{i1}(s)\big)\,\mathrm{d}s + \int_a^t p_{i2}(s)u_2\big(\tau_{i2}(s)\big)\,\mathrm{d}s \leq$$
$$\leq \sum_{k=1}^{2} \delta_k u_k^* \int_a^t p_{ik}(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,\tau^*], \quad i=1,2. \tag{4.150}$$

Now, from the inequalities (4.149) and (4.150) we obtain

$$\delta_i u_i^* \leq \sum_{k=1}^{2} \delta_k u_k^* \int_a^{\tau^*} p_{ik}(s)\,\mathrm{d}s \quad \text{for} \quad i=1,2,$$

whence we get

$$u_i^* \left(\delta_i - \delta_i \int_a^{\tau^*} p_{ii}(s)\,\mathrm{d}s \right) \leq u_{3-i}^* \delta_{3-i} \int_a^{\tau^*} p_{i3-i}(s)\,\mathrm{d}s \quad \text{for} \quad i=1,2.$$

By virtue of the conditions (4.36) and (4.42), the last relation yields $u_i^* \leq u_{3-i}^*$ for $i=1,2$ and thus we have

$$u_1^* = u_2^* \,\big(:= u^*\big). \tag{4.151}$$

Now the inequality (4.149), in view of the conditions (4.36) and (4.148), implies

$$u_i(t) \geq u^* \sum_{k=1}^{2} \delta_k \int_a^t p_{ik}(s)\,\mathrm{d}s + u^*\bigg(\delta_i - \sum_{k=1}^{2} \delta_k \int_a^{\tau^*} p_{ik}(s)\,\mathrm{d}s\bigg) = \\ = u^* f_i(t) \ \text{ for } \ t \in [a,\tau^*], \ \ i = 1,2. \quad (4.152)$$

On the other hand, using the conditions (4.148) and (4.151), the relation (4.150) can be rewritten as

$$u_i(t) \leq u^* \sum_{k=1}^{2} \delta_k \int_a^t p_{ik}(s)\,\mathrm{d}s = u^* f_i(t) \ \text{ for } \ t \in [a,\tau^*], \ \ i = 1,2. \quad (4.153)$$

Hence, the inequalities (4.152) and (4.153) arrive at

$$u_i(t) = u^* f_i(t) \ \text{ for } \ t \in [a,\tau^*], \ \ i = 1,2. \quad (4.154)$$

Finally, the integration of the system (4.66) from a to τ^*, in view of the conditions (4.67) and (4.154), yields that

$$u_i(\tau^*) = \int_a^{\tau^*} p_{i1}(s) u_1\big(\tau_{i1}(s)\big)\,\mathrm{d}s + \int_a^{\tau^*} p_{i2}(s) u_2\big(\tau_{i2}(s)\big)\,\mathrm{d}s = \\ = u^* \sum_{j=1}^{2} \int_a^{\tau^*} p_{ij}(s) f_j\big(\tau_{ij}(s)\big)\,\mathrm{d}s \ \text{ for } \ t \in [a,\tau^*], \ \ i = 1,2,$$

whence we get

$$u^*\bigg[\delta_i - \sum_{j=1}^{2} \int_a^{\tau^*} p_{ij}(s)\bigg(\sum_{k=1}^{2} \delta_k \int_a^{\tau_{ij}(s)} p_{jk}(\xi)\,\mathrm{d}\xi\bigg)\mathrm{d}s\bigg] = 0 \ \text{ for } \ i = 1,2, \quad (4.155)$$

because of the notation (4.147), (4.148) and (4.151).

We have proved that every solution u to the problem (4.66), (4.67) admits the representation

$$u(t) = u^* f(t) \ \text{ for } \ t \in [a,\tau^*],$$

where $f = (f_1, f_2)^T$ and the number u^* satisfies the condition (4.155). Consequently, if there exists $i \in \{1,2\}$ such that the inequality (4.43) is fulfilled, then the homogeneous problem (4.66), (4.67) has only the trivial solution.

It remains to show that if the condition (4.43) is not satisfied for every $i \in \{1,2\}$, i.e.,

$$\sum_{j=1}^{2} \int_a^{\tau^*} p_{ij}(s) f_j\big(\tau_{ij}(s)\big)\,\mathrm{d}s = \delta_i \ \text{ for } \ i = 1,2, \quad (4.156)$$

then the problem (4.66), (4.67) has a nontrivial solution. Indeed, the relations (4.36) and (4.148) yield

$$f_i(t) \le f_i(\tau^*) = \delta_i \quad \text{for} \quad t \in [a, \tau^*], \quad i = 1, 2.$$

Therefore, using the conditions (4.36) and (4.156), it is easy to verify that

$$0 \le \sum_{k=1}^{2} \int_a^t p_{ik}(s)\big[\delta_k - f_k\big(\tau_{ik}(s)\big)\big]\, \mathrm{d}s \le$$

$$\le \sum_{k=1}^{2} \int_a^{\tau^*} p_{ik}(s)\big[\delta_k - f_k\big(\tau_{ik}(s)\big)\big]\, \mathrm{d}s =$$

$$= \delta_i - \sum_{k=1}^{2} \int_a^{\tau^*} p_{ik}(s) f_k\big(\tau_{ik}(s)\big)\, \mathrm{d}s = 0 \quad \text{for} \quad t \in [a, \tau^*], \quad i = 1, 2.$$

Hence we get

$$f_i(t) = \sum_{k=1}^{2} \int_a^t p_{ik}(s) f_k\big(\tau_{ik}(s)\big)\, \mathrm{d}s \quad \text{for} \quad t \in [a, \tau^*], \quad i = 1, 2.$$

Consequently, $f = (f_1, f_2)^T$ is a nontrivial solution to the problem (4.66), (4.67). □

Proof of Theorem 4.30. Let the operator ℓ be defined by the formula (4.31). It is clear that $\ell \in \mathcal{P}_{ab}^2$. According to the conditions (4.44) and (4.45), there exist $x_0 > 0$ and $\varepsilon \in [0, 1[$ such that

$$\int_t^{\tau_{ik}(t)} p(s)\, \mathrm{d}s \le \frac{1}{x_0} \ln\left(x_0 + \frac{\varepsilon x_0}{\mathrm{e}^{x_0 \int_a^{\tau^*} p(s)\, \mathrm{d}s} - \varepsilon} \right)$$

holds for a. e. $t \in [a, b]$, $i, k = 1, 2$. Hence we get

$$\mathrm{e}^{x_0 \int_a^{\tau_{ik}(t)} p(s)\, \mathrm{d}s} - \varepsilon \le x_0\, \mathrm{e}^{x_0 \int_a^t p(s)\, \mathrm{d}s} \quad \text{for a. e.} \quad t \in [a, b], \quad i, k = 1, 2. \tag{4.157}$$

Put

$$\gamma_i(t) = \mathrm{e}^{x_0 \int_a^t p(s)\, \mathrm{d}s} - \varepsilon \quad \text{for} \quad t \in [a, b], \quad i = 1, 2. \tag{4.158}$$

Obviously, $\gamma = (\gamma_1, \gamma_2)^T \in AC([a, b]; \mathbb{R}^2)$ and the inequality (4.7) is satisfied. Moreover, by virtue of the conditions (4.46), (4.157), and (4.158), we get

$$\gamma_i'(t) = x_0 p(t)\, \mathrm{e}^{x_0 \int\limits_a^t p(s)\,\mathrm{d}s} \geq \sum_{k=1}^{2} p_{ik}(t) x_0\, \mathrm{e}^{x_0 \int\limits_a^t p(s)\,\mathrm{d}s} \geq$$
$$\geq \sum_{k=1}^{2} p_{ik}(t) \Bigg(\mathrm{e}^{x_0 \int\limits_a^{\tau_{ik}(t)} p(s)\,\mathrm{d}s} - \varepsilon \Bigg) =$$
$$= \sum_{k=1}^{2} p_{ik}(t) \gamma_k\big(\tau_{ik}(t)\big) = \ell_i(\gamma)(t) \ \text{ for a.e. } \ t \in [a,b], \quad i = 1,2.$$

Therefore, the vector function γ also satisfies the condition (4.8) and thus, using Theorem 4.8, we get $\ell \in \mathcal{S}_{ab}^2(a)$. □

Proof of Corollary 4.31. The validity of the corollary follows immediately from Theorem 4.30 (see also Corollary 4.26 in the case where $p_{ik} \equiv 0$ on the interval $[a, \tau^*]$ for every $i, k \in \{1,2\}$). □

Proof of Theorem 4.32. Let the operator ℓ be defined by the formula (4.31). It is clear that $\ell \in \mathcal{P}_{ab}^2$. Let, moreover, the operator $\bar{\ell} = \big(\bar{\ell}_1, \bar{\ell}_2\big)^T$ be defined by the formula

$$\bar{\ell}_i(v)(t) = \sum_{k=1}^{2} p_{ik}(t) \omega_{ik}(t) \Bigg(\sum_{j=1}^{2} \int\limits_t^{\tau_{ik}(t)} p_{kj}(s) v_j\big(\tau_{kj}(s)\big)\,\mathrm{d}s \Bigg)$$
$$\text{for a.e. } \ t \in [a,b] \ \text{ and all } \ v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2), \quad i = 1,2, \tag{4.159}$$

where the functions ω_{ik} $(i, k = 1,2)$ are given by the relation (4.49). Obviously, $\bar{\ell} \in \mathcal{P}_{ab}^2$ and

$$\ell_i\big(\varphi(v)\big)(t) - \sum_{k=1}^{2} p_{ik}(t) \varphi_k(v)(t) =$$
$$= \sum_{k=1}^{2} p_{ik}(t) \Bigg(\sum_{j=1}^{2} \int\limits_t^{\tau_{ik}(t)} p_{kj}(s) v_j\big(\tau_{kj}(s)\big)\,\mathrm{d}s \Bigg) \leq \bar{\ell}_i(v)(t)$$
$$\text{for a.e. } \ t \in [a,b] \ \text{ and all } \ v = (v_1, v_2)^T \in C_a([a,b];\mathbb{R}_+^2), \quad i = 1,2,$$

where the operator φ is defined by the formula (4.12) and φ_k denotes its kth component. Consequently, the inequality (4.20) holds on the set $C_a([a,b];\mathbb{R}_+^2)$. Therefore, the assumptions of Corollary 4.15 are satisfied. □

Proof of Corollary 4.33. Let the operator ℓ be defined by the formula (4.31). It is clear that $\ell \in \mathcal{P}_{ab}^2$. Analogously to the proof of Theorem 4.32, one can show that the inequality (4.20) holds on the set $C_a([a,b];\mathbb{R}_+^2)$, where the operators φ and $\bar{\ell} = (\bar{\ell}_1, \bar{\ell}_2)^T$ are given by the formulas (4.12) and

(4.159), respectively. Moreover, in view of the condition (4.51), the inequality (4.23) is fulfilled as well. Therefore, by virtue of Corollary 4.16, we get $\ell \in \mathcal{S}_{ab}^2(a)$. $\square$

Proof of Corollary 4.35. Let the operator ℓ be defined by the formula (4.31). It is clear that $\ell \in \mathcal{P}_{ab}^2$. Analogously to the proof of Theorem 4.32, one can show that the inequality (4.20) holds on the set $C_a([a,b];\mathbb{R}_+^2)$, where the operators φ and $\overline{\ell} = (\overline{\ell}_1, \overline{\ell}_2)^T$ are given by the formulas (4.12) and (4.159), respectively. On the other hand, the inequality (4.52) yields the validity of the condition (4.26) and thus, using Corollary 4.18, we get $\ell \in \mathcal{S}_{ab}^2(a)$. $\square$

Proof of Proposition 4.37. Let the operator ℓ be defined by the formula (4.31). It is clear that $\ell \in \mathcal{P}_{ab}^2$. According to the assumptions of the proposition, there exists $\varepsilon > 0$ such that

$$\sum_{k=1}^{n} p_{ik}(t)\Big[\delta_k\big(\tau_{ik}(t) - a\big) + \varepsilon\Big] \leq \delta_i \text{ for a.e. } t \in [a,b], \quad i = 1,2. \tag{4.160}$$

Put

$$\gamma_i(t) = \delta_i(t-a) + \varepsilon \text{ for } t \in [a,b], \quad i = 1,2.$$

Clearly, $\gamma = (\gamma_1, \gamma_2)^T \in AC([a,b];\mathbb{R}^2)$ and the inequality (4.7) is satisfied. Moreover, by virtue of the relation (4.160), we get

$$\begin{aligned} \gamma_i'(t) = \delta_i \geq \sum_{k=1}^{2} p_{ik}(t)\Big[\delta_k\big(\tau_{ik}(t) - a\big) + \varepsilon\Big] = \\ = \sum_{k=1}^{2} p_{ik}(t)\gamma_k\big(\tau_{ik}(t)\big) = \ell_i(\gamma)(t) \text{ for a.e. } t \in [a,b], \quad i = 1,2. \end{aligned}$$

Therefore, the condition (4.8) is satisfied and thus, using Theorem 4.8, we get $\ell \in \mathcal{S}_{ab}^2(a)$. $\square$

Proof of Theorem 4.38. Let the operator ℓ be defined by the formula (4.32). It is clear that $-\ell \in \mathcal{P}_{ab}^2$ and the components ℓ_{12} and ℓ_{21} of the operator ℓ are zero operators. Moreover, by virtue of Lemma 4.47, each of the conditions (a)–(c) in the theorem guarantees the validity of the inclusion $\ell_{ii} \in \mathcal{S}_{ab}(a)$. Consequently, using Theorem 4.21, we get $\ell \in \mathcal{S}_{ab}^2(a)$. $\square$

4.5. **Counterexamples.** In this section, we construct several counterexamples verifying that some of the results presented in Sections 4.1 and 4.2 are unimprovable in a certain sense.

Example 4.48. Let $\varepsilon \in]0,1[$ and functions $p_{ik}, g_i \in L([a,b];\mathbb{R}_+)$ $(i,k = 1,2)$ be such that

$$\int_a^b p_{i1}(s)\,\mathrm{d}s + \int_a^b p_{i2}(s)\,\mathrm{d}s = 1+\varepsilon, \quad \int_a^b g_i(s)\,\mathrm{d}s < 1 \text{ for } i=1,2, \tag{4.161}$$

$$\int_a^b p_{21}(s)\,\mathrm{d}s > \varepsilon. \tag{4.162}$$

Let $\ell = \ell^+ - \ell^-$, where $\ell^+, \ell^- \in \mathcal{P}^2_{ab}$ are defined by the formulas

$$\ell^+(v)(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} v(b)$$
$$\text{for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}^2) \tag{4.163}$$

and

$$\ell^-(v)(t) = \begin{pmatrix} g_1(t) & 0 \\ ;\ 0 & g_2(t) \end{pmatrix} v(a)$$
$$\text{for a.e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}^2). \tag{4.164}$$

According to the condition (4.161) and Corollary 4.26 with $\delta_1 = \delta_2 = 1$, we find

$$(1-\varepsilon)\ell^+ \in \mathcal{S}^2_{ab}(a).$$

Moreover, in view of the condition (4.161) and Theorem 4.38(a), we get

$$-\ell^- \in \mathcal{S}^2_{ab}(a).$$

We first show that the homogeneous problem (3.3) has only the trivial solution. Indeed, let $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)^T$ be a solution to the problem (3.3). Then

$$\widehat{u}_i(b) = \widehat{u}_1(b) \int_a^b p_{i1}(s)\,\mathrm{d}s + \widehat{u}_2(b) \int_a^b p_{i2}(s)\,\mathrm{d}s \text{ for } i=1,2. \tag{4.165}$$

By virtue of the conditions (4.161) and (4.162), the last relations yield $\widehat{u}_1(b) = \widehat{u}_2(b) = 0$. Therefore, from (3.3) we get $\widehat{u} \equiv 0$. According to Proposition 3.1, the problem (3.1), (3.2) with $q \equiv 0$ and $c = (1,0)^T$ has a unique solution $u = (u_1, u_2)^T$. Obviously, the vector function u satisfies the conditions (4.3) and (4.4).

On the other hand, it is easy to verify that

$$u_i(b) - u_i(a) = u_1(b) \int_a^b p_{i1}(s)\,\mathrm{d}s + \\ + u_2(b) \int_a^b p_{i2}(s)\,\mathrm{d}s - u_i(a) \int_a^b g_i(s)\,\mathrm{d}s \quad (4.166)$$

for $i = 1, 2$. Therefore, using the conditions (4.161) and (4.162), from the equalities (4.166) we get

$$- \varepsilon \bigg(\int_a^b p_{12}(s)\,\mathrm{d}s + \int_a^b p_{21}(s)\,\mathrm{d}s - \varepsilon \bigg) u_1(b) = \\ = \bigg(1 - \int_a^b g_1(s)\,\mathrm{d}s \bigg) \bigg(\int_a^b p_{21}(s)\,\mathrm{d}s - \varepsilon \bigg),$$

and thus $u_1(b) < 0$. Consequently, $\ell \notin \mathcal{S}^2_{ab}(a)$.

This example shows that the assumption (4.6) in Theorem 4.5 cannot be replaced by the assumption

$$(1 - \varepsilon)\ell^+ \in \mathcal{S}^2_{ab}(a), \quad -\ell^- \in \mathcal{S}^2_{ab}(a),$$

no matter how small $\varepsilon > 0$ is.

Example 4.49. Let $\varepsilon \in]0, 1[$ and functions $p_{ij}, g_i \in L([a, b]; \mathbb{R}_+)$ $(i, j = 1, 2)$ be such that

$$\int_a^b p_{i1}(s)\,\mathrm{d}s + \int_a^b p_{i2}(s)\,\mathrm{d}s < 1, \quad \int_a^b g_i(s)\,\mathrm{d}s = 1 + \varepsilon \ \text{ for } \ i = 1, 2. \quad (4.167)$$

Let $\ell = \ell^+ - \ell^-$, where $\ell^+, \ell^- \in \mathcal{P}^2_{ab}$ are defined by the relations (4.163) and (4.164), respectively. According to the condition (4.167) and Corollary 4.26 with $\delta_1 = \delta_2 = 1$, we find

$$\ell^+ \in \mathcal{S}^2_{ab}(a).$$

Moreover, in view of the condition (4.167) and Theorem 4.38(a), we get

$$-(1 - \varepsilon)\ell^- \in \mathcal{S}^2_{ab}(a).$$

We first show that the homogeneous problem (3.3) has only the trivial solution. Indeed, let $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)^T$ be a solution to the problem (3.3). Then the equalities (4.165) are fulfilled. By virtue of the condition (4.167), the relations (4.165) yield $\widehat{u}_1(b) = \widehat{u}_2(b) = 0$. Therefore, from (3.3) we get $\widehat{u} \equiv 0$. According to Proposition 3.1, the problem (3.1), (3.2) with $q \equiv 0$ and $c = (1, 0)^T$ has a unique solution $u = (u_1, u_2)^T$. Obviously, the vector function u satisfies the conditions (4.3) and (4.4).

On the other hand, it is easy to verify that the relations (4.166) hold. Therefore, using the conditions (4.167), from the equalities (4.166) we get

$$\left(1-\int_a^b p_{11}(s)\,\mathrm{d}s\right)\left(1-\int_a^b p_{22}(s)\,\mathrm{d}s\right)u_1(b)- \\ -u_1(b)\int_a^b p_{12}(s)\,\mathrm{d}s\int_a^b p_{21}(s)\,\mathrm{d}s=-\varepsilon\left(1-\int_a^b p_{22}(s)\,\mathrm{d}s\right),$$

and thus $u_1(b)<0$. Consequently, $\ell\notin\mathcal{S}^2_{ab}(a)$.

This example shows that the assumption (4.6) in Theorem 4.5 cannot be replaced by the assumption

$$\ell^+\in\mathcal{S}^2_{ab}(a),\quad -(1-\varepsilon)\ell^-\in\mathcal{S}^2_{ab}(a),$$

no matter how small $\varepsilon>0$ is.

Example 4.50. Let $\tau_{ik}\equiv b$ for $i,k=1,2$. Choose $p_{ik}\in L([a,b];\mathbb{R}_+)$ $(i,k=1,2)$ such that

$$p_{11}\equiv p_{22},\quad p_{12}\equiv p_{21},$$

and

$$\int_a^b p_{11}(s)\,\mathrm{d}s+\int_a^b p_{12}(s)\,\mathrm{d}s=1.$$

Let the operator ℓ be defined by the formula (4.31). Obviously, $\ell\in\mathcal{P}^2_{ab}$ and, for any $m>k$, the condition (4.9) with $\alpha=1$ is satisfied, where the functions ϱ^m $(m=2,3,\dots)$ are defined by the relation (4.11) and $\varrho^1=(1,1)^T$. Moreover, the condition (4.14) is fulfilled with $\delta_1=\delta_2=1$ and the inequality (4.20) holds on the set $C_a([a,b];\mathbb{R}^2_+)$, where the operator φ is given by the relation (4.12) and $\bar\ell=\left(\bar\ell_1,\bar\ell_2\right)^T\in\mathcal{P}^2_{ab}$ is defined by the formula

$$\bar\ell_i(v)(t)=\sum_{j=1}^2 p_{ij}(t)\left(\int_t^b\sum_{k=1}^2 p_{jk}(s)v_k(b)\,\mathrm{d}s\right) \\ \text{for a.e. } t\in[a,b] \text{ and all } v=(v_1,v_2)^T\in C([a,b];\mathbb{R}^2),\ \ i=1,2.$$

Since

$$\int_a^b\sum_{j=1}^2 p_{ij}(s)\left(\int_s^b\sum_{k=1}^2 p_{jk}(\xi)\,\mathrm{d}\xi\right)\mathrm{e}^{\int_s^b\sum_{\nu=1}^2 p_{i\nu}(\eta)\,\mathrm{d}\eta}\,\mathrm{d}s=1 \ \text{ for } \ i=1,2,$$

the conditions

$$\mathrm{e}^{\max\left\{\int_a^b\ell_{11}(1)(s)\,\mathrm{d}s,\int_a^b\ell_{22}(1)(s)\,\mathrm{d}s\right\}}\int_a^b h(s)\,\mathrm{e}^{\int_s^b p(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s=1$$

and

$$\max\left\{\lambda_1\, \mathrm{e}^{\int_a^b \ell_{11}(1)(s)\,\mathrm{d}s}, \lambda_2\, \mathrm{e}^{\int_a^b \ell_{22}(1)(s)\,\mathrm{d}s}\right\} = 1$$

are fulfilled, where the functions h, p and the numbers λ_1, λ_2 are given by the relations (4.25), (4.24), and (4.27), respectively (note that $\widetilde{p}_{12} \equiv \widetilde{p}_{21} \equiv p_{12} \equiv p_{21}$ and $\widetilde{q}_i$ $(i = 1, 2)$ are given by the formula (4.22)).

On the other hand, the vector function $u = (u_1, u_2)^T$, where

$$u_i(t) = \int_a^t p_{i1}(s)\,\mathrm{d}s + \int_a^t p_{i2}(s)\,\mathrm{d}s \quad \text{for } t \in [a, b], \quad i = 1, 2,$$

is a nontrivial solution to the problem (3.3). Therefore, by virtue of Proposition 4.3, we get $\ell \notin \mathcal{S}^2_{ab}(a)$.

This example shows that the assumption $\alpha \in [0, 1[$ in Corollary 4.9 and Theorem 4.24 cannot be replaced by the assumption $\alpha \in [0, 1]$, and the strict inequalities (4.13), (4.23), and (4.26) in Corollaries 4.11, 4.16, and 4.18, respectively, cannot be replaced by the nonstrict ones.

Moreover, this example shows that the strict inequalities (4.35), (4.51), and (4.52) in Corollaries 4.26, 4.33, and 4.35, respectively, cannot be replaced by the nonstrict ones.

5. Weak Theorems on Differential Inequalities

In the previous section, we have established conditions sufficient for the validity of the inclusion $\ell \in \mathcal{S}^2_{ab}(a)$ in the case where both components[13] ℓ_{12} and ℓ_{21} of the operator ℓ are positive. A question that naturally arises is what happens in the case where the components indicated are not both positive. Proposition 4.1 claims that for the ordinary differential system

$$u' = P(t)u + q(t)$$

in which $P = (p_{ik})^2_{i,k=1} \colon [a, b] \to \mathbb{R}^{2\times 2}$ is an integrable matrix function and $q \in L([a, b]; \mathbb{R}^2)$, a theorem on differential inequalities holds if

$$p_{12}(t) \geq 0, \quad p_{21}(t) \geq 0 \quad \text{for a.e. } t \in [a, b]. \tag{5.1}$$

In other words, the condition (5.1) is **sufficient** for the validity of the inclusion $\ell \in \mathcal{S}^2_{ab}(a)$, where the operator ℓ is defined by the relation

$$\ell(v)(t) = P(t)v(t) \quad \text{for a.e. } t \in [a, b] \text{ and all } v \in C([a, b]; \mathbb{R}^2).$$

If the coefficients p_{ik} $(i, k = 1, 2)$ are continuous, then the condition (5.1) is not only sufficient but also **necessary** (see, e. g., [42, § 1.7]).

Therefore, the requirement of the validity of the condition (4.5) in Definition 4.2 seems to be too restrictive in the case where the components ℓ_{12} and ℓ_{21} of the operator ℓ are not both positive. We can weaken the condition (4.5) in the following way.

[13] See the item 27 in Section 1.

Definition 5.1. Let $k \in \{1, 2\}$. We say that a linear bounded operator $\ell\colon C([a, b]; \mathbb{R}^2) \to L([a, b]; \mathbb{R}^2)$ belongs to the set $\widehat{\mathcal{S}}^{2,k}_{ab}(a)$ if for an arbitrary function $u = (u_1, u_2)^T \in AC([a, b]; \mathbb{R}^2)$ satisfying the inequalities (4.3) and (4.4) the relation

$$u_k(t) \geq 0 \ \text{ for } \ t \in [a, b] \tag{5.2}$$

is fulfilled.

If $\ell \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$, then we say that the *weak theorem on differential inequalities* holds for the system (3.1).

Remark 5.2. Let $k \in \{1, 2\}$. It follows immediately from Definitions 4.2 and 5.1 that:

(a) $\mathcal{S}^2_{ab}(a)$ is a proper subset of the set $\widehat{\mathcal{S}}^{2,k}_{ab}(a)$.

(b) If the operator ℓ is such that $\ell_{3-kk} \in \mathcal{P}_{ab}$ and $\ell_{3-k3-k} \in \mathcal{S}_{ab}(a)$, then

$$\ell \in \widehat{\mathcal{S}}^{2,k}_{ab}(a) \iff \ell \in \mathcal{S}^2_{ab}(a).$$

In this section, we establish *weak theorems on differential inequalities* for the "anti-diagonal" two-dimensional system

$$\begin{aligned} u_1'(t) &= h_1(u_2)(t) + q_1(t), \\ u_2'(t) &= h_2(u_1)(t) + q_2(t), \end{aligned} \tag{5.3}$$

where $h_1, h_2 \in \mathcal{L}_{ab}$ and $q_1, q_2 \in L([a, b]; \mathbb{R})$, in the case where either h_1 or h_2 is a positive operator.

Remark 5.3. For the sake of convenience, if the weak theorem on differential inequalities (resp., the theorem on differential inequalities) holds for the system (5.3), then we write $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$ (resp., $(h_1, h_2) \in \mathcal{S}^2_{ab}(a)$) instead of $\ell \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$ (resp., $\ell \in \mathcal{S}^2_{ab}(a)$) with

$$\ell(v)(t) = \begin{pmatrix} h_1(v_2)(t) \\ h_2(v_1)(t) \end{pmatrix} \ \text{ for a.e. } t \in [a, b] \ \text{ and all } \ v = (v_1, v_2)^T \in C([a, b]; \mathbb{R}^2).$$

Remark 5.4. It follows immediately from Definition 5.1 that:

(a) $(0, h) \in \widehat{\mathcal{S}}^{2,1}_{ab}(a)$ for every $h \in \mathcal{L}_{ab}$.[14]

(b) $(h, 0) \in \widehat{\mathcal{S}}^{2,2}_{ab}(a)$ for every $h \in \mathcal{L}_{ab}$.

(c) For any $k = 1, 2$ and $h_1, h_2 \in \mathcal{L}_{ab}$, we have

$$(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a) \iff (h_2, h_1) \in \widehat{\mathcal{S}}^{2,3-k}_{ab}(a).$$

(d) For any $k = 1, 2$ and $h_1, h_2 \in \mathcal{P}_{ab}$, we have

$$(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a) \iff (h_2, h_1) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a).$$

Remark 5.5. Let $k \in \{1, 2\}$ and $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$. Then it is clear that the homogeneous problem

$$\begin{gathered} u_1'(t) = h_1(u_2)(t), \quad u_2'(t) = h_2(u_1)(t), \\ u_1(a) = 0, \quad u_2(a) = 0 \end{gathered} \tag{5.4}$$

---

[14] The symbol 0 stands here for the zero operator.

has only the trivial solution. Indeed, if $(u_1,u_2)^T$ is a solution to the problem (5.4), then the inclusion $(h_1,h_2)\in\widehat{\mathcal{S}}_{ab}^{2,k}(a)$ yields that $u_k\equiv 0$. Consequently, $u_{3-k}\equiv 0$ as well and thus the problem (5.4) has only the trivial solution.

Therefore, according to Proposition 3.1, the Cauchy problem

$$u_1(a)=c_1,\quad u_2(a)=c_2 \tag{5.5}$$

subjected to the system (5.3) has a unique solution for all $q_1,q_2\in L([a,b];\mathbb{R})$ and $c_1,c_2\in\mathbb{R}$. However, the inclusion $(h_1,h_2)\in\widehat{\mathcal{S}}_{ab}^{2,k}(a)$ guarantees, in addition, that the solution $(u_1,u_2)^T$ to this problem satisfies the condition (5.2) whenever q_1,q_2 and c_1,c_2 are such that

$$q_k(t)\geq 0 \ \text{ for a. e. } \ t\in[a,b],\ \ c_k\geq 0 \ \ (k=1,2).$$

In Section 5.1, main results are formulated, their proofs being postponed till Section 5.4. Differential systems with argument deviations are studied in more detail in Section 5.2, in which case further results are obtained. In Section 5.5, the counterexamples are constructed verifying that the results obtained are unimprovable in a certain sense.

5.1. **Main results.** The next statement describes a characteristic property of the set $\widehat{\mathcal{S}}_{ab}^{2,k}(a)$.

Theorem 5.6. *Let $k\in\{1,2\}$, $h_k\in\mathcal{P}_{ab}$, and $h_{3-k}=h_{3-k,0}-h_{3-k,1}$ with $h_{3-k,0},h_{3-k,1}\in\mathcal{P}_{ab}$. Assume that*

$$(h_1,h_{2,0})\in\widehat{\mathcal{S}}_{ab}^{2,1}(a),\quad (h_1,-h_{2,1})\in\widehat{\mathcal{S}}_{ab}^{2,1}(a)\ \text{ if }\ k=1,$$

and

$$(h_{1,0},h_2)\in\widehat{\mathcal{S}}_{ab}^{2,2}(a),\quad (-h_{1,1},h_2)\in\widehat{\mathcal{S}}_{ab}^{2,2}(a)\ \text{ if }\ k=2.$$

Then $(h_1,h_2)\in\widehat{\mathcal{S}}_{ab}^{2,k}(a)$.

It is proved in [40, Ch. VII, § 1.2] that $h\in\mathcal{L}_{ab}$ admits the representation $h=h_0-h_1$ with $h_0,h_1\in\mathcal{P}_{ab}$ if and only if the operator h is strongly bounded.[15] Consequently, due to the results given in Sections 5.1.1 and 5.1.2, Theorem 5.6 allows one to obtain several efficient conditions for positive h_k and strongly bounded h_{3-k} that guarantee the validity of the inclusion $(h_1,h_2)\in\widehat{\mathcal{S}}_{ab}^{2,k}(a)$.

5.1.1. *The case $h_1,h_2\in\mathcal{P}_{ab}$.* We first consider the case where both operators h_1 and h_2 are positive. In this case, we have

$$(h_1,h_2)\in\widehat{\mathcal{S}}_{ab}^{2,k}(a)\iff (h_1,h_2)\in\mathcal{S}_{ab}^{2}(a)$$

(see Remark 5.2(b)). We have studied properties of the set $\mathcal{S}_{ab}^2(a)$ in Section 4. For the sake of completeness, we formulate here a general result (see Theorem 5.7) and two of its corollaries. Then we derive two new corollaries of this general theorem (namely, Corollary 5.14 and Proposition 5.15), which cannot be found above.

[15]See Definition 1.1.

Theorem 5.7 (Theorem 4.8)**.** *Let $k \in \{1,2\}$ and $h_1, h_2 \in \mathcal{P}_{ab}$. Then $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$ if and only if there exist functions $\gamma_1, \gamma_2 \in AC([a,b];\mathbb{R})$ such that*

$$\gamma_1(t) > 0, \quad \gamma_2(t) > 0 \;\; \text{for} \;\; t \in [a,b] \tag{5.6}$$

and

$$\gamma_1'(t) \geq h_1(\gamma_2)(t), \quad \gamma_2'(t) \geq h_2(\gamma_1)(t) \;\; \text{for a. e.} \;\; t \in [a,b]. \tag{5.7}$$

Corollary 5.8 (Corollary 4.16)**.** *Let $h_1, h_2 \in \mathcal{P}_{ab}$ and there exist operators $\widetilde{h}_1, \widetilde{h}_2 \in \mathcal{P}_{ab}$ such that the inequalities*

$$h_i\big(\psi\big(h_{3-i}(z)\big)\big)(t) - h_i(1)(t)\psi\big(h_{3-i}(z)\big)(t) \leq \widetilde{h}_i(z)(t)$$
$$\text{for a. e.} \;\; t \in [a,b] \;\; \text{and all} \;\; z \in C([a,b];\mathbb{R}_+), \;\; i = 1,2 \tag{5.8}$$

hold, where

$$\psi(f)(t) = \int_a^t f(s)\,\mathrm{d}s \;\; \text{for} \;\; t \in [a,b], \;\; f \in L([a,b];\mathbb{R}). \tag{5.9}$$

Let, moreover,

$$\int_a^b g(s)\,\mathrm{e}^{\int_s^b \omega(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s < 1, \tag{5.10}$$

where

$$\omega(t) = \max\big\{h_1(1)(t), h_2(1)(t)\big\} \;\; \text{for a. e.} \;\; t \in [a,b], \tag{5.11}$$
$$g(t) = \max\big\{\widetilde{h}_1(1)(t), \widetilde{h}_2(1)(t)\big\} \;\; \text{for a. e.} \;\; t \in [a,b].$$

Then $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,1}_{ab}(a) \cap \widehat{\mathcal{S}}^{2,2}_{ab}(a)$.

Remark 5.9. The strict inequality (5.10) in the previous corollary cannot be replaced by the nonstrict one (see Example 4.50 with $p_{11} \equiv 0$ and $p_{22} \equiv 0$).

Corollary 5.10 (Corollary 4.18)**.** *Let $h_1, h_2 \in \mathcal{P}_{ab}$ and let there exist operators $\widetilde{h}_1, \widetilde{h}_2 \in \mathcal{P}_{ab}$ such that the inequalities* (5.8) *hold, where the operator ψ is defined by the relation* (5.9)*. Let, moreover,*

$$\max\big\{\lambda_1, \lambda_2\big\} < 1, \tag{5.12}$$

where

$$\lambda_i = \int_a^b \cosh\bigg(\int_s^b \omega(\xi)\,\mathrm{d}\xi\bigg)\widetilde{h}_i(1)(s)\,\mathrm{d}s +$$
$$+ \int_a^b \sinh\bigg(\int_s^b \omega(\xi)\,\mathrm{d}\xi\bigg)\widetilde{h}_{3-i}(1)(s)\,\mathrm{d}s \;\; \text{for} \;\; i = 1,2$$

and the function ω is defined by the relation (5.11)*. Then the pair (h_1, h_2) belongs both to $\widehat{\mathcal{S}}^{2,1}_{ab}(a)$ and $\widehat{\mathcal{S}}^{2,2}_{ab}(a)$.*

Remark 5.11. The strict inequality (5.12) in the previous corollary cannot be replaced by the nonstrict one (see Example 4.50 with $p_{11} \equiv 0$ and $p_{22} \equiv 0$).

Now we introduce a simple notation.

Notation 5.12. For any $\ell \in \mathcal{L}_{ab}$, we put

$$b_\ell^* = \inf \mathcal{A}(\ell),$$

where $\mathcal{A}(\ell)$ denotes the set of all $t \in [a,b]$ for which the implication

$$\left.\begin{array}{r} z \in C([a,b];\mathbb{R}), \\ z(\xi) = 0 \ \text{ for } \ \xi \in [a,t] \end{array}\right\} \quad \Longrightarrow \quad \ell(z)(\xi) = 0 \ \text{ for a.e. } \ \xi \in [a,b]$$

holds.

Remark 5.13. It is easy to verify that $b_\ell^* \in \mathcal{A}(\ell)$, i.e.,

$$\left.\begin{array}{r} z \in C([a,b];\mathbb{R}), \\ z(\xi) = 0 \ \text{ for } \ \xi \in [a,b_\ell^*] \end{array}\right\} \quad \Longrightarrow \quad \ell(z)(\xi) = 0 \ \text{ for a.e. } \ \xi \in [a,b].$$

The following statements can also be derived from Theorem 5.7.

Corollary 5.14. *Let $h_1, h_2 \in \mathcal{P}_{ab}$ and there exist $i \in \{1,2\}$ such that*

$$\int_a^{b^*_{h_{3-i}}} h_i\big(\psi\big(h_{3-i}(1)\big)\big)(s)\,\mathrm{d}s < 1, \tag{5.13}$$

where the operator ψ is given by the relation (5.9) *and the number $b^*_{h_{3-i}}$ is defined in Notation 5.12. Then $(h_1,h_2) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a) \cap \widehat{\mathcal{S}}_{ab}^{2,2}(a)$.*

The next proposition can be regarded as a complement of the previous corollary.

Proposition 5.15. *Let $k \in \{1,2\}$, $h_1, h_2 \in \mathcal{P}_{ab}$, and there exist $i \in \{1,2\}$ such that*

$$\int_a^{b^*_{h_{3-i}}} h_i\big(\psi\big(h_{3-i}(1)\big)\big)(s)\,\mathrm{d}s = 1, \tag{5.14}$$

where the operator ψ is given by the relation (5.9) *and the number $b^*_{h_{3-i}}$ is defined in Notation* 5.12. *Then $(h_1,h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$ if and only if the homogeneous problem* (5.4) *has only the trivial solution.*

Example 5.16. On the interval $[0, \pi/4]$, we consider the integro-differential system

$$
\begin{aligned}
u_1'(t) &= d_1 \sin t \int_0^{t/2} s u_2\big(s/2\big)\,\mathrm{d}s + q_1(t),\\
u_2'(t) &= d_2 \cos(2t) \int_0^{t} \cos(2s) u_1\big(\tau(s)\big)\,\mathrm{d}s + q_2(t),
\end{aligned}
\tag{5.15}
$$

where $\tau\colon [0, \pi/4] \to [0, \pi/4]$ is a measurable function, $q_1, q_2 \in L[0, \pi/4]; \mathbb{R})$, and $d_1, d_2 \in \mathbb{R}_+$ are such that

$$
d_1 d_2 < \frac{2^{12}}{4\pi\big(1+2\sqrt{2}\big) - \pi^2\big(1+\sqrt{2}\big) - 24}\,.
$$

It is clear that the system (5.15) is a particular case of the system (5.3) in which $a = 0$, $b = \pi/4$, and h_1, h_2 are given by the formulas

$$
\begin{aligned}
h_1(z)(t) &= d_1 \sin t \int_0^{t/2} s z\big(s/2\big)\,\mathrm{d}s,\\
h_2(z)(t) &= d_2 \cos(2t) \int_0^{t} \cos(2s) z\big(\tau(s)\big)\,\mathrm{d}s
\end{aligned}
\tag{5.16}
$$

for a.e. $t \in [0, \pi/4]$ and all $z \in C([0, \pi/4]; \mathbb{R})$. It is not difficult to verify that $b_{h_2}^* = \text{ess sup}\big\{\tau(t) : t \in [0, \pi/4]\big\}$ (see Notation 5.12) and

$$
\begin{aligned}
\psi\big(h_2(1)\big)(t) = \int_0^t h_2(1)(s)\,\mathrm{d}s = \int_0^t d_2 \cos(2s) \int_0^s \cos(2\xi)\,\mathrm{d}\xi\,\mathrm{d}s =\\
= \frac{d_2}{16}\big(1 - \cos(4t)\big) \quad \text{for} \quad t \in [0, \pi/4].
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\int_0^{b_{h_2}^*} h_1\big(\psi\big(h_2(1)\big)\big)(s)\,\mathrm{d}s \le \int_0^{\pi/4} h_1\big(\psi\big(h_2(1)\big)\big)(s)\,\mathrm{d}s =\\
= \int_0^{\pi/4} d_1 \sin s \int_0^{s/2} \xi\, \frac{d_2}{16}\big(1 - \cos(2\xi)\big)\,\mathrm{d}\xi\,\mathrm{d}s =\\
= \frac{d_1 d_2}{2^{12}}\Big(4\pi\big(1+2\sqrt{2}\big) - \pi^2\big(1+\sqrt{2}\big) - 24\Big) < 1.
\end{aligned}
$$

Therefore, according to Corollary 5.14 with $i = 1$ and Remark 5.5, the Cauchy problem

$$
u_1(0) = c_1, \quad u_2(0) = c_2 \tag{5.17}
$$

subjected to the system (5.15) has a unique solution for arbitrary $q_1, q_2 \in L([0,\pi/4];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if q_1, q_2 and c_1, c_2 fulfil the additional condition

$$q_k(t) \geq 0 \ \text{ for a.e. } \ t \in [0,\pi/4], \quad c_k \geq 0 \ \ (k=1,2), \tag{5.18}$$

then the unique solution $(u_1, u_2)^T$ to this problem satisfies the relation

$$u_1(t) \geq 0, \quad u_2(t) \geq 0 \ \text{ for } \ t \in [0,\pi/4].$$

Example 5.17. On the interval $[0,1]$, we consider the Cauchy problem

$$u''(t) = \frac{d}{(1-t)^\lambda} \int_0^t \frac{u\big(\tau(s)\big)}{(1-s)^\lambda} \, \mathrm{d}s + q(t); \quad u(0) = c_1, \ \ u'(0) = c_2, \tag{5.19}$$

where $\lambda < 1$, $0 \leq d < (3-2\lambda)(2-\lambda)$, $\tau \colon [0,1] \to [0,1]$ is a measurable function, $q \in L([0,1];\mathbb{R})$, and $c_1, c_2 \in \mathbb{R}$.

It is clear that the problem (5.19) can be regarded as a particular case of the problem (5.3), (5.5) in which $a = 0$, $b = 1$, $q_1 \equiv 0$, $q_2 \equiv q$, and h_1, h_2 are given by the formulas

$$h_1(z)(t) = z(t), \quad h_2(z)(t) = \frac{d}{(1-t)^\lambda} \int_0^t \frac{z\big(\tau(s)\big)}{(1-s)^\lambda} \, \mathrm{d}s \tag{5.20}$$

for a.e. $t \in [0,1]$ and all $z \in C([0,1];\mathbb{R})$. It is not difficult to verify that $b^*_{h_2} = \operatorname{ess\,sup}\big\{\tau(t) : \ t \in [0,1]\big\}$ (see Notation 5.12) and

$$\begin{aligned}\int_0^{b^*_{h_2}} h_1\Big(\psi\big(h_2(1)\big)\Big)(s) \, \mathrm{d}s &= \int_0^{b^*_{h_2}} (b^*_{h_2} - s) h_2(1)(s) \, \mathrm{d}s \leq \\ &\leq \int_0^1 (1-s) h_2(1)(s) \, \mathrm{d}s = d \int_0^1 (1-s)^{1-\lambda} \int_0^s \frac{\mathrm{d}\xi}{(1-\xi)^\lambda} \, \mathrm{d}s = \\ &= \frac{d}{(3-2\lambda)(2-\lambda)} < 1.\end{aligned}$$

Therefore, according to Corollary 5.14 with $i = 1$ and Remark 5.5, the problem (5.19) has a unique solution for arbitrary $q \in L([0,1];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if q and c_1, c_2 fulfil the additional condition

$$q(t) \geq 0 \ \text{ for a.e. } \ t \in [0,1], \quad c_1 \geq 0, \quad c_2 \geq 0, \tag{5.21}$$

then the unique solution u to this problem satisfies the relation

$$u(t) \geq 0, \quad u'(t) \geq 0 \ \text{ for } \ t \in [0,1].$$

5.1.2. *The case $h_k \in \mathcal{P}_{ab}$ and $-h_{3-k} \in \mathcal{P}_{ab}$.* Now we consider the case where the operators h_k and h_{3-k} are positive and negative, respectively. Here we have a sufficient and necessary condition for the validity of the inclusion $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$ provided that both operators h_1 and h_2 are a-Volterra ones.

Theorem 5.18. *Let $k \in \{1,2\}$, $-h_{3-k}, h_k \in \mathcal{P}_{ab}$ and let the operators h_1, h_2 be a-Volterra ones. Then $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$ if and only if there exist functions $\gamma_1, \gamma_2 \in AC_{loc}([a,b[\,;\mathbb{R})$ such that $\gamma_k \in C([a,b];\mathbb{R})$,*

$$\gamma_k'(t) \le h_k(\gamma_{3-k})(t) \ \textit{ for a. e. } \ t \in [a,b],^{16} \tag{5.22}$$

$$\gamma_{3-k}'(t) \le h_{3-k}(\gamma_k)(t) \ \textit{ for a. e. } \ t \in [a,b], \tag{5.23}$$

$$\gamma_k(t) \ge 0 \ \textit{ for } \ t \in [a,b], \tag{5.24}$$

$$\gamma_k(a) > 0, \quad \gamma_{3-k}(a) \le 0, \tag{5.25}$$

and

$$|\gamma_1(t)| + |\gamma_2(t)| \ne 0 \ \textit{ for } \ t \in]a,b[\,. \tag{5.26}$$

Remark 5.19. Since possibly $\gamma_{3-k}(t) \to -\infty$ as $t \to b-$, the condition (5.22) of the previous theorem is understood in the sense that for any $b_0 \in]a,b[$ the relation

$$\gamma_k'(t) \le h_k^{ab_0}(\gamma_{3-k})(t) \ \text{ for a. e. } \ t \in [a,b_0]$$

holds, where $h_k^{ab_0}$ denotes the restriction of the operator h_k to the space $C([a,b_0];\mathbb{R})$.[17]

Remark 5.20. Observe that the function γ_{3-k} in Theorem 5.18 necessarily satisfies the condition

$$\gamma_{3-k}(t) \le 0 \ \text{ for } \ t \in [a,b[\,. \tag{5.27}$$

Theorem 5.18 yields the following corollary.

Corollary 5.21. *Let $k \in \{1,2\}$, $-h_{3-k}, h_k \in \mathcal{P}_{ab}$ and let the operators h_1, h_2 be a-Volterra ones. If, moreover, the inequality*

$$\int_a^b \big|h_k\big(\psi\big(h_{3-k}(1)\big)\big)(s)\big| \,\mathrm{d}s \le 1 \tag{5.28}$$

holds, where the operator ψ is defined by (5.9)*, then $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$.*

Remark 5.22. The inequality (5.28) of the previous corollary cannot be replaced by the inequality

$$\int_a^b \big|h_k\big(\psi\big(h_{3-k}(1)\big)\big)(s)\big| \,\mathrm{d}s \le 1 + \varepsilon, \tag{5.29}$$

[16] See Remark 5.19.
[17] See Definition 1.8.

no matter how small $\varepsilon > 0$ is (see Example 5.40).

Example 5.23. On the interval $[0, \pi/4]$, we consider the integro-differential system (5.15), where $\tau\colon [0,\pi/4] \to [0,\pi/4]$ is a measurable function, $\tau(t) \le t$ for a. e. $t \in [0,\pi/4]$, $q_1, q_2 \in L([0,\pi/4];\mathbb{R})$, and $d_1 \ge 0$, $d_2 \le 0$ are such that

$$d_1|d_2| \le \frac{2^{12}}{4\pi\big(1+2\sqrt{2}\big) - \pi^2\big(1+\sqrt{2}\big) - 24}\,.$$

It is clear that the system (5.15) is a particular case of the system (5.3) in which $a = 0$, $b = \pi/4$, and h_1, h_2 are given by the formulas (5.16). Analogously to Example 5.16, we get the relation

$$\int_0^{\pi/4} \big|h_1\big(\psi\big(h_2(1)\big)\big)(s)\big|\,\mathrm{d}s = \frac{d_1|d_2|}{2^{12}}\Big(4\pi\big(1+2\sqrt{2}\big) - \pi^2\big(1+\sqrt{2}\big) - 24\Big) \le 1.$$

Therefore, according to Corollary 5.21 with $k = 1$ and Remark 5.5, the problem (5.15), (5.17) has a unique solution for every $q_1, q_2 \in L([0,\pi/4];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if q_1, q_2 and c_1, c_2 fulfil the additional condition (5.18), then the unique solution $(u_1,u_2)^T$ to this problem satisfies the relation

$$u_1(t) \ge 0 \ \text{ for } \ t \in [0,\pi/4].$$

Example 5.24. On the interval $[0,1]$, we consider the problem (5.19), where $\lambda < 1$, $d \le 0$, $|d| \le (3-2\lambda)(2-\lambda)$, $\tau\colon [0,1] \to [0,1]$ is a measurable function, $\tau(t) \le t$ for a. e. $t \in [0,1]$, $q \in L([0,1];\mathbb{R})$, and $c_1, c_2 \in \mathbb{R}$.

It is clear that the problem (5.19) can be regarded as a particular case of the problem (5.3), (5.5) in which $a = 0$, $b = 1$, $q_1 \equiv 0$, $q_2 \equiv q$, and h_1, h_2 are given by the formulas (5.20). Analogously to Example 5.17, we get the relation

$$\int_0^1 \big|h_1\big(\psi\big(h_2(1)\big)\big)(s)\big|\,\mathrm{d}s = \frac{|d|}{(3-2\lambda)(2-\lambda)} \le 1.$$

Therefore, according to Corollary 5.21 with $k = 1$ and Remark 5.5, the problem (5.19) has a unique solution for arbitrary $q \in L([0,1];\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$. Moreover, if q and c_1, c_2 fulfil the additional condition (5.21), then the unique solution u to the this problem satisfies the relation

$$u(t) \ge 0 \ \text{ for } \ t \in [0,1].$$

5.2. **Systems with argument deviations.** In this part, we establish some corollaries of the results stated in the previous section for the differential system with argument deviations (3.1′) in which $p_{11} \equiv 0$ and $p_{22} \equiv 0$. More precisely, efficient conditions are found for the validity of the inclusion $(h_1,h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$ whenever the operators h_1, h_2 are defined by the formula

$$h_k(z)(t) = f_k(t)z\big(\tau_k(t)\big)$$
$$\text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),\quad k = 1,2, \tag{5.30}$$

where $f_1, f_2 \in L([a,b];\mathbb{R})$ and $\tau_1, \tau_2 \colon [a,b] \to [a,b]$ are measurable functions. Throughout this section, the following notation is used:

$$\tau_k^* = \operatorname{ess\,sup}\big\{\tau_k(t) : t \in [a,b]\big\} \quad \text{for} \quad k = 1, 2.$$

As was mentioned above, if the operators h_1 and h_2 are both positive, then $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$ if and only if $(h_1, h_2) \in \mathcal{S}_{ab}^{2}(a)$ (see Remarks 5.2(b) and 5.3). Therefore, having the operators h_1 and h_2 given by the relation (5.30), the efficient conditions guaranteeing the validity of the inclusion $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$ can be immediately derived from those stated in Section 4.2 provided that

$$f_1(t) \geq 0, \quad f_2(t) \geq 0 \quad \text{for a. e. } t \in [a,b]. \tag{5.31}$$

For the sake of completeness, we reformulate here three main results stated in Section 4.2 and then we establish two new statements which can be derived from Corollary 5.14 and Proposition 5.15 (namely, Theorems 5.30 and 5.31).

Theorem 5.25 (Theorem 4.30)**.** *Let h_1, h_2 be the operators defined by the relation* (5.30) *and let the condition* (5.31) *hold. Put*

$$\omega(t) = \max\big\{f_1(t), f_2(t)\big\} \quad \textit{for a. e. } t \in [a,b] \tag{5.32}$$

and assume that $\omega \not\equiv 0$ on $[a, \tau^*]$ and

$$\operatorname{ess\,sup}\left\{ \int_t^{\tau_i(t)} \omega(s)\,\mathrm{d}s : t \in [a,b] \right\} < \eta^* \quad \textit{for} \quad i = 1, 2,$$

where $\tau^* = \max\{\tau_1^*, \tau_2^*\}$ and

$$\eta^* = \sup\left\{ \frac{1}{x} \ln\left(x + \frac{x}{\exp\big(x \int_a^{\tau^*} \omega(s)\,\mathrm{d}s\big) - 1} \right) : \; x > 0 \right\}.$$

Then the pair (h_1, h_2) belongs both to $\widehat{\mathcal{S}}_{ab}^{2,1}(a)$ and $\widehat{\mathcal{S}}_{ab}^{2,2}(a)$.

Theorem 5.26 (Corollary 4.33)**.** *Let h_1, h_2 be the operators defined by the relation* (5.30) *and let the condition* (5.31) *hold. Assume that the inequality* (5.10) *holds, where the function ω is defined by the relation* (5.32) *and*

$$g(t) = \max\big\{g_1(t), g_2(t)\big\} \quad \textit{for a. e. } t \in [a,b]$$

in which

$$g_i(t) = f_i(t)\sigma_i(t) \int_t^{\tau_i(t)} f_{3-i}(s)\,\mathrm{d}s \quad \textit{for a. e. } t \in [a,b], \quad i = 1, 2,$$

and

$$\sigma_i(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}\big(\tau_i(t) - t\big)\Big) \quad \textit{for a. e. } t \in [a,b], \quad i = 1, 2. \tag{5.33}$$

Then $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a) \cap \widehat{\mathcal{S}}_{ab}^{2,2}(a)$.

Remark 5.27. The strict inequality (5.10) in the last theorem cannot be replaced by the nonstrict one (see Example 4.50 with $p_{11} \equiv 0$ and $p_{22} \equiv 0$).

Theorem 5.28 (Corollary 4.35)**.** *Let h_1, h_2 be the operators defined by the relation* (5.30) *and let the condition* (5.31) *hold. Assume that the inequality* (5.12) *holds, where*

$$\lambda_i = \int_a^b \cosh\left(\int_s^b \omega(\xi)\,\mathrm{d}\xi\right) f_i(s)\sigma_i(s)\left(\int_s^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s+ \\ + \int_a^b \sinh\left(\int_s^b \omega(\xi)\,\mathrm{d}\xi\right) f_{3-i}(s)\sigma_{3-i}(s)\left(\int_s^{\tau_{3-i}(s)} f_i(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s$$

for $i = 1, 2$ in which the functions ω and σ_1, σ_2 are defined, respectively, by the relations (5.32) *and* (5.33)*. Then the pair (h_1, h_2) belongs both to $\widehat{\mathcal{S}}^{2,1}_{ab}(a)$ and $\widehat{\mathcal{S}}^{2,2}_{ab}(a)$.*

Remark 5.29. The strict inequality (5.12) in the previous theorem cannot be replaced by the nonstrict one (see Example 4.50 with $p_{11} \equiv 0$ and $p_{22} \equiv 0$).

For the operator h_k given by the relation (5.30), according to Notation 5.12, we have $b^*_{h_k} \leq \tau^*_k$. It is however easy to see that the equality $b^*_{h_k} = \tau^*_k$ does not hold in general. On the other hand, it is clear that the number τ^*_k is easier to compute than $b^*_{h_k}$. Therefore, the results obtained below by using Corollary 5.14 and Proposition 5.15 are formulated in terms of the number τ^*_k instead of $b^*_{h_k}$.

Theorem 5.30. *Let h_1, h_2 be the operators defined by the relation* (5.30) *and let the condition* (5.31) *hold. If the inequality*

$$\int_a^{\tau^*_{3-i}} f_i(s)\left(\int_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s < 1$$

holds for some $i \in \{1, 2\}$, then the pair (h_1, h_2) belongs both to $\widehat{\mathcal{S}}^{2,1}_{ab}(a)$ and $\widehat{\mathcal{S}}^{2,2}_{ab}(a)$.

The next theorem can be regarded as a complement of the previous one.

Theorem 5.31. *Let $i, k \in \{1, 2\}$, h_1, h_2 be the operators defined by the relation* (5.30)*, the condition* (5.31) *hold and let the equality*

$$\int_a^{\tau^*_{3-i}} f_i(s)\left(\int_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s = 1 \tag{5.34}$$

be satisfied. Then $(h_1,h_2)\in\widehat{\mathcal{S}}^{2,k}_{ab}(a)$ *if and only if*

$$\int\limits_a^{\tau^*_{3-i}} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)x_i\big(\tau_{3-i}(\xi)\big)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s<1, \tag{5.35}$$

where

$$x_i(t)=\int\limits_a^{t} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \ \ \textit{for}\ \ t\in[a,b]. \tag{5.36}$$

In what follows, we give two statements dealing with the validity of the inclusion $(h_1,h_2)\in\widehat{\mathcal{S}}^{2,k}_{ab}(a)$ in the case where one of the operators h_1 and h_2 given by the relation (5.30) is positive and the second one is negative.

The next result follows from Corollary 5.21.

Theorem 5.32. *Let* $k\in\{1,2\}$, h_1 *and* h_2 *be the operators defined by the relation* (5.30),

$$f_k(t)\geq 0,\quad f_{3-k}(t)\leq 0 \ \ \textit{for a. e.}\ \ t\in[a,b], \tag{5.37}$$

and

$$|f_i(t)|\big(\tau_i(t)-t\big)\leq 0 \ \ \textit{for a. e.}\ \ t\in[a,b],\quad i=1,2. \tag{5.38}$$

If, moreover, the inequality

$$\int\limits_a^{b} f_k(s)\bigg(\int\limits_a^{\tau_k(s)} |f_{3-k}(\xi)|\,\mathrm{d}\xi\bigg)\,\mathrm{d}s\leq 1 \tag{5.39}$$

is satisfied, then the pair (h_1,h_2) *belongs to the set* $\widehat{\mathcal{S}}^{2,k}_{ab}(a)$.

Remark 5.33. The inequality (5.39) in the previous theorem cannot be replaced by the inequality

$$\int\limits_a^{b} f_k(s)\bigg(\int\limits_a^{\tau_k(s)} |f_{3-k}(\xi)|\,\mathrm{d}\xi\bigg)\,\mathrm{d}s\leq 1+\varepsilon, \tag{5.40}$$

no matter how small $\varepsilon>0$ is (see Example 5.40).

The next statement contains the so-called Vallée-Poussin type conditions.

Theorem 5.34. *Let* $k\in\{1,2\}$, h_1 *and* h_2 *be the operators defined by the relation* (5.30), *and let the conditions* (5.37) *and* (5.38) *be satisfied. Assume that there exist numbers* $\alpha_1,\alpha_2\in\mathbb{R}_+$, $\alpha_3>0$, $\lambda\in[0,1[$, *and* $\nu\in[0,\lambda]$ *such* *that*

$$\int_0^{+\infty} \frac{\mathrm{d}s}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} \geq \frac{(b-a)^{1-\lambda}}{1-\lambda}\,, \tag{5.41}$$

$$(b-t)^{\lambda-\nu} f_k(t) \leq$$

$$\leq \alpha_3 \left[1 + \delta_k(t) \int_{\tau_k(t)}^{t} \Big(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^\lambda}\Big)\,\mathrm{d}s\right] \text{ for a. e. } t \in [a,b], \tag{5.42}$$

$$(b-t)^{\lambda+\nu} |f_{3-k}(t)| \leq \alpha_1 \text{ for a. e. } t \in [a,b], \tag{5.43}$$

and

$$\alpha_3 (b-t)^\nu |f_{3-k}(t)| \big(t - \tau_{3-k}(t)\big) \leq \alpha_2 + \frac{\nu}{(b-t)^{1-\lambda}} \text{ for a. e. } t \in [a,b], \tag{5.44}$$

where

$$\delta_k(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}\big(t - \tau_k(t)\big)\Big) \text{ for a. e. } t \in [a,b].$$

Then the pair (h_1, h_2) *belongs to the set* $\widehat{\mathcal{S}}^{2,k}_{ab}(a)$.

Remark 5.35. The inequality (5.41) in the previous theorem cannot be replaced by the inequality

$$\int_0^{+\infty} \frac{\mathrm{d}s}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} \geq (1-\varepsilon) \frac{(b-a)^{1-\lambda}}{1-\lambda}\,, \tag{5.45}$$

no matter how small $\varepsilon > 0$ is (see Example 5.41).

Remark 5.36. Using Theorem 5.6 and combining the results stated above, we can immediately derive several conditions sufficient for the validity of the inclusion $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$ if the operators h_1 and h_2 are defined by the formulas

$$h_k(z)(t) = f_k(t) z\big(\tau_k(t)\big) \text{ for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R})$$

and

$$h_{3-k}(z)(t) = f_{3-k,0}(t) z\big(\tau_{3-k,0}(t)\big) - f_{3-k,1}(t) z\big(\tau_{3-k,1}(t)\big) \\ \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),$$

where $f_k, f_{3-k,0}, f_{3-k,1} \in L([a,b];\mathbb{R}_+)$ and $\tau_k, \tau_{3-k,0}, \tau_{3-k,1} \colon [a,b] \to [a,b]$ are measurable functions. However, we do not formulate them here in detail.

5.3. **Auxiliary lemmas.** In this part we give several lemmas that we will need in the proofs of the results stated in Sections 5.1 and 5.2.

Lemma 5.37. *Let* $k \in \{1,2\}$, $-h_{3-k}, h_k \in \mathcal{P}_{ab}$ *and let the operators* h_1 *and* h_2 *be* a*-Volterra ones. Assume that there exist functions* $\gamma_1, \gamma_2 \in$

$AC_{loc}([a,b[\,;\mathbb{R})$ *such that* $\gamma_k \in C([a,b];\mathbb{R})$ *and the relations* (5.22)–(5.25) *hold. Then, for any* $u_1, u_2 \in AC([a,b];\mathbb{R})$ *fulfilling the inequalities*

$$u_1'(t) \geq h_1(u_2)(t), \quad u_2'(t) \geq h_2(u_1)(t) \;\; \textit{for a. e. } \; t \in [a,b] \tag{5.46}$$

and

$$u_1(a) \geq 0, \quad u_2(a) \geq 0, \tag{5.47}$$

the condition

$$u_k(t) \geq 0 \;\; \textit{for } \; t \in [a, b_k] \tag{5.48}$$

holds, where

$$b_k = \sup\big\{x \in \,]a,b] : \; \gamma_k(t) > 0 \; \textit{for} \; t \in [a,x]\big\}. \tag{5.49}$$

Proof. Let functions $u_1, u_2 \in AC([a,b];\mathbb{R})$ satisfy the inequalities (5.46) and (5.47). Define the number b_k by the relation (5.49). It is clear that $b_k > a$ and

$$\gamma_1(t) > 0 \;\; \text{for} \;\; t \in [a, b_k[\,. \tag{5.50}$$

Assume that, on the contrary, the relation (5.48) does not hold. Then there exists $t_0 \in \,]a, b_k[$ such that

$$u_k(t_0) < 0. \tag{5.51}$$

Put

$$\lambda = \max\left\{\frac{u_k(t)}{\gamma_k(t)} : \; t \in [a,t_0]\right\}. \tag{5.52}$$

It is clear that

$$0 \leq \lambda < +\infty. \tag{5.53}$$

Define the functions w_1 and w_2 by setting

$$w_i(t) = \lambda\gamma_i(t) - u_i(t) \;\; \text{for} \;\; t \in [a,t_0], \;\; i = 1,2. \tag{5.54}$$

Since the operators h_1, h_2 are a-Volterra ones, using the conditions (5.46), (5.22), (5.23), (5.53) and Remark 5.19, we get

$$\begin{aligned} w_i'(t) = \lambda\gamma_i'(t) - u_i'(t) &\leq h_i^{at_0}(\lambda\gamma_{3-i} - u_{3-i})(t) = \\ &= h_i^{at_0}(w_{3-i})(t) \;\; \text{for a. e.} \;\; t \in [a,t_0], \;\; i = 1,2, \end{aligned} \tag{5.55}$$

where $h_1^{at_0}$ and $h_2^{at_0}$ are the restrictions of the operators h_1 and h_2 to the space $C([a,t_0];\mathbb{R})$.[18]

On the other hand, by virtue of the relation (5.52), it is clear that

$$w_k(t) \geq 0 \;\; \text{for} \;\; t \in [a,t_0] \tag{5.56}$$

and there exists $t_1 \in [a, t_0[$ such that

$$w_k(t_1) = 0. \tag{5.57}$$

Since we suppose that $-h_{3-k} \in \mathcal{P}_{ab}$, we get from (5.25), (5.47) and (5.53)–(5.56) that

$$w_{3-k}'(t) \leq h_{3-k}^{at_0}(w_k)(t) \leq 0 \;\; \text{for a. e.} \;\; t \in [a,t_0]$$

[18]See Definition 1.8.

and

$$w_{3-k}(a) = \lambda\gamma_{3-k}(a) - u_{3-k}(a) \le 0.$$

Hence we obtain

$$w_{3-k}(t) \le 0 \quad \text{for} \quad t \in [a, t_0]. \tag{5.58}$$

However, we suppose that $h_k \in \mathcal{P}_{ab}$, and thus we get from (5.55) and (5.58) that

$$w_k'(t) \le h_k^{at_0}(w_{3-k})(t) \le 0 \quad \text{for a. e.} \quad t \in [a, t_0]. \tag{5.59}$$

Finally, by virtue of (5.24), (5.53) and (5.57), the relation (5.59) yields

$$0 = w_k(t_1) \ge w_k(t_0) = \lambda\gamma_k(t_0) - u_k(t_0) \ge -u_k(t_0),$$

which contradicts the inequality (5.51).

The contradiction obtained proves the validity of the desired relation (5.48). □

Lemma 5.38. *Let $i \in \{1,2\}$, $f_1, f_2 \in L([a,b];\mathbb{R}_+)$, and $\tau_1, \tau_2 \colon [a,b] \to [a,b]$ be measurable functions such that the equality* (5.34) *holds. Then the homogeneous problem*

$$u_1'(t) = f_1(t)u_2\big(\tau_1(t)\big), \quad u_2'(t) = f_2(t)u_1\big(\tau_2(t)\big), \tag{5.60}$$

$$u_1(a) = 0, \quad u_2(a) = 0 \tag{5.61}$$

has only the trivial solution if and only if the inequality (5.35) *is satisfied, where the function x_i is defined by the formula* (5.36)*.*

Proof. Let $(u_1, u_2)^T$ be a solution to the problem (5.60), (5.61). We first show that the function u_i does not change its sign on the interval $[a, \tau_{3-i}^*]$. Assume that, on the contrary, u_i changes its sign on $[a, \tau_{3-i}^*]$. Put

$$M = \max\big\{u_i(t) : \ t \in [a, \tau_{3-i}^*]\big\}, \quad m = -\min\big\{u_i(t) : \ t \in [a, \tau_{3-i}^*]\big\}, \tag{5.62}$$

and choose $t_M, t_m \in [a, \tau_{3-i}^*]$ such that

$$u_i(t_M) = M, \quad u_i(t_m) = -m. \tag{5.63}$$

Clearly,

$$M > 0, \quad m > 0, \tag{5.64}$$

and we can assume without loss of generality that $t_m < t_M$. By virtue of the relations (5.62), it follows from the $(3-i)$th condition in (5.61) and the $(3-i)$th equation in (5.60) that

$$u_{3-i}(t) = \int_a^t f_{3-i}(s)u_i\big(\tau_{3-i}(s)\big)\,\mathrm{d}s \le M \int_a^t f_{3-i}(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a,b]. \tag{5.65}$$

Therefore, the integration of the ith equation in (5.60) from t_m to t_M, in view of the conditions (5.34), (5.63) and (5.65), yields

$$M + m = \int_{t_m}^{t_M} f_i(s)u_{3-i}\big(\tau_i(s)\big)\,\mathrm{d}s \le M \int_{t_m}^{t_M} f_i(s)\bigg(\int_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \le M,$$

which contradicts the second inequality in (5.64).

The contradiction obtained proves that the function u_i does not change its sign on the interval $[a,\tau^*_{3-i}]$. Therefore, we can assume without loss of generality that

$$u_i(t)\geq 0 \ \text{ for } \ t\in[a,\tau^*_{3-i}].$$

It follows from (5.60) and (5.61) that

$$u_i(t)=\int\limits_a^t f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)u_i\big(\tau_{3-i}(\xi)\big)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \ \text{ for } \ t\in[a,b]. \tag{5.66}$$

Since $\tau_{3-i}(t)\leq\tau^*_{3-i}$ for a. e. $t\in[a,b]$ and the function u_i is non-negative on $[a,\tau^*_{3-i}]$, the last relation yields

$$u_i\big(\tau_{3-i}(t)\big)\leq u(\tau^*_{3-i}) \ \text{ for a. e. } \ t\in[a,b].$$

Therefore, in view of the notation (5.36), the representation (5.66) implies that

$$\begin{aligned} u_i(t)\leq u(\tau^*_{3-i})\int\limits_a^t f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s=& \\ =u_i(\tau^*_{3-i})x_i(t) \ \text{ for } \ t\in[a,b]& \end{aligned} \tag{5.67}$$

and

$$\begin{aligned} u_i(\tau^*_{3-i})-u_i(t)=\int\limits_t^{\tau^*_{3-i}} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)u_i\big(\tau_{3-i}(\xi)\big)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s\leq& \\ \leq u_i(\tau^*_{3-i})\int\limits_t^{\tau^*_{3-i}} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \ \text{ for } \ t\in[a,\tau^*_{3-i}].& \end{aligned} \tag{5.68}$$

Using the equality (5.34) and the notation (5.36) in the relation (5.68), we get

$$u_i(\tau^*_{3-i})x_i(t)=u_i(\tau^*_{3-i})\Bigg(1-\int\limits_t^{\tau^*_{3-i}} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s\Bigg)\leq u_i(t)$$

for $t\in[a,\tau^*_{3-i}]$, which together with the above-proved relation (5.67) yields

$$u_i(t)=u_i(\tau^*_{3-i})x_i(t) \ \text{ for } \ t\in[a,\tau^*_{3-i}]. \tag{5.69}$$

Finally, (5.66) and (5.69) result in

$$u_i(t)=u_i(\tau^*_{3-i})\int\limits_a^t f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)x_i\big(\tau_{3-i}(\xi)\big)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \ \text{ for } \ t\in[a,b], \tag{5.70}$$

whence we obtain

$$u_i(\tau^*_{3-i})\left[1-\int\limits_a^{\tau^*_{3-i}} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)x_i\big(\tau_{3-i}(\xi)\big)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s\right]=0. \tag{5.71}$$

We have proved that every solution $(u_1,u_2)^T$ to the problem (5.60), (5.61) satisfies the relation (5.70), where $u_i(\tau^*_{3-i})$ fulfils the equality (5.71). Consequently, if the inequality (5.35) holds, then $u_i\equiv 0$ on the interval $[a,b]$, and thus the homogeneous problem (5.60), (5.61) has only the trivial solution.

It remains to show that if the inequality (5.35) is not satisfied, i. e.,

$$\int\limits_a^{\tau^*_{3-i}} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)x_i\big(\tau_{3-i}(\xi)\big)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s=1, \tag{5.72}$$

then the homogeneous problem (5.60), (5.61) has a nontrivial solution. Indeed, in view of the equality (5.34), the relation (5.36) yields

$$x_i\big(\tau_{3-i}(t)\big)\leq x_i(\tau^*_{3-i})=1 \quad \text{for a. e. } t\in[a,b].$$

Therefore, using the equality (5.72), it is easy to verify that

$$\begin{aligned}0\leq \int\limits_a^{t} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)\Big[1-x_i\big(\tau_{3-i}(\xi)\big)\Big]\,\mathrm{d}\xi\bigg)\,\mathrm{d}s\leq\\ \leq \int\limits_a^{\tau^*_{3-i}} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)\Big[1-x_i\big(\tau_{3-i}(\xi)\big)\Big]\,\mathrm{d}\xi\bigg)\,\mathrm{d}s=\\ =1-\int\limits_a^{\tau^*_{3-i}} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)x_i\big(\tau_{3-i}(\xi)\big)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s=0\end{aligned}$$

for $t\in[a,\tau^*_{3-i}]$. Hence we get

$$x_i(t)=\int\limits_a^{t} f_i(s)\bigg(\int\limits_a^{\tau_i(s)} f_{3-i}(\xi)x_i\big(\tau_{3-i}(\xi)\big)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \quad\text{for } t\in[a,\tau^*_{3-i}]. \tag{5.73}$$

Put

$$u_{3-i}(t)=\int\limits_a^{t} f_{3-i}(s)x_i\big(\tau_{3-i}(s)\big)\,\mathrm{d}s,\qquad u_i(t)=\int\limits_a^{t} f_i(s)u_{3-i}\big(\tau_i(s)\big)\,\mathrm{d}s$$

for $t\in[a,b]$. By virtue of the equality (5.73), it is clear that $u_i(t)=x_i(t)$ for $t\in[a,\tau^*_{3-i}]$, and thus

$$u_{3-i}(t)=\int\limits_a^{t} f_{3-i}(s)u_i\big(\tau_{3-i}(s)\big)\,\mathrm{d}s \quad\text{for } t\in[a,b].$$

Consequently, $(u_1, u_2)^T$ is a nontrivial solution to the problem (5.60), (5.61). $\square$

Lemma 5.39. *Let the numbers* $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_3 > 0$ *and* $\lambda \in [0,1[$ *be such that*

$$\int\limits_0^{+\infty} \frac{\mathrm{d}s}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} = \frac{(b-a)^{1-\lambda}}{1-\lambda}\,. \tag{5.74}$$

Then for an arbitrary $\nu \in [0, \lambda]$ *there exist functions* $\gamma_k \in C([a,b];\mathbb{R})$ *and* $\gamma_{3-k} \in C_{loc}([a,b[\,;\mathbb{R})$ *such that* $\gamma_k', \gamma_k'', \gamma_{3-k}' \in C_{loc}([a,b[\,;\mathbb{R})$,

$$\gamma_k(t) > 0 \ \ \textit{for} \ \ t \in [a,b[\,, \tag{5.75}$$

$$\gamma_{3-k}(a) = 0, \quad \gamma_{3-k}(t) < 0 \ \ \textit{for} \ \ t \in]a,b[\,, \tag{5.76}$$

$$\gamma_k'(t) = \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\, \gamma_{3-k}(t) \ \ \textit{for} \ \ t \in [a,b[\,, \tag{5.77}$$

$$\gamma_{3-k}'(t) = -\frac{\alpha_1}{(b-t)^{\lambda+\nu}}\, \gamma_k(t) + \\ + \Big(\frac{\nu}{b-t} + \frac{\alpha_2}{(b-t)^{\lambda}}\Big)\gamma_{3-k}(t) \ \ \textit{for} \ \ t \in [a,b[\,, \tag{5.78}$$

and

$$\gamma_k''(t) \le 0 \ \ \textit{for} \ \ t \in [a,b[\,. \tag{5.79}$$

Proof. Define the function $\varrho\colon [a,b[\, \to \mathbb{R}_+$ by setting

$$\int\limits_{\varrho(t)}^{+\infty} \frac{\mathrm{d}s}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} = \frac{(b-t)^{1-\lambda}}{1-\lambda} \ \ \text{for} \ \ t \in [a,b[\,.$$

In view of the equality (5.74), we get

$$\varrho(a) = 0, \quad \varrho(t) > 0 \ \ \text{for} \ \ t \in]a,b[\,, \tag{5.80}$$

and

$$\varrho'(t) = \frac{\alpha_1 + \alpha_2\varrho(t) + \alpha_3\varrho^2(t)}{(b-t)^{\lambda}} \ \ \text{for} \ \ t \in [a,b[\,.$$

Put

$$\gamma_k(t) = \exp\bigg(-\int\limits_a^t \frac{\alpha_3\varrho(s)}{(b-s)^{\lambda}}\,\mathrm{d}s\bigg), \quad \gamma_{3-k}(t) = -\frac{\varrho(t)\gamma_k(t)}{(b-t)^{\nu}} \ \ \text{for} \ \ t \in [a,b[\,.$$

It is not difficult to verify that $\gamma_k, \gamma_{3-k} \in C_{loc}([a,b[\,;\mathbb{R})$ and the conditions (5.77) and (5.78) are satisfied. Therefore, $\gamma_k', \gamma_{3-k}' \in C_{loc}([a,b[\,;\mathbb{R})$, as well. Moreover, in view of the relations (5.80), it is clear that the conditions (5.75) and (5.76) are fulfilled. Consequently, by direct calculation we can verify that $\gamma_k'' \in C_{loc}([a,b[\,;\mathbb{R})$ and that the inequality (5.79) is satisfied. Since the function γ_k is positive and non-increasing on $[a,b[$, there exists a finite limit $\lim_{t\to b-}\gamma_k(t)$. Therefore, $\gamma_k \in C([a,b];\mathbb{R})$ when we put $\gamma_k(b) = \lim_{t\to b-}\gamma_k(t)$. $\square$

5.4. **Proofs.** First recall (see Remark 5.3) that we write $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$ instead of $\ell \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$ with ℓ given by the formula

$$\ell(v)(t) = \begin{pmatrix} h_1(v_2)(t) \\ h_2(v_1)(t) \end{pmatrix} \quad \text{for a. e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2).$$

Therefore, the inequalities (4.3) and (4.4) appearing in Definition 5.1, expressed in terms of its components, have the form

$$u_1'(t) \geq h_1(u_2)(t), \quad u_2'(t) \geq h_2(u_1)(t) \text{ for a. e. } t \in [a,b] \tag{5.81}$$

and

$$u_1(a) \geq 0, \quad u_2(a) \geq 0 \tag{5.82}$$

in the case indicated.

Proof of Theorem 5.6. In view of Remark 5.4(c), we can assume without loss of generality that $k = 1$. Let $(u_1, u_2)^T \in AC([a,b];\mathbb{R}^2)$ be a vector function satisfying the inequalities (5.81) and (5.82). We will show that the function u_1 is non-negative.

According to the inclusion $(h_1, -h_{2,1}) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$ and Remark 5.5, the problem

$$\alpha_1'(t) = h_1(\alpha_2)(t), \quad \alpha_2'(t) = -h_{2,1}(\alpha_1)(t) + h_{2,0}([u_1]_-)(t), \tag{5.83}$$

$$\alpha_1(a) = 0, \quad \alpha_2(a) = 0 \tag{5.84}$$

has a unique solution $(\alpha_1, \alpha_2)^T$ and

$$\alpha_1(t) \geq 0 \text{ for } t \in [a,b]. \tag{5.85}$$

In view the conditions (5.81), (5.82), (5.83), (5.84) and the assumption $h_{2,0} \in \mathcal{P}_{ab}$, we get

$$\alpha_1'(t) + u_1'(t) \geq h_1(\alpha_2 + u_2)(t) \text{ for a. e. } t \in [a,b],$$

$$\alpha_2'(t) + u_2'(t) \geq -h_{2,1}(\alpha_1 + u_1)(t) + h_{2,0}(u_1 + [u_1]_-)(t) \geq$$
$$\geq -h_{2,1}(\alpha_1 + u_1)(t) \text{ for a. e. } t \in [a,b],$$

and

$$\alpha_1(a) + u_1(a) \geq 0, \quad \alpha_2(a) + u_2(a) \geq 0.$$

Consequently, the inclusion $(h_1, -h_{2,1}) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$ yields

$$\alpha_1(t) + u_1(t) \geq 0 \text{ for } t \in [a,b]. \tag{5.86}$$

Now, the inequalities (5.85) and (5.86) imply that

$$[u_1(t)]_- \leq \alpha_1(t) \text{ for } t \in [a,b]. \tag{5.87}$$

On the other hand, by virtue of the conditions (5.83), (5.85), (5.87) and the assumptions $h_{2,0}, h_{2,1} \in \mathcal{P}_{ab}$, we obtain

$$\alpha_1'(t) = h_1(\alpha_2)(t), \quad \alpha_2'(t) \leq h_{2,0}(\alpha_1)(t) \text{ for a. e. } t \in [a,b].$$

Hence, on account of the equalities (5.84), the inclusion $(h_1, h_{2,0}) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$ yields that

$$\alpha_1(t) \le 0 \quad \text{for} \quad t \in [a, b].$$

The latter relation, together with the inequality (5.86), guarantees that the function u_1 is non-negative and thus $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$. □

Proof of Corollary 5.14. According to the inequality (5.13) and the assumption $h_1 \in \mathcal{P}_{ab}$, there exists $\varepsilon > 0$ such that

$$\varepsilon\Bigg(1 + \int_a^{b^*_{h_{3-i}}} h_i(1)(s)\,\mathrm{d}s\Bigg) + \int_a^{b^*_{h_{3-i}}} h_i\big(\psi\big(h_{3-i}(1)\big)\big)(s)\,\mathrm{d}s \le 1. \tag{5.88}$$

Put

$$\gamma_{3-i}(t) = \varepsilon + \int_a^t h_{3-i}(1)(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a, b], \tag{5.89}$$

$$\gamma_i(t) = \varepsilon + \int_a^t h_i(\gamma_{3-i})(s)\,\mathrm{d}s \quad \text{for} \quad t \in [a, b]. \tag{5.90}$$

It is clear that $\gamma_1, \gamma_2 \in AC([a, b]; \mathbb{R})$ satisfy the inequalities (5.6) because the operators h_1 and h_2 are positive. Put

$$\widetilde{\gamma}_i(t) = \begin{cases} \gamma_i(t) & \text{for} \quad t \in [a, b^*_{h_{3-i}}[\,, \\ \gamma_i\big(b^*_{h_{3-i}}\big) & \text{for} \quad t \in [b^*_{h_{3-i}}, b]. \end{cases} \tag{5.91}$$

Then the relations (5.88)–(5.90) yield

$$\begin{aligned} \widetilde{\gamma}_i(t) \le \gamma_i\big(b^*_{h_{3-i}}\big) = \varepsilon + \int_a^{b^*_{h_{3-i}}} h_i\big(\varepsilon + \psi(h_{3-i}(1))\big)(s)\,\mathrm{d}s = \\ = \varepsilon\Bigg(1 + \int_a^{b^*_{h_{3-i}}} h_i(1)(s)\,\mathrm{d}s\Bigg) + \int_a^{b^*_{h_{3-i}}} h_i\big(\psi\big(h_{3-i}(1)\big)\big)(s)\,\mathrm{d}s \le 1 \quad \text{for} \quad t \in [a, b]. \end{aligned} \tag{5.92}$$

On the other hand, in view of the relations (5.91), (5.92), the assumption $h_{3-i} \in \mathcal{P}_{ab}$ and Remark 5.13, it follows from the equalities (5.89) and (5.90) that

$$\gamma_i'(t) = h_i(\gamma_{3-i})(t) \quad \text{for a.e.} \quad t \in [a, b]$$

and

$$\gamma_{3-i}'(t) = h_{3-i}(1)(t) \ge h_{3-i}\big(\widetilde{\gamma}_i\big)(t) = h_{3-i}(\gamma_i)(t) \quad \text{for a.e.} \quad t \in [a, b],$$

i.e., the inequalities (5.7) are fulfilled. Consequently, using Theorem 5.7, we get that $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a) \cap \widehat{\mathcal{S}}_{ab}^{2,2}(a)$. □

Proof of Proposition 5.15. Suppose that the equality (5.14) holds and the problem (5.4) has only the trivial solution. We will show that the pair (h_1, h_2) belongs both to $\widehat{\mathcal{S}}_{ab}^{2,1}(a)$ and $\widehat{\mathcal{S}}_{ab}^{2,2}(a)$. According to Proposition 3.1, the problem

$$\gamma_1'(t) = h_1(\gamma_2)(t), \quad \gamma_2'(t) = h_2(\gamma_1)(t), \tag{5.93}$$

$$\gamma_1(a) = 1, \quad \gamma_2(a) = 1 \tag{5.94}$$

has a unique solution $(\gamma_1, \gamma_2)^T$. Put

$$m = \min\big\{\gamma_i(t) : \ t \in \big[a, b^*_{h_{3-i}}\big]\big\} \tag{5.95}$$

and choose $t_m \in \big[a, b^*_{h_{3-i}}\big]$ such that $\gamma_i(t_m) = m$.

Assume that

$$m \le 0 \tag{5.96}$$

By virtue of the notation (5.95) and the assumption $h_{3-i} \in \mathcal{P}_{ab}$, the relations (5.93) and (5.94) yield

$$\gamma_{3-i}(t) = 1 + \int_a^t h_{3-i}(\gamma_i)(s)\,\mathrm{d}s \ge m \int_a^t h_{3-i}(1)(s)\,\mathrm{d}s = m\,\psi\big(h_{3-i}(1)\big)(t)$$

for $t \in [a, b]$. Consequently, in view of the inequality (5.96) and the assumption $h_i \in \mathcal{P}_{ab}$, the relations (5.93) and (5.94) imply that

$$m = 1 + \int_a^{t_m} h_i(\gamma_{3-i})(s)\,\mathrm{d}s \ge 1 + m \int_a^{t_m} h_i\big(\psi\big(h_{3-i}(1)\big)\big)(s)\,\mathrm{d}s \ge$$

$$\ge 1 + m \int_a^{b^*_{h_{3-i}}} h_i\big(\psi\big(h_{3-i}(1)\big)\big)(s)\,\mathrm{d}s.$$

Using the equality (5.14) in the last relation, we get the contradiction $m \ge m + 1$.

The contradiction obtained proves that $m > 0$, i. e.,

$$\gamma_i(t) > 0 \ \text{ for } \ t \in \big[a, b^*_{h_{3-i}}\big]. \tag{5.97}$$

Now we define the function $\widetilde{\gamma}_i$ by the formula (5.91). Obviously, $\widetilde{\gamma}_i(t) > 0$ holds for $t \in [a, b]$ and therefore, by virtue of the assumption $h_{3-i} \in \mathcal{P}_{ab}$ and Remark 5.13, the $(3-i)$th equation in (5.93) yields that

$$\gamma_{3-i}'(t) = h_{3-i}(\gamma_i)(t) = h_{3-i}\big(\widetilde{\gamma}_i\big)(t) \ge 0 \ \text{ for a. e. } \ t \in [a, b].$$

Since $\gamma_{3-i}(a) > 0$, the last relation guarantees that $\gamma_{3-i}(t) > 0$ holds for $t \in [a, b]$. Now, the ith equation in (5.93) implies

$$\gamma_i'(t) = h_i(\gamma_{3-i})(t) \ge 0 \ \text{ for a. e. } \ t \in [a, b],$$

which, together with the inequality (5.97), results in $\gamma_i(t) > 0$ for $t \in [a, b]$. Consequently, Theorem 5.7 guarantees that $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a) \cap \widehat{\mathcal{S}}_{ab}^{2,2}(a)$.

Now suppose that $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$ for some $k \in \{1, 2\}$. Then, according to Remark 5.5, the homogeneous problem (5.4) has only the trivial solution. $\square$

Proof of Theorem 5.18. First suppose that $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$. According to Remark 5.5, the system (5.93) has a unique solution $(\gamma_1, \gamma_2)^T$ satisfying the initial conditions

$$\gamma_k(a) = 1, \quad \gamma_{3-k}(a) = 0, \tag{5.98}$$

and, moreover, the relation

$$\gamma_k(t) \geq 0 \ \text{ for } \ t \in [a, b] \tag{5.99}$$

holds. It is clear that $\gamma_1, \gamma_2 \in AC([a, b]; \mathbb{R})$ satisfy the relations (5.22)–(5.25). We will show that the condition (5.26) also holds. Assume that, on the contrary, the relation (5.26) is not satisfied. Then there exists $t_0 \in]a, b[$ such that $\gamma_{3-k}(t_0) = 0$ and

$$\gamma_k(t_0) = 0. \tag{5.100}$$

By virtue of the inequality (5.99) and the assumption $-h_{3-k} \in \mathcal{P}_{ab}$, the conditions (5.23), (5.98) and $\gamma_{3-k}(t_0) = 0$ imply that $\gamma_{3-k}(t) = 0$ holds for $t \in [a, t_0]$. Since h_k is an a-Volterra operator, the kth equation in (5.93) implies that

$$\gamma_k'(t) = 0 \ \text{ for a.e. } \ t \in [a, t_0].$$

This relation, together with the condition $\gamma_k(a) = 1$, yields that $\gamma_k(t_0) = 1$, which contradicts the equality (5.100). The contradiction obtained proves the validity of the desired relation (5.26).

Now suppose that there exist functions $\gamma_1, \gamma_2 \in AC_{loc}([a, b[\,; \mathbb{R})$ such that $\gamma_k \in C([a, b]; \mathbb{R})$ and the relations (5.22)–(5.25) hold. We will show that $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,k}(a)$. Let a vector function $(u_1, u_2)^T \in AC([a, b]; \mathbb{R}^2)$ satisfy the inequalities (5.81) and (5.82). By virtue of Lemma 5.37, the relation (5.48) holds, where the number b_k is defined by the formula (5.49).

If $b_k = b$, then the proof is complete. Assume that $b_k < b$ and let $b_0 \in]b_k, b[$ be arbitrary but fixed. We will show that

$$u_k(t) \geq 0 \ \text{ for } \ t \in [a, b_0]. \tag{5.101}$$

It follows from the relations (5.24) and (5.49) that the inequality (5.50) holds and

$$\gamma_k(t) = 0 \ \text{ for } \ t \in [b_k, b].$$

Consequently, by virtue of the assumptions (5.26) and (5.27), there exist $a_0 \in]a, b_k[$ and $\lambda_1 \in \mathbb{R}_+$ such that

$$u_{3-k}(t) \geq \lambda_1 \gamma_{3-k}(t) \ \text{ for } \ t \in [a_0, b_0]. \tag{5.102}$$

On the other hand, in view of the inequality (5.50), there exist $\lambda_2 \in \mathbb{R}_+$ such that

$$u_k(t) \leq \lambda_2 \gamma_k(t) \ \text{ for } \ t \in [a, a_0]. \tag{5.103}$$

Since the negative operator h_{3-k} is an a-Volterra one, using the conditions (5.23), (5.81) and (5.103), we get

$$u'_{3-k}(t) - \lambda_2\gamma'_{3-k}(t) \geq h^{aa_0}_{3-k}(u_k - \lambda_2\gamma_k)(t) \geq 0 \text{ for a.e. } t \in [a, a_0],$$

where $h^{aa_0}_{3-k}$ is the restriction of the operator h_{3-k} to the space $C\big([a, a_0]; \mathbb{R}\big)$.[19] However, the functions u_{3-k} and γ_{3-k} satisfy the inequalities (5.82) and (5.25), and thus the previous relation yields

$$u_{3-k}(t) \geq \lambda_2\gamma_{3-k}(t) \text{ for } t \in [a, a_0]. \tag{5.104}$$

Therefore, if we put $\lambda = \max\{\lambda_1, \lambda_2\}$, then, in view of the inequality (5.27), we get from the relations (5.102) and (5.104) that

$$u_{3-k}(t) \geq \lambda\gamma_{3-k}(t) \text{ for } t \in [a, b_0]. \tag{5.105}$$

Since the positive operator h_1 is an a-Volterra one, the inequalities (5.22) and (5.27) imply

$$\gamma'_k(t) \leq h^{ab_0}_k(\gamma_{3-k})(t) \leq 0 \text{ for a.e. } t \in [a, b_0],$$

where $h^{ab_0}_k$ is the restriction of the operator h_k to the space $C\big([a, b_0]; \mathbb{R}\big)$.[20] The function γ_k vanishes on the interval $[b_k, b]$, and thus we have

$$h^{ab_0}_k(\gamma_{3-k})(t) = 0 \text{ for a.e. } t \in [b_k, b_0]. \tag{5.106}$$

Now the relations (5.105) and (5.106) imply

$$h^{ab_0}_k(u_{3-k})(t) \geq \lambda h^{ab_0}_k(\gamma_{3-k})(t) = 0 \text{ for a.e. } t \in [b_k, b_0],$$

which, together with the inequalities (5.81) and (5.48), results in the desired relation (5.101). Since the point b_0 was chosen arbitrarily, we have proved that the function u_k is non-negative on $[a, b]$. Consequently, the inclusion $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$ holds. □

Proof of Corollary 5.21. Put

$$\gamma_k(t) = 1 - \int_a^t \big|h_k\big(\psi\big(h_{3-k}(1)\big)\big)(s)\big| \,\mathrm{d}s \text{ for } t \in [a, b] \tag{5.107}$$

and

$$\gamma_{3-k}(t) = \int_a^t h_{3-k}(1)(s)\,\mathrm{d}s \text{ for } t \in [a, b]. \tag{5.108}$$

Obviously, we have $\gamma_1, \gamma_2 \in AC([a, b]; \mathbb{R})$. In view of the inequality (5.28), it is clear that the conditions (5.24) and (5.25) are satisfied and

$$\gamma_k(t) \leq 1 \text{ for } t \in [a, b]. \tag{5.109}$$

Since $-h_{3-k}, h_k \in \mathcal{P}_{ab}$, we get from the relations (5.107)–(5.109) that

$$\gamma'_k(t) = h_k\big(\psi\big(h_{3-k}(1)\big)\big)(t) = h_k(\gamma_{3-k})(t) \text{ for a.e. } t \in [a, b]$$

[19] See Definition 1.8.
[20] See Definition 1.8.

and

$$\gamma_2'(t) = h_{3-k}(1)(t) \le h_{3-k}(\gamma_k)(t) \ \text{ for a. e. } \ t \in [a,b],$$

i. e., the functions γ_1, γ_2 satisfy the inequalities (5.22) and (5.23).

We will show that the condition (5.26) is satisfied. Assume that, on the contrary, the relation (5.26) does not hold. Then there exists $t_0 \in]a,b[$ such that $\gamma_{3-k}(t_0) = 0$ and

$$\gamma_k(t_0) = 0. \tag{5.110}$$

Therefore, the equality (5.108) yields

$$h_{3-k}(1)(t) = 0 \ \text{ for a. e. } \ t \in [a, t_0].$$

Since the operator h_k is an a–Volterra one, the last relation results in

$$h_k\big(\psi\big(h_{3-k}(1)\big)\big)(t) = 0 \ \text{ for a. e. } \ t \in [a, t_0].$$

Hence, the equality (5.107) implies that $\gamma_k(t_0) = 1$, which contradicts the relation (5.110). The contradiction obtained proves the validity of the desired condition (5.26).

Consequently, using Theorem 5.18, we obtain $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$. □

Proof of Theorem 5.30. It is clear that $h_1, h_2 \in \mathcal{P}_{ab}$. According to Notation 5.12, we get $b^*_{h_{3-i}} \le \tau^*_{3-i}$. Therefore, the validity of the theorem follows immediately from Corollary 5.14. □

Proof of Theorem 5.31. It is clear that $h_1, h_2 \in \mathcal{P}_{ab}$.

First suppose that $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$. In view of Remark 5.5, the homogeneous problem (5.60), (5.61) has only the trivial solution and thus Lemma 5.38 guarantees that the inequality (5.35) is satisfied.

Now suppose that the inequality (5.35) is fulfilled. According to Notation 5.12, we have $b^*_{h_{3-i}} \le \tau^*_{3-i}$. If

$$\int_a^{b^*_{h_{3-i}}} h_i\big(\psi\big(h_{3-i}(1)\big)\big)(s)\,\mathrm{d}s = \int_a^{b^*_{h_{3-i}}} f_i(s)\left(\int_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s < 1,$$

then Corollary 5.14 yields that $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$. If

$$\int_a^{b^*_{h_{3-i}}} h_i\big(\psi\big(h_{3-i}(1)\big)\big)(s)\,\mathrm{d}s = \int_a^{b^*_{h_{3-i}}} f_i(s)\left(\int_a^{\tau_i(s)} f_{3-i}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s = 1,$$

then, by virtue of Proposition 5.15, the inclusion $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$ holds provided that the the homogeneous problem (5.60), (5.61) has only the trivial solution. But the absence of nontrivial solutions of this problem is guaranteed by the inequality (5.35) (see Lemma 5.38). Consequently, we have $(h_1, h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$, as well. □

Proof of Theorem 5.32. It is clear that $h_k \in \mathcal{P}_{ab}$ and $-h_{3-k} \in \mathcal{P}_{ab}$. Moreover, the inequalities (5.38) guarantee that the operators h_1 and h_2 are both a-Volterra ones. Therefore, the validity of the theorem follows immediately from Corollary 5.21. $\square$

Proof of Theorem 5.34. It is clear that $h_k \in \mathcal{P}_{ab}$ and $-h_{3-k} \in \mathcal{P}_{ab}$. Moreover, the inequalities (5.38) guarantee that the operators h_1 and h_2 are both a-Volterra ones. According to the inequalities (5.41) and (5.43), the number α_1 can be increased so that the equality (5.74) is satisfied instead of the inequality (5.41), and the condition (5.43) is still true. Then, by virtue of Lemma 5.39, there exist functions $\gamma_k \in C([a,b];\mathbb{R})$ and $\gamma_{3-k} \in C_{loc}([a,b[\,;\mathbb{R})$ such that $\gamma_k', \gamma_k'', \gamma_{3-k}' \in C_{loc}([a,b[\,;\mathbb{R})$, and the conditions (5.75)–(5.79) are satisfied. Obviously, $\gamma_1, \gamma_2 \in AC_{loc}([a,b[\,;\mathbb{R})$. Using the inequalities (5.75)–(5.78), we get

$$\gamma_k'(t) \le 0, \quad \gamma_{3-k}'(t) \le 0 \text{ for } t \in [a,b[\,. \tag{5.111}$$

Put

$$\mathcal{A} = \big\{t \in [a,b] : \ f_k(t) > 0\big\}, \quad \mathcal{B} = \big\{t \in [a,b] : \ f_{3-k}(t) < 0\big\}.$$

If we take the conditions (5.38) into account, by direct calculation we obtain

$$\begin{aligned}
&\gamma_{3-k}\big(\tau_k(t)\big) = \gamma_{3-k}(t) - \int_{\tau_k(t)}^{t} \gamma_{3-k}'(s)\,\mathrm{d}s = \\
&= \gamma_{3-k}(t) + \int_{\tau_k(t)}^{t} \frac{\alpha_1}{(b-s)^{\lambda+\nu}}\,\gamma_k(s)\,\mathrm{d}s - \int_{\tau_k(t)}^{t} \left[\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}}\right]\gamma_{3-k}(s)\,\mathrm{d}s \ge \\
&\qquad \ge \gamma_{3-k}(t) - \gamma_{3-k}\big(\tau_k(t)\big) \int_{\tau_k(t)}^{t} \left[\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}}\right]\mathrm{d}s \ \text{ for a.e. } t \in \mathcal{A}
\end{aligned}$$

and

$$\begin{aligned}
&-\gamma_k\big(\tau_{3-k}(t)\big) = -\gamma_k(t) + \int_{\tau_{3-k}(t)}^{t} \gamma_k'(s)\,\mathrm{d}s \ge -\gamma_k(t) + \gamma_k'(t)\Big(t - \tau_{3-k}(t)\Big) = \\
&\qquad = -\gamma_k(t) + \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\Big(t - \tau_{3-k}(t)\Big)\gamma_{3-k}(t) \ \text{ for a.e. } t \in \mathcal{B}.
\end{aligned}$$

By virtue of the inequalities (5.42), (5.43), (5.44) and (5.75)–(5.78), we get from the last relations that

$$\begin{aligned}
f_k(t)\gamma_{3-k}\big(\tau_k(t)\big) &\ge \frac{f_k(t)}{1 + \int_{\tau_k(t)}^{t} \left[\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}}\right]\mathrm{d}s}\,\gamma_{3-k}(t) \ge \\
&\ge \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,\gamma_{3-k}(t) = \gamma_k'(t) \ \text{ for a.e. } t \in \mathcal{A}
\end{aligned}$$

and

$$
\begin{aligned}
-\,|f_{3-k}(t)|\gamma_k\big(\tau_{3-k}(t)\big) &\geq \\
&\geq -|f_{3-k}(t)|\gamma_k(t) + \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,|f_{3-k}(t)|\Big(t-\tau_{3-k}(t)\Big)\gamma_{3-k}(t) \geq \\
&\geq -\frac{\alpha_1}{(b-t)^{\lambda+\nu}}\,\gamma_k(t) + \left(\frac{\nu}{b-t} + \frac{\alpha_2}{(b-t)^{\lambda}}\right)\gamma_{3-k}(t) = \\
&\qquad\qquad = \gamma'_{3-k}(t) \ \text{ for a.e. } \ t \in \mathcal{B},
\end{aligned}
$$

which, together with the inequalities (5.37) and (5.111), guarantees that

$$
\gamma'_k(t) \leq f_k(t)\gamma_{3-k}\big(\tau_k(t)\big), \quad \gamma'_{3-k}(t) \leq f_{3-k}(t)\gamma_k\big(\tau_{3-k}(t)\big)
$$

for a.e. $t \in [a,b]$, i.e., γ_1 and γ_2 satisfies the inequalities (5.22) and (5.23). Consequently, using Theorem 5.18, we get $(h_1,h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$. $\square$

5.5. **Counterexamples.**

Example 5.40. Let $k \in \{1,2\}$, the operators h_1 and h_2 be defined by the relations (5.30), where the inequalities (5.37) hold and $\tau_{3-k} \equiv a$. Then the condition (5.28) (i.e., (5.39)) is not only sufficient but also necessary for the validity of the inclusion $(h_1,h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$.

Indeed, let $(h_1,h_2) \in \widehat{\mathcal{S}}^{2,k}_{ab}(a)$. Then, according to Remark 5.5, the problem

$$
u'_k(t) = f_k(t)u_{3-k}\big(\tau_k(t)\big), \quad u'_{3-k}(t) = f_{3-k}(t)u_k(a), \tag{5.112}
$$

$$
u_k(a) = 1, \quad u_{3-k}(a) = 0 \tag{5.113}
$$

has a unique solution $(u_1,u_2)^T$ and, moreover, the inequality (5.2) is satisfied. It follows from (5.112) and (5.113) that

$$
u_k(t) = 1 + \int_a^t f_k(s)u_{3-k}\big(\tau_k(s)\big)\,\mathrm{d}s = 1 + \int_a^t f_k(s)\bigg(\int_a^{\tau_k(s)} f_{3-k}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s
$$

for $t \in [a,b]$. Hence, we get

$$
u_k(b) = 1 - \int_a^b f_k(s)\bigg(\int_a^{\tau_k(s)} |f_{3-k}(\xi)|\,\mathrm{d}\xi\bigg)\,\mathrm{d}s,
$$

which, in view of the relation (5.2), guarantees the validity of the inequality (5.39).

This example shows that the inequalities (5.28) and (5.39) in Corollary 5.21 and Theorem 5.32 cannot be replaced by the inequalities (5.29) and (5.40), respectively, no matter how small $\varepsilon > 0$ is.

Example 5.41. Let $k \in \{1,2\}$, $\varepsilon > 0$, $\alpha = \frac{\pi}{2(1-\varepsilon)(b-a)}$, and the operators h_1 and h_2 be defined by the relations (5.30), where $f_k \equiv \alpha$, $f_{3-k} \equiv -\alpha$, and $\tau_i(t) = t$ for $t \in [a,b]$, $i = 1,2$. It is clear that the conditions (5.37), (5.38), (5.42)–(5.44), and (5.45) are fulfilled with $\alpha_1 = \alpha_3 = \alpha$, $\alpha_2 = 0$, and $\lambda = \nu = 0$. On the other hand, the functions

$$u_k(t) = \cos\alpha(t-a), \quad u_{3-k}(t) = -\sin\alpha(t-a) \ \text{ for } \ t \in [a,b]$$

fulfils the inequalities (5.81) and (5.82). However, the function u_k is not non-negative on the entire interval $[a,b]$, and thus $(h_1,h_2) \notin \widehat{\mathcal{S}}_{ab}^{2,k}(a)$.

This example shows that the inequality (5.41) cannot be replaced by the inequality (5.45), no matter how small $\varepsilon > 0$ is.

6. Existence and Uniqueness Theorems

In this section, the question on the existence and uniqueness to a solution of the linear problem (3.1), (3.2) is investigated. Unlike the case of an ordinary differential system, where the Cauchy problem (4.1) is uniquely solvable, the unique solvability of the initial value problem for functional differential systems is not guaranteed in general even for the rather simple system $(3.1')$. The reason lays in a non-local character of functional differential systems, i.e., in the presence of argument deviations τ_{ik} in the system $(3.1')$. That is why the notions of local solution and extendability of solutions have no sense for the problem (3.1), (3.2).

The question on the solvability of various boundary value problems for systems of linear functional differential equations was studied by many authors (see, e.g., [3–5, 7, 13, 19, 22, 23, 25–27, 34, 43–45, 53, 54, 56, 57, 59–62, 64, 69, 70, 76–78, 83, 85] and references therein). As for the initial value problem (3.1), (3.2), we mention the monograph [44], where the system (3.1) with a strongly bounded operator ℓ is considered. Among the rest, in [44], the authors prove the unique solvability of the initial value problem for linear delay differential systems, i.e., for the system $(3.1')$ in which deviations τ_{ik} satisfy the inequality $\tau_{ik}(t) \le t$ for almost every $t \in [a,b]$. Efficient solvability conditions for the problem $(3.1')$, (3.2) with arbitrary deviations τ_{ik} can also be found in [49, 69].

The results presented in this section complement the previously known results. To prove them, we apply theorems on functional differential inequalities stated in Sections 4 and 5. In the first part, we consider the general two-dimensional systems. In the second part, the so-called anti-diagonal systems are studied, the special case of which is also the second-order functional differential equation, and the last part deals with the two-dimensional differential systems with monotone operators. All the results are applied to differential systems with argument deviations $(3.1')$, in which case further results are established. The counterexamples constructed in Section 6.3 show that some of the results obtained are unimprovable in a certain sense.

6.1. **General two-dimensional systems.** In this part, we consider the general two-dimensional differential system (3.1) in which the operator $\ell\colon C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$ is linear and bounded (not strongly bounded in general).

6.1.1. *Main results.* We first give a rather simple statement (namely, Theorem 6.1), which implies, in particular, the unique solvability of the Cauchy problem for linear delay differential systems (see Corollary 6.12 below).

Theorem 6.1. *Let there exist an operator*

$$\varphi_0 \in \mathcal{S}^2_{ab}(a) \tag{6.1}$$

such that the inequality

$$\operatorname{Sgn}\big(v(t)\big)\ell(v)(t) \le \varphi_0(|v|)(t) \quad \text{for a. e. } t \in [a,b] \tag{6.2}$$

holds on the set $C_a([a,b];\mathbb{R}^2)$. Then the problem (3.1), (3.2) *has a unique solution.*

Remark 6.2. The assumption (6.1) of the last theorem cannot be replaced by the assumption

$$(1-\varepsilon)\,\varphi_0 \in \mathcal{S}^2_{ab}(a), \tag{6.3}$$

no matter how small $\varepsilon > 0$ is (see Example 6.36).

Theorem 6.3. *Let there exist an operator*

$$\psi_0 \in \mathcal{S}_{ab}(a) \tag{6.4}$$

such that the inequality

$$\ell(v)(t)\cdot \operatorname{sgn}\big(v(t)\big) \le \psi_0(\|v\|)(t) \quad \text{for a. e. } t \in [a,b] \tag{6.5}$$

holds on the set $C_a([a,b];\mathbb{R}^2)$. Then the problem (3.1), (3.2) *has a unique solution.*

Remark 6.4. Sufficient conditions for the validity of the inclusion (6.4) are established in the paper [24].

The assumption (6.4) of Theorem 6.3 cannot be replaced by the assumption

$$(1-\varepsilon)\,\psi_0 \in \mathcal{S}_{ab}(a), \tag{6.6}$$

no matter how small $\varepsilon > 0$ is (see Example 6.36).

Example 6.5. Consider the two-dimensional system

$$\begin{aligned} u_1'(t) &= p_1(t)u_1\big(\tau(t)\big) + g(t)u_2(t) + q_1(t),\\ u_2'(t) &= p_2(t)u_1\big(\tau(t)\big) - g(t)u_2(t) + q_2(t), \end{aligned} \tag{6.7}$$

where $p_1, p_2, q_1, q_2 \in L([a,b];\mathbb{R})$, $g \in L([a,b];\mathbb{R}_+)$, and $\tau\colon [a,b] \to [a,b]$ is a measurable function. It is clear that the system (6.7) is a particular case of the system (3.1) in which the operator ℓ is defined by the formula

$$\ell(v)(t) = \begin{pmatrix} p_1(t)v_1\big(\tau(t)\big) + g(t)v_2(t) \\ p_2(t)v_1\big(\tau(t)\big) - g(t)v_2(t) \end{pmatrix}$$
$$\text{for a. e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2).$$

Then:

(a) The inequality (6.2) is fulfilled on the set $C([a,b];\mathbb{R}^2)$, where

$$\varphi_0(v)(t) = \begin{pmatrix} |p_1(t)|v_1\big(\tau(t)\big) + g(t)v_2(t) \\ |p_2(t)|v_1\big(\tau(t)\big) \end{pmatrix}$$
$$\text{for a. e. } t \in [a,b] \text{ and all } v = (v_1,v_2)^T \in C([a,b];\mathbb{R}^2).$$

Using Corollary 4.31, we obtain $\varphi_0 \in \mathcal{S}^2_{ab}(a)$ if the condition

$$\int_t^{\tau(t)} \widehat{p}(s)\,\mathrm{d}s \le \frac{1}{\mathrm{e}} \quad \text{for a. e. } t \in [a,b] \tag{6.8}$$

holds, where $\widehat{p}(t) = \max\big\{|p_1(t)| + g(t), |p_2(t)|\big\}$ for a. e. $t \in [a,b]$. Therefore, Theorem 6.1 guarantees the unique solvability of the problem (6.7), (3.2) provided that the condition (6.8) is satisfied.

(b) The inequality (6.5) is satisfied on the set $C([a,b];\mathbb{R}^2)$, where

$$\psi_0(z)(t) = \widetilde{p}(t)z\big(\tau(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R})$$

in which $\widetilde{p}(t) = |p_1(t)| + |p_2(t)|$ for a. e. $t \in [a,b]$. Using Lemma 6.18 below (with $m=1$), we get $\psi_0 \in \mathcal{S}_{ab}(a)$ provided that

$$\int_t^{\tau(t)} \widetilde{p}(s)\,\mathrm{d}s \le \frac{1}{\mathrm{e}} \quad \text{for a. e. } t \in [a,b]. \tag{6.9}$$

Therefore, by virtue of Theorem 6.3, the problem (6.7), (3.2) has a unique solution under the condition (6.9).

Consequently, if we apply Theorem 6.1 to obtain the unique solvability of the considered problem in the case where

$$g(t) \ge |p_2(t)| \quad \text{for a. e. } t \in [a,b],$$

then a stronger condition is required in comparison with Theorem 6.3.

The next theorem contains a certain both-sided restriction imposed on the right-hand side of the system (3.1).

Theorem 6.6. *Let there exist operators $\varphi_0 \in \mathcal{P}^2_{ab}$ and $\varphi_1 \in \mathcal{S}^2_{ab}(a)$ such that the inequality*

$$\big|\ell(v)(t) - \varphi_1(v)(t)\big| \le \varphi_0(|v|)(t) \quad \textit{for a. e. } t \in [a,b] \tag{6.10}$$

holds on the set $C_a([a,b];\mathbb{R}^2)$. If, moreover,

$$\varphi_0 + \varphi_1 \in \mathcal{S}^2_{ab}(a),$$

then the problem (3.1), (3.2) *has a unique solution.*

Remark 6.7. The assumption

$$\varphi_1 \in \mathcal{S}^2_{ab}(a), \quad \varphi_0 + \varphi_1 \in \mathcal{S}^2_{ab}(a) \tag{6.11}$$

in the last theorem can be replaced neither by the assumption

$$(1-\varepsilon)\,\varphi_1 \in \mathcal{S}^2_{ab}(a), \quad \varphi_0 + \varphi_1 \in \mathcal{S}^2_{ab}(a) \tag{6.12}$$

nor by the assumption

$$\varphi_1 \in \mathcal{S}^2_{ab}(a), \quad (1-\varepsilon)\big(\varphi_0 + \varphi_1\big) \in \mathcal{S}^2_{ab}(a), \tag{6.13}$$

no matter how small $\varepsilon > 0$ is (see Examples 6.36 and 6.37).

Theorem 6.6 yields Corollary 6.9 below in which we assume that the linear operator ℓ on the right-hand side of the system (3.1) is not only bounded but it is strongly bounded (see Definition 1.1). It is well-known (see, e. g., [40, Ch. VII, § 1.2]) that a linear operator $\ell\colon C([a,b];\mathbb{R}^2) \to L([a,b];\mathbb{R}^2)$ is strongly bounded if and only if it is regular, i. e., it admits the representation $\ell = \ell^0 - \ell^1$ with $\ell^0, \ell^1 \in \mathcal{P}^2_{ab}$. Therefore, we assume in the corollary indicated that ℓ can be expressed in such a form. Before formulation of the corollary we introduce the following notation.

Notation 6.8. Let $\ell = \ell^0 - \ell^1$, where $\ell^0, \ell^1 \in \mathcal{P}^2_{ab}$. Having the components ℓ^0_{ik} and ℓ^1_{ik} (see the item 27 in Section 1) of the operators ℓ^0 and ℓ^1, for any $\delta \in \mathbb{R}$, we put

$$\ell^{0,\delta}(v)(t) = \begin{pmatrix} \big(\ell^0_{11} + [1-2\delta]\ell^1_{11}\big)(v_1)(t) + \big(\ell^0_{12} + \ell^1_{12}\big)(v_2)(t) \\ \big(\ell^0_{21} + \ell^1_{21}\big)(v_1)(t) + \big(\ell^0_{22} + [1-2\delta]\ell^1_{22}\big)(v_2)(t) \end{pmatrix}$$
$$\text{for a. e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2)$$

and

$$\ell^{1,\delta}(v)(t) = \delta \begin{pmatrix} \ell^1_{11}(v_1)(t) \\ \ell^1_{22}(v_2)(t) \end{pmatrix}$$
$$\text{for a. e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2).$$

Corollary 6.9. *Let $\ell = \ell^0 - \ell^1$, where $\ell^0, \ell^1 \in \mathcal{P}^2_{ab}$ and the components ℓ^1_{11} and ℓ^1_{22} of the operator ℓ^1 are a-Volterra operators. If there exists a number $\gamma \in [0, 1/2]$ such that*

$$\ell^{0,\gamma} \in \mathcal{S}^2_{ab}(a), \quad -\ell^{1,\gamma} \in \mathcal{S}^2_{ab}(a),$$

then the problem (3.1), (3.2) *has a unique solution.*

Remark 6.10. The assumption on the operators ℓ^1_{11} and ℓ^1_{22} to be a-Volterra ones cannot be omitted in the previous corollary, because it is necessary for the validity of the inclusion $-\ell^{1,\gamma} \in \mathcal{S}^2_{ab}(a)$ (see Remark 4.23).

6.1.2. *Systems with argument deviations.* In the sequel, general results of the previous section are applied to one of special types of the system (3.1), namely, the differential system with argument deviations (3.1′), i. e., the system

$$u_1'(t) = p_{11}(t)u_1\big(\tau_{11}(t)\big) + p_{12}(t)u_2\big(\tau_{12}(t)\big) + q_1(t),$$
$$u_2'(t) = p_{21}(t)u_1\big(\tau_{21}(t)\big) + p_{22}(t)u_2\big(\tau_{22}(t)\big) + q_2(t)$$

in which $p_{ik}, q_k \in L([a,b];\mathbb{R})$ and $\tau_{ik}\colon [a,b] \to [a,b]$ are measurable functions $(i,k = 1,2)$.

The next statement can be derived from Theorem 6.1.

Theorem 6.11. *Let*

$$\int_t^{\tau_{ik}(t)} p(s)\,\mathrm{d}s \le \frac{1}{\mathrm{e}} \quad \text{for a. e. } t \in [a,b], \quad i,k = 1,2, \tag{6.14}$$

where

$$p(t) = \max\Big\{|p_{11}(t)| + |p_{12}(t)|, |p_{21}(t)| + |p_{22}(t)|\Big\} \quad \text{for a. e. } t \in [a,b].$$

Then the problem $(3.1'), (3.2)$ *has a unique solution.*

Theorem 6.11 yields

Corollary 6.12. *Let*

$$\tau_{ik}(t) \le t \quad \text{for a. e. } t \in [a,b], \quad i,k = 1,2. \tag{6.15}$$

Then the problem $(3.1'), (3.2)$ *has a unique solution.*

If the deviations τ_{11} and τ_{22} are delays, then the following statement can be derived from Corollary 6.9.

Theorem 6.13. *Let, for each* $i \in \{1,2\}$ *the functions* $[p_{ii}]_-$ *and* τ_{ii} *satisfy the inequality*

$$\tau_{ii}(t) \le t \quad \text{for a. e. } t \in [a,b]$$

and at least one of the conditions:

(a)
$$\int_{\tau_{ii}(t)}^{t} [p_{ii}(s)]_-\,\mathrm{d}s \le \frac{2}{\mathrm{e}} \quad \text{for a. e. } t \in [a,b];$$

(b)
$$\int_a^b [p_{ii}(s)]_- \int_{\tau_{ii}(s)}^{s} [p_{ii}(\xi)]_- \exp\left(\frac{1}{2}\int_{\tau_{ii}(\xi)}^{s} [p_{ii}(\eta)]_-\,\mathrm{d}\eta\right) \mathrm{d}\xi\,\mathrm{d}s \le 4;$$

(c)
$$\int_a^b [p_{ii}(s)]_-\,\mathrm{d}s \le 2.$$

On the other hand, assume that for the functions $[p_{ii}]_+$, p_{i3-i}, *and* τ_{i3-i} $(i = 1,2)$ *at least one of the following conditions is fulfilled:*

(A) *the inequality*

$$\int_t^{\tau_{i3-i}(t)} p(s)\,\mathrm{d}s \le \frac{1}{\mathrm{e}} \quad \text{for a. e. } t \in [a,b], \quad i = 1,2$$

holds, where

$$p(t) = \max\Big\{[p_{11}(t)]_+ + |p_{12}(t)|, |p_{21}(t)| + [p_{22}(t)]_+\Big\}$$

for a. e. $t \in [a,b]$*;*

(B) *the inequality*

$$\mathrm{e}^{\max\left\{\int\limits_a^b [p_{11}(s)]_+\,\mathrm{d}s,\int\limits_a^b [p_{22}(s)]_+\,\mathrm{d}s\right\}} \int\limits_a^b h(s)\,\mathrm{e}^{\int\limits_s^b p(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s < 1$$

is satisfied, where

$$p(t) = \max\big\{\widetilde{p}_{12}(t), \widetilde{p}_{21}(t)\big\} \ \ \text{for a. e. } t \in [a,b],$$

$$h(t) = \max\big\{\widetilde{q}_1(t), \widetilde{q}_2(t)\big\} \ \ \text{for a. e. } t \in [a,b]$$

in which

$$\widetilde{p}_{i3-i}(t) = |p_{i3-i}(t)|\,\mathrm{e}^{\int\limits_a^t ([p_{3-i3-i}(s)]_+ - [p_{ii}(s)]_+)\,\mathrm{d}s} \quad \text{for a. e. } t \in [a,b], \ \ i = 1,2 \tag{6.16}$$

and

$$\widetilde{q}_i(t) = |p_{i3-i}(t)|\,\omega_{i3-i}(t) z_i(t)\,\mathrm{e}^{-\int\limits_a^t [p_{ii}(\eta)]_+\,\mathrm{d}\eta} \quad \text{for a. e. } t \in [a,b], \ \ i = 1,2 \tag{6.17}$$

with

$$z_i(t) = \int\limits_t^{\tau_{i3-i}(t)} \big(|p_{3-ii}(s)| + [p_{3-i3-i}(s)]_+\big)\,\mathrm{d}s \ \ \text{for a. e. } t \in [a,b], \ \ i = 1,2 \tag{6.18}$$

and

$$\omega_{i3-i}(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}\big(\tau_{i3-i}(t) - t\big)\Big) \ \ \text{for a. e. } t \in [a,b], \ \ i = 1,2; \tag{6.19}$$

(C) *the inequality*

$$\max\Big\{\lambda_1\,\mathrm{e}^{\int\limits_a^b [p_{11}(s)]_+\,\mathrm{d}s}, \lambda_2\,\mathrm{e}^{\int\limits_a^b [p_{22}(s)]_+\,\mathrm{d}s}\Big\} < 1$$

holds, where

$$\lambda_i = \int\limits_a^b \cosh\Big(\int\limits_s^b p(\xi)\,\mathrm{d}\xi\Big)\widetilde{q}_i(s)\,\mathrm{d}s + \int\limits_a^b \sinh\Big(\int\limits_s^b p(\xi)\,\mathrm{d}\xi\Big)\widetilde{q}_{3-i}(s)\,\mathrm{d}s$$

for $i = 1,2$ *and*

$$p(t) = \max\big\{\widetilde{p}_{12}(t), \widetilde{p}_{21}(t)\big\} \ \ \text{for a. e. } t \in [a,b]$$

in which the functions $\widetilde{p}_{12}$, $\widetilde{p}_{21}$ *and* $\widetilde{q}_1$, $\widetilde{q}_2$ *are defined, respectively, by the relations* (6.16) *and* (6.17) *with* z_1, z_2 *and* ω_{12}, ω_{21} *given by the formulas* (6.18) *and* (6.19).

Then the problem $(3.1'), (3.2)$ *has a unique solution.*

6.1.3. *Modified pantograph equation.* In this section, we consider the linear two-dimensional system

$$u'(t) = P(t)u(t) + G(t)u\big(\tau(t)\big) + q(t), \tag{6.20}$$

where $P, G\colon [0,T] \to \mathbb{R}^{2\times 2}$ are integrable matrix functions, $\tau\colon [0,T] \to [0,T]$ is a measurable function, $q \in L([0,T];\mathbb{R}^2)$, and $T > 0$. The system of the type (6.20) arises in applications and has been studied by many authors. We mention the *problem of the motion of a pantograph head on an electric locomotive*, where the system of the type (6.20) arises in the dimension 4, on the unbounded interval $[0,+\infty[$, and when

$$\tau(t) = \lambda t \ \text{ for } \ t \geq 0$$

with $0 < \lambda < 1$, and it is referred to as *a pantograph equation* (see, e. g., [41,51,58,67,68] and references therein). Recently, the pantograph equation has been generalized in a number of ways (see, e. g., [11–13, 38, 49, 50, 52]).

It is well-known (see, e. g., [44, § 1.3.2]) that if the deviation τ in the system (6.20) is a delay, then, without any additional assumption, the system (6.20) has a unique solution u satisfying the initial condition

$$u(0) = c, \tag{6.21}$$

where $c \in \mathbb{R}^2$. In any case, we can derive from Theorem 6.3 the following statements.

Theorem 6.14. *Let the deviation τ and the matrix functions $P = (p_{ik})_{i,k=1}^2$ and $G = (g_{ik})_{i,k=1}^2$ satisfy the condition*

$$\int_t^{\tau(t)} \big([p(s)]_+ + g(s)\big)\,\mathrm{d}s \leq \frac{1}{\mathrm{e}} \ \text{ for a. e. } \ t \in [0,T], \tag{6.22}$$

where

$$p(t) = \max\Big\{p_{11}(t) + |p_{21}(t)|, |p_{12}(t)| + p_{22}(t)\Big\} \ \text{ for a. e. } \ t \in [0,T] \tag{6.23}$$

and

$$g(t) = \max\Big\{|g_{11}(t)| + |g_{21}(t)|, |g_{12}(t)| + |g_{22}(t)|\Big\} \ \text{ for a. e. } \ t \in [0,T]. \tag{6.24}$$

Then the problem (6.20), (6.21) *has a unique solution.*

Theorem 6.14 yields

Corollary 6.15. *The problem* (6.20), (6.21) *is uniquely solvable provided that the condition*

$$\int_t^{\tau(t)} \big(\|P(s)\| + \|G(s)\|\big)\,\mathrm{d}s \leq \frac{1}{\mathrm{e}} \ \text{ for a. e. } \ t \in [0,T]$$

holds.

Theorem 6.16. *Let the deviation τ and the matrix functions $P = (p_{ik})_{i,k=1}^2$ and $G = (g_{ik})_{i,k=1}^2$ satisfy the condition*

$$\int_0^T g(s)\sigma(s)\bigg(\int_s^{\tau(s)} f(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{e}^{\int_s^T f(\eta)\,\mathrm{d}\eta}\,\mathrm{d}s < 1, \tag{6.25}$$

where

$$\sigma(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}\big(\tau(t) - t\big)\Big) \text{ for a. e. } t \in [0, T]$$

and $f \equiv [p]_+ + g$ with p and g given by the relations (6.23) *and* (6.24)*, respectively. Then the problem* (6.20), (6.21) *has a unique solution.*

6.1.4. *Auxiliary lemmas.* Let us first formulate the following obvious lemma.

Lemma 6.17. *Let $z \in AC([a,b];\mathbb{R})$. Then $|z| \in AC([a,b];\mathbb{R})$ and the relation*

$$|z(t)|' = z'(t)\operatorname{sgn} z(t) \text{ for a. e. } t \in [a,b]$$

is satisfied.

Now we give two lemmas established in the paper [20].

Lemma 6.18 ([20, Cor. 1.1(ii)])**.** *Let $f_k \in L([a,b];\mathbb{R}_+)$, $\tau_k\colon [a,b] \to [a,b]$ be measurable functions $(k = 1,\dots,m)$, and let*

$$\int_t^{\tau_i(t)} \sum_{k=1}^m f_k(s)\,\mathrm{d}s \le \frac{1}{\mathrm{e}} \text{ for a. e. } t \in [a,b], \quad i = 1,\dots,m.$$

Then the operator ℓ defined by the formula

$$\ell(z)(t) = \sum_{k=1}^m f_k(t) z\big(\tau_k(t)\big) \text{ for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}) \tag{6.26}$$

belongs to the set $\mathcal{S}_{ab}(a)$.

Lemma 6.19 ([20, Cor. 1.1(iii)]; see also [32, Thm. 3.1(c)])**.** *Let $f_k \in L([a,b];\mathbb{R}_+)$, $\tau_k\colon [a,b] \to [a,b]$ be measurable functions $(k = 1,\dots,m)$, and let the inequality*

$$\int_a^b \sum_{k=1}^m f_k(s)\sigma_k(s)\bigg(\int_s^{\tau_k(s)} \sum_{j=1}^m f_j(\xi)\,\mathrm{d}\xi\bigg)\exp\bigg(\int_s^b \sum_{i=1}^m f_i(\eta)\,\mathrm{d}\eta\bigg)\,\mathrm{d}s < 1$$

be satisfied, where

$$\sigma_k(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}\big(\tau_k(t) - t\big)\Big) \text{ for a. e. } t \in [a,b], \quad k = 1,\dots,m.$$

Then the operator ℓ defined by the formula (6.26) *belongs to the set $\mathcal{S}_{ab}(a)$.*

6.1.5. *Proofs.*

Proof of Theorem 6.1. Let u be a solution to the homogeneous problem (3.3). Then, in view of the condition (6.2) and Lemma 6.17, we get

$$|u(t)|' = \operatorname{Sgn}\big(u(t)\big)u'(t) = \operatorname{Sgn}\big(u(t)\big)\ell(u)(t) \leq \\ \leq \varphi_0(|u|)(t) \ \text{ for a. e. } \ t \in [a,b].$$

Since $|u(a)| = 0$, the inclusion $\varphi_0 \in \mathcal{S}^2_{ab}(a)$ yields

$$|u(t)| \leq 0 \ \text{ for } \ t \in [a,b]$$

and thus $u \equiv 0$. We have proved that the homogeneous problem (3.3) has only the trivial solution. Hence, Proposition 3.1 guarantees the unique solvability of the problem (3.1), (3.2). □

Proof of Theorem 6.3. Let u be a solution to the homogeneous problem (3.3). Then $\|u(a)\| = 0$ and, by virtue of the condition (6.5) and Lemma 6.17, we get

$$\|u(t)\|' = u'(t) \cdot \operatorname{sgn}\big(u(t)\big) = \ell(u)(t) \cdot \operatorname{sgn}\big(u(t)\big) \leq \\ \leq \psi_0(\|u\|)(t) \ \text{ for a. e. } \ t \in [a,b].$$

Therefore, the inclusion $\psi_0 \in \mathcal{S}_{ab}(a)$ implies

$$\|u(t)\| \leq 0 \ \text{ for } \ t \in [a,b],$$

and thus $u \equiv 0$. We have proved that the homogeneous problem (3.3) has only the trivial solution. Hence, Proposition 3.1 guarantees the unique solvability of the problem (3.1), (3.2). □

Proof of Theorem 6.6. Let u be a solution to the homogeneous problem (3.3). By virtue of the inclusion $\varphi_1 \in \mathcal{S}^2_{ab}(a)$ and Proposition 4.3, the problem

$$v'(t) = \varphi_1(v)(t) + \varphi_0(|u|)(t), \quad v(a) = 0 \tag{6.27}$$

has a unique solution v. Combining the relations (3.3), (6.10) and (6.27), we get

$$u'(t) - v'(t) = \varphi_1(u - v)(t) + \ell(u)(t) - \varphi_1(u)(t) - \varphi_0(|u|)(t) \leq \\ \leq \varphi_1(u - v)(t) \ \text{ for a. e. } \ t \in [a,b],$$

$$u'(t) + v'(t) = \varphi_1(u + v)(t) + \ell(u)(t) - \varphi_1(u)(t) + \varphi_0(|u|)(t) \geq \\ \geq \varphi_1(u + v)(t) \ \text{ for a. e. } \ t \in [a,b],$$

and

$$u(a) - v(a) = 0, \quad u(a) + v(a) = 0.$$

Consequently, the inclusion $\varphi_1 \in \mathcal{S}^2_{ab}(a)$ implies

$$u(t) - v(t) \leq 0, \quad u(t) + v(t) \geq 0 \ \text{ for } \ t \in [a,b],$$

that is

$$|u(t)| \leq v(t) \ \text{ for } \ t \in [a,b]. \tag{6.28}$$

Taking now the assumption $\varphi_0 \in \mathcal{P}^2_{ab}$ into account, we get from the differential equation in (6.27) that

$$v'(t) \le \big(\varphi_0 + \varphi_1\big)(v)(t) \quad \text{for a.e. } t \in [a,b]. \tag{6.29}$$

However, we also suppose that $\varphi_0 + \varphi_1 \in \mathcal{S}^2_{ab}(a)$ and thus the inequality (6.29) results in $v(t) \le 0$ for $t \in [a,b]$. Therefore, the inequality (6.28) yields $u \equiv 0$, i.e., the homogeneous problem (3.3) has only the trivial solution. Hence, Proposition 3.1 guarantees the unique solvability of the problem (3.1), (3.2). □

Proof of Corollary 6.9. It follows from Notation 6.8 that the inequality

$$\big|\ell^1(v)(t) - \ell^{1,\gamma}(v)(t)\big| \le \ell^{0,\frac{\gamma}{2}}(|v|)(t) - \ell^0(|v|)(t) \quad \text{for a.e. } t \in [a,b]$$

holds on the set $C([a,b];\mathbb{R}^2)$. Furthermore, it is clear that

$$\ell^{0,\gamma} = \ell^{0,\frac{\gamma}{2}} - \ell^{1,\gamma}.$$

Consequently, the assumptions of Theorem 6.6 with $\varphi_0 = \ell^{0,\frac{\gamma}{2}}$ and $\varphi_1 = -\ell^{1,\gamma}$ are satisfied. □

Proof of Theorem 6.11. It is clear that the system (3.1′) is a particular case of the system (3.1) in which the operator ℓ is defined by the formula

$$\ell(v)(t) = \begin{pmatrix} p_{11}(t)v_1\big(\tau_{11}(t)\big) + p_{12}(t)v_2\big(\tau_{12}(t)\big) \\ p_{21}(t)v_1\big(\tau_{21}(t)\big) + p_{22}(t)v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$
$$\text{for a.e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2). \tag{6.30}$$

Obviously, the condition (6.2) is fulfilled on the set $C([a,b];\mathbb{R}^2)$, where the operator φ_0 is given by the relation

$$\varphi_0(v)(t) = \begin{pmatrix} |p_{11}(t)|v_1\big(\tau_{11}(t)\big) + |p_{12}(t)|v_2\big(\tau_{12}(t)\big) \\ |p_{21}(t)|v_1\big(\tau_{21}(t)\big) + |p_{22}(t)|v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$
$$\text{for a.e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2).$$

Moreover, by virtue of the condition (6.14), Corollary 4.31 yields the validity of the inclusion $\varphi_0 \in \mathcal{S}^2_{ab}(a)$. Consequently, the assumptions of Theorem 6.1 are satisfied. □

Proof of Corollary 6.12. The assertion of the corollary follows immediately from Theorem 6.11, because the assumption (6.15) implies the validity of the condition (6.14). □

Proof of Theorem 6.13. It is clear that the system (3.1′) is a particular case of the system (3.1) in which the operator ℓ is given by the formula (6.30).

Let the operators ℓ^0 and ℓ^1 be defined by the relations

$$\ell^0(v)(t) = \begin{pmatrix} [p_{11}(t)]_+ v_1\big(\tau_{11}(t)\big) + [p_{12}(t)]_+ v_2\big(\tau_{12}(t)\big) \\ [p_{21}(t)]_+ v_1\big(\tau_{21}(t)\big) + [p_{22}(t)]_+ v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$
$$\text{for a. e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2)$$

and

$$\ell^1(v)(t) = \begin{pmatrix} [p_{11}(t)]_- v_1\big(\tau_{11}(t)\big) + [p_{12}(t)]_- v_2\big(\tau_{12}(t)\big) \\ [p_{21}(t)]_- v_1\big(\tau_{21}(t)\big) + [p_{22}(t)]_- v_2\big(\tau_{22}(t)\big) \end{pmatrix}$$
$$\text{for a. e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2),$$

respectively. Obviously, we have $\ell^0, \ell^1 \in \mathcal{P}^2_{ab}$ and $\ell = \ell^0 - \ell^1$.

By virtue of the conditions (a)–(c) of the theorem, Theorem 4.38 yields that

$$-\ell^{1,\frac{1}{2}} \in \mathcal{S}^2_{ab}(a).$$

On the other hand, it follows from Corollaries 4.31, 4.33, and 4.35 that each of the conditions (A)–(C) of the theorem guarantees the validity of the inclusion

$$\ell^{0,\frac{1}{2}} \in \mathcal{S}^2_{ab}(a).$$

Consequently, the assumptions of Corollary 6.9 with $\gamma = \frac{1}{2}$ are satisfied. □

Proof of Theorem 6.14. It is clear that the system (6.20) is a particular case of the system (3.1) in which $a = 0$, $b = T$, and the operator ℓ is defined by the formula

$$\ell(v)(t) = P(t)v(t) + G(t)v\big(\tau(t)\big)$$
$$\text{for a. e. } t \in [0,T] \text{ and all } v \in C([0,T];\mathbb{R}^2). \tag{6.31}$$

Obviously, the condition (6.5) is fulfilled on the set $C([0,T];\mathbb{R}^2)$, where the operator ψ_0 is given by the relation

$$\psi_0(z)(t) = [p(t)]_+ z(t) + g(t)z\big(\tau(t)\big)$$
$$\text{for a. e. } t \in [0,T] \text{ and all } z \in C([0,T];\mathbb{R}) \tag{6.32}$$

and the functions p and g are given by the formulas (6.23) and (6.24), respectively. Moreover, by virtue of Lemma 6.18 (with $m = 2$), the condition (6.22) yields the validity of the inclusion $\psi_0 \in \mathcal{S}_{0T}(0)$. Consequently, the assumptions of Theorem 6.3 are satisfied. □

Proof of Corollary 6.15. In view of the inequality

$$[p(t)]_+ + g(t) \le \|P(t)\| + \|G(t)\| \quad \text{for a. e. } t \in [0,T]$$

in which the functions p and g are given by the formulas (6.23) and (6.24), the assertion of the corollary follows immediately from Theorem 6.14. □

Proof of Theorem 6.16. It is clear that the system (6.20) is a particular case of the system (3.1) in which $a = 0$, $b = T$, and the operator ℓ is defined by the formula (6.31). Moreover, the condition (6.5) is fulfilled on the set $C([0, T]; \mathbb{R}^2)$, where the operator ψ_0 is given by the relation (6.32) and the functions p and g are given by the formulas (6.23) and (6.24), respectively. Therefore, the assumptions of Theorem 6.3 are satisfied, because the inequality (6.25) guarantees the validity of the inclusion $\psi_0 \in \mathcal{S}_{0T}(0)$ (see Lemma 6.19 with $m = 2$). $\square$

6.2. **Anti-diagonal systems.** In this part, we consider the two-dimensional linear system with the so-called anti-diagonal right-hand side, i. e., the system (3.1) in which $\ell_{11} = 0$ and $\ell_{22} = 0$.[21] More precisely, in what follows we consider the system

$$\begin{aligned} u_1'(t) &= h_1(u_2)(t) + q_1(t), \\ u_2'(t) &= h_2(u_1)(t) + q_2(t), \end{aligned} \tag{6.33}$$

where $h_1, h_2 \in \mathcal{L}_{ab}$ and $q_1, q_2 \in L([a, b]; \mathbb{R})$. Obviously, the initial condition (3.2) expressed in terms of its components has the form

$$u_1(a) = c_1, \quad u_2(a) = c_2, \tag{6.34}$$

where $c_1, c_2 \in \mathbb{R}$.

It should be noted here that the second-order functional differential equation

$$u''(t) = h(u)(t) + q(t), \tag{6.35}$$

where $h \in \mathcal{L}_{ab}$ and $q \in L([a, b]; \mathbb{R})$, can be also regarded as a particular case of (6.33). Therefore, the results stated below can be immediately reformulated in order to guarantee the unique solvability of the Cauchy problem subjected to the second-order equation (6.35).

6.2.1. *Main results.* We first give two existence results for the problem (6.33), (6.34) the proofs of which are based on the technique of differential inequalities.

Theorem 6.20. *Let $k \in \{1, 2\}$, $m \in \{0, 1\}$, and $h_i = h_{i,0} - h_{i,1}$ with $h_{i,j} \in \mathcal{P}_{ab}$ $(i = 1, 2,\ j = 0, 1)$. Assume that there exist functions $\beta_1, \beta_2 \in AC([a, b]; \mathbb{R})$ such that*

$$\beta_1(t) > 0, \quad \beta_2(t) > 0 \ \textit{ for } \ t \in [a, b], \tag{6.36}$$

$$\beta_1'(t) \geq h_{k,0}(\beta_2)(t) + h_{k,1}(\beta_2)(t) \ \textit{ for a. e. } \ t \in [a, b], \tag{6.37}$$

$$\beta_2'(t) \leq -h_{3-k,0}(\beta_1)(t) - h_{3-k,1}(\beta_1)(t) \ \textit{ for a. e. } \ t \in [a, b], \tag{6.38}$$

$$\int_a^b h_{k,1-m}(\beta_2)(s)\, ds \leq \beta_1(a), \tag{6.39}$$

[21] The symbol 0 stands here for the zero operator.

and

$$\int_a^b h_{3-k,m}(\beta_1)(s)\,\mathrm{d}s + \int_a^b h_{3-k,1-m}\big(\chi\big(h_{k,1-m}(\beta_2)\big)\big)(s)\,\mathrm{d}s \le \beta_2(b), \quad (6.40)$$

whereas the inequality (6.40) *is supposed to be strict if* $h_{3-k,m} = 0$.[22] *Here, the operator* χ *is defined by the formula*

$$\chi(f)(t) = \int_a^t f(s)\,\mathrm{d}s \;\; \text{for} \;\; t \in [a,b], \quad f \in L([a,b];\mathbb{R}).$$

Then the problem (6.33), (6.34) *has a unique solution.*

If the operators h_1 and h_2 are monotone and one of them is an a-Volterra operator, then the assumption $\beta_1 \in AC([a,b];\mathbb{R})$ in the previous theorem can be weakened (see Theorem 6.21). On the other hand, if the operators h_1 and h_2 are both a-Volterra ones, then the problem (6.33), (6.34) is uniquely solvable without any additional assumptions (see, e. g., [44, § 1.3.2]).

Theorem 6.21. *Let* $k \in \{1,2\}$, $m \in \{0,1\}$, $(-1)^m h_k, (-1)^{1-m} h_{3-k} \in \mathcal{P}_{ab}$, *and the operator* h_{3-k} *be an* a*-Volterra one. Assume that there exist functions* $\gamma_1 \in AC_{loc}([a,b[\,;\mathbb{R})$ *and* $\gamma_2 \in AC([a,b];\mathbb{R})$ *such that*

$$\gamma_1(t) > 0 \;\; \text{for} \;\; t \in [a,b[\,, \quad \gamma_2(t) > 0 \;\; \text{for} \;\; t \in [a,b], \quad (6.41)$$

$$\gamma_1'(t) \ge (-1)^m h_k(\gamma_2)(t) \;\; \text{for a. e.} \;\; t \in [a,b], \quad (6.42)$$

and

$$\gamma_2'(t) \le (-1)^m h_{3-k}(\gamma_1)(t) \;\; \text{for a. e.} \;\; t \in [a,b].^{23} \quad (6.43)$$

Then the problem (6.33), (6.34) *has a unique solution.*

Remark 6.22. Since possibly $\gamma_1(t) \to +\infty$ as $t \to b-$, the condition (6.43) of the previous theorem is understood in the sense that for any $b_0 \in \,]a,b[$ the relation

$$\gamma_2'(t) \le (-1)^m h_{3-k}^{ab_0}(\gamma_1)(t) \;\; \text{for a. e.} \;\; t \in [a,b_0] \quad (6.44)$$

holds, where $h_{3-k}^{ab_0}$ is the restriction of the operator h_{3-k} to the space $C([a,b_0];\mathbb{R})$.[24]

The next statement is proved by using weak theorems on differential inequalities discussed in Section 5.

Theorem 6.23. *Let* $k \in \{1,2\}$, $m \in \{0,1\}$, $(-1)^m h_k \in \mathcal{P}_{ab}$, *and let there exist operators* $g_0 \in \mathcal{L}_{ab}$ *and* $g_1 \in \mathcal{P}_{ab}$ *such that*

$$\big((-1)^m h_k, g_0\big) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \quad \big((-1)^m h_k, g_0 + g_1\big) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \quad (6.45)$$

[22] The symbol 0 stands here for the zero operator.
[23] See Remark 6.22.
[24] See Definition 1.8.

and the inequality

$$\big|h_{3-k}(z)(t) + (-1)^{1-m} g_0(z)(t)\big| \le g_1\big(|z|\big)(t) \ \text{ for a. e. } \ t \in [a,b] \tag{6.46}$$

holds on the set $C_a([a,b];\mathbb{R})$. *Then the problem* (6.33), (6.34) *has a unique solution.*

Remark 6.24. The assumption (6.45) in the previous theorem can be replaced neither by the assumption

$$\begin{aligned} \big((-1)^m h_k, g_0\big) &\in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \\ \big((-1)^m (1-\varepsilon_1) h_k, (1-\varepsilon_2)(g_0+g_1)\big) &\in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \end{aligned} \tag{6.47}$$

nor by the assumption

$$\begin{aligned} \big((-1)^m (1-\varepsilon_1) h_k, (1-\varepsilon_2) g_0\big) &\in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \\ \big((-1)^m h_k, g_0+g_1\big) &\in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \end{aligned} \tag{6.48}$$

no matter how small $\varepsilon_1, \varepsilon_2 \in [0,1[$ with $\varepsilon_1+\varepsilon_2>0$ are (see Examples 6.38 and 6.39).

Theorem 6.23 yields

Corollary 6.25. *Let* $k \in \{1,2\}$, $m \in \{0,1\}$, $(-1)^m h_k \in \mathcal{P}_{ab}$, *and let* $h_{3-k} = h_{3-k,0} - h_{3-k,1}$ *with* $h_{3-k,0}, h_{3-k,1} \in \mathcal{P}_{ab}$. *If*

$$\begin{aligned} \big((-1)^m h_k, h_{3-k,m}\big) &\in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \\ \Big((-1)^m h_k, -\frac{1}{2}\, h_{3-k,1-m}\Big) &\in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \end{aligned} \tag{6.49}$$

then the problem (6.33), (6.34) *has a unique solution.*

Remark 6.26. In Section 5, there is proved the following assertion (see Theorem 5.6): *If* $h_1 \in \mathcal{P}_{ab}$ *and* $h_2 = h_{2,0} - h_{2,1}$ *with* $h_{2,0}, h_{2,1} \in \mathcal{P}_{ab}$ *are such that*

$$(h_1, h_{2,0}) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \quad (h_1, -h_{2,1}) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a),$$

then $(h_1, h_2) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$, *as well.* It is easy to find operators $h_1, h_{2,0}, h_{2,1} \in \mathcal{P}_{ab}$ such that under the assumption

$$\big(h_1, h_{2,0}\big) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \quad \Big(h_1, -\frac{1}{2}\, h_{2,1}\Big) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a), \tag{6.50}$$

the weak theorem on differential inequalities does not hold for the system (6.33) with $h_2 = h_{2,0} - h_{2,1}$, i.e., $(h_1, h_2) \notin \widehat{\mathcal{S}}_{ab}^{2,1}(a)$. However, Corollary 6.25 guarantees that the problem (6.33), (6.34) remains to be uniquely solvable, if the inclusions (6.50) are satisfied.

As it was said above, the Cauchy problem for the second order functional differential equations can be regarded as a particular case of (6.33), (6.34).

As an example, we consider on the interval $[0,1]$ the problem

$$x''(t) = \frac{1}{(1-t)^\nu} \int\limits_0^t \frac{d_1 x\big(\tau(s)\big) - d_2 x(\lambda s)}{(1-s)^\nu}\, \mathrm{d}s + q(t), \tag{6.51}$$

$$x(0) = c_1, \quad x'(0) = c_2, \tag{6.52}$$

where $d_1, d_2 \in \mathbb{R}_+$, $\nu < 1$, $\lambda \in [0,1]$, $\tau\colon [0,1] \to [0,1]$ is a measurable function, $q \in L([0,1];\mathbb{R})$, and $c_1, c_2 \in \mathbb{R}$.

Corollary 6.27. *Let at least one of the following conditions be fulfilled:*

(a) *The deviation τ is a delay, i. e.,*

$$\tau(t) \le t \ \text{ for a. e. } \ t \in [0,1];$$

(b) *The numbers d_1 and d_2 satisfy the inequalities*

$$d_1 < (3-2\nu)(2-\nu), \quad d_2 \le 2(3-2\nu)(2-\nu). \tag{6.53}$$

Then the problem (6.51), (6.52) *has a unique solution.*

6.2.2. *Systems with argument deviations.* Now we give some corollaries of Theorems 6.20 and 6.21 for two-dimensional differential systems with argument deviations. Consider the system

$$\begin{aligned} u_1'(t) &= f_1(t) u_2\big(\tau_1(t)\big) + q_1(t), \\ u_2'(t) &= f_2(t) u_1\big(\tau_2(t)\big) + q_2(t), \end{aligned} \tag{6.54}$$

where $f_1, f_2, q_1, q_2 \in L([a,b];\mathbb{R})$ and $\tau_1, \tau_2\colon [a,b] \to [a,b]$ are measurable functions.

In order to simplify formulation of the next statement, we put

$$f_{i,0} \equiv [f_i]_+\,, \quad f_{i,1} \equiv [f_i]_- \ \text{ for } \ i = 1,2. \tag{6.55}$$

Theorem 6.20 implies the following Vallée-Poussin type result.

Theorem 6.28. *Let $k \in \{1,2\}$, $m \in \{0,1\}$ and the functions $f_{i,j}$ $(i = 1,2,$ $j = 0,1)$ be defined by the relations* (6.55). *Assume that there exist numbers $\alpha_i \in \mathbb{R}_+$ $(i = 1,\dots,4)$, at least one of which is positive, and $\lambda \in [0,1[$ such that*

$$\int\limits_{\omega_1}^{\omega_2} \frac{\mathrm{d}s}{\alpha_1 + (\alpha_2+\alpha_3)s + \alpha_4 s^2} > \frac{(b-a)^{1-\lambda}}{1-\lambda}\,, \tag{6.56}$$

$$\alpha_1 (b-t)^\lambda \Bigg(\int\limits_t^{\tau_{3-k}(t)} \frac{\mathrm{d}s}{(b-s)^\lambda} \Bigg) |f_{3-k}(t)| \le$$

$$\le \alpha_2 \Bigg(1 + \int\limits_{\tau_{3-k}(t)}^t \frac{\alpha_3}{(b-s)^\lambda}\, \mathrm{d}s \Bigg) \ \text{for a. e. } \ t \in [a,b], \tag{6.57}$$

$$(b-t)^{\lambda}|f_{3-k}(t)| \le \alpha_4\left(1+\int\limits_{\tau_{3-k}(t)}^{t}\frac{\alpha_3}{(b-s)^{\lambda}}\,\mathrm{d}s\right) \ \textit{for a. e. } t\in[a,b], \tag{6.58}$$

$$(b-t)^{\lambda}|f_{k}(t)| \le \alpha_1\left(1+\int\limits_{t}^{\tau_k(t)}\frac{\alpha_2}{(b-s)^{\lambda}}\,\mathrm{d}s\right) \ \textit{for a. e. } t\in[a,b], \tag{6.59}$$

and

$$\alpha_4(b-t)^{\lambda}\left(\int\limits_{\tau_k(t)}^{t}\frac{\mathrm{d}s}{(b-s)^{\lambda}}\right)|f_k(t)| \le$$

$$\le \alpha_3\left(1+\int\limits_{t}^{\tau_k(t)}\frac{\alpha_2}{(b-s)^{\lambda}}\,\mathrm{d}s\right) \ \textit{for a. e. } t\in[a,b], \tag{6.60}$$

where $\omega_1 = \|f_{k,1-m}\|_L$ *and the number* ω_2 *has the following properties:*

(i) *If* $f_{k,1-m}\equiv 0$ *and* $f_{3-k,m}\equiv 0$ *then* $\omega_2=+\infty$*;*
(ii) *If* $f_{k,1-m}\equiv 0$ *and* $f_{3-k,m}\not\equiv 0$ *then* $\omega_2=\|f_{3-k,m}\|_L^{-1}$*;*
(iii) *If* $f_{k,1-m}\not\equiv 0$ *and* $f_{3-k,m}\not\equiv 0$ *then* $\|f_{k,1-m}\|_L<\omega_2\le\|f_{3-k,m}\|_L^{-1}$ *and*

$$\int\limits_a^b f_{3-k,1-m}(s)\left(\int\limits_a^{\tau_{3-k}(s)} f_{k,1-m}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s \le$$

$$\le \Big(1-\omega_2\|f_{3-k,m}\|_L\Big)\exp\left(-\int\limits_a^b\frac{\alpha_2+\alpha_4\omega_2}{(b-s)^{\lambda}}\,\mathrm{d}s\right); \tag{6.61}$$

(iv) *If* $f_{k,1-m}\not\equiv 0$ *and* $f_{3-k,m}\equiv 0$ *then* $\|f_{k,1-m}\|_L<\omega_2<+\infty$ *and*

$$\int\limits_a^b f_{3-k,1-m}(s)\left(\int\limits_a^{\tau_{3-k}(s)} f_{k,1-m}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s<\exp\left(-\int\limits_a^b\frac{\alpha_2+\alpha_4\omega_2}{(b-s)^{\lambda}}\,\mathrm{d}s\right). \tag{6.62}$$

Then the problem (6.54), (6.34) *has a unique solution.*

If neither of the functions f_1 and f_2 changes its sign and at least one of the deviations τ_1 and τ_2 is a delay, then we can derive the following statement from Theorem 6.21.

Theorem 6.29. *Let* $k\in\{1,2\}$, $m\in\{0,1\}$,

$$(-1)^m f_k(t)\ge 0, \quad (-1)^{1-m}f_{3-k}(t)\ge 0 \ \textit{for a. e. } t\in[a,b], \tag{6.63}$$

and

$$|f_{3-k}(t)|\big(\tau_{3-k}(t)-t\big)\le 0 \ \textit{for a. e. } t\in[a,b]. \tag{6.64}$$

Assume that there exist numbers $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_+$, at least one of which is positive, $\lambda \in [0,1[$, and $\nu \in [0, \lambda]$ such that

$$\int_0^{+\infty} \frac{\mathrm{d}s}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} > \frac{(b-a)^{1-\lambda}}{1-\lambda}\,, \tag{6.65}$$

$$(b-t)^{\lambda+\nu} |f_k(t)| \le \alpha_1 \quad \textit{for a. e. } t \in [a,b], \tag{6.66}$$

$$\alpha_3 (b-t)^{\nu} |f_k(t)| \big(t - \tau_k(t)\big) \le \alpha_2 + \frac{\nu}{(b-t)^{1-\lambda}} \quad \textit{for a. e. } t \in [a,b], \tag{6.67}$$

and

$$(b-t)^{\lambda-\nu} |f_{3-k}(t)| \le \alpha_3 \left(1 + \sigma_{3-k}(t) \int_{\tau_{3-k}(t)}^{t} \Big(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}}\Big) \mathrm{d}s\right) \quad \textit{for a. e. } t \in [a,b], \tag{6.68}$$

where

$$\sigma_{3-k}(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}\big(t - \tau_{3-k}(t)\big)\Big) \quad \textit{for a. e. } t \in [a,b].$$

Then the problem (6.54), (6.34) *has a unique solution.*

Theorem 6.29 implies

Corollary 6.30. *Let*

$$f_1(t) \ge 0, \quad f_2(t) \le 0 \quad \textit{for a. e. } t \in [a,b].$$

Assume that there exist numbers $\alpha, \beta \in \mathbb{R}_+$, $\lambda \in [0,1[$ and $\nu \in [0,\lambda]$ such that

$$\int_0^{+\infty} \frac{\mathrm{d}s}{\alpha + \beta s^2} > \frac{(b-a)^{1-\lambda}}{1-\lambda}\,,$$

and let either the conditions

$$f_1(t)\big(\tau_1(t) - t\big) \le 0, \quad |f_2(t)|\big(\tau_2(t) - t\big) \ge 0 \quad \textit{for a. e. } t \in [a,b],$$
$$(b-t)^{\lambda-\nu} f_1(t) \le \beta, \quad (b-t)^{\lambda+\nu} |f_2(t)| \le \alpha \quad \textit{for a. e. } t \in [a,b],$$

or the inequalities

$$f_1(t)\big(\tau_1(t) - t\big) \ge 0, \quad |f_2(t)|\big(\tau_2(t) - t\big) \le 0 \quad \textit{for a. e. } t \in [a,b],$$
$$(b-t)^{\lambda+\nu} f_1(t) \le \alpha, \quad (b-t)^{\lambda-\nu} |f_2(t)| \le \beta \quad \textit{for a. e. } t \in [a,b]$$

be satisfied. Then the problem (6.54), (6.34) *has a unique solution.*

In order to illustrate Theorem 6.23, we consider the differential system

$$\begin{aligned} u_1'(t) &= f_1(t) u_2\big(\tau_1(t)\big) + q_1(t), \\ u_2'(t) &= f_{2,0}(t) u_1\big(\tau_{2,0}(t)\big) - f_{2,1}(t) u_1\big(\tau_{2,1}(t)\big) + q_2(t), \end{aligned} \tag{6.69}$$

where $f_1, f_{2,0}, f_{2,1} \in L([a,b];\mathbb{R}_+)$, $\tau_1, \tau_{2,0}, \tau_{2,1} \colon [a,b] \to [a,b]$ are measurable functions, and $q_1, q_2 \in L([a,b];\mathbb{R})$.

Corollary 6.31. *Let*

$$\tau_1(t) \le t, \quad \tau_{2,1}(t) \le t \quad \textit{for a. e. } t \in [a,b]$$

and the functions f_1, τ_1, $f_{2,0}$, $\tau_{2,0}$ satisfy at least one of the following conditions:

(a)

$$\int\limits_t^{\tau_{2,0}(t)} \omega(s)\,\mathrm{d}s \le \frac{1}{\mathrm{e}} \quad \textit{for a. e. } t \in [a,b],$$

where

$$\omega(t) = \max\big\{f_1(t), f_{2,0}(t)\big\} \quad \textit{for a. e. } t \in [a,b]; \tag{6.70}$$

(b)

$$\int\limits_a^b \cosh\bigg(\int\limits_s^b \omega(\xi)\,\mathrm{d}\xi\bigg) f_{2,0}(s)\alpha(s)\bigg(\int\limits_s^{\tau_{2,0}(s)} f_1(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s < 1,$$

where the function ω is defined by the formula (6.70) *and*

$$\alpha(t) = \frac{1}{2}\Big(1 + \operatorname{sgn}\big(\tau_{2,0}(t) - t\big)\Big) \quad \textit{for a. e. } t \in [a,b];$$

(c) *either*

$$\int\limits_a^{\tau_{2,0}^*} f_1(s)\bigg(\int\limits_a^{\tau_1(s)} f_{2,0}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s < 1$$

or

$$\int\limits_a^{\tau_1^*} f_{2,0}(s)\bigg(\int\limits_a^{\tau_{2,0}(s)} f_1(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s < 1,$$

where

$$\tau_1^* = \operatorname{ess\,sup}\big\{\tau_1(t) : t \in [a,b]\big\}, \quad \tau_{2,0}^* = \operatorname{ess\,sup}\big\{\tau_{2,0}(t) : t \in [a,b]\big\}.$$

Furthermore, assume that the functions f_1, τ_1, $f_{2,1}$, $\tau_{2,1}$ satisfy at least one of the following conditions:

(A)

$$\int\limits_a^b f_1(s)\bigg(\int\limits_a^{\tau_1(s)} f_{2,1}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \le 2;$$

(B) *there exist numbers* $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_3 > 0$, $\lambda \in [0,1[\,$, *and* $\nu \in [0,\lambda]$ *such that*

$$\int_0^{+\infty} \frac{\mathrm{d}s}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} \geq \frac{(b-a)^{1-\lambda}}{1-\lambda}\,,$$

$$(b-t)^{\lambda-\nu} f_1(t) \leq \alpha_3 \left[1 + \int_{\tau_1(t)}^{t} \left(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^\lambda} \right) \mathrm{d}s \right] \ \text{for a. e. } t \in [a,b],$$

$$(b-t)^{\lambda+\nu} f_{2,1}(t) \leq 2\alpha_1 \ \text{for a. e. } t \in [a,b],$$

and

$$\alpha_3 (b-t)^\nu f_{2,1}(t) \big(t - \tau_{2,1}(t) \big) \leq 2 \Big(\alpha_2 + \frac{\nu}{(b-t)^{1-\lambda}} \Big) \ \text{for a. e. } t \in [a,b].$$

Then the problem (6.69), (6.34) *has a unique solution.*

6.2.3. *Auxiliary lemmas.* Consider the homogeneous problem

$$u_1'(t) = h_1(u_2)(t), \quad u_2'(t) = h_2(u_1)(t), \tag{6.71}$$

$$u_1(a) = 0, \quad u_2(a) = 0 \tag{6.72}$$

corresponding to the problem (6.33), (6.34). For the sake of convenience, we formulate the following obvious lemma.

Lemma 6.32. $(u_1, u_2)^T$ *is a solution to the problem* (6.71), (6.72) *if and only if* $(-u_1, u_2)^T$ *is a solution to the problem*

$$v_1'(t) = -h_1(v_2)(t), \quad v_2'(t) = -h_2(v_1)(t),$$
$$v_1(a) = 0, \quad v_2(a) = 0.$$

Lemma 6.33. *Let* $h_i = h_{i,0} - h_{i,1}$ *with* $h_{i,0}, h_{i,1} \in \mathcal{P}_{ab}$ $(i = 1,2)$. *Assume that there exist functions* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in AC([a,b];\mathbb{R})$ *such that*

$$\alpha_1(t) \leq \beta_1(t), \quad \alpha_2(t) \leq \beta_2(t) \ \text{for } t \in [a,b], \tag{6.73}$$

$$\alpha_1'(t) \leq h_{1,0}(\alpha_2)(t) - h_{1,1}(\beta_2)(t) \ \text{for a. e. } t \in [a,b], \tag{6.74}$$

$$\alpha_2'(t) \geq h_{2,0}(\beta_1)(t) - h_{2,1}(\alpha_1)(t) \ \text{for a. e. } t \in [a,b], \tag{6.75}$$

$$\beta_1'(t) \geq h_{1,0}(\beta_2)(t) - h_{1,1}(\alpha_2)(t) \ \text{for a. e. } t \in [a,b], \tag{6.76}$$

and

$$\beta_2'(t) \leq h_{2,0}(\alpha_1)(t) - h_{2,1}(\beta_1)(t) \ \text{for a. e. } t \in [a,b]. \tag{6.77}$$

Then, for arbitrary $c_1 \in \big[\alpha_1(a), \beta_1(a)\big]$ *and* $c_2 \in \big[\alpha_2(b), \beta_2(b)\big]$, *the system* (6.71) *has at least one solution* $(u_1, u_2)^T$ *satisfying the conditions* $u_1(a) = c_1$, $u_2(b) = c_2$ *and*

$$\alpha_i(t) \leq u_i(t) \leq \beta_i(t) \ \text{for } t \in [a,b], \quad i = 1,2. \tag{6.78}$$

Proof. For any $k = 1, 2$ and $z \in C([a,b];\mathbb{R})$, we put

$$\chi_k(z)(t) = \frac{1}{2}\Big(|z(t)-\alpha_k(t)| - |z(t)-\beta_k(t)| + \alpha_k(t) + \beta_k(t)\Big) \text{ for } t \in [a,b].$$

It is clear that $\chi_1, \chi_2\colon C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ are continuous operators and

$$\alpha_k(t) \le \chi_k(z)(t) \le \beta_k(t) \text{ for } t \in [a,b], \quad z \in C([a,b];\mathbb{R}), \quad k = 1, 2. \tag{6.79}$$

Put

$$T_1(z)(t) = c_1 + \int_a^t h_1\big(\chi_2(z)\big)(s)\,\mathrm{d}s \text{ for } t \in [a,b], \quad z \in C([a,b];\mathbb{R}),$$

$$T_2(z)(t) = c_2 - \int_t^b h_2\big(\chi_1(z)\big)(s)\,\mathrm{d}s \text{ for } t \in [a,b], \quad z \in C([a,b];\mathbb{R}).$$

By virtue of the inequalities (6.79) and the assumptions $h_{i,0}, h_{i,1} \in \mathcal{P}_{ab}$ $(i = 1, 2)$, for any $z \in C([a,b];\mathbb{R})$ the functions $T_1(z)$ and $T_2(z)$ belong to the set $AC([a,b];\mathbb{R})$,

$$|T_k(z)(t)| \le M_k \text{ for } t \in [a,b], \quad k = 1, 2, \tag{6.80}$$

and

$$\begin{aligned} h_{k,0}(\alpha_{3-k})(t) - h_{k,1}(\beta_{3-k})(t) \le \frac{\mathrm{d}}{\mathrm{d}t}\, T_k(z)(t) \le \\ \le h_{k,0}(\beta_{3-k})(t) - h_{k,1}(\alpha_{3-k})(t) \text{ for a.e. } t \in [a,b], \quad k = 1, 2, \end{aligned} \tag{6.81}$$

where

$$M_k = |c_k| + \int_a^b \big(h_{k,0} + h_{k,1}\big)\big(|\alpha_{3-k}| + |\beta_{3-k}|\big)(s)\,\mathrm{d}s \text{ for } k = 1, 2.$$

Now we define the operator $T\colon C([a,b];\mathbb{R}^2) \to C([a,b];\mathbb{R}^2)$ by the formula

$$T(v)(t) = \begin{pmatrix} T_1(v_2)(t) \\ T_2(v_1)(t) \end{pmatrix} \text{ for } t \in [a,b], \quad v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2).$$

In view of the conditions (6.80) and (6.81), it is clear that the operator T maps continuously the Banach space $C([a,b];\mathbb{R}^2)$ into its relatively compact subset. Therefore, using the Schauder fixed point theorem, we conclude that the operator T has a fixed point, i.e., there exist $u_1, u_2 \in C([a,b];\mathbb{R})$ such that

$$u_1(t) = T_1(u_2)(t), \quad u_2(t) = T_2(u_1)(t) \text{ for } t \in [a,b]. \tag{6.82}$$

Obviously, $u_1, u_2 \in AC([a,b];\mathbb{R})$, $u_1(a) = c_1$, $u_2(b) = c_2$, and thus we have

$$\alpha_1(a) \le u_1(a) \le \beta_1(a), \quad \alpha_2(b) \le u_2(b) \le \beta_2(b). \tag{6.83}$$

On the other hand, by virtue of the conditions (6.76), (6.81) and (6.82), we get

$$\begin{aligned} u_1'(t) - \beta_1'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\, T_1(u_2)(t) - \beta_1'(t) \le \\ \le h_{1,0}(\beta_2)(t) - h_{1,1}(\alpha_2)(t) - \beta_1'(t) \le 0 \ \text{ for a. e. } \ t \in [a,b], \end{aligned}$$

which, together with the first relation in (6.83), implies that $u_1(t) \le \beta_1(t)$ holds for $t \in [a,b]$. The other inequalities in the condition (6.78) can be proved analogously using the inequalities (6.74), (6.75) and (6.77). However, this means that

$$u_1(t) = c_1 + \int_a^t h_1(u_2)(s)\,\mathrm{d}s, \quad u_2(t) = c_2 - \int_t^b h_2(u_1)(s)\,\mathrm{d}s \ \text{ for } \ t \in [a,b],$$

i. e., the vector function $(u_1,u_2)^T$ is a solution to the system (6.71) satisfying the conditions $u_1(a) = c_1$, $u_2(b) = c_2$, and (6.78). □

Lemma 6.34. *Let $h_i = h_{i,0} - h_{i,1}$ with $h_{i,0}, h_{i,1} \in \mathcal{P}_{ab}$ $(i = 1,2)$. Assume that there exist positive functions $\beta_1, \beta_2 \in AC([a,b];\mathbb{R})$ satisfying the conditions*

$$\beta_1'(t) \ge h_{1,0}(\beta_2) + h_{1,1}(\beta_2) \quad \text{for a. e. } \ t \in [a,b], \tag{6.84}$$

$$\beta_2'(t) \le -h_{2,0}(\beta_1) - h_{2,1}(\beta_1) \ \text{ for a. e. } \ t \in [a,b]. \tag{6.85}$$

Then the problem

$$u_1(a) = 0, \quad u_2(b) = 0 \tag{6.86}$$

subjected to the system (6.71) *has only the trivial solution.*

Proof. Let the operator $\omega\colon C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ be defined by the formula

$$\omega(z)(t) = z(a+b-t) \ \text{ for } \ t \in [a,b], \ \ z \in C([a,b];\mathbb{R}).$$

For any $z \in C([a,b];\mathbb{R})$ and $m = 0,1$, we put

$$\widetilde{h}_{1,m}(z)(t) = h_{1,m}\big(\omega(z)\big)(t) \ \text{ for a. e. } \ t \in [a,b]$$

and

$$\widetilde{h}_{2,m}(z)(t) = h_{2,m}(z)(a+b-t) \ \text{ for a. e. } \ t \in [a,b].$$

It is clear that $\widetilde{h}_{k,m} \in \mathcal{P}_{ab}$ for $k = 1,2$ and $m = 0,1$.

If $(u_1,u_2)^T$ is a solution to the problem (6.71), (6.86), then the vector function $(u_1,\omega(u_2))^T$ is a solution to the problem

$$v_1'(t) = \widetilde{h}_{1,0}(v_2)(t) - \widetilde{h}_{1,1}(v_2)(t), \ \ v_2'(t) = \widetilde{h}_{2,1}(v_1)(t) - \widetilde{h}_{2,0}(v_1)(t), \tag{6.87}$$

$$v_1(a) = 0, \quad v_2(a) = 0, \tag{6.88}$$

and vice versa, if $(v_1,v_2)^T$ is a solution to the problem (6.87), (6.88), then the vector function $(v_1,\omega(v_2))^T$ is a solution to the problem (6.71), (6.86).

On the other hand, it follows from the inequalities (6.36), (6.84) and (6.85) that the functions $\gamma_1 \equiv \beta_1$ and $\gamma_2 \equiv \omega(\beta_2)$ are positive and satisfy the inequalities

$$\gamma_1'(t) \geq \widetilde{h}_{1,0}(\gamma_2)(t) + \widetilde{h}_{1,1}(\gamma_2)(t) \quad \text{for a. e. } t \in [a,b]$$

and

$$\gamma_2'(t) \geq \widetilde{h}_{2,0}(\gamma_1)(t) + \widetilde{h}_{2,1}(\gamma_1)(t) \quad \text{for a. e. } t \in [a,b].$$

Consequently, using Theorem 4.8, we obtain

$$\widetilde{h} \in \mathcal{S}^2_{ab}(a), \tag{6.89}$$

where the operator $\widetilde{h}$ is defined by the relation

$$\widetilde{h}(w)(t) = \begin{pmatrix} \widetilde{h}_{1,0}(w_2)(t) + \widetilde{h}_{1,1}(w_2)(t) \\ \widetilde{h}_{2,0}(w_1)(t) + \widetilde{h}_{2,1}(w_1)(t) \end{pmatrix}$$
$$\text{for a. e. } t \in [a,b] \text{ and all } w = (w_1, w_2)^T \in C([a,b];\mathbb{R}^2).$$

It is also easy to verify that the inequality

$$\operatorname{Sgn}\big(w(t)\big) \begin{pmatrix} \widetilde{h}_{1,0}(w_2)(t) - \widetilde{h}_{1,1}(w_2)(t) \\ \widetilde{h}_{2,1}(w_1)(t) - \widetilde{h}_{2,0}(w_1)(t) \end{pmatrix} \leq \widetilde{h}(|w|)(t) \quad \text{for a. e. } t \in [a,b]$$

holds for every $w = (w_1, w_2)^T \in C([a,b];\mathbb{R}^2)$. Therefore, by virtue of Theorem 6.1, the inclusion (6.89) and the above-mentioned equivalence, we get the assertion of the lemma. □

Lemma 6.35. *Let numbers $\alpha_i \in \mathbb{R}_+$ $(i = 1, \dots, 4)$, at least one of which is positive, $\varrho_a, \varrho_b \in]0, +\infty[$, and $\lambda \in [0, 1[$ be such that $\varrho_a < \varrho_b$ and*

$$\int_{\varrho_a}^{\varrho_b} \frac{\mathrm{d}s}{\alpha_1 + (\alpha_2 + \alpha_3)s + \alpha_4 s^2} = \frac{(b-a)^{1-\lambda}}{1-\lambda}\,. \tag{6.90}$$

Then there exist positive functions $\beta_1, \beta_2 \in AC([a,b];\mathbb{R})$ satisfying $\beta_1', \beta_2' \in C_{loc}([a,b[\,;\mathbb{R})$ and the conditions

$$\beta_1'(t) = \frac{\alpha_3}{(b-t)^\lambda}\,\beta_1(t) + \frac{\alpha_1}{(b-t)^\lambda}\,\beta_2(t) \quad \textit{for } t \in [a,b[\,, \tag{6.91}$$

$$\beta_2'(t) = -\frac{\alpha_4}{(b-t)^\lambda}\,\beta_1(t) - \frac{\alpha_2}{(b-t)^\lambda}\,\beta_2(t) \quad \textit{for } t \in [a,b[\,, \tag{6.92}$$

$$\beta_1(a) = \varrho_a\,, \quad \beta_1(b) = \varrho_b \beta_2(b), \quad \beta_2(a) = 1 \tag{6.93}$$

and

$$\beta_2(b) \geq \exp\bigg(- \int_a^b \frac{\alpha_2 + \alpha_4 \varrho_b}{(b-s)^\lambda}\,\mathrm{d}s \bigg). \tag{6.94}$$

Proof. Define the function $\varrho\colon [a,b] \to \mathbb{R}_+$ by setting

$$\int\limits_{\varrho(t)}^{\varrho_b} \frac{\mathrm{d}s}{\alpha_1 + (\alpha_2+\alpha_3)s + \alpha_4 s^2} = \frac{(b-t)^{1-\lambda}}{1-\lambda} \quad \text{for } \; t \in [a,b].$$

In view of the equality (6.90), we get

$$\varrho(t) > 0 \quad \text{for } \; t \in [a,b], \qquad \varrho(a) = \varrho_a, \qquad \varrho(b) = \varrho_b\,, \tag{6.95}$$

and

$$\varrho'(t) = \frac{\alpha_1 + (\alpha_2+\alpha_3)\varrho(t) + \alpha_4\varrho^2(t)}{(b-t)^\lambda} \quad \text{for } \; t \in [a,b[\,. \tag{6.96}$$

Put

$$\beta_2(t) = \exp\bigg(-\int\limits_a^t \frac{\alpha_2 + \alpha_4\varrho(s)}{(b-s)^\lambda}\,\mathrm{d}s \bigg), \qquad \beta_1(t) = \varrho(t)\beta_2(t) \quad \text{for } \; t \in [a,b].$$

It is not difficult to verify that $\beta_1, \beta_2 \in AC([a,b];\mathbb{R})$ and, in view of the condition (6.96), the equalities (6.91) and (6.92) are satisfied. Therefore, we have $\beta_1', \beta_2' \in C_{loc}([a,b[\,;\mathbb{R})$. Moreover, by virtue of the relations (6.95) and (6.96), it is clear that the conditions (6.36), (6.93) and (6.94) hold as well. □

6.2.4. *Proofs.*

Proof of Theorem 6.20. According to Proposition 3.1, to prove the theorem it is sufficient to show that the homogeneous problem (6.71), (6.72) has only the trivial solution. In view of Lemma 6.32, we can assume without loss of generality that $k = 1$ and $m = 0$. Let $(u_1,u_2)^T$ be a solution to the problem (6.71), (6.72).

It follows from the conditions (6.36)–(6.38) that

$$\beta_1(t) \ge \beta_1(a) + \chi\big(h_{1,0}(\beta_2)\big)(t), \qquad \beta_2(t) \ge \beta_2(b) \quad \text{for } \; t \in [a,b],$$

and thus the inequality (6.40) yields

$$\beta_1(a)\int\limits_a^b h_{2,0}(1)(s)\,\mathrm{d}s + \beta_2(b)\int\limits_a^b h_{2,0}\big(\chi\big(h_{1,0}(1)\big)\big)(s)\,\mathrm{d}s + {} \\ {} + \beta_2(b)\int\limits_a^b h_{2,1}\big(\chi\big(h_{1,1}(1)\big)\big)(s)\,\mathrm{d}s \le \beta_2(b).$$

Consequently, using the condition (6.36), we get

$$\int\limits_a^b h_{2,0}\big(\chi\big(h_{1,0}(1)\big)\big)(s)\,\mathrm{d}s + \int\limits_a^b h_{2,1}\big(\chi\big(h_{1,1}(1)\big)\big)(s)\,\mathrm{d}s < 1, \tag{6.97}$$

because we suppose that the inequality (6.40) is strict if $h_{2,0}(1) \equiv 0$.

Put

$$\alpha_1(t) = -\int_a^t h_{1,1}(\beta_2)(s)\,\mathrm{d}s \quad \text{for } t \in [a,b] \tag{6.98}$$

and

$$\alpha_2(t) = \int_a^t h_{2,0}(\beta_1)(s)\,\mathrm{d}s - \int_a^t h_{2,1}(\alpha_1)(s)\,\mathrm{d}s \quad \text{for } t \in [a,b]. \tag{6.99}$$

It is clear that

$$\alpha_1(t) \le 0, \quad \alpha_2(t) \ge 0 \quad \text{for } t \in [a,b] \tag{6.100}$$

and, using the inequality (6.39), it is easy to verify that

$$-\alpha_1(t) = \int_a^t h_{1,1}(\beta_2)(s)\,\mathrm{d}s \le \beta_1(a) \le \beta_1(t) \quad \text{for } t \in [a,b]. \tag{6.101}$$

By virtue of the conditions (6.100) and (6.101), from the relations (6.37), (6.38), (6.98) and (6.99) we get

$$\begin{gathered}
\alpha_1'(t) = -h_{1,1}(\beta_2)(t) \le h_{1,0}(\alpha_2)(t) - h_{1,1}(\beta_2)(t) \quad \text{for a.e. } t \in [a,b], \\
\alpha_2'(t) = h_{2,0}(\beta_1)(t) - h_{2,1}(\alpha_1)(t) \quad \text{for a.e. } t \in [a,b], \\
\beta_1'(t) \ge h_{1,0}(\beta_2)(t) \ge h_{1,0}(\beta_2)(t) - h_{1,1}(\alpha_2)(t) \quad \text{for a.e. } t \in [a,b],
\end{gathered} \tag{6.102}$$

and

$$\begin{aligned}
\beta_2'(t) \le -h_{2,0}(\beta_1)(t) - h_{2,1}(\beta_1)(t) \le \\
\le h_{2,0}(\alpha_1)(t) - h_{2,1}(\beta_1)(t) \quad \text{for a.e. } t \in [a,b],
\end{aligned} \tag{6.103}$$

i.e., the inequalities (6.74)–(6.77) are satisfied. Moreover, in view of the first inequality in (6.100), it is clear that

$$\alpha_1(t) \le \beta_1(t) \quad \text{for } t \in [a,b]. \tag{6.104}$$

On the other hand, the relations (6.40), (6.98) and (6.99) result in

$$\alpha_2(b) = \int_a^b h_{2,0}(\beta_1)(s)\,\mathrm{d}s + \int_a^b h_{2,1}\big(\chi\big(h_{1,1}(\beta_2)\big)\big)(s)\,\mathrm{d}s \le \beta_2(b).$$

Furthermore, the conditions (6.102)–(6.104) yield that

$$\begin{aligned}
\alpha_2'(t) = h_{2,0}(\beta_1)(t) - h_{2,1}(\alpha_1)(t) \ge \\
\ge h_{2,0}(\alpha_1)(t) - h_{2,1}(\beta_1)(t) \ge \beta_2'(t) \quad \text{for a.e. } t \in [a,b].
\end{aligned}$$

Hence, the last two relations guarantee that $\alpha_2(t) \le \beta_2(t)$ holds for $t \in [a,b]$, and thus the condition (6.73) is satisfied.

Therefore, by virtue of Lemma 6.33, the system (6.71) has a solution $(x_1,x_2)^T$ satisfying the conditions

$$x_1(a) = 0, \quad x_2(b) = \beta_2(b), \tag{6.105}$$

and

$$\alpha_k(t) \le x_k(t) \le \beta_k(t) \ \text{ for } \ t \in [a,b], \quad k = 1,2. \tag{6.106}$$

We will show that

$$x_2(a) > 0. \tag{6.107}$$

Indeed, the inequalities (6.100) and (6.106) imply that $x_2(t) \ge 0$ for $t \in [a,b]$, and since $(x_1, x_2)^T$ is a solution to the problem (6.71), (6.105), the first equation in (6.71) yields

$$x_1(t) \le \int_a^t h_{1,0}(x_2)(s)\,\mathrm{d}s, \quad -x_1(t) \le \int_a^t h_{1,1}(x_2)(s)\,\mathrm{d}s \ \text{ for } \ t \in [a,b].$$

Using these relations in the second equation of the system (6.71), we get

$$\begin{aligned} x_2'(t) \le h_{2,0}\big(\chi\big(h_{1,0}(x_2)\big)\big)(t) + {}& \\ + h_{2,1}\big(\chi\big(h_{1,1}(x_2)\big)\big)(t) & \ \text{ for a.e. } \ t \in [a,b]. \end{aligned} \tag{6.108}$$

Put $M = \max\big\{x_2(t) : t \in [a,b]\big\}$ and choose $t_M \in [a,b]$ such that $x_2(t_M) = M$. The integration of the inequality (6.108) from a to t_M yields

$$\begin{aligned} M \le x_2(a) + \int_a^{t_M} h_{2,0}\big(\chi\big(h_{1,0}(x_2)\big)\big)(s)\,\mathrm{d}s + \int_a^{t_M} h_{2,1}\big(\chi\big(h_{1,1}(x_2)\big)\big)(s)\,\mathrm{d}s \le \\ \le x_2(a) + M\bigg(\int_a^b h_{2,0}\big(\chi\big(h_{1,0}(1)\big)\big)(s)\,\mathrm{d}s + \int_a^b h_{2,1}\big(\chi\big(h_{1,1}(1)\big)\big)(s)\,\mathrm{d}s\bigg). \end{aligned} \tag{6.109}$$

In view of the conditions (6.36) and (6.105), we have $M > 0$. Therefore, the relations (6.97) and (6.109) arrive at $M < x_2(a) + M$, and thus the desired inequality (6.107) holds.

At last, we put

$$w_k(t) = x_2(b)u_k(t) - x_k(t)u_2(b) \ \text{ for } \ t \in [a,b], \quad k = 1,2.$$

Obviously, the vector function $(w_1, w_2)^T$ is a solution to the problem (6.71), (6.86). Therefore, Lemma 6.34 yields that $w_1 \equiv 0$ and $w_2 \equiv 0$. Consequently, we have

$$0 = w_2(a) = -x_2(a)u_2(b),$$

which, together with the inequality (6.107), implies $u_2(b) = 0$. However, this means that the vector function $(u_1, u_2)^T$ is also a solution to the problem (6.71), (6.86), and thus Lemma 6.34 yields that $u_1 \equiv 0$ and $u_2 \equiv 0$.

Consequently, the homogeneous problem (6.71), (6.72) has only the trivial solution. □

Proof of Theorem 6.21. According to Proposition 3.1, to prove the theorem it is sufficient to show that the homogeneous problem (6.71), (6.72) has only the trivial solution. In view of Lemma 6.32, we can assume without loss of generality that $k = 1$ and $m = 0$. Assume that, on the contrary, $(u_1, u_2)^T$

is a nontrivial solution to the problem (6.71), (6.72). Then it is clear that $u_1 \not\equiv 0$ and $u_2 \not\equiv 0$.

First suppose that u_2 does not change its sign. Then we can assume without loss of generality that $u_2(t) \geq 0$ holds for $t \in [a,b]$. Since the operator h_1 is positive, the first equation in the system (6.71) implies that $u_1'(t) \geq 0$ is satisfied for a. e. $t \in [a,b]$. Therefore, by virtue of the initial condition (6.72), we have $u_1(t) \geq 0$ for $t \in [a,b]$. On the other hand, the operator h_2 is negative, and thus the second equation in the system (6.71) yields that $u_2'(t) \leq 0$ holds for a. e. $t \in [a,b]$. Consequently, using the condition $u_2(a) = 0$, we get the contradiction $u_2 \equiv 0$.

Now suppose that the function u_2 changes its sign. Put

$$\lambda_1 = \inf \mathcal{A}, \quad \lambda_2 = \max\left\{\frac{u_2(t)}{\gamma_2(t)} : t \in [a,b]\right\}, \tag{6.110}$$

where

$$\mathcal{A} = \Big\{\lambda > 0 : \lambda\gamma_1(t) - u_1(t) \geq 0 \ \text{ for } \ t \in [a,b[\ \Big\}. \tag{6.111}$$

It is clear that

$$0 \leq \lambda_1 < +\infty, \ \ 0 < \lambda_2 < +\infty, \tag{6.112}$$

and there exists $t_0 \in]a,b]$ such that

$$\frac{u_2(t_0)}{\gamma_2(t_0)} = \lambda_2\,. \tag{6.113}$$

Without loss of generality, we can assume that $t_0 < b$ and that there exists $b_0 \in]t_0,b[$ such that

$$u_2(b_0) = 0. \tag{6.114}$$

Indeed, if either $t_0 = b$ or $u_2(t) > 0$ for $t \in [t_0,b[$, then there exists $t^* \in]a,t_0[$ with the properties

$$u_2(t) > 0 \ \text{ for } \ t \in]t^*,b[\,, \quad u_2(t^*) = 0.$$

Then we can redefine the numbers $\lambda_1, \lambda_2, t_0$ for the solution $(-u_1,-u_2)^T$ to the problem (6.71), (6.72), and we can take $b_0 = t^*$.

Now we put

$$w_1(t) = \lambda_1\gamma_1(t) - u_1(t) \ \text{ for } \ t \in [a,b[\,, \ \ w_2(t) = \lambda_2\gamma_2(t) - u_2(t) \ \text{ for } \ t \in [a,b].$$

Obviously, we have $w_1 \in AC_{loc}([a,b[\,;\mathbb{R})$ and $w_2 \in AC([a,b];\mathbb{R})$. By virtue of the conditions (6.41), (6.110) and (6.113), it is clear that

$$w_1(t) \geq 0 \ \text{ for } \ t \in [a,b[\,, \quad w_2(t) \geq 0 \ \text{ for } \ t \in [a,b], \tag{6.115}$$

and

$$w_2(t_0) = 0. \tag{6.116}$$

Moreover, either the relation $\lambda_1 < \lambda_2$ or $\lambda_1 \geq \lambda_2$ holds.

First suppose that $\lambda_1 < \lambda_2$. Then, in view of the conditions (6.41), (6.44), (6.71), (6.112), (6.115) and the fact that h_2 is a negative a-Volterra operator, we get

$$w_2'(t) \leq h_2^{ab_0}(\lambda_2\gamma_1 - u_1)(t) \leq h_2^{ab_0}(w_1)(t) \leq 0 \ \text{ for a. e. } \ t \in [a,b_0].$$

Therefore, by virtue of the conditions (6.41), (6.112) and (6.114), the last relation yields

$$w_2(t_0) \geq w_2(b_0) = \lambda_2\gamma_2(b_0) > 0,$$

which contradicts the equality (6.116).

Now suppose that $\lambda_1 \geq \lambda_2$. Then the second relation in (6.112) implies

$$\lambda_1 > 0. \tag{6.117}$$

Using the conditions (6.41), (6.42), (6.71), (6.72), (6.115), (6.117) and the assumption $h_1 \in \mathcal{P}_{ab}$, we get

$$w_1'(t) \geq h_1(\lambda_1\gamma_2 - u_2)(t) \geq h_1(w_2)(t) \geq 0 \quad \text{for a. e. } t \in [a,b]$$

and

$$w_1(a) = \lambda_1\gamma_1(a) > 0.$$

Consequently, the inequality $w_1(t) > 0$ holds for $t \in [a,b[$. Therefore, there exists $\varepsilon > 0$ such that

$$w_1(t) \geq \varepsilon u_1(t) \quad \text{for } \ t \in [a,b[\,,$$

i. e.,

$$\frac{\lambda_1}{1+\varepsilon}\,\gamma_1(t) - u_1(t) \geq 0 \quad \text{for } \ t \in [a,b[\,.$$

However, in view of the definition (6.111) of the set $\mathcal{A}$, the last relation implies that $\frac{\lambda_1}{1+\varepsilon} \in \mathcal{A}$, which contradicts the first equality in (6.110).

The contradictions obtained prove that the homogeneous problem (6.71), (6.72) has only the trivial solution. $\square$

Proof of Theorem 6.23. According to Proposition 3.1, to prove the theorem it is sufficient to show that the homogeneous problem (6.71), (6.72) has only the trivial solution. In view of Lemma 6.32, we can assume without loss of generality that $k = 1$ and $m = 0$.

Let $(u_1, u_2)^T$ be a solution to the problem (6.71), (6.72). By virtue of the assumption $(h_1, g_0) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$ and Remark 5.5, the problem

$$x_1'(t) = h_1(x_2)(t), \quad x_2'(t) = g_0(x_1)(t) + g_1(|u_1|)(t), \tag{6.118}$$

$$x_1(a) = 0, \quad x_2(a) = 0 \tag{6.119}$$

has a unique solution $(x_1, x_2)^T$. Combining the conditions (6.46), (6.71), (6.72), (6.118) and (6.119), we get

$$x_1'(t) + u_1'(t) = h_1(x_2 + u_2)(t) \quad \text{for a. e. } t \in [a,b],$$
$$x_2'(t) + u_2'(t) = g_0(x_1 + u_1)(t) + h_2(u_1)(t) - g_0(u_1)(t) + g_1(|u_1|)(t) \geq$$
$$\geq g_0(x_1 + u_1)(t) \quad \text{for a. e. } t \in [a,b],$$
$$x_1(a) + u_1(a) = 0,$$

and

$$\begin{gathered}
x_1'(t) - u_1'(t) = h_1(x_2 - u_2)(t) \text{ for a.e. } t \in [a,b], \\
x_2'(t) - u_2'(t) = g_0(x_1 - u_1)(t) - h_2(u_1)(t) + g_0(u_1)(t) + g_1(|u_1|)(t) \geq \\
\geq g_0(x_1 - u_1)(t) \text{ for a.e. } t \in [a,b], \\
x_1(a) - u_1(a) = 0.
\end{gathered}$$

Consequently, the inclusion $(h_1, g_0) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$ guarantees that

$$x_1(t) + u_1(t) \geq 0, \quad x_1(t) - u_1(t) \geq 0 \text{ for } t \in [a,b],$$

and thus

$$|u_1(t)| \leq x_1(t) \text{ for } t \in [a,b]. \tag{6.120}$$

Taking now the assumption $g_1 \in \mathcal{P}_{ab}$ into account, we get from the equalities (6.118) that

$$x_1'(t) = h_1(x_2)(t), \quad x_2'(t) \leq (g_0 + g_1)(x_1)(t) \text{ for a.e. } t \in [a,b]. \tag{6.121}$$

However, we also suppose that $\big(h_1, g_0 + g_1\big) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$, and thus the relations (6.119) and (6.121) guarantee that $x_1(t) \leq 0$ holds for $t \in [a,b]$. Therefore, the inequality (6.120) yields that $u_1 \equiv 0$. Consequently, (6.71) and (6.72) imply $u_2 \equiv 0$, i.e., the homogeneous problem (6.71), (6.72) has only the trivial solution. □

Proof of Corollary 6.25. It is not difficult to verify that the assumptions of Theorem 6.23 are satisfied with $g_0 = -\frac{1}{2}\, h_{3-k,1-m}$ and $g_1 = h_{3-k,m} + \frac{1}{2}\, h_{3-k,1-m}$. □

Proof of Corollary 6.27. It is clear that the problem (6.51), (6.52) is a particular case of (6.33), (6.34) with $a = 0$, $b = 1$, $q_1 \equiv 0$, $q_2 \equiv q$ and $h_2 = h_{2,0} - h_{2,1}$, where h_1, $h_{2,0}$, $h_{2,1}$ are defined by the formulas $h_1(z)(t) \stackrel{\text{def}}{=} z(t)$ and

$$h_{2,0}(z)(t) = \frac{d_1}{(1-t)^\nu} \int_0^t \frac{z\big(\tau(s)\big)}{(1-s)^\nu}\, ds, \quad h_{2,1}(z)(t) = \frac{d_2}{(1-t)^\nu} \int_0^t \frac{x(\lambda s)}{(1-s)^\nu}\, ds$$

for a.e. $t \in [0,1]$ and all $z \in C\big([0,1];\mathbb{R}\big)$. Obviously, $h_1, h_{2,0}, h_{2,1} \in \mathcal{P}_{01}$ and the operators $h_1, h_{2,1}$ are 0-Volterra ones.

Case (a): Since τ is a delay, the operator $h_{2,0}$ is a 0-Volterra one. Therefore, the operator ℓ defined by the formula

$$\varphi(v)(t) = \begin{pmatrix} h_1(v_2)(t) \\ h_{2,0}(v_1)(t) + h_{2,1}(v_1)(t) \end{pmatrix}$$
$$\text{for a.e. } t \in [a,b] \text{ and all } v = (v_1, v_2)^T \in C([a,b];\mathbb{R}^2).$$

is a 0-Volterra one, and thus Proposition 4.20 yields that $\varphi \in \mathcal{S}_{01}^2(0)$. On the other hand, for any $z = (z_1, z_2)^T \in C([0,1];\mathbb{R}^2)$ we have

$$\operatorname{Sgn}\big(z(t)\big) \begin{pmatrix} h_1(z_2)(t) \\ h_2(z_1)(t) \end{pmatrix} \leq \varphi(|z|)(t) \text{ for a.e. } t \in [a,b].$$

Consequently, the validity of the corollary follows from Theorem 6.1.

Case (b): Analogously to Example 5.17, using the inequalities (6.53) we get

$$\int_0^1 h_1\big(\psi\big(h_{2,0}(1)\big)\big)(s)\,\mathrm{d}s < 1, \qquad \int_0^1 h_1\big(\psi\big(h_{2,1}(1)\big)\big)(s)\,\mathrm{d}s \le 2,$$

where the operator ψ is defined by the relation (5.9). Therefore, Corollaries 5.14 and 5.21 guarantee that

$$\big(h_1, h_{2,0}\big) \in \widehat{\mathcal{S}}_{01}^{2,1}(0), \qquad \Big(h_1, -\frac{1}{2}\,h_{2,1}\Big) \in \widehat{\mathcal{S}}_{01}^{2,1}(0).$$

Consequently, the assumptions of Corollary 6.25 with $k = 1$ and $m = 0$ are satisfied. □

Proof of Theorem 6.28. It is clear that the system (6.54) is a particular case of the system (6.33) in which the operators h_1, h_2 are given by the formula

$$h_i(z)(t) = f_i(t) z\big(\tau_i(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}), \quad i = 1,2. \tag{6.122}$$

Let the operators $h_{i,j}$ be defined by the relations

$$h_{i,j}(z)(t) = f_{i,j}(t) z\big(\tau_i(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}), \quad i = 1,2, \quad j = 0,1,$$

where the functions $f_{i,j}$ are given by the formula (6.55). It is clear that $h_{i,j} \in \mathcal{P}_{ab}$ ($i = 1,2,\ j = 0,1$) and $h_i = h_{i,0} - h_{i,1}$ for $i = 1,2$. By virtue of the inequality (6.56), there exist $\varrho_a, \varrho_b \in \mathbb{R}_+$ such that $\omega_1 < \varrho_a < \varrho_b < \omega_2$ and the equality (6.90) is fulfilled. According to Lemma 6.35, we can find functions $\beta_1, \beta_2 \in AC([a,b];\mathbb{R})$ satisfying the relations (6.36) and (6.91)–(6.94). Using these conditions, we get

$$\beta_1'(t) \ge 0, \quad \beta_2'(t) \le 0 \ \text{ for } \ t \in [a,b[\,. \tag{6.123}$$

Put

$$A_i = \big\{t \in [a,b] : \ f_i(t) \ne 0\big\} \ \text{ for } \ i = 1,2. \tag{6.124}$$

If we take the relations (6.36), (6.91), (6.92) and (6.123) into account, by direct calculation we obtain

$$\beta_2\big(\tau_k(t)\big) = \beta_2(t) - \int_{\tau_k(t)}^{t} \beta_2'(s)\,\mathrm{d}s =$$

$$= \beta_2(t) + \int_{\tau_k(t)}^{t} \frac{\alpha_4}{(b-s)^\lambda}\,\beta_1(s)\,\mathrm{d}s + \int_{\tau_k(t)}^{t} \frac{\alpha_2}{(b-s)^\lambda}\,\beta_2(s)\,\mathrm{d}s \le$$

$$\leq \beta_2(t)+\beta_1(t)\int\limits_{\tau_k(t)}^{t}\frac{\alpha_4}{(b-s)^{\lambda}}\,\mathrm{d}s+\beta_2\big(\tau_k(t)\big)\int\limits_{\tau_k(t)}^{t}\frac{\alpha_2}{(b-s)^{\lambda}}\,\mathrm{d}s \ \text{ for a.e. } t\in[a,b]$$

and

$$-\beta_1\big(\tau_{3-k}(t)\big)=-\beta_1(t)+\int\limits_{\tau_{3-k}(t)}^{t}\beta_1'(s)\,\mathrm{d}s=$$

$$=-\beta_1(t)+\int\limits_{\tau_{3-k}(t)}^{t}\frac{\alpha_3}{(b-s)^{\lambda}}\,\beta_1(s)\,\mathrm{d}s+\int\limits_{\tau_{3-k}(t)}^{t}\frac{\alpha_1}{(b-s)^{\lambda}}\,\beta_2(s)\,\mathrm{d}s\geq$$

$$\geq-\beta_1(t)+\beta_1\big(\tau_{3-k}(t)\big)\int\limits_{\tau_{3-k}(t)}^{t}\frac{\alpha_3}{(b-s)^{\lambda}}\,\mathrm{d}s+\beta_2(t)\int\limits_{\tau_{3-k}(t)}^{t}\frac{\alpha_1}{(b-s)^{\lambda}}\,\mathrm{d}s$$

$$\text{for a.e. } t\in[a,b].$$

Therefore, by virtue of the conditions (6.36), (6.28)–(6.60), (6.91) and (6.92), we get from the last two relations that

$$|f_k(t)|\beta_2\big(\tau_k(t)\big)\leq$$

$$\leq\frac{|f_k(t)|\int\limits_{\tau_k(t)}^{t}\frac{\alpha_4}{(b-s)^{\lambda}}\,\mathrm{d}s}{1+\int\limits_{t}^{\tau_k(t)}\frac{\alpha_2}{(b-s)^{\lambda}}\,\mathrm{d}s}\,\beta_1(t)+\frac{|f_k(t)|}{1+\int\limits_{t}^{\tau_k(t)}\frac{\alpha_2}{(b-s)^{\lambda}}\,\mathrm{d}s}\,\beta_2(t)\leq$$

$$\leq\frac{\alpha_3}{(b-t)^{\lambda}}\,\beta_1(t)+\frac{\alpha_1}{(b-t)^{\lambda}}\,\beta_2(t)=\beta_1'(t)\ \text{ for a.e. } t\in A_k$$

and

$$-|f_{3-k}(t)|\beta_1\big(\tau_{3-k}(t)\big)\geq$$

$$\geq-\frac{|f_{3-k}(t)|}{1+\int\limits_{\tau_{3-k}(t)}^{t}\frac{\alpha_3}{(b-s)^{\lambda}}\,\mathrm{d}s}\,\beta_1(t)-\frac{|f_{3-k}(t)|\int\limits_{t}^{\tau_{3-k}(t)}\frac{\alpha_1}{(b-s)^{\lambda}}\,\mathrm{d}s}{1+\int\limits_{\tau_{3-k}(t)}^{t}\frac{\alpha_3}{(b-s)^{\lambda}}\,\mathrm{d}s}\,\beta_2(t)\geq$$

$$\geq-\frac{\alpha_4}{(b-t)^{\lambda}}\,\beta_1(t)-\frac{\alpha_2}{(b-t)^{\lambda}}\,\beta_2(t)=\beta_2'(t)\ \text{ for a.e. } t\in A_{3-k}\,,$$

which, together with the inequalities (6.123), guarantees that

$$\beta_1'(t)\geq|f_k(t)|\beta_2\big(\tau_k(t)\big)\ \text{ for a.e. } t\in[a,b]$$

and

$$\beta_2'(t) \le -|f_{3-k}(t)|\beta_1\big(\tau_{3-k}(t)\big) \quad \text{for a.e. } t \in [a,b].$$

Consequently, the functions β_1, β_2 satisfy the conditions (6.37) and (6.38).

On the other hand, in view of the relations (6.93) and (6.123), we get

$$\int_a^b f_{k,1-m}(s)\beta_2\big(\tau_k(s)\big)\,\mathrm{d}s \le \beta_2(a)\|f_{k,1-m}\|_L = \omega_1 < \varrho_a = \beta_1(a),$$

and thus the inequality (6.39) holds. Finally, we will show that the inequality (6.40) is satisfied in all cases (i)–(iv). Note that in view of (6.93) and (6.123) we have

$$\begin{aligned}\Phi := \int_a^b f_{3-k,m}(s)\beta_1\big(\tau_{3-k}(s)\big)\,\mathrm{d}s+ \\ + \int_a^b f_{3-k,1-m}(s)\Bigg(\int_a^{\tau_{3-k}(s)} f_{k,1-m}(\xi)\beta_2\big(\tau_k(\xi)\big)\,\mathrm{d}\xi\Bigg)\,\mathrm{d}s \le \\ \le \varrho_b\beta_2(b)\|f_{3-k,m}\|_L + \int_a^b f_{3-k,1-m}(s)\Bigg(\int_a^{\tau_{3-k}(s)} f_{k,1-m}(\xi)\,\mathrm{d}\xi\Bigg)\,\mathrm{d}s. \quad (6.125)\end{aligned}$$

Case (i): $f_{k,1-m} \equiv 0$ *and* $f_{3-k,m} \equiv 0$. In this case, we have $\Phi = 0$ and thus the inequality (6.40) trivially holds as the strict one.

Case (ii): $f_{k,1-m} \equiv 0$ *and* $f_{3-k,m} \not\equiv 0$. The relation (6.125) yields

$$\Phi \le \varrho_b\beta_2(b)\|f_{3-k,m}\|_L < \omega_2\|f_{3-k,m}\|_L\beta_2(b) = \beta_2(b),$$

i.e., the inequality (6.40) holds.

Case (iii): $f_{k,1-m} \not\equiv 0$ *and* $f_{3-k,m} \not\equiv 0$. In view of the conditions (6.61) and (6.94), the relation (6.125) implies

$$\begin{aligned}\Phi \le \varrho_b\beta_2(b)\|f_{3-k,m}\|_L + \Big(1-\omega_2\|f_{3-k,m}\|_L\Big)\exp\Bigg(-\int_a^b \frac{\alpha_2+\alpha_4\omega_2}{(b-s)^\lambda}\,\mathrm{d}s\Bigg) \le \\ \le \varrho_b\beta_2(b)\|f_{3-k,m}\|_L + \Big(1-\varrho_b\|f_{3-k,m}\|_L\Big)\exp\Bigg(-\int_a^b \frac{\alpha_2+\alpha_4\varrho_b}{(b-s)^\lambda}\,\mathrm{d}s\Bigg) \le \\ \le \varrho_b\beta_2(b)\|f_{3-k,m}\|_L + \Big(1-\varrho_b\|f_{3-k,m}\|_L\Big)\beta_2(b) = \beta_2(b),\end{aligned}$$

i.e., the inequality (6.40) is satisfied.

Case (iv): $f_{k,1-m} \not\equiv 0$ *and* $f_{3-k,m} \equiv 0$. Using (6.62) and (6.94), we get from (6.125) the relation

$$\Phi < \exp\bigg(-\int_a^b \frac{\alpha_2 + \alpha_4 \omega_2}{(b-s)^\lambda}\, \mathrm{d}s \bigg) \le \exp\bigg(-\int_a^b \frac{\alpha_2 + \alpha_4 \varrho_b}{(b-s)^\lambda}\, \mathrm{d}s \bigg) \le \beta_2(b),$$

and thus the inequality (6.40) holds as the strict one.

Consequently, the assumptions of Theorem 6.20 are satisfied in all cases (i)–(iv). □

Proof of Theorem 6.29. It is clear that the system (6.54) is a particular case of the system (6.33) in which the operators h_1, h_2 are defined by the formula (6.122). Obviously, the inclusions $(-1)^m h_k, (-1)^{1-m} h_{3-k} \in \mathcal{P}_{ab}$ hold, and the operator h_{3-k} is an a-Volterra one. By virtue of the inequality (6.65), there exist $\varrho_a, \varrho_b \in \,]0, +\infty[$ such that $\varrho_a < \varrho_b$ and

$$\int_{\varrho_a}^{\varrho_b} \frac{\mathrm{d}s}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} = \frac{(b-a)^{1-\lambda}}{1-\lambda}\,.$$

Therefore, according to Lemma 6.35, we can find $\omega_1, \omega_2 \in AC([a,b];\mathbb{R})$ such that $\omega_1', \omega_2' \in C_{loc}([a,b[\,;\mathbb{R})$ and the conditions

$$\omega_1'(t) = \frac{\alpha_2}{(b-t)^\lambda}\,\omega_1(t) + \frac{\alpha_1}{(b-t)^\lambda}\,\omega_2(t) \quad \text{for } t \in [a,b[\,, \tag{6.126}$$

$$\omega_2'(t) = -\frac{\alpha_3}{(b-t)^\lambda}\,\omega_1(t) \qquad \text{for } t \in [a,b[\tag{6.127}$$

and

$$\omega_i(t) > 0 \quad \text{for } t \in [a,b], \quad i = 1,2,$$

are satisfied. Put

$$\gamma_1(t) = \frac{\omega_1(t)}{(b-t)^\nu} \quad \text{for } t \in [a,b[\,, \qquad \gamma_2(t) = \omega_2(t) \quad \text{for } t \in [a,b].$$

It is easy to see that $\gamma_1 \in AC_{loc}([a,b[\,;\mathbb{R})$, $\gamma_2 \in AC([a,b];\mathbb{R})$, and the condition (6.41) holds. Using the equalities (6.126) and (6.127), we get

$$\gamma_1'(t) = \Big(\frac{\nu}{b-t} + \frac{\alpha_2}{(b-t)^\lambda}\Big)\gamma_1(t) + \frac{\alpha_1}{(b-t)^{\lambda+\nu}}\,\gamma_2(t) \quad \text{for } t \in [a,b[\tag{6.128}$$

and

$$\gamma_2'(t) = -\frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,\gamma_1(t) \quad \text{for } t \in [a,b[\,. \tag{6.129}$$

Consequently, it is clear that γ_2' is continuous and non-increasing on the interval $[a,b[$ and

$$\gamma_1'(t) \ge 0, \quad \gamma_2'(t) \le 0 \quad \text{for } t \in [a,b[\,. \tag{6.130}$$

Let the set A_{3-k} be defined by the formula (6.124). If we take the conditions (6.41), (6.64) and (6.128)–(6.130) into account, by direct calculation we obtain

$$\gamma_2\big(\tau_k(t)\big) = \gamma_2(t) + \int\limits_t^{\tau_k(t)} \gamma_2'(s)\,\mathrm{d}s \le \gamma_2(t) + \gamma_2'(t)\big(\tau_k(t) - t\big) =$$
$$= \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,\big(t - \tau_k(t)\big)\gamma_1(t) + \gamma_2(t) \quad \text{for a.e. } t \in [a,b]$$

and

$$-\gamma_1\big(\tau_{3-k}(t)\big) = -\gamma_1(t) + \int\limits_{\tau_{3-k}(t)}^{t} \gamma_1'(s)\,\mathrm{d}s =$$
$$= -\gamma_1(t) + \int\limits_{\tau_{3-k}(t)}^{t} \Big(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^\lambda}\Big)\gamma_1(s)\,\mathrm{d}s + \int\limits_{\tau_{3-k}(t)}^{t} \frac{\alpha_1}{(b-s)^{\lambda+\nu}}\,\gamma_2(s)\,\mathrm{d}s \ge$$
$$\ge -\gamma_1(t) + \gamma_1\big(\tau_{3-k}(t)\big) \int\limits_{\tau_{3-k}(t)}^{t} \Big(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^\lambda}\Big)\,\mathrm{d}s \quad \text{for a.e. } t \in A_{3-k}\,.$$

Therefore, by virtue of the conditions (6.41), (6.63), (6.64), (6.66)–(6.68), (6.128) and (6.129), from the last relations we get

$$(-1)^m f_k(t)\gamma_2\big(\tau_k(t)\big) = |f_k(t)|\gamma_2\big(\tau_k(t)\big) \le$$
$$\le \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,|f_k(t)|\big(t - \tau_k(t)\big)\gamma_1(t) + |f_k(t)|\gamma_2(t) \le$$
$$\le \Big(\frac{\nu}{b-t} + \frac{\alpha_2}{(b-t)^\lambda}\Big)\gamma_1(t) + \frac{\alpha_1}{(b-t)^{\lambda+\nu}}\,\gamma_2(t) = \gamma_1'(t) \quad \text{for a.e. } t \in [a,b]$$

and

$$(-1)^m f_{3-k}(t)\gamma_1\big(\tau_{3-k}(t)\big) = -|f_{3-k}(t)|\gamma_1\big(\tau_{3-k}(t)\big) \ge$$
$$\ge -\frac{|f_{3-k}(t)|}{1 + \int\limits_{\tau_{3-k}(t)}^{t} \Big(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^\lambda}\Big)\,\mathrm{d}s}\,\gamma_1(t) \ge$$
$$\ge -\frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,\gamma_1(t) = \gamma_2'(t) \quad \text{for a.e. } t \in A_{3-k}\,,$$

which, together with the second inequality in (6.130), guarantees that

$$\gamma_1'(t) \ge (-1)^m f_k(t)\gamma_2\big(\tau_k(t)\big) \quad \text{for a.e. } t \in [a,b]$$

and

$$\gamma_2'(t) \le (-1)^m f_{3-k}(t)\gamma_1\big(\tau_{3-k}(t)\big) \quad \text{for a.e. } t \in [a,b],$$

and thus the functions γ_1, γ_2 satisfy the conditions (6.42) and (6.43).

Consequently, the assumptions of Theorem 6.21 are fulfilled. $\square$

Proof of Corollary 6.30. The validity of the corollary follows from Theorem 6.29 with $\alpha_1 = \alpha$, $\alpha_2 = 0$, $\alpha_3 = \beta$ and $k = 2$, $m = 1$ (resp. $k = 1$, $m = 0$). $\square$

Proof of Corollary 6.31. It is clear that the system (6.69) is a particular case of the system (6.33) in which

$$h_1(z)(t) = f_1(t)z\big(\tau_1(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R})$$

and $h_2 = h_{2,0} - h_{2,1}$, where

$$h_{2,i}(z)(t) = f_{2,i}(t)z\big(\tau_{2,i}(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}), \quad i = 0, 1.$$

Obviously, $h_1, h_{2,0}, h_{2,1} \in \mathcal{P}_{ab}$. By virtue of the assumption (a) (resp., (b), resp., (c)) of the corollary, it follows from Theorem 5.25 (resp., Theorem 5.28, resp., Theorem 5.30) that

$$\big(h_1, h_{2,0}\big) \in \widehat{\mathcal{S}}^{2,1}_{ab}(a).$$

On the other hand, in view of the assumption (A) (resp., (B)) of the corollary, Theorem 5.32 (resp., Theorem 5.34) yields that

$$\Big(h_1, -\frac{1}{2}\, h_{2,1}\Big) \in \widehat{\mathcal{S}}^{2,1}_{ab}(a).$$

Consequently, the assumptions of Corollary 6.25 with $k = 1$ and $m = 0$ are satisfied. $\square$

6.3. **Counterexamples.** In this section, we construct counterexamples verifying that some of the results stated above are unimprovable in a certain sense.

Example 6.36. Let $\varepsilon \in \,]0,1[$ and the operator ℓ be defined by the formula

$$\ell(v)(t) = \frac{1}{2(b-a)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} v(b) \quad \text{for } t \in [a,b], \quad v \in C([a,b];\mathbb{R}^2).$$

It is clear that the inequalities (6.2), (6.5) and (6.10) are satisfied on the set $C([a,b];\mathbb{R}^2)$, where $\varphi_1 = 0$,[25]

$$\varphi_0(v)(t) = \frac{1}{2(b-a)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v(b) \quad \text{for } t \in [a,b], \quad v \in C([a,b];\mathbb{R}^2)$$

and

$$\psi_0(z)(t) = \frac{z(b)}{b-a} \quad \text{for } t \in [a,b], \quad z \in C([a,b];\mathbb{R}).$$

Moreover, using Corollary 4.26 (with $\delta_1 = \delta_2 = 1$) and Lemma 4.45, we get the inclusions

$$0 \in \mathcal{S}^2_{ab}(a), \quad (1-\varepsilon)\,\varphi_0 \in \mathcal{S}^2_{ab}(a), \quad (1-\varepsilon)\,\psi_0 \in \mathcal{S}_{ab}(a).$$

[25] The symbol 0 stands here for the zero operator.

On the other hand, the problem (3.1), (3.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1, 2)$ has a nontrivial solution $(u_1, u_2)^T$, where

$$u_1(t) = t - a, \quad u_2(t) = a - t \ \text{ for } \ t \in [a, b].$$

This example shows that the assumptions (6.1), (6.4) and (6.11) of Theorems 6.1, 6.3 and 6.6 cannot be replaced by the assumptions (6.3), (6.6) and (6.13), respectively, no matter how small $\varepsilon > 0$ is.

Example 6.37. Let $\varepsilon \in]0, 1[$, $\alpha \in]0, 1]$, and $a < t_0 < t_1 < b$. Choose functions $p_{ik} \in L([a, b]; \mathbb{R})$ $(i, k = 1, 2)$ such that

$$p_{ik}(t)(t - t_0)(t - t_1) \leq 0 \ \text{ for a.e. } \ t \in [a, b], \quad i, k = 1, 2,$$
$$\sum_{k=1}^{2} \int_a^{t_0} |p_{ik}(s)| \, \mathrm{d}s = \frac{\alpha}{1 + \varepsilon}\,, \qquad \int_{t_1}^{b} |p_{ii}(s)| \, \mathrm{d}s = 2 + \varepsilon \ \text{ for } \ i = 1, 2,$$

$p_{12} \equiv p_{21} \equiv 0$ on the interval $[t_1, b]$, and

$$\sum_{k=1}^{2} \int_{t_0}^{t_1} p_{ik}(s) \, \mathrm{d}s = 1 - \alpha \ \text{ for } \ i = 1, 2.$$

Let the operator ℓ be defined by the formula

$$\ell(v)(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} v\big(\tau(t)\big) \quad \text{for a.e. } \ t \in [a, b] \ \text{ and all } \ v \in C([a, b]; \mathbb{R}^2),$$

where

$$\tau(t) = \begin{cases} b & \text{for } \ t \in [a, t_0[\,, \\ t_1 & \text{for } \ t \in [t_0, b]. \end{cases}$$

It is easy to verify that the inequality (6.10) is satisfied on the set $C([a, b]; \mathbb{R}^2)$, where

$$\varphi_0(v)(t) = \begin{pmatrix} \widetilde{p}_1(t) & |p_{12}(t)| \\ |p_{21}(t)| & \widetilde{p}_2(t) \end{pmatrix} v\big(\tau(t)\big) \quad \text{for a.e. } \ t \in [a, b] \ \text{ and all } \ v \in C([a, b]; \mathbb{R}^2),$$

$$\varphi_1(v)(t) = -\begin{pmatrix} g_1(t) & 0 \\ 0 & g_2(t) \end{pmatrix} v\big(\mu(t)\big) \quad \text{for a.e. } \ t \in [a, b] \ \text{ and all } \ v \in C([a, b]; \mathbb{R}^2),$$

in which

$$\widetilde{p}_i(t) = \begin{cases} |p_{ii}(t)| & \text{for } t \in [a, t_1[\,, \\ \dfrac{1+\varepsilon}{2+\varepsilon}\,|p_{ii}(t)| & \text{for } t \in [t_1, b], \end{cases}$$

$$g_i(t) = \begin{cases} 0 & \text{for } t \in [a, t_1[\,, \\ \dfrac{1+\varepsilon}{2+\varepsilon}\,|p_{ii}(t)| & \text{for } t \in [t_1, b], \end{cases}$$

and

$$\mu(t) = \begin{cases} a & \text{for } t \in [a, t_1[\,, \\ t_1 & \text{for } t \in [t_1, b]. \end{cases}$$

Obviously, $\varphi_0 + \varphi_1 \in \mathcal{P}^2_{ab}$, $-\varphi_1 \in \mathcal{P}^2_{ab}$, and φ_1 is an a-Volterra operator. Moreover,

$$(1-\varepsilon)\int_a^b g_i(s)\,\mathrm{d}s = \frac{1-\varepsilon^2}{2+\varepsilon}\int_{t_1}^b |p_{ii}(s)|\,\mathrm{d}s = 1-\varepsilon^2 < 1 \quad \text{for } i = 1, 2,$$

and thus, by Theorem 4.38(a), we get the inclusion

$$(1-\varepsilon)\varphi_1 \in \mathcal{S}^2_{ab}(a).$$

Furthermore, since

$$\int_a^b \big(\widetilde{p}_i(s) + |p_{i3-i}(s)|\big)\,\mathrm{d}s = \int_a^{t_1} \big(|p_{i1}(s)| + |p_{i2}(s)|\big)\,\mathrm{d}s =$$

$$= \frac{\alpha}{1+\varepsilon} + 1 - \alpha = \frac{1+(1-\alpha)\varepsilon}{1+\varepsilon} < 1$$

for $i = 1, 2$, using Corollary 4.26 (with $\delta_1 = \delta_2 = 1$) we obtain the inclusion

$$\varphi_0 + \varphi_1 \in \mathcal{S}^2_{ab}(a).$$

On the other hand, the problem (3.1), (3.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1, 2)$ has a nontrivial solution $(u_1, u_2)^T$, where

$$u_i(t) = \begin{cases} (1+\varepsilon)\displaystyle\int_a^t |p_{i1}(s)|\,\mathrm{d}s + (1+\varepsilon)\int_a^t |p_{i2}(s)|\,\mathrm{d}s & \text{for } t \in [a, t_0[\,, \\ \alpha + \displaystyle\int_{t_0}^t p_{i1}(s)\,\mathrm{d}s + \int_{t_0}^t p_{i2}(s)\,\mathrm{d}s & \text{for } t \in [t_0, b]. \end{cases}$$

This example shows that the assumption (6.11) of Theorem 6.6 cannot be replaced by the assumption (6.12), no matter how small $\varepsilon > 0$ is.

Example 6.38. Let $\varepsilon_1, \varepsilon_2 \in [0,1[\,$, $\varepsilon_1 + \varepsilon_2 > 0$, and the operators h_1, h_2 be defined by the relations

$$h_1(z)(t) = f_1(t) z\big(\tau(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R})$$

and

$$h_2(z)(t) = f_2(t) z(b) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),$$

where $f_1, f_2 \in L([a,b];\mathbb{R}_+)$ and $\tau\colon [a,b] \to [a,b]$ is a measurable function such that

$$\int_a^b f_1(s) \bigg(\int_a^{\tau(s)} f_2(\xi)\,\mathrm{d}\xi \bigg) \mathrm{d}s = 1.$$

It is clear that $h_1, h_2 \in \mathcal{P}_{ab}$ and, for any $z \in C([a,b];\mathbb{R})$, the inequality (6.46) with $k = 1$ and $m = 0$ is satisfied, where $g_0 = 0$[26] and $g_1 = h_2$. Moreover,

$$(1-\varepsilon_1)(1-\varepsilon_2) \int_a^b f_1(s) \bigg(\int_a^{\tau(s)} f_2(\xi)\,\mathrm{d}\xi \bigg) \mathrm{d}s < 1,$$

and thus, using Theorem 5.30, we get

$$\big((1-\varepsilon_1)\, h_1, (1-\varepsilon_2)\, h_2\big) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a).$$

It is also clear that $(h_1, 0) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a)$ (see, e. g., Theorem 5.30). Consequently, the assumptions of Theorem 6.23 with $k = 1$ and $m = 0$ are satisfied, except the condition (6.45), instead of which the condition (6.47) is fulfilled. On the other hand, the problem (6.33), (6.34) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1, 2)$ has a nontrivial solution $(u_1, u_2)^T$, where

$$u_1(t) = \int_a^t f_1(s) \bigg(\int_a^{\tau(s)} f_2(\xi)\,\mathrm{d}\xi \bigg) \mathrm{d}s, \quad u_2(t) = \int_a^t f_2(s)\,\mathrm{d}s \quad \text{for } t \in [a,b].$$

This example shows that the assumption (6.45) of Theorem 6.23 cannot be replaced by the assumption (6.47), no matter how small $\varepsilon_1, \varepsilon_2 \in [0,1[$ with $\varepsilon_1 + \varepsilon_2 > 0$ are.

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[26] The symbol 0 stands here for the zero operator.

Example 6.39. Let $\alpha \in]0,1[\,$, $\varepsilon_1, \varepsilon_2 \in [0,1[\,$, $\varepsilon_1 + \varepsilon_2 > 0$, and $a < t_1 < t_2 < b$. Put $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and choose $f_1, f_2 \in L([a,b];\mathbb{R})$ such that

$$f_1(t) \geq 0, \quad (t-t_1)(t-t_2)f_2(t) \leq 0 \ \text{ for a.e. } t \in [a,b],$$

$$\int_a^{t_1} f_1(s)\bigg(\int_a^s |f_2(\xi)|\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = \frac{\alpha}{1+\varepsilon}\,,$$

$$\int_{t_1}^{t_2} f_1(s)\bigg((1+\varepsilon)\int_a^{t_1} |f_2(\xi)|\,\mathrm{d}\xi + \int_{t_1}^s f_2(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = 1-\alpha,$$

$$\int_{t_2}^b f_1(s)\,\mathrm{d}s = \varepsilon \min\left\{\frac{\int_{t_1}^{t_2} f_1(s)\,\mathrm{d}s \int_a^{t_1}|f_2(s)|\,\mathrm{d}s}{\int_a^{t_2}|f_2(s)|\,\mathrm{d}s}\,,\ \frac{1}{(1+\varepsilon)\int_a^{t_1}|f_2(s)|\,\mathrm{d}s + \int_{t_1}^{t_2} f_2(s)\,\mathrm{d}s}\right\},$$

and

$$\int_{t_2}^b f_1(s)\bigg(\int_{t_2}^s |f_2(\xi)|\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = 2(1+\varepsilon).$$

Furthermore, we put

$$u_2(t) = \begin{cases} (1+\varepsilon)\displaystyle\int_a^t |f_2(s)|\,\mathrm{d}s & \text{for } t \in [a,t_1[\,, \\ (1+\varepsilon)\displaystyle\int_a^{t_1} |f_2(s)|\,\mathrm{d}s + \int_{t_1}^t f_2(s)\,\mathrm{d}s & \text{for } t \in [t_1,b], \end{cases}$$

and

$$u_1(t) = \int_a^t f_1(s)u_2(s)\,\mathrm{d}s \ \text{ for } t \in [a,b].$$

It is clear that $u_1, u_2 \in AC([a,b];\mathbb{R})$, $u_1(t_2) = 1$ and $u_1(b) \leq -(1+\varepsilon)$, and thus there exists $t_0 \in [t_2, b]$ such that $u_1(t_0) = -(1+\varepsilon)$. Let the operators h_1 and h_2 be defined by the relations

$$h_1(z)(t) = f_1(t)z(t) \ \text{ for a.e. } t \in [a,b] \ \text{ and all } z \in C([a,b];\mathbb{R})$$

and

$$h_2(z)(t) = f_2(t)z\big(\tau(t)\big) \ \text{ for a.e. } t \in [a,b] \ \text{ and all } z \in C([a,b];\mathbb{R}),$$

where

$$\tau(t) = \begin{cases} t_0 & \text{for } t \in [a,t_1[\,, \\ t_2 & \text{for } t \in [t_1,b]. \end{cases}$$

It is not difficult to verify that for any $z \in C([a,b];\mathbb{R})$ the inequality (6.46) with $k=1$ and $m=0$ is satisfied, where

$$g_0(z)(t) = -p_0(t)z\big(\tau_0(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),$$
$$g_1(z)(t) = p_1(t)z\big(\tau(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),$$
$$p_0(t) = \begin{cases} 0 & \text{for } t \in [a,t_2[\,, \\ \dfrac{1}{2}\,|f_2(t)| & \text{for } t \in [t_2,b], \end{cases} \qquad p_1(t) = \begin{cases} |f_2(t)| & \text{for } t \in [a,t_2[\,, \\ \dfrac{1}{2}\,|f_2(t)| & \text{for } t \in [t_2,b], \end{cases}$$

and

$$\tau_0(t) = \begin{cases} a & \text{for } t \in [a,t_2[\,, \\ t_2 & \text{for } t \in [t_2,b]. \end{cases}$$

Obviously, we have $h_1 \in \mathcal{P}_{ab}$ and

$$\big(g_0+g_1\big)(z)(t) = \widetilde{f}(t)z\big(\tau(t)\big) \quad \text{for a. e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}),$$

where

$$\widetilde{f}(t) = \begin{cases} |f_2(t)| & \text{for } t \in [a,t_2[\,, \\ 0 & \text{for } t \in [t_2,b]. \end{cases}$$

Therefore, $g_0+g_1 \in \mathcal{P}_{ab}$ and

$$\begin{aligned} \int_a^b f_1(s)\bigg(\int_a^s \widetilde{f}(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s &= \int_a^{t_1} f_1(s)\bigg(\int_a^s |f_2(\xi)|\,\mathrm{d}\xi\bigg)\,\mathrm{d}s + \\ &+ \int_{t_1}^{t_2} f_1(s)\bigg((1+\varepsilon)\int_a^{t_1} |f_2(\xi)|\,\mathrm{d}\xi + \int_{t_1}^s f_2(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s - \\ &- \bigg(\varepsilon\int_{t_1}^{t_2} f_1(s)\,\mathrm{d}s\int_a^{t_1} |f_2(s)|\,\mathrm{d}s - \int_{t_2}^b f_1(s)\,\mathrm{d}s\int_a^{t_2} |f_2(s)|\,\mathrm{d}s\bigg) \leq \\ &\leq \frac{\alpha}{1+\varepsilon} + 1 - \alpha = \frac{1+\varepsilon(1-\alpha)}{1+\varepsilon} < 1. \end{aligned}$$

Hence, Theorem 5.30 yields that

$$\big(h_1, g_0+g_1\big) \in \widehat{\mathcal{S}}_{ab}^{2,1}(a).$$

Furthermore, $-g_0 \in \mathcal{P}_{ab}$, the operators h_1, g_0 are a-Volterra ones, and since

$$\begin{aligned} (1-\varepsilon_1)(1-\varepsilon_2)\int_a^b f_1(s)\bigg(\int_a^s p_0(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s &\leq \\ &\leq \frac{1-\varepsilon}{2}\int_{t_2}^b f_1(s)\bigg(\int_{t_2}^s |f_2(\xi)|\,\mathrm{d}\xi\bigg)\,\mathrm{d}s = 1-\varepsilon^2 < 1, \end{aligned}$$

using Theorem 5.32 we get

$$\Big((1-\varepsilon_1)\,h_1,(1-\varepsilon_2)\,g_0\Big)\in\widehat{\mathcal{S}}^{2,1}_{ab}(a).$$

Consequently, the assumptions of Theorem 6.23 with $k=1$ and $m=0$ are satisfied, except the condition (6.45), instead of which the condition (6.48) is fulfilled. On the other hand, $(u_1,u_2)^T$ is a nontrivial solution to the problem $(6.33),(6.34)$ with $q_i\equiv 0$ and $c_i=0$ $(i=1,2)$.

This example shows that the assumption (6.45) of Theorem 6.23 cannot be replaced by the assumption (6.48), no matter how small $\varepsilon_1,\varepsilon_2\in[0,1[$ with $\varepsilon_1+\varepsilon_2>0$ are.

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