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S. Kharibegashvili

BOUNDARY VALUE PROBLEMS FOR SOME CLASSES OF NONLINEAR WAVE EQUATIONS

Abstract. For some classes of nonlinear wave equations, the boundary value problems (the first Darboux problem and their multi-dimensional versions, the characteristic Cauchy problem, and so on) are considered in angular and conic domains. Depending on the exponent of nonlinearity and the spatial dimension of equations, the issues of the global and local solvability as well as of the smoothness and uniqueness of solutions of these problems are studied.

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Key words and phrases. Characteristic Cauchy problem, characteristic boundary value problems, Darboux problems, nonlinear wave equations, global and local solvability, blow-up of solutions.

რეზიუმე. არაწრფივი ტალღის განტოლებათა ზოგიერთი კლასისათვის განხილულია სასაზღვრო ამოცანები (დარბუს პირველი ამოცანა და მისი მრავალგანზომილებიანი ვარიანტები, კოშის მახასიათებელი ამოცანა და სხვა) კუთხოვან და კონუსურ არეებში. არაწრფივობის მაჩვენებლის და განტოლების სივრცითი განზომილების მიხედვით შესწავლილია გლობალური და ლოკალური ამოხსნადობის, აგრეთვე ამ ამოცანათა ამონახსნთა სიგლუვის და ერთადერთობის საკითხები.

Introduction

In mathematical modelling of many physical processes there arise wave equations involving nonlinearities which are, in particular, represented by source terms. The Cauchy problem and the mixed problems for these equations have been studied with sufficient thoroughness (see, for e.g., [9], [11], [16], [18], [19], [21], [23], [45], [50]–[53], [60]–[62], [64], [65], [75]–[77]). But as for the boundary value problems for these equations such as, for example, the characteristic Cauchy problems, the Darboux problems in angular and conic domains, arising in mathematical modelling of: (i) small harmonic oscillations of a wedge in a supersonic flow; (ii) string oscillation in a viscous liquid (see [13], [67], [68]), they are at the initial stage of investigation.

The goal of the present work is to fill in this gap to a certain extent. The presence in equations of even weak nonlinearities may violate the correctness of the problems, which may show itself in the destruction of solutions in a finite time interval or the non-existence of solvability or uniqueness of solutions of the problems under consideration.

The work consists of five chapters. In Chapter I we investigate the first Darboux problem for a weakly nonlinear wave equation with one spatial variable when, depending on the type of nonlinearity, the problem is globally solvable in some cases and only locally solvable in other cases. Herein we consider the issues of the uniqueness and smoothness of the solution ([1], [20]).

Chapter II studies the characteristic Cauchy problem for a multidimensional nonlinear wave equation in a light cone of the future. Depending on the exponent of nonlinearity and the spatial dimension of the equation, we investigate the issues of the global and local solvability of the problem ([25]–[27], [29], [32]).

Chapter III is devoted to Sobolev's problem for a multidimensional nonlinear wave equation in a conic domain of time type, while in Chapter IV we consider multidimensional versions of the first Darboux problem ([6], [28], [30]).

Finally, the last Chapter V studies the characteristic boundary value problems for a multidimensional hyperbolic equation with power nonlinearity and the iterated wave operator in the principal part. Depending on the exponent of nonlinearity and the spatial dimension of the equation, we investigate the issues on the existence and uniqueness of solutions of the boundary value problems ([31], [33]).

When investigating the above-mentioned problems, the use will be made of the classical methods of characteristics and integral equations, as well as the methods of the modern nonlinear analysis (the method of a priori estimates, the Schauder and Leray–Schauder fixed point principles and the principle of contracting mappings, the method of test-functions, embedding theorems, etc.).

Note that the problems we consider in the present work for linear wave equations are well-posed in the corresponding function spaces ([2]–[8], [12], [14], [17], [24], [25], [34], [54], [55], [57], [58], [63], [70], [71]).

CHAPTER 1

The First Darboux Problem for a Weakly Nonlinear Wave Equation with One Spatial Variable

1. Statement of the Problem

In the plane of the variables x and t we consider a nonlinear wave equation of the type

$$L_f u := u_{tt} - u_{xx} + f(x,t,u) = F(x,t), \tag{1.1}$$

where $f = f(x,t,u)$ is a given nonlinear with respect to u real function, $F = F(x,t)$ is a given and $u = u(x,t)$ is an unknown real function.

By $D_T: \ -kt < x < t,\ 0 < t < T\ (0 < k = \text{const} < 1,\ T \le \infty)$ we denote a triangular domain lying inside the characteristic angle $\big\{(x,t) \in \mathbb{R}^2 : \ t > |x|\big\}$ and bounded by the characteristic segment $\gamma_{1,T}: \ x = t$, $0 \le t \le T$, and the segments $\gamma_{2,T}: \ x = -kt$, $0 \le t \le T$ and $\gamma_{3,T}: \ t = T$, $-kT \le x \le T$ of time and spatial type, respectively.

For the equation (1.1), we consider the first Darboux problem: find in the domain D_T a solution $u(x,t)$ of that equation according to the boundary conditions [**2**, p. 228]

$$u\big|_{\gamma_{i,T}} = \varphi_i, \ \ i = 1,2, \tag{1.2}$$

where φ_i, $i = 1,2$, are given real functions satisfying the compatibility condition $\varphi_1(O) = \varphi_2(O)$ at the common point $O = O(0,0)$.

Remark 1.1. Below it will be assumed that the functions $f : \overline{D}_T \times \mathbb{R} \to \mathbb{R}$ and $F : \overline{D}_T \to \mathbb{R}$ are continuous. Moreover, and without restriction of generality we may assume that

$$f(x,t,0) = 0, \ \ (x,t) \in \overline{D}_T.$$

Definition 1.1. Let $f \in C(\overline{D}_T \times \mathbb{R})$, $F \in C(\overline{D}_T)$ and $\varphi_i \in C^1(\gamma_{i,T})$, $i = 1,2$. A function u is said to be a strong generalized solution of the problem (1.1), (1.2) of the class C in the domain D_T, if $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in C^2(\overline{D}_T)$ such that $u_n \to u$ and $L_f u_n \to F$ in the space $C(\overline{D}_T)$ and $u_n|_{\gamma_{i,T}} \to \varphi_i$ in the space $C^1(\gamma_{i,T})$, $i = 1,2$.

Remark 1.2. Obviously, a classical solution of the problem (1.1), (1.2) from the space $C^2(\overline{D}_T)$ is a strong generalized solution of that problem of the class C in the domain D_T. In its turn, if a strong generalized solution of the problem (1.1), (1.2) of the class C in the domain D_T belongs to the space $C^2(\overline{D}_T)$, then that solution will also be a classical solution of that problem. It should be noted that a strong generalized solution of the problem (1.1), (1.2) of the class C in the domain D_T satisfies the boundary conditions (1.2) in the usual classical sense.

Definition 1.2. Let $f \in C(\overline{D}_\infty \times \mathbb{R})$, $F \in C(\overline{D}_\infty)$ and $\varphi_i \in C^1(\gamma_{i,\infty})$, $i = 1, 2$. We say that the problem (1.1), (1.2) is globally solvable in the class C if for every finite $T > 0$ this problem has a strong generalized solution of the class C in the domain D_T.

2. An a Priori Estimate of a Solution of the Problem (1.1), (1.2)

Let

$$g(x,t,u) = \int_0^u f(x,t,s)\,ds, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}. \tag{2.1}$$

Consider the following conditions imposed on the function $g = g(x,t,u)$ from (2.1):

$$g(x,t,u) \geq -M_1 - M_2 u^2, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{2.2}$$

$$g_t(x,t,u) \leq M_3 + M_4 u^2, \quad (x,t,u) \in \overline{D}_T \times \mathbb{R}, \tag{2.3}$$

where $M_i = M_i(T) = \text{const} \geq 0$, $i = 1, 2, 3, 4$.

Lemma 2.1. *Let f, $f_u \in C(\overline{D}_\infty \times \mathbb{R})$, $F \in C(\overline{D}_T)$, $\varphi_i \in C^1(\gamma_{i,T})$, $i = 1, 2$, and the conditions* (2.2) *and* (2.3) *be fulfilled. Then for a strong generalized solution $u = u(x,t)$ of the problem* (1.1), (1.2) *of the class C in the domain D_T the a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \leq c_1 \Big(\|F\|_{C(\overline{D}_T)} + \sum_{i=1}^{2} \|\varphi_i\|_{C^1(\gamma_{i,T})} \Big) + c_2 \tag{2.4}$$

is valid with nonnegative constants $c_i = c_i(f,T)$, $i = 1, 2$, not depending on u and F, φ_1, φ_2, where $c_1 > 0$.

Proof. Let u be a strong generalized solution of the problem (1.1), (1.2) of the class C in the domain D_T. By Definition 1.1, there exists a sequence of functions $u_n \in C^2(\overline{D}_T)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|L_f u_n - F\|_{C(\overline{D}_T)} = 0, \tag{2.5}$$

$$\lim_{n\to\infty} \big\| u_n\big|_{\gamma_{i,T}} - \varphi_i \big\|_{C^1(\gamma_{i,T})} = 0, \tag{2.6}$$

and hence

$$\lim_{n\to\infty} \big\| f(x,t,u_n) - f(x,t,u) \big\|_{C(\overline{D}_T)} = 0. \tag{2.7}$$

Consider the function $u_n \in C^2(\overline{D}_T)$ as a solution of the problem

$$L_f u_n = F_n, \tag{2.8}$$

$$u_n\big|_{\gamma_{i,T}} = \varphi_{in}, \quad i = 1, 2. \tag{2.9}$$

Here

$$F_n := L_f u_n. \tag{2.10}$$

Multiplying both parts of the equation (2.8) by $\frac{\partial u_n}{\partial t}$ and integrating over the domain $D_\tau := \big\{(x,t) \in D_T : \ t < \tau\big\}$, $0 < \tau \le T$, by virtue of (2.1) we obtain

$$\frac{1}{2} \int\limits_{D_\tau} \frac{\partial}{\partial t} \Big(\frac{\partial u_n}{\partial t}\Big)^2 \, dx\, dt - \int\limits_{D_\tau} \frac{\partial^2 u_n}{\partial x^2} \, \frac{\partial u_n}{\partial t} \, dx\, dt +$$
$$+ \int\limits_{D_\tau} \frac{\partial}{\partial t} \big(g(x,t,u_n(x,t)\big) \, dx\, dt - \int\limits_{D_\tau} g_t\big(x,t,u_n(x,t)\big) \, dx\, dt =$$
$$= \int\limits_{D_\tau} F_n \, \frac{\partial u_n}{\partial t} \, dx\, dt. \tag{2.11}$$

Let $\Omega_\tau := D_\infty \cap \{t = \tau\}$, $0 < \tau \le T$. Then, taking into account the equalities (2.9) and integrating by parts the left-hand side of the equality (2.11), we obtain

$$\int\limits_{D_\tau} F_n \, \frac{\partial u_n}{\partial t} \, dx\, dt =$$
$$= \sum_{i=1}^{2} \int\limits_{\gamma_{i,T}} \frac{1}{2\nu_t} \bigg[\Big(\frac{\partial u_n}{\partial x}\, \nu_t - \frac{\partial u_n}{\partial t}\, \nu_x\Big)^2 + \Big(\frac{\partial u_n}{\partial t}\Big)^2 (\nu_t^2 - \nu_x^2)\bigg] \, ds +$$
$$+ \frac{1}{2} \int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_n}{\partial t}\Big)^2 + \Big(\frac{\partial u_n}{\partial x}\Big)^2\bigg] \, dx + \int\limits_{\Omega_\tau} g\big(x,\tau,u_n(x,\tau)\big) \, dx +$$
$$+ \sum_{i=1}^{2} \int\limits_{\gamma_{i,T}} g\big(x,t,\varphi_{in}(x,t)\big)\nu_t \, ds - \int\limits_{D_\tau} g_t\big(x,t,u_n(x,t)\big) \, dx\, dt, \tag{2.12}$$

where $\nu = (\nu_x, \nu_t)$ is the unit vector of the outer normal to ∂D_τ, $\gamma_{i,\tau} := \gamma_{i,T} \cap \{t \le \tau\}$.

Since $\big(\nu_t \frac{\partial}{\partial x} - \nu_x \frac{\partial}{\partial t}\big)$ is an inner differential operator on $\gamma_{i,\tau}$, owing to (2.9) we have

$$\bigg|\Big(\frac{\partial u_n}{\partial x}\, \nu_t - \frac{\partial u_n}{\partial t}\, \nu_x\Big)\Big|_{\gamma_{i,\tau}}\bigg| \le \|\varphi_{in}\|_{C^1(\gamma_{i,\tau})}, \quad i = 1, 2. \tag{2.13}$$

Taking into account that $D_\tau: \ -kt < x < t, 0 < t < \tau$, where $0 < k < 1$, it can be easily seen that

$$\begin{gathered}
(\nu_x,\nu_t)\big|_{\gamma_{1,\tau}} = \Big(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\Big),\\
(\nu_x,\nu_t)\big|_{\gamma_{2,\tau}} = \Big(-\frac{1}{\sqrt{1+k^2}},-\frac{1}{\sqrt{1+k^2}}\Big),
\end{gathered} \tag{2.14}$$

$$\begin{gathered}
(\nu_t^2-\nu_x^2)\big|_{\gamma_{1,\tau}} = 0, (\nu_t^2-\nu_x^2)\big|_{\gamma_{2,\tau}} = \frac{k^2-1}{k^2+1} < 0,\\
\nu_t\big|_{\gamma_{i,\tau}} < 0, \ \ i=1,2.
\end{gathered} \tag{2.15}$$

Due to the Cauchy inequality, by (2.2), (2.3), (2.13), (2.14) and (2.15) it follows from (2.12) that

$$\begin{gathered}
\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_n}{\partial t}\Big)^2 + \Big(\frac{\partial u_n}{\partial x}\Big)^2\bigg]\,dx = \\
= -\sum_{i=1}^{2}\int\limits_{\gamma_{i,T}} \frac{1}{\nu_t}\bigg[\Big(\frac{\partial u_n}{\partial x}\,\nu_t - \frac{\partial u_n}{\partial t}\,\nu_x\Big)^2 + \Big(\frac{\partial u_n}{\partial t}\Big)^2(\nu_t^2-\nu_x^2)\bigg]\,ds- \\
-2\int\limits_{\Omega_\tau} g\big(x,\tau,u_n(x,\tau)\big)\,dx - 2\sum_{i=1}^{2}\int\limits_{\gamma_{i,\tau}} g\big(x,t,\varphi_{in}(x,t)\big)\nu_t\,ds+ \\
+2\int\limits_{D_\tau} g_t\big(x,t,u_n(x,t)\big)\,dx\,dt + 2\int\limits_{D_\tau} F_n\,\frac{\partial u_n}{\partial t}\,dx\,dt \le \\
\le \sqrt{2}\int\limits_{\gamma_{1,\tau}} \|\varphi_{1n}\|^2_{C^1(\gamma_{1,T})}\,ds + \frac{\sqrt{1+k^2}}{k}\int\limits_{\gamma_{2,\tau}} \|\varphi_{2n}\|^2_{C^1(\gamma_{2,T})}\,ds+ \\
+2\sum_{i=1}^{2}\int\limits_{\gamma_{i,\tau}} \big(M_1+M_2\varphi_{in}^2(x,t)\big)\,ds + 2\int\limits_{\Omega_\tau} \big(M_1+M_2u_n^2(x,\tau)\big)\,dx+ \\
+2\int\limits_{D_\tau} \big(M_3+M_4u_n^2(x,t)\big)\,dx\,dt + 2\int\limits_{D_\tau} F_n\,\frac{\partial u_n}{\partial t}\,dx\,dt \le \\
\le M_5 + M_6\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})} + M_7\int\limits_{\Omega_\tau} u_n^2\,dx + M_8\int\limits_{D_\tau} u_n^2\,dx\,dt+ \\
+\int\limits_{D_\tau} \Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt + \int\limits_{D_\tau} F_n^2\,dx\,dt,
\end{gathered} \tag{2.16}$$

where we have used the fact that $\|\varphi_{in}\|_{C(\gamma_{i,T})} \le \|\varphi_{in}\|_{C^1(\gamma_{i,T})}$.

Here

$$
\begin{gathered}
M_5=2M_1\Big(\sum_{i=1}^{2}\operatorname{mes}\gamma_{i,T}+\operatorname{mes}\Omega_T\Big)+2M_3\operatorname{mes}D_T,\\
M_6=\sqrt{2}\,\operatorname{mes}\gamma_{1,T}+\frac{\sqrt{1+k^2}}{k}\,\operatorname{mes}\gamma_{2,T}+2M_2\sum_{i=1}^{2}\operatorname{mes}\gamma_{i,T},\\
M_7=2M_2,\quad M_8=2M_4.
\end{gathered}
\tag{2.17}
$$

Since $\gamma_{1,\tau}:\ t=x,\ 0\le x\le\tau$ and $\gamma_{2,\tau}:\ t=-\frac{1}{k}\,x,\ -k\tau\le x\le 0$, by virtue of (2.9) and the Newton–Leibnitz formula we have

$$
\begin{gathered}
u_n(x,\tau)=\varphi_{2n}(x)+\int\limits_{-\frac{1}{k}\,x}^{\tau}\frac{\partial u_n(x,\sigma)}{\partial t}\,d\sigma,\quad -k\tau\le x\le 0,\\
u_n(x,\tau)=\varphi_{1n}(x)+\int\limits_{x}^{\tau}\frac{\partial u_n(x,\sigma)}{\partial t}\,d\sigma,\quad 0\le x\le \tau.
\end{gathered}
\tag{2.18}
$$

Using the Cauchy and Schwartz inequalities, from (2.18) we get

$$
\begin{aligned}
u_n^2(x,\tau)&\le 2\varphi_{2n}^2(x)+2\Big(\int\limits_{-\frac{1}{k}\,x}^{\tau}\frac{\partial u_n(x,\sigma)}{\partial t}\,d\sigma\Big)^2\le\\
&\le 2\varphi_{2n}^2(x)+2\int\limits_{-\frac{1}{k}\,x}^{\tau}1^2\,d\sigma\int\limits_{-\frac{1}{k}\,x}^{\tau}\Big(\frac{\partial u_n(x,\sigma)}{\partial t}\Big)^2\,d\sigma\le\\
&\le 2\varphi_{2n}^2(x)+2T\int\limits_{-\frac{1}{k}\,x}^{\tau}\Big(\frac{\partial u_n(x,\sigma)}{\partial t}\Big)^2\,d\sigma
\end{aligned}
\tag{2.19}
$$

for $-k\tau\le x\le 0$. Analogously, for $0\le x\le\tau$ from (2.18) we have

$$
u_n^2(x,\tau)\le 2\varphi_{1n}^2(x)+2T\int\limits_{x}^{\tau}\Big(\frac{\partial u_n(x,\sigma)}{\partial t}\Big)^2\,d\sigma. \tag{2.20}
$$

It follows from (2.19) and (2.20) that

$$
\begin{aligned}
\int\limits_{\Omega_\tau}u_n^2\,dx&=\int\limits_{\Omega_\tau\cap\{x\le 0\}}u_n^2\,dx+\int\limits_{\Omega_\tau\cap\{x>0\}}u_n^2\,dx\le\\
&\le\int\limits_{\Omega_\tau\cap\{x\le 0\}}\Big[2\varphi_{2n}^2(x)+2T\int\limits_{-\frac{1}{k}\,x}^{\tau}\Big(\frac{\partial u_n(x,\sigma)}{\partial t}\Big)^2\,d\sigma\Big]\,dx+
\end{aligned}
$$

$$+ \int\limits_{\Omega_\tau \cap \{x>0\}} \Big[2\varphi_{1n}^2(x) + 2T \int\limits_x^\tau \Big(\frac{\partial u_n(x,\sigma)}{\partial t} \Big)^2 d\sigma \Big] dx \le$$

$$\le 2T \sum_{i=1}^2 \|\varphi_{in}\|^2_{C(\gamma_{i,T})} + 2T \int\limits_{D_\tau} \Big(\frac{\partial u_n}{\partial t} \Big)^2 dx\,dt. \quad (2.21)$$

By (2.21), we have

$$\int\limits_{D_\tau} u_n^2\,dx\,dt = \int\limits_0^\tau d\sigma \int\limits_{\Omega_\sigma} u_n^2\,dx \le$$

$$\le \int\limits_0^\tau \Big[2T \sum_{i=1}^2 \|\varphi_{in}\|^2_{C(\gamma_{i,T})} + 2T \int\limits_{D_\sigma} \Big(\frac{\partial u_n}{\partial t} \Big)^2 dx\,dt \Big] d\sigma \le$$

$$\le 2T^2 \Big[\sum_{i=1}^2 \|\varphi_{in}\|^2_{C(\gamma_{i,T})} + \int\limits_{D_\tau} \Big(\frac{\partial u_n}{\partial t} \Big)^2 dx\,dt \Big]. \quad (2.22)$$

Taking into account (2.21), (2.22) and the fact that $\|\varphi_{in}\|_{C(\gamma_{i,T})} \le \|\varphi_{in}\|_{C^1(\gamma_{i,T})}$, from (2.16) we obtain

$$\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial u_n}{\partial t} \Big)^2 + \Big(\frac{\partial u_n}{\partial x} \Big)^2 \Big] dx \le M_5 + M_9 \sum_{i=1}^2 \|\varphi_{in}\|^2_{C^1(\gamma_{i,T})} +$$

$$+ M_{10} \int\limits_{D_\tau} \Big(\frac{\partial u_n}{\partial t} \Big)^2 dx\,dt + \int\limits_{D_\tau} F_n^2\,dx\,dt, \quad (2.23)$$

where

$$M_9 = M_6 + 2TM_7 + 2T^2M_8, \quad M_{10} = 2TM_7 + 2T^2M_8 + 1. \quad (2.24)$$

Putting

$$w(\tau) = \int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial u_n}{\partial t} \Big)^2 + \Big(\frac{\partial u_n}{\partial x} \Big)^2 \Big] dx \quad (2.25)$$

and taking into account that

$$\int\limits_{D_\tau} \Big(\frac{\partial u_n}{\partial t} \Big)^2 dx\,dt = \int\limits_0^\tau d\sigma \int\limits_{\Omega_\sigma} \Big(\frac{\partial u_n}{\partial t} \Big)^2 dx,$$

from (2.23) we have

$$w(\tau) \le M_{10} \int\limits_0^\tau w(\sigma)\,d\sigma + M_5 +$$

$$+(M_9+1)\Big(\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\int\limits_{D_T}\|F_n\|^2_{C(\overline{D}_T)}\,dx\,dt\Big)\leq$$

$$\leq M_{10}\int\limits_0^\tau w(\sigma)\,d\sigma+M_5+$$

$$+(M_9+1)\Big(\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\|F_n\|^2_{C(\overline{D}_T)}\operatorname{mes}D_T\Big)\leq$$

$$\leq M_{10}\int\limits_0^\tau w(\sigma)\,d\sigma+M_{11}\Big(\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\|F_n\|^2_{C(\overline{D}_T)}\Big)+M_5, \quad (2.26)$$

where

$$M_{11}=(M_9+1)\max(1,\operatorname{mes}D_T). \quad (2.27)$$

By Gronwall's lemma [**15**, p. 13], from (2.26) we find that

$$w(\tau)\leq\Big[M_{11}\Big(\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\|F_n\|^2_{C(\overline{D}_T)}\Big)+M_5\Big]\exp M_{10}\tau. \quad (2.28)$$

If $(x,t)\in\overline{D}_T$, then owing to (2.9) the equality

$$u_n(x,t)=u_n(-kt,t)+\int\limits_{-kt}^{x}\frac{\partial u_n(\sigma,t)}{\partial x}\,d\sigma=\varphi_{2n}(t)+\int\limits_{-kt}^{x}\frac{\partial u_n(\sigma,t)}{\partial x}\,d\sigma$$

holds, whence with regard for (2.25), (2.28) and the Cauchy and Schwartz inequalities we obtain

$$|u_n(x,t)|^2\leq 2\varphi^2_{2n}(t)+2\Big(\int\limits_{-kt}^{x}\frac{\partial u_n(\sigma,t)}{\partial x}\,d\sigma\Big)^2\leq$$

$$\leq 2\|\varphi_{2n}\|^2_{C(\gamma_{2,T})}+2\int\limits_{-kt}^{x}1^2\,d\sigma\int\limits_{-kt}^{x}\Big(\frac{\partial u_n(\sigma,t)}{\partial x}\Big)^2\,d\sigma\leq$$

$$\leq 2\|\varphi_{2n}\|^2_{C^1(\gamma_{2,T})}+2(x+kt)\int\limits_{\Omega_t}\Big(\frac{\partial u_n(\sigma,t)}{\partial x}\Big)^2\,d\sigma\leq$$

$$\leq 2\|\varphi_{2n}\|^2_{C^1(\gamma_{2,T})}+2(1+k)tw(t)\leq 2\|\varphi_{2n}\|^2_{C^1(\gamma_{2,T})}+$$

$$\leq 2(1+k)T\Big[M_{11}\Big(\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\|F_n\|^2_{C(\overline{D}_T)}\Big)+M_5\Big]\exp M_{10}T. \quad (2.29)$$

Taking into account (2.17), (2.24), (2.27) and using the obvious inequality $\big(\sum\limits_{i=1}^{n} a_i^2\big)^{1/2} \le \sum\limits_{i=1}^{n} |a_i|$, from (2.29) we find

$$\|u_n\|_{C(\overline{D}_T)} \le c_1\Big(\|F_n\|_{C(\overline{D}_T)} + \sum_{i=1}^{2} \|\varphi_{in}\|_{C^1(\gamma_{i,T})}\Big) + c_2 \tag{2.30}$$

with nonnegative constants $c_i = c_i(f,T)$, $i=1,2$, not depending on u_n, F_n, φ_{1n} and φ_{2n}; here $c_1>0$. Finally, owing to (2.5)–(2.10) and passing in the inequality (2.30) to limit as $n\to\infty$, we obtain the a priori estimate (2.4). □

3. Reduction of the Problem (1.1), (1.2) to a Nonlinear Integral Equation of Volterra Type

Let $P=P(x,t)$ be an arbitrary point of the domain D_T. By $G_{x,t}$ we denote the characteristic quadrangle with the vertices at the point $P(x,t)$ as well as at the points P_1 and P_2, P_3 lying, respectively, on the supports of the data $\gamma_{1,T}$ and $\gamma_{2,T}$ of the problem (1.1), (1.2), i.e.,

$$\begin{aligned}
&P_1 := P_1\Big(\frac{k(x-t)}{k+1}, \frac{t-x}{k+1}\Big),\\
&P_2 := P_2\Big(\frac{1}{2}\,\frac{1-k}{1+k}\,(t-x), \frac{1}{2}\,\frac{1-k}{1+k}\,(t-x)\Big),\\
&P_3 := P_3\Big(\frac{x+t}{2}, \frac{x+t}{2}\Big).
\end{aligned}$$

Let $u\in C^2(\overline{D}_T)$ be a classical solution of the problem (1.1), (1.2). Integrating the equality (1.1) with respect to the domain $G_{x,t}$ which is the characteristic quadrangle of that equation and using the boundary conditions (1.2), we can easily get the following equality [**1**]; [**2**, p. 65]:

$$\begin{aligned}
&u(x,t) + \frac{1}{2}\int\limits_{G_{x,t}} f\big(x',t',u(x',t')\big)\,dx'\,dt' =\\
&= \varphi_2(P_1) + \varphi_1(P_3) - \varphi_1(P_1) + \frac{1}{2}\int\limits_{G_{x,t}} F(x',t')\,dx'\,dt', \quad (x,t)\in D_T.
\end{aligned} \tag{3.1}$$

Remark 3.1. The equality (3.1) can be considered as a nonlinear integral equation of Volterra type which we rewrite in the form

$$\begin{aligned}
&u(x,t) + \big(L_0^{-1} f\big|_{u=u(x,t)}\big)(x,t) =\\
&= \big(\ell_0^{-1}(\varphi_1,\varphi_2)\big)(x,t) + (L_0^{-1}F)(x,t), \quad (x,t)\in D_T.
\end{aligned} \tag{3.2}$$

Here L_0^{-1} and ℓ_0^{-1} are the linear operators acting by the formulas

$$(L_0^{-1}v)(x,t) = \frac{1}{2}\int\limits_{G_{x,t}} v(x',t')\,dx'\,dt', \tag{3.3}$$

$$\big(\ell_0^{-1}(\varphi_1,\varphi_2)\big)(x,t) = \varphi_2(P_1) + \varphi_1(P_3) - \varphi_1(P_1). \tag{3.4}$$

Note that $L_0^{-1}v$ $(\ell_0^{-1}(\varphi_1,\varphi_2))$ from (3.3), (3.4) is a solution of the corresponding to (1.1), (1.2) homogeneous linear problem, i.e., for $f=0$, when $F=v$, $\varphi_1=\varphi_2=0$ $(F=0)$. Moreover, $L_0^{-1}v \in C^{k+1}(\overline{D}_T)$ if $v\in C^k(\overline{D}_T)$ and $\ell_0^{-1}(\varphi_1,\varphi_2)\in C^k(\overline{D}_T)$ for $\varphi_i\in C^k(\gamma_{i,T})$, $i=1,2$; $k=0,1,2,\dots$.

Lemma 3.1. *Let $f\in C^1(\overline{D}_T\times\mathbb{R})$. The function $u\in C(\overline{D}_T)$ is a strong generalized solution of the problem* (1.1), (1.2) *of the class C in the domain D_T if and only if it is a continuous solution of the nonlinear integral equation* (3.2).

Proof. Indeed, let $u\in C(\overline{D}_T)$ be a solution of the equation (3.2). Since $F\in C(\overline{D}_T)$ $(\varphi_i\in C^1(\gamma_{i,T}))$ and the space $C^2(\overline{D}_T)$ $(C^2(\gamma_{i,T}))$ is dense in $C(\overline{D}_T)$ $(C^1(\gamma_{i,T}))$ [**56**, p. 37], there exists a sequence of functions $F_n\in C^2(\overline{D}_T)$ $(\varphi_{in}\in C^2(\gamma_{in}))$ such that $\lim\limits_{n\to\infty}\|F_n-F\|_{C(\overline{D}_T)}=0$ $(\lim\limits_{n\to\infty}\|\varphi_{in}-\varphi_i\|_{C^1(\gamma_{i,T})}=0$, $i=1,2)$. Analogously, since $u\in C(\overline{D}_T)$, there exists a sequence of functions $w_n\in C^2(\overline{D}_T)$ such that $w_n\to u$ in the space $C(\overline{D}_T)$. Assume

$$u_n = -L_0^{-1}f\big|_{u=w_n} + \ell_0^{-1}(\varphi_{1n},\varphi_{2n}) + L_0^{-1}F_n. \tag{3.5}$$

Since $f\in C^1(\overline{D}_T\times\mathbb{R})$, according to Remark 3.1 we have $u_n\in C^2(\overline{D}_T)$ and $u_n|_{\gamma_{i,T}}=\varphi_{in}$, $i=1,2$. Taking now into account that the linear operators $L_0^{-1}: C(\overline{D}_T)\to C(\overline{D}_T)$ and $\ell_0^{-1}: C^1(\gamma_{1,T})\times C^1(\gamma_{2,T})\to C(\overline{D}_T)$ are continuous and that by our assumption

$$\lim_{n\to\infty}\|w_n-u\|_{C(\overline{D}_T)} = \lim_{n\to\infty}\|F_n-F\|_{C(\overline{D}_T)} = \\ = \lim_{n\to\infty}\|\varphi_{in}-\varphi_i\|_{C^1(\gamma_{i,T})} = 0, \tag{3.6}$$

by virtue of (3.5) we have

$$u_n(x,t)\longrightarrow\Big[-\big(L_0^{-1}f\big|_{u=u(x,t)}\big)(x,t)+\big(\ell_0^{-1}(\varphi_1,\varphi_2)\big)(x,t)+(L_0^{-1}F)(x,t)\Big]$$

in the space $C(\overline{D}_T)$. But it follows from the equality (3.2) that

$$-\big(L_0^{-1}f\big|_{u=u(x,t)}\big)(x,t) + \big(\ell_0^{-1}(\varphi_1,\varphi_2)\big)(x,t) + (L_0^{-1}F)(x,t) = u(x,t).$$

Thus we have

$$\lim_{n\to\infty}\|u_n-u\|_{C(\overline{D}_T)} = 0. \tag{3.7}$$

On the other hand, by Remark 3.1 and (3.5), we have

$$L_0 u_n = -f\big|_{u=w_n} + F_n, \tag{3.8}$$

$$u_n\big|_{\gamma_{i,T}} = \varphi_{in}, \quad i = 1, 2. \tag{3.9}$$

From (3.6)–(3.9) it follows $\lim\limits_{n\to\infty} \|u_n|_{\gamma_{i,T}} - \varphi_i\|_{C^1(\gamma_{i,T})} = 0$, $i = 1, 2$, and since

$$\begin{aligned} L_f u_n = L_0 u_n + f\big|_{u=u_n} &= -f\big|_{u=w_n} + F_n + f\big|_{u=u_n} = \\ &= -\big(f(\cdot\,, w_n) - f(\cdot\,, u)\big) + \big(f(\cdot\,, u_n) - f(\cdot\,, u)\big) + F_n, \end{aligned}$$

we have $L_f u_n \to F$ in the space $C(\overline{D}_T)$, as $n \to \infty$. The converse is obvious. □

4. Global Solvability of the Problem (1.1), (1.2) in the Class of Continuous Functions

As is mentioned above, L_0^{-1} from (3.3) is a linear continuous operator acting in the space $C(\overline{D}_T)$. Let us show that this operator acts in fact linearly and continuously from the space $C(\overline{D}_T)$ to the space of continuously differentiable functions $C^1(\overline{D}_T)$. Towards this end, by means of the linear non-singular transformation of independent variables $t = \xi + \eta$, $x = \xi - \eta$ we pass to the plane of the variables ξ, η. As a result, the triangular domain D_T transforms into the triangle $\widetilde{D}_T$ with vertices at the points $O(0,0)$, $N_1(T,0)$, $N_2\big(\frac{1-k}{2}\,T, \frac{1+k}{2}\,T\big)$, and the characteristic quadrangle $G_{x,t}$ from the previous section transforms into the rectangle $\widetilde{G}_{x,t}$ with the vertices $\widetilde{P}\big(\frac{t+x}{2}, \frac{t-x}{2}\big)$, $\widetilde{P}_1\big(\frac{1}{2}\,\frac{1-k}{1+k}\,(t-x), \frac{t-x}{2}\big)$, $\widetilde{P}_2\big(\frac{1}{2}\,\frac{1-k}{1+k}\,(t-x), 0\big)$, $\widetilde{P}_3\big(\frac{t+x}{2}, 0\big)$, i.e., in the variables ξ, η: $\widetilde{P}(\xi,\eta)$, $\widetilde{P}_1\big(\frac{1-k}{1+k}\,\eta, \eta\big)$, $\widetilde{P}_2\big(\frac{1-k}{1+k}\,\eta, 0\big)$ and $\widetilde{P}_3(\xi, 0)$. Moreover, the operator L_0^{-1} from (3.3) transforms into the operator $\widetilde{L}_0^{-1}$ acting in the space $C(\overline{\widetilde{D}}_T)$ by the formula

$$\begin{aligned} (\widetilde{L}_0^{-1} w)(\xi,\eta) &= \int\limits_{\widetilde{G}_{x,t}} w(\xi',\eta')\, d\xi'\, d\eta' = \\ &= \int\limits_{\frac{1-k}{1+k}\,\eta}^{\xi} d\xi' \int\limits_0^{\eta} w(\xi',\eta')\, d\xi'\, d\eta', \quad (\xi,\eta) \in \widetilde{D}_T. \end{aligned} \tag{4.1}$$

If $w \in C(\overline{\widetilde{D}}_T)$, then it immediately follows from (4.1) that

$$\frac{\partial}{\partial \xi}\, (\widetilde{L}_0^{-1} w)(\xi,\eta) = \int\limits_0^{\eta} w(\xi,\eta')\, d\eta', \tag{4.2}$$

$$\frac{\partial}{\partial\eta}\,(\widetilde{L}_0^{-1}w)(\xi,\eta)=-\frac{1-k}{1+k}\int\limits_0^{\eta}w\Big(\xi',\frac{1-k}{1+k}\,\eta\Big)\,d\xi'+\int\limits_{\frac{1-k}{1+k}\,\eta}^{\xi}w(\xi',\eta')\,d\xi'. \quad (4.3)$$

Taking now into account that for $(\xi,\eta)\in\widetilde{D}_T$ we have $0\le\xi\le T$ and $0\le\eta\le\frac{1+k}{2}\,T$, by virtue of (4.1), (4.2), (4.3) and the fact that $0<k<1$ we have

$$\|\widetilde{L}_0^{-1}w\|_{C(\overline{\widetilde{D}}_T)}+\Big\|\frac{\partial}{\partial\xi}\,\widetilde{L}_0^{-1}w\Big\|_{C(\overline{\widetilde{D}}_T)}+\Big\|\frac{\partial}{\partial\eta}\,\widetilde{L}_0^{-1}w\Big\|_{C(\overline{\widetilde{D}}_T)}\le$$
$$\le\Big(\xi-\frac{1-k}{1+k}\,\eta\Big)\eta\|w\|_{C(\overline{\widetilde{D}}_T)}+\eta\|w\|_{C(\overline{\widetilde{D}}_T)}+\frac{1-k}{1+k}\,\eta\|w\|_{C(\overline{\widetilde{D}}_T)}+$$
$$+\Big(\xi-\frac{1-k}{1+k}\,\eta\Big)\|w\|_{C(\overline{\widetilde{D}}_T)}\le(T^2+3T)\|w\|_{C(\overline{\widetilde{D}}_T)},$$

i.e.,

$$\|\widetilde{L}_0^{-1}\|_{C(\overline{\widetilde{D}}_T)\to C^1(\overline{\widetilde{D}}_T)}\le(T^2+3T), \quad (4.4)$$

which was to be demonstrated.

Further, since the space $C^1(\overline{\widetilde{D}}_T)$ is embedded compactly into the space $C(\overline{\widetilde{D}}_T)$ [**10**, p. 135], the operator $\widetilde{L}_0^{-1}:C(\overline{\widetilde{D}}_T)\to C(\overline{\widetilde{D}}_T)$ is, by virtue of (4.4), linear and compact. Thus getting now back from the variables ξ and η to the variables x and t, for the operator L_0^{-1} from (3.3) we obtain the validity of the following statement.

Lemma 4.1. *The operator $L_0^{-1}:C(D_T)\to C(D_T)$ acting by the formula* (3.3) *is linear and compact.*

We rewrite the equation (3.2) in the form

$$u=Au:=-\big(L_0^{-1}f\big|_{u=u(x,t)}\big)+\ell_0^{-1}(\varphi_1,\varphi_2)+L_0^{-1}F, \quad (4.5)$$

where the operator $A:C(\overline{D}_T)\to C(\overline{D}_T)$ is continuous and compact since the nonlinear operator $K:C(\overline{D}_T)\to C(\overline{D}_T)$ acting by the formula $Ku:=-f(x,t,u)$ is bounded and continuous and the linear operator $L_0^{-1}:C(\overline{D}_T)\to C(\overline{D}_T)$ is, by Lemma 4.1, compact. We have taken here into account that the component $A_1u:=\ell_0^{-1}(\varphi_1,\varphi_2)+L_0^{-1}F$ of A from (4.5) is a constant and hence a continuous and compact operator acting in the space $C(\overline{D}_T)$. At the same time, by Lemmas 2.1 and 3.1 as well as by (2.2), (2.3), (2.17), (2.24) and (2.27), for any parameter $\tau\in[0,1]$ and every solution $u\in C(\overline{D}_T)$ of the equation $u=\tau Au$ the a priori estimate (2.4) is valid with the same constants c_1 and c_2, not depending on u, F, φ_1, φ_2 and τ. Therefore, by Leray–Schauder's theorem [**66**, p. 375] the equation (4.5) under the conditions of Lemmas 2.1 and 3.1 has at least one solution $u\in C(\overline{D}_T)$. Thus, by Lemmas 2.1 and 3.1, we proved the following [**1**]

Theorem 4.1. *Let $f\in C^1(\overline{D}_\infty\times\mathbb{R})$ and the conditions* (2.2) *and* (2.3) *be fulfilled for every $T>0$. Then the problem* (1.1), (1.2) *is globally solvable* *in the class C in the sense of Definition* 1.2*, i.e., for every* $\varphi_i \in C^1(\gamma_{i,\infty})$*,* $i = 1, 2$*, and* $F \in C(\overline{D}_\infty)$ *and for every* $T > 0$ *the problem* (1.1)*,* (1.2) *has a strong generalized solution of the class C in the domain D_T in the sense of Definition* 1.1*.*

We now cite certain classes of functions $f = f(x,t,u)$, frequently encountered in applications, for which the conditions (2.2) and (2.3) are fulfilled:

1. $f(x,t,u) = f_0(x,t)\psi(u)$, where f_0, $\frac{\partial}{\partial t} f_0 \in C(\overline{D}_\infty)$ and $\psi \in C(\mathbb{R})$.

In this case $g(x,t,u) = f_0(x,t) \int\limits_0^u \psi(s)\,ds$, and if the inequality $|\psi(u)| \leq d_1|u| + d_2$ is fulfilled, then the conditions (2.2) and (2.3) will be fulfilled.

2. $f(x,t,u) = f_0(x,t)|u|^\alpha \operatorname{sgn} u$, where f_0, $\frac{\partial}{\partial t} f_0 \in C(\overline{D}_\infty)$, $\alpha > 1$.

In this case $g(x,t,u) = f_0(x,t)|u|^{\alpha+1}$, and if the inequalities $f_0(x,t) \geq 0$, $\frac{\partial}{\partial t} f_0(x,t) \leq 0$ are fulfilled, then the conditions (2.2) and (2.3) will be fulfilled.

3. $f(x,t,u) = f_0(x,t)e^u$, where f_0, $\frac{\partial}{\partial t} f_0 \in C(\overline{D}_\infty)$.

In this case $g(x,t,u) = f(x,t,u)$, and if the inequalities $f_0(x,t) \geq 0$, $\frac{\partial}{\partial t} f_0(x,t) \leq 0$ are fulfilled, then the conditions (2.2) and (2.3) will be fulfilled.

Thus, if the function $f \in C^1(\overline{D}_\infty \times \mathbb{R})$ belongs to one of the abovementioned classes, then according to Theorem 4.1 the problem (1.1), (1.2) is globally solvable in the class C in the sense of Definition 1.2.

We present here an example of a function f which is also encountered in applications, when at least one of the conditions (2.2) or (2.3) is violated. Such a function is

$$f(x,t,u) = f_0(x,t)|u|^\alpha, \quad \alpha > 1, \tag{4.6}$$

where f_0, $\frac{\partial}{\partial t} f_0 \in C(\overline{D}_\infty)$ and $f_0 \neq 0$. In this case, by virtue of (4.6) we have $g(x,t,u) = f_0(x,t)|u|^{\alpha+1} \operatorname{sgn} u$, and since $\alpha > 1$ and $f_0 \neq 0$, the condition (2.2) is violated. If $\frac{\partial}{\partial t} f_0 \neq 0$, then the condition (2.3) will also be violated.

Below, it will be shown that if the conditions (2.2) and (2.3) are violated, then the problem (1.1), (1.2) fails to be globally solvable.

5. The Smoothness and Uniqueness of the Solution of the Problem (1.1), (1.2). The Existence of a Global Solution in D_∞

According to Remark 3.1, by virtue of the equalities (3.2), (3.3) and (3.4), if the conditions of Theorem 4.1 except possibly (2.2) and (2.3) are fulfilled, then a strong generalized solution of the problem (1.1), (1.2) belongs in fact to the space $C^1(\overline{D}_T)$. The same reasoning leads us to

Lemma 5.1. *Let u be a strong generalized solution of the problem* (1.1)*,* (1.2) *of the class C in the domain D_T in the sense of Definition* 1.1*.*

Then if $f \in C^k(\overline{D}_T \times \mathbb{R})$*,* $F \in C^k(\overline{D}_T)$ *and* $\varphi_i \in C^{k+1}(\gamma_{i,T})$*,* $i = 1, 2$*,* $k \geq 0$*, then we have* $u \in C^{k+1}(\overline{D}_T)$*.*

From the above lemma it follows, in particular, that for $k \geq 1$ a strong generalized solution of the problem (1.1), (1.2) of the class C in the domain D_T is a classical solution of that problem in the sense of Definition 1.1.

It is said that a function $f = f(x, t, u)$ satisfies the local Lipschitz condition on the set $\overline{D}_\infty \times \mathbb{R}$ if

$$\big|f(x,t,u_2) - f(x,t,u_1)\big| \leq \\ \leq M(T,R)|u_2 - u_1|, \quad (x,t) \in \overline{D}_T, \quad |u_i| \leq R, \quad i = 1, 2, \quad (5.1)$$

where $M = M(T, R) = \text{const} \geq 0$.

Lemma 5.2. *If the function* $f \in C(\overline{D}_T \times \mathbb{R})$ *satisfies the condition* (5.1)*, then the problem* (1.1), (1.2) *cannot have more than one strong generalized solution of the class* C *in the domain* D_T*.*

Proof. Indeed, assume that the problem (1.1), (1.2) has two strong generalized solutions u_1 and u_2 of the class C in the domain D_T. By Definition 1.1, there exists a sequence of functions $u_{jn} \in C^2(\overline{D}_T)$, $j = 1, 2$, such that

$$\lim_{n\to\infty} \|u_{jn} - u_j\|_{C(\overline{D}_T)} = \lim_{n\to\infty} \|L_f u_{jn} - F\|_{C(\overline{D}_T)} = \\ = \lim_{n\to\infty} \big\|u_{jn}\big|_{\gamma_{i,n}} - \varphi_i\|_{C^1(\gamma_{i,T})} = 0, \quad i, j = 1, 2. \quad (5.2)$$

Let $\omega_n = u_{2n} - u_{1n}$. It can be easily seen that the function $\omega_n \in C^2(\overline{D}_T)$ is a classical solution of the problem

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\Big)\omega_n + g_n = F_n, \quad (5.3)$$

$$\omega_n\big|_{\gamma_{i,T}} = \varphi_{in}, \quad i = 1, 2. \quad (5.4)$$

Here

$$g_n = f(x,t,u_{2n}) - f(x,t,u_{1n}), \quad (5.5)$$

$$F_n = L_f u_{2n} - L_f u_{1n}, \quad (5.6)$$

$$\varphi_{in} = (u_{2n} - u_{1n})\big|_{\gamma_{i,T}}, \quad i = 1, 2. \quad (5.7)$$

By virtue of (5.2), there exists a number $m = \text{const} > 0$, not depending on the indices j and n, such that $\|u_{jn}\|_{C(\overline{D}_T)} \leq m$, whence, in its turn, by (5.1) and (5.5) it follows that

$$|g_n| \leq M(T, 2m)|u_{2n} - u_{1n}|. \quad (5.8)$$

The equalities (5.2), (5.6) and (5.7) imply that

$$\lim_{n\to\infty} \|F_n\|_{C(\overline{D}_T)} = 0, \quad \lim_{n\to\infty} \|\varphi_{in}\|_{C^1(\gamma_{i,T})} = 0, \quad i = 1, 2. \quad (5.9)$$

Multiplying both parts of the equation (5.3) by $\frac{\partial \omega_n}{\partial t}$ and integrating with respect to the domain $D_\tau := \big\{(x,t) \in D_T : \ t < \tau\big\}$, $0 < \tau \leq T$, due to the boundary conditions (5.4), just as in obtaining the inequality (2.16), from (2.12)–(2.15) and (5.8) we have

$$\int_{\Omega_\tau}\left[\Big(\frac{\partial\omega_n}{\partial t}\Big)^2+\Big(\frac{\partial\omega_n}{\partial x}\Big)^2\right]dx=$$

$$=-\sum_{i=1}^{2}\int_{\gamma_{i,\tau}}\frac{1}{\nu_t}\left[\Big(\frac{\partial\omega_n}{\partial x}\,\nu_t-\frac{\partial\omega_n}{\partial t}\,\nu_x\Big)^2+\Big(\frac{\partial\omega_n}{\partial t}\Big)^2(\nu_t^2-\nu_x^2)\right]ds+$$

$$+2\int_{D_\tau}(F_n-g_n)\,\frac{\partial\omega_n}{\partial t}\,dx\,dt\le$$

$$\le\sqrt{2}\int_{\gamma_{1,\tau}}\|\varphi_{1n}\|^2_{C^1(\gamma_{1,T})}\,ds+\frac{\sqrt{1+k^2}}{k}\int_{\gamma_{2,\tau}}\|\varphi_{2n}\|^2_{C^1(\gamma_{2,T})}\,ds+$$

$$+\int_{D_\tau}(F_n-g_n)^2\,dx\,dt+\int_{D_\tau}\Big(\frac{\partial\omega_n}{\partial t}\Big)^2\,dx\,dt\le$$

$$\le\widetilde{M}\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\int_{D_\tau}\Big(\frac{\partial\omega_n}{\partial t}\Big)^2\,dx\,dt+$$

$$+2\int_{D_\tau}g_n^2\,dx\,dt+2\int_{D_\tau}F_n^2\,dx\,dt\le$$

$$\le\widetilde{M}\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\int_{D_\tau}\Big(\frac{\partial\omega_n}{\partial t}\Big)^2\,dx\,dt+$$

$$+2M^2(T,2m)\int_{D_\tau}\omega_n^2\,dx\,dt+2\int_{D_\tau}F_n^2\,dx\,dt. \tag{5.10}$$

By the inequalities (2.21) and (2.22) which, with regard for (5.4), are likewise valid for the function ω_n, from (5.10) we find that

$$\int_{\Omega_\tau}\left[\omega_n^2+\Big(\frac{\partial\omega_n}{\partial t}\Big)^2+\Big(\frac{\partial\omega_n}{\partial x}\Big)^2\right]dx\le$$

$$\le 2T\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+2T\int_{D_\tau}\Big(\frac{\partial\omega_n}{\partial t}\Big)^2\,dx\,dt+$$

$$+\widetilde{M}\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\int_{D_\tau}\Big(\frac{\partial\omega_n}{\partial t}\Big)^2\,dx\,dt+$$

$$+4T^2M^2(T,2m)\left[\sum_{i=1}^{2}\|\varphi_{in}\|^2_{C^1(\gamma_{i,T})}+\int_{D_\tau}\Big(\frac{\partial\omega_n}{\partial t}\Big)^2\,dx\,dt\right]+2\int_{D_\tau}F_n^2\,dx\,dt\le$$

$$\le \widetilde{M}_1 \int\limits_{D_\tau} \Big[\omega_n^2 + \Big(\frac{\partial \omega_n}{\partial t}\Big)^2 + \Big(\frac{\partial \omega_n}{\partial x}\Big)^2\Big]\, dx\, dt +$$

$$+\widetilde{M}_2 \Big[\sum_{i=1}^{2} \|\varphi_{in}\|^2_{C^1(\gamma_{i,T})} + \|F_n\|^2_{C(\overline{D}_T)}\Big], \quad 0 < \tau \le T, \tag{5.11}$$

where

$$\widetilde{M}_1 = 1 + 2T + 4T^2 M^2(T, 2m),$$
$$\widetilde{M}_2 = 2 \operatorname{mes} D_T + 2T + \widetilde{M} + 4T^2 M^2(T, 2m).$$

Assuming

$$v_n(\tau) = \int\limits_{\Omega_\tau} \Big[\omega_n^2 + \Big(\frac{\partial \omega_n}{\partial t}\Big)^2 + \Big(\frac{\partial \omega_n}{\partial x}\Big)^2\Big]\, dx$$

and taking into account the equality

$$\int\limits_{\Omega_\tau} \Big[\omega_n^2 + \Big(\frac{\partial \omega_n}{\partial t}\Big)^2 + \Big(\frac{\partial \omega_n}{\partial x}\Big)^2\Big]\, dx\, dt = \int\limits_0^\tau v_n(\sigma)\, d\sigma,$$

from (5.11) we obtain that

$$v_n(\tau) \le \widetilde{M}_1 \int\limits_0^\tau v_n(\sigma)\, d\sigma + \widetilde{M}_2 \Big[\sum_{i=1}^{2} \|\varphi_{in}\|^2_{C^1(\gamma_{i,T})} + \|F_n\|^2_{C(\overline{D}_T)}\Big]. \tag{5.12}$$

By Gronwall's lemma, from (5.12) it follows

$$v_n(\tau) \le \widetilde{M}_2 \Big[\sum_{i=1}^{2} \|\varphi_{in}\|^2_{C^1(\gamma_{i,T})} + \|F_n\|^2_{C(\overline{D}_T)}\Big] \exp \widetilde{M}_1 T, \quad 0 < \tau \le T. \tag{5.13}$$

Since $\omega_n = u_{2n} - u_{1n}$, from (5.2) and (5.9) it also follows that

$$\begin{aligned} &\lim_{n\to\infty} \|\omega_n\|_{C(\overline{D}_T)} = \|u_2 - u_1\|_{C(\overline{D}_T)}, \\ &\lim_{n\to\infty} \big\|\omega_n - (u_2 - u_1)\big\|_{C(\overline{D}_T)} = 0. \end{aligned} \tag{5.14}$$

In particular, from (5.13) for $\tau = T$ we have

$$\int\limits_{D_T} \omega_n^2\, dx\, dt \le \widetilde{M}_2 \Big[\sum_{i=1}^{2} \|\varphi_{in}\|^2_{C^1(\gamma_{i,T})} + \|F_n\|^2_{C(\overline{D}_T)}\Big] \exp \widetilde{M}_1 T. \tag{5.15}$$

Passing now in the inequality (5.15) to limit as $n \to \infty$ and taking into account the equalities (5.9) and (5.14) as well as the theorem on the passage to limit under the integral sign, we obtain

$$\int\limits_{D_T} |u_2 - u_1|^2\, dx\, dt \le 0,$$

whence it immediately follows that $u_2 = u_1$, and hence the proof of Lemma 5.2 is complete. □

Theorem 4.1 and Lemmas 5.1 and 5.2 imply the following

Theorem 5.1. *Let* $\varphi_i \in C^2(\gamma_{i,\infty})$, $i = 1, 2$, $F \in C^1(\overline{D}_\infty)$, $f \in C^1(\overline{D}_\infty \times \mathbb{R})$, *and the conditions* (2.2) *and* (2.3) *be fulfilled. Then the problem* (1.1), (1.2) *has a unique global classical solution* $u \in C(\overline{D}_\infty)$ *in the domain* D_∞.

Proof. Since the function f from the space $C^1(\overline{D}_\infty \times \mathbb{R})$ satisfies the local Lipschitz condition (5.1), according to Theorem 4.1 and Lemmas 5.1 and 5.2, in the domain D_T for $T = n$ there exists a unique classical solution $u_n \in C^2(\overline{D}_T)$ of the problem (1.1), (1.2). Since u_{n+1} is likewise a classical solution of the problem (1.1), (1.2) in the domain D_n, by virtue of Lemma 2.5 we have $u_{n+1}|_{D_n} = u_n$. Therefore the function u constructed in the domain D_∞ by the rule $u(x,t) = u_n(x,t)$ for $n = [t] + 1$, where $[t]$ is the integer part of the number t and $(x,t) \in \overline{D}_\infty$, will be the unique classical solution of the problem (1.1), (1.2) in the domain D_∞ of the class $C^2(\overline{D}_\infty)$.

Thus the proof of Theorem 5.1 is complete. □

6. The Cases of the Non-Existence of a Global Solution of the Problem (1.1), (1.2)

Below it will be shown that in case the conditions (2.2) or (2.3) are violated, the problem (1.1), (1.2) cannot be globally solvable in the class C in the sense of Definition 1.2.

Lemma 6.1. *Let u be a strong generalized solution of the problem* (1.1), (1.2) *of the class C in the domain D_T in the sense of Definition* 1.1 *under the homogeneous boundary conditions, i.e., for $\varphi_i = 0$, $i = 1, 2$. Then the following integral equality*

$$\int\limits_{D_T} u\Box\varphi \, dx\, dt = -\int\limits_{D_T} f(x,t,u)\varphi \, dx\, dt + \int\limits_{D_T} F\varphi \, dx\, dt \tag{6.1}$$

is valid for any function φ such that

$$\varphi \in C^2(\overline{D}_T), \ \ \varphi\big|_{t=T} = 0, \ \ \varphi_t\big|_{t=T} = 0, \ \ \varphi\big|_{\gamma_{2,T}} = 0, \tag{6.2}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$.

Proof. By the definition of a strong generalized solution u of the problem (1.1), (1.2) of the class C in the domain D_T, we have $u \in C(\overline{D}_T)$, and there exists a sequence of functions $u_n \in C^2(\overline{D}_T)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = \lim_{n\to\infty} \|L_f u_n - F\|_{C(\overline{D}_T)} = \\ = \lim_{n\to\infty} \big\|u_n\big|_{\gamma_{i,T}} - 0\big\|_{C^1(\gamma_{i,T})} = 0, \ \ i = 1, 2. \tag{6.3}$$

Let $F_n = L_f u_n$, $\varphi_{in} = u_n|_{\gamma_{i,T}}$, $i = 1, 2$. We multiply both parts of the equality $L_f u_n = F_n$ by the function φ and integrate the obtained equality over the domain D_T. After integration of the left-hand side of the above equality by parts, we obtain

$$\int\limits_{D_T} u\Box\varphi\,dx\,dt + \int\limits_{\partial D_T} \frac{\partial u_n}{\partial N}\,\varphi\,ds - \int\limits_{\partial D_T} u_n\,\frac{\partial\varphi}{\partial N}\,ds + \\ + \int\limits_{D_T} f(x,t,u_n)\varphi\,dx\,dt = \int\limits_{D_T} F_n\varphi\,dx\,dt, \quad (6.4)$$

where $\frac{\partial}{\partial N} = \nu_t\frac{\partial}{\partial t} - \nu_x\frac{\partial}{\partial x}$ is the derivative with respect to the conormal, and $\nu = (\nu_x, \nu_t)$ is the unit vector of the outer normal to ∂D_T.

Taking into account that the operator of differentiation with respect to the conormal $\frac{\partial}{\partial N}$ is an outer differential operator on the characteristic curve $\gamma_{1,T}$, and hence $\frac{\partial u_n}{\partial N}\big|_{\gamma_{1,T}} = \frac{\partial \varphi_{1n}}{\partial N}$, by the equalities from (6.2) we have

$$\begin{gathered}\int\limits_{\partial D_T} \frac{\partial u_n}{\partial N}\,\varphi\,ds = \int\limits_{\gamma_{1,T}} \frac{\partial \varphi_{1n}}{\partial N}\,\varphi\,ds, \\ \int\limits_{\partial D_T} u_n\,\frac{\partial\varphi}{\partial N}\,ds = \sum_{i=1}^{2}\int\limits_{\gamma_{i,T}} \varphi_{in}\,\frac{\partial\varphi}{\partial N}\,ds.\end{gathered} \quad (6.5)$$

Since $\varphi_{in} = u_n|_{\gamma_{i,T}}$, $i = 1, 2$, by virtue of (6.3) we find that

$$\lim_{n\to\infty}\Big\|\frac{\partial\varphi_{1n}}{\partial N}\Big\|_{C(\gamma_{1,T})} = 0, \quad \lim_{n\to\infty}\|\varphi_{in}\|_{C(\gamma_{i,T})} = 0, \;\; i = 1, 2. \quad (6.6)$$

By (6.3) and (6.6), passing in the equality (6.4) to limit as $n \to \infty$ we obtain the equality

$$\int\limits_{D_T} u\Box\varphi\,dx\,dt + \int\limits_{D_T} f(x,t,u)\varphi\,dx\,dt = \int\limits_{D_T} F\varphi\,dx\,dt.$$

Thus the lemma is proved. □

Consider the following condition imposed on the function f:

$$f(x,t,u) \le -\lambda|u|^{\alpha+1}, \;\; (x,t,u) \in \overline{D}_\infty \times \mathbb{R}; \;\; \lambda, \alpha = \text{const} > 0. \quad (6.7)$$

It can be easily verified that if the condition (6.7) is fulfilled, then the condition (2.2) is violated.

Introduce into consideration a function $\varphi^0 = \varphi^0(x,t)$ such that

$$\varphi^0 \in C^2(\overline{D}_\infty), \;\; \varphi^0\big|_{D_{T=1}} > 0, \;\; \varphi^0\big|_{\gamma_{2,\infty}} = 0, \;\; \varphi^0\big|_{t\ge 1} = 0 \quad (6.8)$$

and let

$$\varkappa_0 = \int\limits_{D_{T=1}} \frac{|\Box\varphi^0|^{p'}}{|\varphi^0|^{p'-1}}\,dx\,dt < +\infty, \;\; p' = 1 + \frac{1}{\alpha}\,. \quad (6.9)$$

It is not difficult to verify that in the capacity of the function φ_0 satisfying the conditions (6.8) and (6.9) we can take the function

$$\varphi^0(x,t)=\begin{cases}(x+kt)^n(1-t)^m, & (x,t)\in D_{T=1},\\ 0, & t\geq 1,\end{cases}$$

for sufficiently large positive constants n and m.

Putting $\varphi_T(x,t)=\varphi^0\big(\frac{x}{T},\frac{t}{T}\big)$, $T>0$, by virtue of (6.8) we can see that

$$\begin{gathered}\varphi_T\in C^2(\overline{D}_\infty),\ \ \varphi_T\big|_{D_T}>0,\\ \varphi_T\big|_{\gamma_{2,T}}=0,\ \ \varphi_T\big|_{t=T}=0,\ \ \frac{\partial\varphi_T}{\partial t}\Big|_{t=T}=0.\end{gathered}\tag{6.10}$$

Assuming the function F is fixed, we interoduce into consideration the function of one variable T,

$$\zeta(T)=\int\limits_{D_T}F\varphi_T\,dx\,dt,\ \ T>0.\tag{6.11}$$

There takes place the following theorem on the nonexistence of global solvability of the problem (1.1), (1.2) [**1**].

Theorem 6.1. *Let the function $f\in C(\overline{D}_\infty\times\mathbb{R})$ satisfy the condition* (6.7)*, $F\in C(\overline{D}_\infty)$, $F\geq 0$, and the boundary conditions* (1.2) *be homogeneous , i.e., $\varphi_i=0$, $i=1,2$. Let, moreover,*

$$\liminf_{T\to+\infty}\zeta(T)>0.\tag{6.12}$$

Then there exists a positive number $T_0=T_0(F)$ such that for $T>T_0$ the problem (1.1), (1.2) *cannot have a strong generalized solution u of the class C in the domain D_T.*

Proof. Assume that under the conditions of the above theorem there exists a strong generalized solution u of the problem (1.1), (1.2) of the class C in the domain D_T. Then, by Lemma 6.1, we have the equality (6.1) in which, owing to (6.10), we can take in the capacity of the function φ the function $\varphi=\varphi_T$, i.e.,

$$-\int\limits_{D_T}f(x,t,u)\varphi_T\,dx\,dt+\int\limits_{D_T}F\varphi_T\,dx\,dt=\int\limits_{D_T}u\Box\varphi_T\,dx\,dt.\tag{6.13}$$

Since $\varphi_T>0$ in the domain D_T, by the condition (6.7) and the designation (6.11), from (6.13) we have

$$\lambda\int\limits_{D_T}|u|^p\varphi_T\,dx\,dt\leq\int\limits_{D_T}|u|\,|\Box\varphi|\,dx\,dt-\zeta(T),\ \ p=\alpha+1.\tag{6.14}$$

If in Young's inequality with parameter $\varepsilon>0$,

$$ab\leq\frac{\varepsilon}{p}\,a^p+\frac{1}{p'\varepsilon^{p'-1}}\,b^{p'};\ \ a,b\geq 0,\ \ \frac{1}{p}+\frac{1}{p'}=1,\ \ p=\alpha+1>1,$$

we take $a = |u|\varphi_T^{1/p}$, $b = \frac{|\Box\varphi_T|}{\varphi_T^{1/p}}$, then taking into account that $p'/p = p'-1$ we will obtain

$$|u\Box\varphi_T| = |u|\varphi_T^{1/p}\,\frac{|\Box\varphi_T|}{\varphi_T^{1/p}} \le \frac{\varepsilon}{p}\,|u|^p\varphi_T + \frac{1}{p'\varepsilon^{p'-1}}\,\frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,. \tag{6.15}$$

It follows from (6.14) and (6.15) that

$$\Big(\lambda - \frac{\varepsilon}{p}\Big)\int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \le \frac{1}{p'\varepsilon^{p'-1}}\int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \zeta(T),$$

whence for $\varepsilon < \lambda p$ we find that

$$\int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \le \frac{p}{(|\lambda|p-\varepsilon)p'\varepsilon^{p'-1}}\int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \frac{p}{\lambda p-\varepsilon}\,\zeta(T). \tag{6.16}$$

Bearing in mind that $p' = \frac{p}{p-1}$, $p = \frac{p'}{p'-1}$ and $\min\limits_{0<\varepsilon<\lambda p} \frac{p}{(\lambda p-\varepsilon)p'\varepsilon^{p'-1}} = \frac{1}{\lambda^p}$, which is achieved for $\varepsilon = \lambda$, from (6.16) we get

$$\int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \le \frac{1}{\lambda^{p'}}\int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt - \frac{p'}{\lambda}\,\zeta(T). \tag{6.17}$$

Since $\varphi_T(x,t) = \varphi^0\big(\frac{x}{T},\frac{t}{T}\big)$, by virtue of (6.8) and (6.9), after the change of variables $t = Tt'$, $x = Tx'$, we can easily verify that

$$\int\limits_{D_T} \frac{|\Box\varphi_T|^{p'}}{\varphi_T^{p'-1}}\,dx\,dt =$$

$$= T^{-2(p'-1)}\int\limits_{D_{T=1}} \frac{|\Box\varphi^0|^{p'}}{(\varphi^0)^{p'-1}}\,dx'\,dt' = T^{-2(p'-1)}\varkappa_0 < +\infty. \tag{6.18}$$

By virtue of (6.10) and (6.18), the inequality (6.17) yields

$$0 \le \int\limits_{D_T} |u|^p\varphi_T\,dx\,dt \le \frac{1}{\lambda^{p'}}\,T^{-2(p'-1)}\varkappa_0 - \frac{p'}{\lambda}\,\zeta(T). \tag{6.19}$$

Because of the fact that $p' = \frac{p}{p-1} > 1$ we have $-2(p'-1) < 0$, and by (6.9) we get

$$\lim_{T\to\infty} \frac{1}{\lambda^{p'}}\,T^{-2(p'-1)}\varkappa_0 = 0.$$

Therefore, by (6.12) there exists a positive number $T_0 = T_0(F)$ such that for $T > T_0$ the right-hand side of the inequality (6.19) is negative, while the left-hand side of that inequality is nonnegative. This means that if there exists a strong generalized solution u of the problem (1.1), (1.2) of the class C in the domain D_T, then necessarily $T \le T_0$, which proves Theorem 6.1. $\Box$

Remark 6.1. It is not difficult to verify that if $F \in C(\overline{D}_\infty)$, $F \geq 0$ and $F(x,t) \geq ct^{-m}$ for $t \geq 1$, where $c = \text{const} > 0$ and $0 \leq m = \text{const} \leq 2$, then the condition (6.12) is fulfilled, and according to Theorem 6.1 in this case the problem (1.1), (1.2) has no strong generalized solution u of the class C in the domain D_T for large T [**1**].

7. The Local Solvability of the Problem (1.1), (1.2)

Theorem 7.1. *Let $f \in C^1(\overline{D}_\infty \times \mathbb{R})$, $F \in C(\overline{D}_\infty)$ and $\varphi_i \in C^1(\gamma_{i,\infty})$, $i = 1,2$. Then there exists a positive number $T_0 = T_0(F, \varphi_1, \varphi_2)$ such that for $T \leq T_0$ the problem* (1.1), (1.2) *has a unique strong generalized solution u of the class C in the domain D_T.*

Proof. By Lemma 3.1, the existence of a strong generalized solution of the problem (1.1), (1.2) of the class C in the domain D_T is equivalent to that of a continuous solution u of the nonlinear integral equation (3.2), or what is the same thing, of the equation (4.5), i.e.,

$$u = Au := -\big(L_0^{-1} f\big|_{u=u(x,t)}\big) + \ell_0^{-1}(\varphi_1, \varphi_2) + L_0^{-1} F, \tag{7.1}$$

where $A : C(\overline{D}_T) \to C(\overline{D}_T)$ is a continuous and compact operator. Therefore, to prove that the equation (7.1) is solvable, it suffices, by Schauder's theorem, to show that the operator A transforms some ball $B(0,R) := \big\{v \in C(\overline{D}_T) :\ \|v\|_{C(\overline{D}_T)} \leq R\big\}$ of radius $R > 0$ (which is a closed and convex set in the Banach space $C(\overline{D}_T)$) into itself for sufficiently small T.

Owing to (3.3) and (3.4), we can easily see that

$$\|L_0^{-1}\|_{C(\overline{D}_T)\to C(\overline{D}_T)} \leq \frac{1}{2}\,\text{mes}\, D_T = \frac{1}{4}\,(1+k)T^2, \tag{7.2}$$

$$\|\ell_0^{-1}\|_{C^1(\gamma_{1,T})\times C^1(\gamma_{2,T})\to C(\overline{D}_T)} \leq 3. \tag{7.3}$$

We fix now an arbitrary positive number T_*, and let $T \leq T_*$. By (7.1), (7.2) and (7.3), for

$$\|u\|_{C(\overline{D}_T)} \leq R = 4\sum_{i=1}^{2} \|\varphi_i\|_{C^1(\gamma_{i,T_*})}, \quad M_* = \sup_{\substack{(x,t)\in \overline{D}_{T_*}\\ |u|\leq R}} \big|f(x,t,u)\big| \tag{7.4}$$

we have

$$\|Au\|_{C(\overline{D}_T)} \leq \|L_0^{-1}\|_{C(\overline{D}_T)\to C(\overline{D}_T)} \sup_{\substack{(x,t)\in \overline{D}_{T_*}\\ |u|\leq R}} \big|f(x,t,u)\big| +$$
$$+\|\ell_0^{-1}\|_{C^1(\gamma_{1,T})\times C^1(\gamma_{2,T})\to C(\overline{D}_T)} \Big[\sum_{i=1}^{2} \|\varphi_i\|_{C^1(\gamma_{i,T})}\Big] +$$
$$+\|L_0^{-1}\|_{C(\overline{D}_T)\to C(\overline{D}_T)} \|F\|_{C(\overline{D}_T)} \leq$$

$$\leq \frac{1}{4}\,(1+k)M_*T^2 + \frac{1}{4}\,(1+k)T^2\|F\|_{C(\overline{D}_{T_*})} + 3\sum_{i=1}^{2}\|\varphi_i\|_{C^1(\gamma_{i,T})} =$$

$$= \Big[\frac{1}{4}\,(1+k)M_* + \frac{1}{4}\,(1+k)\|F\|_{C(\overline{D}_{T_*})}\Big]T^2 + 3\sum_{i=1}^{2}\|\varphi_i\|_{C^1(\gamma_{i,T_*})}. \qquad (7.5)$$

From (7.4) and (7.5), in its turn, it follows that if $T \leq T_0$, where

$$T_0 := \min\bigg[T_*, \bigg\{\Big(\frac{1}{4}\,(1+k)M_* + \frac{1}{4}\,(1+k)\|F\|_{C(\overline{D}_{T_*})}\Big)^{-1}\sum_{i=1}^{2}\|\varphi_i\|_{C^1(\gamma_{i,T_*})}\bigg\}^{1/2}\bigg],$$

then $\|Au\|_{C(\overline{D}_T)} \leq R$ for $\|u\|_{C(\overline{D}_T)} \leq R$. Thus Theorem 7.1 is proved completely, since the uniqueness of a solution follows directly from Lemma 3.1. □

CHAPTER 2

The Characteristic Cauchy Problem for a Class of Nonlinear Wave Equations in the Light Cone of the Future

1. Statement of the Problem

Consider the nonlinear wave equation of the type

$$L_f u := \frac{\partial^2 u}{\partial t^2} - \Delta u + f(u) = F, \tag{1.1}$$

where f and F are given real functions, f is a nonlinear function, and u is an unknown real function, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$.

For the equation (1.1) we consider the characteristic Cauchy problem: find in the frustrum of the light cone of future $D_T: \ |x| < t < T$, $x = (x_1, \dots, x_n)$, $n > 1$, $T = \text{const} > 0$, a solution $u(x,t)$ according the boundary condition

$$u\big|_{S_T} = 0, \tag{1.2}$$

where $S_T: \ t = |x|$, $t \leq T$, is the characteristic conic surface. Considering the case $T = +\infty$, we assume that $D_\infty: \ t > |x|$ and $S_\infty = \partial D_\infty: \ t = |x|$.

Below we will consider the following conditions imposed on the function f:

$$f \in C(\mathbb{R}), \ \ |f(u)| \leq M_1 + M_2|u|^\alpha, \ \ \alpha = \text{const} > 0, \tag{1.3}$$

$$\int\limits_0^u f(s)\,ds \geq -M_3 - M_4 u^2, \tag{1.4}$$

where $M_i = \text{const} \geq 0$, $i = 1, 2, 3, 4$.

Remark 1.1. Note that in case $\alpha \leq 1$ the inequality (1.3) results in the inequality (1.4).

Let $\overset{\circ}{W}{}_2^1(D_T, S_T) := \big\{u \in W_2^1(D_T): \ u|_{S_T} = 0\big\}$, where $W_2^k(D_T)$ is the well-known Sobolev's space consisting of the functions $u \in L_2(D_T)$ whose all generalized derivatives up to the k-th order, inclusive, also belong to the space $L_2(D_T)$, while the equality $u|_{S_T} = 0$ is understood in the sense of the trace theory [**49**, p. 70].

Definition 1.1. Let $F \in L_2(D_T)$. A function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is said to be a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T if there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ and $L_f u_m \to F$ in the space $L_2(D_T)$.

Definition 1.2. Let $F \in L_{2,loc}(D_\infty)$ and $F \in L_2(D_T)$ for any $T > 0$. We say that the problem (1.1), (1.2) is globally solvable in the class W_2^1 if for every $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the space D_T.

2. A Priori Estimate of a Solution of the Problem (1.1), (1.2) in the Class W_2^1

Lemma 2.1. *Let $F \in L_2(D_T)$, and let the function $f \in C(\mathbb{R})$ satisfy the condition* (1.4). *Then for every strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem* (1.1), (1.2) *of the class W_2^1 in the domain D_T the estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq c_1 \|F\|_{L_2(D_T)} + c_2 \tag{2.1}$$

is valid with nonnegative constants $c_i = c_i(f, T)$, $i = 1, 2$, independent of u and F.

Proof. Let $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ be a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T. By Definition 1.1, there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \quad \lim_{m\to\infty} \|L_f u_m - F\|_{L_2(D_T)} = 0. \tag{2.2}$$

Consider the function $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ as a solution of the problem

$$L_f u_m = F_m, \tag{2.3}$$

$$u_m\big|_{S_m} = 0. \tag{2.4}$$

Here

$$F_m := L_f u_m. \tag{2.5}$$

Putting

$$g(u) := \int\limits_0^u f(s)\, ds \tag{2.6}$$

and multiplying both parts of the equation (2.3) by $\frac{\partial u_m}{\partial t}$, after integration over the domain D_τ, $0 < \tau \leq T$, we obtain

$$\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt-\int\limits_{D_\tau}\Delta u_m\,\frac{\partial u_m}{\partial t}\,dx\,dt+
+\int\limits_{D_\tau}\frac{\partial}{\partial t}\,g(u_m)\,dx\,dt=\int\limits_{D_\tau}F_m\,\frac{\partial u_m}{\partial t}\,dx\,dt. \quad (2.7)$$

Let $\Omega_\tau := D_\infty \cap \{t=\tau\}$ and denote by $\nu=(\nu_1,\dots,\nu_n,\nu_0)$ the unit vector of the outer normal to $S_T\setminus\big\{(0,\dots,0)\big\}$. Integrating by parts and taking into account the equality (2.4) and $\nu|_{\Omega_\tau}=(0,\dots,0,1)$, we easily get

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt=$$
$$=\int\limits_{\partial D_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\nu_0\,ds=\int\limits_{\Omega_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx+\int\limits_{S_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\nu_0\,ds,$$
$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\,g(u_m)\,dx\,dt=\int\limits_{\partial D_\tau}g(u_m)\nu_0\,ds=\int\limits_{\Omega_\tau}g(u_m)\,dx,$$
$$\int\limits_{D_\tau}\frac{\partial^2 u_m}{\partial x_i^2}\,\frac{\partial u_m}{\partial t}\,dx\,dt=\int\limits_{\partial D_\tau}\frac{\partial u_m}{\partial x_i}\,\frac{\partial u_m}{\partial t}\,\nu_i\,ds-\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\,dx\,dt=$$
$$=\int\limits_{\partial D_\tau}\frac{\partial u_m}{\partial x_i}\,\frac{\partial u_m}{\partial t}\,\nu_i\,ds-\frac{1}{2}\int\limits_{\partial D_\tau}\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\nu_0\,ds=$$
$$=\int\limits_{\partial D_\tau}\frac{\partial u_m}{\partial x_i}\,\frac{\partial u_m}{\partial t}\,\nu_i\,ds-\frac{1}{2}\int\limits_{S_\tau}\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\nu_0\,ds-\frac{1}{2}\int\limits_{\Omega_\tau}\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\,dx,$$

whence by virtue of (2.7), it follows that

$$\int\limits_{D_\tau}F_m\,\frac{\partial u_m}{\partial t}\,dx\,dt=$$
$$=\int\limits_{S_\tau}\frac{1}{2\nu_0}\bigg[\sum_{i=1}^{n}\Big(\frac{\partial u_m}{\partial x_i}\,\nu_0-\frac{\partial u_m}{\partial t}\,\nu_i\Big)^2+\Big(\frac{\partial u_m}{\partial t}\Big)^2\Big(\nu_0^2-\sum_{j=1}^{n}\nu_j^2\Big)\bigg]\,ds+$$
$$\frac{1}{2}\int\limits_{\Omega_\tau}\bigg[\Big(\frac{\partial u_m}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\bigg]\,dx+\int\limits_{\Omega_\tau}g(u_m)\,dx. \quad (2.8)$$

Since S_τ is a characteristic surface, we have

$$\Big(\nu_0^2-\sum_{j=1}^{n}\nu_j^2\Big)\Big|_{S_\tau}=0. \quad (2.9)$$

Taking into account the fact that $\big(\nu_0\,\frac{\partial}{\partial x_i}-\nu_i\,\frac{\partial}{\partial t}\big)$, $i=1,\dots,n$, is an inner differential operator on S_τ, by virtue of (2.4) we have

$$\Big(\frac{\partial u_m}{\partial x_i}\,\nu_0-\frac{\partial u_m}{\partial t}\,\nu_i\Big)\Big|_{S_\tau}=0,\ \ i=1,\dots,n. \tag{2.10}$$

Bearing in mind (2.9) and (2.10), it follows from (2.8) that

$$\int\limits_{\Omega_\tau}\bigg[\Big(\frac{\partial u_m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\bigg]\,dx+2\int\limits_{\Omega_\tau}g(u_m)\,dx=$$
$$=2\int\limits_{D_\tau}F_m\,\frac{\partial u_m}{\partial t}\,dx\,dt. \tag{2.11}$$

By (1.4) and (2.6) as well as by the Cauchy inequality $2F_m\,\frac{\partial u_m}{\partial t}\le F_m^2+\big(\frac{\partial u_m}{\partial t}\big)^2$, from (2.11) we have

$$\int\limits_{\Omega_\tau}\bigg[\Big(\frac{\partial u_m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\bigg]\,dx\le$$
$$\le 2M_3\operatorname{mes}\Omega_\tau+2M_4\int\limits_{\Omega_\tau}u_m^2\,dx+\int\limits_{D_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt+\int\limits_{D_\tau}F_m^2\,dx\,dt\le$$
$$\le 2M_3\operatorname{mes}\Omega_\tau+2M_4\int\limits_{\Omega_\tau}u_m^2\,dx+\int\limits_{D_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt+\int\limits_{D_\tau}F_m^2\,dx\,dt. \tag{2.12}$$

From the equalities $v|_{S_T}=0$ and $v(x,t)=\int\limits_{|x|}^{t}\frac{\partial v(x,\tau)}{\partial t}\,d\tau$, $(x,t)\in\overline{D}_T$, valid for every function $v\in\overset{\circ}{C}{}^2(\overline{D}_T,S_T)$, reasoning in a standard way we obtain the following inequalities [**49**, p. 63]:

$$\int\limits_{\Omega_\tau}v^2\,dx\le T\int\limits_{D_\tau}\Big(\frac{\partial v}{\partial t}\Big)^2\,dx\,dt,\ \ 0<\tau\le T, \tag{2.13}$$
$$\int\limits_{D_\tau}v^2\,dx\,dt\le T^2\int\limits_{D_\tau}\Big(\frac{\partial v}{\partial t}\Big)^2\,dx\,dt,\ \ 0<\tau\le T. \tag{2.14}$$

By virtue of (2.13) and (2.14), from (2.12) we get

$$\int\limits_{\Omega_\tau}\bigg[u_m^2+\Big(\frac{\partial u_m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\bigg]\,dx\le 2M_3\operatorname{mes}\Omega_\tau+$$
$$+(2M_4+1)T\int\limits_{D_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt+\int\limits_{D_\tau}\Big(\frac{\partial u_m}{\partial t}\Big)^2\,dx\,dt+\int\limits_{D_\tau}F_m^2\,dx\,dt\le$$
$$\le\big[2(M_4+1)T+1\big]\int\limits_{D_\tau}\bigg[u_m^2+\Big(\frac{\partial u_m}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u_m}{\partial x_i}\Big)^2\bigg]\,dx+$$

$$+2M_3 \operatorname{mes} \Omega_T + \|F_m\|^2_{L_2(D_T)}. \tag{2.15}$$

Putting

$$w(\tau) = \int\limits_{\Omega_\tau} \Big[u_m^2 + \Big(\frac{\partial u_m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 \Big]\, dx \tag{2.16}$$

and taking into account the equality

$$\int\limits_{D_\tau} \Big[u_m^2 + \Big(\frac{\partial u_m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 \Big]\, dx\, dt = \int\limits_0^\tau w(\sigma)\, d\sigma,$$

from (2.15) we have

$$w(\tau) \le M_5 \int\limits_0^\tau w(\sigma)\, d\sigma + \big(\|F_m\|^2_{L_2(D_T)} + M_6\big). \tag{2.17}$$

Here

$$M_5 = (2M_4 + 1)T + 1, \quad M_6 = 2M_3 \operatorname{mes} \Omega_T. \tag{2.18}$$

From (2.17), by Gronwall's lemma [**15**, p. 13] it follows that

$$w(\tau) \le \big(\|F_m\|^2_{L_2(D_T)} + M_6\big) \exp M_5 \tau \le \\ \le \big(\|F_m\|^2_{L_2(D_T)} + M_6\big) \exp M_5 T. \tag{2.19}$$

The inequality (2.19) with regard for (2.16) implies that

$$\|u_m\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = \int\limits_{D_T} \Big[u_m^2 + \Big(\frac{\partial u_m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 \Big]\, dx\, dt = \\ = \int\limits_0^T w(\sigma)\, d\sigma \le T\big(\|F_m\|^2_{L_2(D_T)} + M_6\big) \exp M_5 T,$$

that is,

$$\|u_m\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \le c_1 \|F_m\|_{L_2(D_T)} + c_2. \tag{2.20}$$

Here

$$c_1 = \sqrt{T} \exp \frac{1}{2} M_5 T, \quad c_2 = \sqrt{T M_6} \exp \frac{1}{2} M_5 T. \tag{2.21}$$

By (2.2) and (2.5), passing in the inequality (2.20) to limit as $m \to \infty$, we obtain the required inequality (2.1). □

3. The Global Solvability of the Problem (1.1), (1.2) in the Class W_2^1

Remark 3.1. Before we proceed to considering the issue of the solvability of the nonlinear problem (1.1), (1.2), let us consider the same issue for the linear case in which in the equation (1.1) the function $f = 0$, i.e., for the problem

$$L_0 u := \frac{\partial^2 u}{\partial t^2} - \Delta u = F(x,t), \quad (x,t) \in D_T, \tag{3.1}$$

$$u(x,t) = 0, \quad (x,t) \in S_T. \tag{3.2}$$

In this case, for $F \in L_2(D_T)$ we introduce analogously the notion of a strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (3.1), (3.2) of the class W_2^1 in the domain $\overline{D}_T$ for which there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ such that $\lim\limits_{m\to\infty} \|u_m - u\|_{W_2^1(D_T)} = 0$, $\lim\limits_{m\to\infty} \|L_0 u_m - F\|_{L_2(D_T)} = 0$. It should be noted that as is seen from the proof of Lemma 2.1, for the solution of the problem (3.1), (3.2) the a priori estimate (2.1) is also valid in which, by virtue of (1.3), (1.4) for $M_i = 0$, $i = 1,2,3,4$, the constant M_6 from (2.18) is equal to zero, and hence c_2, by virtue of (2.21), is also equal to zero. Thus for a strong generalized solution u of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T the estimate

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq c_1 \|F\|_{L_2(D_T)}, \quad c_1 = \sqrt{T} \exp \frac{1}{2} M_5 T \tag{3.3}$$

holds by virtue of (2.20).

The constant M_5 here is defined from (2.18), and since for $f = 0$ in the inequality (1.4) the constant $M_4 = 0$, therefore $M_5 = T + 1$, and hence

$$c_1 = \sqrt{T} \exp \frac{1}{2} T(1+T). \tag{3.4}$$

As far as the space $C_0^\infty(D_T)$ of finitary infinitely differentiable in D_T functions is dense in $L_2(D_T)$, for a given $F \in L_2(D_T)$ there exists a sequence of functions $F_m \in C_0^\infty(D_T)$ such that $\lim\limits_{m\to\infty} \|F_m - F\|_{L_2(D_T)} = 0$. For a fixed m, extending the values of the function F_m by zero beyond the domain D_T and leaving the same notation, we will have $F_m \in C^\infty(\mathbb{R}_+^{n+1})$ for which the support $\operatorname{supp} F_m \subset D_\infty$, where $\mathbb{R}_+^{n+1} = \mathbb{R}^{n+1} \cap \{t \geq 0\}$. Denote by u_m the solution of the Cauchy problem: $L_0 u_m = F_m$, $u_m|_{t=0} = 0$, $\frac{\partial u_m}{\partial t}\big|_{t=0} = 0$, which, as is known, exists, is unique and belongs to the space $C^\infty(\mathbb{R}_+^{n+1})$ [**17**, p. 192]. Moreover, since $\operatorname{supp} F_m \subset D_\infty$, $u_m|_{t=0} = 0$ and $\frac{\partial u_m}{\partial t}\big|_{t=0} = 0$, taking into account the geometry of the domain of dependence of a solution of the linear wave equation we will have $\operatorname{supp} u_m \subset D_\infty$ [**17**, p. 191]. Leaving for the restriction of the function u_m to the domain D_T the same notation, we can easily see that $u_m \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$, and in view of (3.3) the inequality

$$\|u_m - u_k\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le c_1 \|F_m - F_k\|_{L_2(D_T)} \tag{3.5}$$

holds.

Since the sequence $\{F_m\}$ is fundamental in $L_2(D_T)$, the sequence $\{u_m\}$ is likewise fundamental in the entire space $\overset{\circ}{W}{}_2^1(D_T, S_T)$. Therefore there exists a function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ such that $\lim\limits_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0$, and since $L_0 u_m = F_m \to F$ in the space $L_2(D_T)$, this function is, according to Remark 3.1, a strong generalized solution of the problem (3.1), (3.2). The uniqueness of this solution in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ follows from the estimate (3.3). Thus for the solution u of the problem (3.1), (3.2) we can write $u = L_0^{-1} F$, where $L_0^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a linear continuous operator whose norm admits, by virtue of (3.3) and (3.4), the estimate

$$\|L_0^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}_2^1(D_T,S_T)} \le \sqrt{T} \exp \frac{1}{2} T(1+T). \tag{3.6}$$

Remark 3.2. The embedding operator $I : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_q(D_T)$ is linear continuous and compact for $1 < q < \frac{2(n+1)}{n-1}$, when $n \ge 2$ [**49**, p. 81]. At the same time, the Nemytski operator $K : L_q(D_T) \to L_2(D_T)$, acting by the formula $Ku = f(u)$, where the function f satisfies the condition (1.3), is continuous and bounded if $q \ge 2\alpha$ [**47**, p. 349], [**48**, pp. 66, 57]. Thus if $\alpha < \frac{n+1}{n-1}$, i.e., $2\alpha < \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Therefore, in this case the operator

$$K_0 = KI : \overset{\circ}{W}{}_2^1(D_T, S_T) \longrightarrow L_2(D_T) \tag{3.7}$$

will be continuous and compact. Moreover, from $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ it follows that $f(u) \in L_2(D_T)$, and if $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, then $f(u_m) \to f(u)$ in the space $L_2(D_T)$.

By Remarks 3.1 and 3.2, for $F \in L_2(D_T)$ and $\alpha < \frac{n+1}{n-1}$ the function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T if and only if u is a solution of the following functional equation

$$u = L_0^{-1}\big(-f(u) + F\big)$$

or, what is the same thing, of the equation

$$u = Au := L_0^{-1}(-K_0 u + F) \tag{3.8}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$. Since the operator $K_0 : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T)$ from (3.7) is, by Remark 3.2, continuous and compact, the operator A :

$\overset{\circ}{W}{}_2^1(D_T,S_T) \to \overset{\circ}{W}{}_2^1(D_T,S_T)$ is, owing to (3.6), likewise continuous and compact. At the same time, by Lemma 2.1 and (1.4), (2.18), (2.20), for any parameter $\tau \in [0,1]$ and every solution $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ of the equation $u = \tau A u$ with the parameter τ, the a priori estimate (2.1) is valid with the same nonnegative constants c_1 and c_2 not depending on u, F and the parameter τ. Therefore, by the Leray–Schauder theorem [**66**, p. 375] the equation (3.8), and hence the problem (1.1), (1.2), has at least one solution $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$.

Thus the following theorem is valid.

Theorem 3.1. *Let $F \in L_{2,loc}(D_\infty)$ and $F \in L_2(D_T)$ for any $T > 0$. Let $0 < \alpha < \frac{n+1}{n-1}$ and the function f satisfy the inequality* (1.3)*. Moreover, in case $\alpha > 1$, let the function f satisfy also the condition* (1.4)*. Then the problem* (1.1), (1.2) *is globally solvable in the class W_2^1 in the sense of Definition* 1.2*, i.e., for any $T > 0$ this problem has at least one strong generalized solution of the class W_2^1 in the domain D_T.*

Remark 3.3. Note that under the conditions of Theorem 3.1 the problem (1.1), (1.2) may have more than one solution. Indeed, if $F = 0$ and $f(u) = -|u|^\alpha$, where $0 < \alpha < 1$, then the conditions of Theorem 3.1 are fulfilled and the problem (1.1), (1.2) has, besides a trivial solution, an infinite set of global solutions $u_\sigma(x,t)$ in the domain D_∞ depending on the parameter $\sigma \geq 0$ and given by the formula

$$u_\sigma(x,t) = \begin{cases} \beta\big[(t-\sigma)^2 - |x|^2\big]^{\frac{1}{1-\alpha}}, & t > \sigma + |x|, \\ 0, & |x| \leq t \leq \sigma + |x|, \end{cases}$$

where

$$\beta = \lambda^{\frac{1}{1-\alpha}} \Big[\frac{4\alpha}{(1-\alpha)^2} + \frac{2(n+1)}{1-\alpha}\Big]^{-\frac{1}{1-\alpha}}.$$

It can be easily seen that $u_\sigma(x,t) \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ for any $T > 0$. Moreover, $u_\sigma(x,t) \in C^1(\overline{D}_\infty)$, and for $1/2 < \alpha < 1$ the function $u_\sigma(x,t)$ belongs to the space $C^2(\overline{D})$.

4. The Local Solvability of the Problem (1.1), (1.2) in the Class W_2^1 in Case the Condition (1.4) is Violated

As it will be shown, when the condition (1.4) is violated the problem (1.1), (1.2) is unable to be globally solvable in the sense of Definition 1.2, although, as we will see below, there takes place the local solvability.

We restrict ourselves to the consideration of the case

$$1 < \alpha < \frac{n+1}{n-1}, \tag{4.1}$$

since for $\alpha \leq 1$ from (1.3) it follows (1.4).

In [**27**] it is shown that if the condition (4.1) is fulfilled, then we have the inequality

$$\|u\|_{L_{2\alpha}(D_T)} \leq c_0 \ell_{\alpha,n} T^{\delta_{\alpha,n}} \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \quad \forall\, u \in \overset{\circ}{W}{}_2^1(D_T,S_T), \tag{4.2}$$

where

$$\ell_{\alpha,n} = \Big(\frac{\omega_n}{n+1}\Big)^{\frac{\delta_{\alpha,n}}{n+1}}, \quad \delta_{\alpha,n} = \Big(\frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}\Big)(n+1),$$

a positive constant c_0 does not depend on u and T, and, as is easily seen, the condition $\delta_{\alpha,n} > 0$ is equivalent to the condition $\alpha < \frac{n+1}{n-1}$; ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 4.1. Let $B(0,R) := \big\{u \in \overset{\circ}{W}{}_2^1(D_T,S_T) :\ \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq R\big\}$ be a closed (convex) ball in the Hilbert space $\overset{\circ}{W}{}_2^1(D_T,S_T)$ with radius $R > 0$ and center in the zero element. Since the problem (1.1), (1.2) is equivalent to the equation (3.8) in the class $\overset{\circ}{W}{}_2^1(D_T,S_T)$ and by Remark 3.2 the operator $A : \overset{\circ}{W}{}_2^1(D_T,S_T) \to \overset{\circ}{W}{}_2^1(D_T,S_T)$ from (3.8) is (if the condition (4.1) is fulfilled) continuous and compact, according to the Schauder's principle to prove the solvability of the equation (3.8) it suffices to prove that the operator A transforms the ball $B(0,R)$ into itself [**66**, p. 370]. Towards this end, on the basis of the inequality (4.2) we estimate the value $\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}$.

For the operator K_0 from (3.7), by means of (1.3) and (4.2), we have

$$\begin{aligned}
\|K_0 u\|_{L_2(D_T)} &= \|f(u)\|_{L_2(D_T)} \leq \big\|(M_1 + M_2|u|^\alpha)\big\|_{L_2(D_T)} \leq \\
&\leq M_1 (\operatorname{mes} D_T)^{1/2} + M_2 \big\||u|^\alpha\big\|_{L_2(D_T)} = \\
&= M_1 (\operatorname{mes} D_T)^{1/2} + M_2 \|u\|^\alpha_{L_{2\alpha}(D_T)} \leq \\
&\leq M_1 (\operatorname{mes} D_T)^{1/2} + M_2 c_0 \ell_{\alpha,n} T^{\delta_{\alpha,n}} \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}
\end{aligned} \tag{4.3}$$

for any $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$.

Next, for the operator A from (3.8) by virtue of (3.6) and (4.3) we have

$$\begin{gathered}
\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \\
\leq \|L_0^{-1}\|_{L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T,S_T)} \big[\|K_0 u\|_{L_2(D_T)} + \|F\|_{L_2(D_T)}\big] \leq \\
\leq \sqrt{T}\,\Big(\exp \frac{1}{2}\, T(1+T)\Big) \times \\
\times \Big[M_1 (\operatorname{mes} D_T)^{1/2} + M_2 c_0 \ell_{\alpha,n} T^{\delta_{\alpha,n}} \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} + \|F\|_{L_2(D_T)}\Big]
\end{gathered} \tag{4.4}$$

for any $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$.

Fix the numbers $R > 0$ and $T_0 > 0$, and let $T \leq T_0$. Then for $\forall\, u \in B(0,R)$, by virtue of (4.4) and the fact that $\delta_{\alpha,n} > 0$, if the condition (4.1) is fulfilled, then we have

$$\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{T_0}\Big(\exp\frac{1}{2}\,T_0(1+T_0)\Big)\times$$
$$\times\Big[M_1(\operatorname{mes} D_{T_0})^{1/2}+M_2c_0\ell_{\alpha,n}T_0^{\delta_{\alpha,n}}R+\|F\|_{L_2(D_{T_0})}\Big],$$

whence it follows that for sufficiently small $T_0>0$

$$\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq R \ \ \forall\, u\in B(0,R),\ \ T\leq T_0. \tag{4.5}$$

From (4.5), by Remark 4.1, we find that the problem (1.1), (1.2) is locally solvable in the class W_2^1.

Thus the following theorem is valid.

Theorem 4.1. *Let $F\in L_{2,loc}(D_\infty)$ and $F\in L_2(D_T)$ for any $T>0$. Let $1<\alpha<\frac{n+1}{n-1}$. For the function f let the condition* (1.3) *be fulfilled but the condition* (1.4) *may be violated. Then the problem* (1.1), (1.2) *is locally solvable in the class W_2^1, i.e., there exists a number $T_0=T_0(F)>0$ such that for $T\leq T_0$ this problem has at leat one strong generalized solution of the class W_2^1 in the domain D_T.*

5. The Non-Existence of the Global Solvability of the Problem (1.1), (1.2) in the Class W_2^1 in Case the Condition (1.4) is Violated

We restrict ourselves to the consideration of the case where

$$1<\alpha<\frac{n+1}{n-1} \tag{5.1}$$

and

$$f(u)\leq -\lambda|u|^{\alpha},\ \ \lambda=\text{const}>0. \tag{5.2}$$

It can be easily verified that if the conditions (5.1) and (5.2) are fulfilled, then the condition (1.4) is violated. It will be shown that if for the function F the condition

$$F\in L_{2,loc}(D_\infty),\ \ F\in L_2(D_T)\ \ \forall\, T>0,\ \ F>0 \tag{5.3}$$

is fulfilled, then the problem (1.1), (1.2) fails to be globally solvable in the class W_2^1.

Assume that if the conditions (5.1), (5.2) and (5.3) are fulfilled, then the problem (1.1), (1.2) is globally solvable in the class W_2^1, i.e., for any $T>0$ this problem has a strong generalized solution u of the class W_2^1 in the domain D_T. By Definition 1.1, this means that $u\in \overset{\circ}{W}{}_2^1(D_T,S_T)$ and there exists a sequence of functions $u_m\in \overset{\circ}{C}{}^2(\overline{D}_T,S_T)$ such that

$$\lim_{m\to\infty}\|u_m-u\|_{\overset{\circ}{W}{}_2^1(D_T)}=0,\quad \lim_{m\to\infty}\|L_fu_m-F\|_{L_2(D_T)}=0. \tag{5.4}$$

We use here the method of test functions [**53**, pp. 10–12]. Let the function φ be such that

$$\varphi\in C^2(\overline{D}_T),\ \ \varphi\big|_{t=T}=0,\ \ \frac{\partial\varphi}{\partial t}\Big|_{t=T}=0,\ \ \varphi\big|_{D_T}>0. \tag{5.5}$$

Then putting $F_m := L_f u_m$ and integrating the integral equality

$$\int\limits_{D_T}(L_f u_m)\varphi\,dx\,dt=\int\limits_{D_T}F_m\varphi\,dx\,dt$$

by parts, we obtain

$$\int\limits_{D_T}u_m\Box\varphi\,dx\,dt+\int\limits_{S_T}\Big[\frac{\partial u_m}{\partial N}\,\varphi-\frac{\partial\varphi}{\partial N}\,u_m\Big]\,ds+ \\ +\int\limits_{D_T}f(u_m)\varphi\,dx\,dt=\int\limits_{D_T}F_m\varphi\,dx\,dt, \tag{5.6}$$

where $\Box:=\frac{\partial^2}{\partial t^2}-\Delta$, $\frac{\partial}{\partial N}=\nu_0\,\frac{\partial}{\partial t}-\sum_{i=1}^n\nu_i\,\frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal, and $\nu=(\nu_1,\dots,\nu_0,\nu_0)$ is the unit vector of the outer normal to ∂D_T.

Since on the characteristic conic surface S_T the derivative with respect to the conormal $\frac{\partial}{\partial N}$ is an inner differential operator, by virtue of the fact that $u_m|_{S_T}=0$ we have $\frac{\partial u_m}{\partial N}\big|_{S_T}=0$.

Therefore the equality (5.6) takes the form

$$\int\limits_{D_T}u_m\Box\varphi\,dx\,dt+\int\limits_{D_T}f(u_m)\varphi\,dx\,dt=\int\limits_{D_T}F_m\varphi\,dx\,dt. \tag{5.7}$$

Further, by (5.4), passing in the equality (5.7) to limit as $m\to\infty$, we obtain

$$\int\limits_{D_T}u_m\Box\varphi\,dx\,dt+\int\limits_{D_T}f(u)\varphi\,dx\,dt=\int\limits_{D_T}F\varphi\,dx\,dt. \tag{5.8}$$

Assuming

$$\gamma(T)=\int\limits_{D_T}F\varphi\,dx\,dt, \tag{5.9}$$

by (5.2) and (5.5) we get from (5.8) that

$$\lambda\int\limits_{D_T}|u|^\alpha\varphi\,dx\,dt\le\int\limits_{D_T}u\Box\varphi\,dx\,dt-\gamma(T). \tag{5.10}$$

If in Young's inequality

$$ab\le\frac{\varepsilon}{\alpha}\,a^\alpha+\frac{1}{\alpha'\varepsilon^{\alpha'-1}}\,b^{\alpha'},\ \ a,b\ge 0,\ \ \alpha'=\frac{\alpha}{\alpha-1}$$

with the parameter $\varepsilon > 0$ we take $a = |u|\varphi^{1/\alpha}$ and $b = \frac{|\Box\varphi|}{\varphi^{1/\alpha}}$, then taking into account that $\frac{\alpha'}{\alpha} = \alpha' - 1$ we have

$$|u\Box\varphi| = |u|\varphi^{1/\alpha} \cdot \frac{|\Box\varphi|}{\varphi^{1/\alpha}} \leq \frac{\varepsilon}{\alpha}\,|u|^{\alpha}\varphi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\,\frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,. \tag{5.11}$$

By virtue of (5.11), from (5.10) it follows the inequality

$$\Big(\lambda - \frac{\varepsilon}{\alpha}\Big)\int\limits_{D_T} |u|^{\alpha}\varphi\,dx\,dt \leq \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int\limits_{D_T}\frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt - \gamma(T),$$

whence for $\varepsilon < \lambda\alpha$ we find that

$$\int\limits_{D_T} |u|^{\alpha}\varphi\,dx\,dt \leq \frac{\alpha}{(\lambda\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}}\int\limits_{D_T}\frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt - \gamma(T). \tag{5.12}$$

Taking into account the equalities

$$\alpha' = \frac{\alpha}{\alpha-1}, \;\; \alpha = \frac{\alpha'}{\alpha'-1} \;\text{ and }\; \min_{0<\varepsilon<\lambda\alpha}\frac{\alpha}{(\lambda\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = \frac{1}{\lambda^{\alpha'}}$$

which is achieved for $\varepsilon = \lambda$, we obtain from (5.11) that

$$\int\limits_{D_T} |u|^{\alpha}\varphi\,dx\,dt \leq \frac{1}{\lambda^{\alpha'}}\int\limits_{D_T}\frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt - \frac{\alpha'}{\lambda}\,\gamma(T). \tag{5.13}$$

In the capacity of the test function φ we take now the function $\varphi(x,t) = \varphi_0\big[\frac{2}{T^2}\,(t^2+|x|^2)\big]$, where the function $\varphi_0 = \varphi_0(\sigma)$ of one variable σ is such that [**53**, p. 22]

$$\begin{gathered}\varphi_0 \in C^2(\mathbb{R}), \;\; \varphi_0 \geq 0, \;\; \varphi_0' \leq 0;\\ \varphi_0\big|_{[0,1]} = 1, \;\; \varphi_0\big|_{[2,\infty)} = 0, \;\; \varphi_0\big|_{(1,2)} > 0.\end{gathered} \tag{5.14}$$

By (5.14), the test function $\varphi(x,t) = \varphi_0\big[\frac{2}{T^2}\,(t^2+|x|^2)\big] = 0$ for $r = (t^2+|x|^2)^{1/2} \geq T$. Therefore, after the change of variables $t = \frac{1}{\sqrt{2}}\,T\xi_0$ and $x = \frac{1}{\sqrt{2}}\,T\xi$ it is not difficult to verify that

$$\int\limits_{D_T}\frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt = \int\limits_{\substack{r=(t^2+|x|^2)<T,\\ t>|x|}}\frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,dx\,dt = \Big(\frac{1}{\sqrt{2}}\,T\Big)^{n+1-2\alpha'}\varkappa_0, \tag{5.15}$$

where

$$\varkappa_0 = \int\limits_{\substack{1<|\xi_0|^2+|\xi|^2<2,\\ \xi_0>|\xi|}}\frac{|2(1-n)\varphi_0' + 4(\xi_0^2-|\xi|^2)\varphi_0''|^{\alpha'}}{\varphi_0^{\alpha'-1}}\,d\xi\,d\xi_0 < +\infty.$$

Due to (5.15), from the inequality (5.13) with regard for the fact that $\varphi_0(\sigma) = 1$ for $0 \le \sigma \le 1$ we obtain the inequality

$$\int\limits_{r \le \frac{1}{\sqrt{2}} T} |u|^\alpha \, dx\, dt \le \int\limits_{D_T} |u|^\alpha \varphi \, dx\, dt \le \frac{1}{\lambda^{\alpha'}} \Big(\frac{1}{\sqrt{2}} T\Big)^{n+1-2\alpha'} \varkappa_0 - \frac{\alpha'}{\lambda}\, \gamma(T). \quad (5.16)$$

In case $\alpha < \frac{n+1}{n-1}$, i.e., for $n + 1 - 2\alpha' < 0$, the equation

$$g(T) = \frac{1}{\lambda^{\alpha'}} \Big(\frac{1}{\sqrt{2}} T\Big)^{n+1-2\alpha'} \varkappa_0 - \frac{\alpha'}{\lambda}\, \gamma(T) = 0 \quad (5.17)$$

has a unique positive root $T = T_0 > 0$ because

$$g_1(T) = \frac{1}{\lambda^{\alpha'}} \Big(\frac{1}{\sqrt{2}} T\Big)^{n+1-2\alpha'} \varkappa_0$$

is a positive, continuous, strictly decreasing function on the interval $(0, +\infty)$ satisfying $\lim\limits_{T\to 0} g_1(T) = +\infty$ and $\lim\limits_{T\to +\infty} g_1(T) = 0$, and the function $\gamma(T)$, $T > 0$, is, by virtue of (5.9), (5.14) and the fact that $F|_{D_\infty} > 0$, positive, continuous and decreasing with $\lim\limits_{T\to+\infty} \gamma(T) > 0$. Moreover, $g(T) < 0$ for $T > T_0$, and $g(T) > 0$ for $0 < T < T_0$. Consequently, for $T > T_0$, the right-hand side of (5.16) is negative, but this is impossible. The obtained contradiction proves that if the conditions (5.1), (5.2) and (5.3) are fulfilled, the problem (1.1), (1.2) is not globally solvable in the class W_2^1. Incidentally, we have obtained an estimate of T when the problem (1.1), (1.2) (which is, as shown in the previous section, locally solvable) has a strong generalized solution of the class W_2^1 in the domain D_T. The estimate is $T \le T_0$, where T_0 is the unique positive root of the equation (5.17).

6. The Global Solvability of the Problem (1.1), (1.2) in the Class W_2^2

Below, in considering the problem (1.1), (1.2) we will restrict ourselves to the case of three spatial variables, i.e., $n = 3$. The increase of the smoothness of the solution of the problem (1.1), (1.2) allows us to widen the interval (5.1) in which the exponent α varies.

Instead of the conditions (1.3) and (1.4) imposed on the function f, we consider the following conditions:

$$f \in C^1(\mathbb{R}), \ \ f(0) = 0, \ \ |f'(u)| \le M(1 + |u|^2), \ \ u \in \mathbb{R}, \quad (6.1)$$

$$g(u) = \int\limits_0^u f(\tau)\, d\tau, \ \ \inf_{u\in\mathbb{R}} g(u) > -\infty, \ \ g(u) \ge -M_* u^2, \ \ u \in \mathbb{R}, \quad (6.2)$$

where $M, M_* = \mathrm{const} > 0$.

Obviously, the function $f(u) = m^2 u + u^3$ satisfies the conditions (6.1) and (6.2) [**58**]. At the same time, for $n = 3$, the interval of variation (5.1) of the exponent α is $1 < \alpha < 2$.

Assume $\overset{\circ}{W}{}_2^k(D_T,S_T) := \big\{W_2^*(D_T):\ u|_{S_T}=0\big\}$, where $W_2^k(D_T)$ is the well-known Sobolev's space [**49**, p. 56] consisting of the elements $L_2(D_T)$ having generalized derivatives up to the order k, inclusive, from $L_2(D_T)$, while the equality $u|_{S_T}=0$ is understood in the sense of the trace theory [**49**, p. 70].

Definition 6.1. Let $F\in \overset{\circ}{W}{}_2^1(D_T,S_T)$. A function $u=u(x,t)$ is said to be a solution of the problem $(1.1),(1.2)$ of the class W_2^2 in the domain D_T, if $u\in \overset{\circ}{W}{}_2^2(D_T,S_T)$ and it satisfies both the equation (1.1) almost everywhere in the domain D_T and the boundary condition (1.2) in the sense of the trace theory (and hence $u\in \overset{\circ}{W}{}_2^1(D_T,S_T)$).

Definition 6.2. Let $F\in \overset{\circ}{W}{}_2^1(D_T,S_T)$. The function $u\in \overset{\circ}{W}{}_2^2(D_T,S_T)$ is said to be a strong generalized solution of the problem $(1.1),(1.2)$ of the class W_2^2 in the domain D_T if there exists a sequence of functions $u_n\in C^\infty(\overline{D}_T)$ satisfying the boundary condition (1.2) and

$$\lim_{n\to\infty}\|u_n-u\|_{W_2^2(D_T)}=0,\quad \lim_{n\to\infty}\|F_n-F\|_{W_2^1(D_T)}=0, \tag{6.3}$$

where

$$F_n=L_f u_n \ \text{ and } \ \operatorname{supp}F_n\cap S_T=\varnothing. \tag{6.4}$$

Since $f(0)=0$, it is evident that $F_n\in \overset{\circ}{W}{}_2^1(D_T,S_T)$.

Remark 6.1. A strong generalized solution of the problem $(1.1),(1.2)$ of the class W_2^2 in the sense of Definition 6.2 is likewise a solution of the problem $(1.1),(1.2)$ of the class W_2^2 since, as it will be shown below, the first equality of (6.3) implies that $f(u_n)\to f(u)$ in $L_2(D_T)$. On the other hand, we will show the solvability of the problem $(1.1),(1.2)$ in the sense of Definition 6.2 and the uniqueness of the solution of the problem in the sense of Definition 6.1. Obviously, this implies the uniqueness of the solution of the problem in the sense of Definition 6.2, and hence the equivalence of these definitions.

Definition 6.3. Let $F\in L_{2,loc}(D_\infty)$ and $F\in \overset{\circ}{W}{}_2^1(D_T,S_T)$ for any $T>0$. We say that the problem $(1.1),(1.2)$ is globally solvable in the class W_2^2 if for any $T>0$ this problem has a solution of the class W_2^2 in the domain D_T in the sense of Definition 6.1.

Lemma 6.1. *Let $n=3$ and the conditions* (6.1), (6.2) *and $F\in \overset{\circ}{W}{}_2^1(D_T,S_T)$ be fulfilled. Then for every strong generalized solution u of the problem* $(1.1),(1.2)$ *of the class W_2^2 in the domain D_T in the sense of Definition* 6.2 *the a priori estimate*

$$\|u\|_{W_2^2(D_T)}\le$$

$$\leq c\Big[1+\|F\|_{L_2(D_T)}+\|F\|^3_{L_2(D_T)}+\|F\|_{W_2^1(D_T)}\exp\big(c\|F\|^2_{L_2(D_T)}\big)\Big] \quad (6.5)$$

is valid with a positive constant c not depending on u and F.

Proof. By Definition 6.2 of a strong generalized solution u of the problem (1.1), (1.2) of the class W_2^2 in the domain D_T, there exists a sequence of functions $u_n \in C^\infty(\overline{D}_T)$ satisfying the conditions (1.2), (6.3) and (6.4) and, hence,

$$L_f u_n = F_n, \quad u_n \in C^\infty(\overline{D}_T), \quad (6.6)$$

$$u_n\big|_{S_T} = 0. \quad (6.7)$$

The proof of the above lemma runs in a few steps.

1^0. Putting $\Omega_\tau := D_\infty \cap \{t=\tau\}$, we first show the validity of the a priori estimate

$$\int\limits_{\Omega_t}\Big[u_n^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2+\sum_{i=1}^3\Big(\frac{\partial u_n}{\partial x_i}\Big)^2\Big]\,dx \leq c_1\Big(1+\int\limits_{D_t}F_n^2\,dx\,dt\Big), \quad 0<t\leq T, \quad (6.8)$$

with a positive constant c_1 not depending on u_n and F_n. Indeed, multiplying both parts of the equation (6.6) by $\frac{\partial u_n}{\partial t}$ and integrating over the domain D_τ, $0<\tau\leq T$, with regard for (6.2) we obtain

$$\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial t}\Big)^2 dx\,dt-\int\limits_{D_\tau}\Delta u_n\,\frac{\partial u_n}{\partial t}\,dx\,dt+\int\limits_{D_\tau}\frac{\partial}{\partial t}\,g(u_n)\,dx\,dt=$$

$$=\int\limits_{D_\tau}F_n\,\frac{\partial u_n}{\partial t}\,dx\,dt \quad \Big(\Delta=\sum_{i=1}^3\frac{\partial^2}{\partial x_i^2}\Big). \quad (6.9)$$

Denote by $\nu=(\nu_1,\nu_2,\nu_3,\nu_0)$ the unit vector of the outer normal to $S_T\setminus\{(0,0,0,0)\}$. The integration by parts, with regard for $g(0)=0$ from (6.2), the inequality (2.7) and $\nu|_{\Omega_\tau}=(0,0,0,1)$, provides us with

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial t}\Big)^2 dx\,dt=$$

$$=\int\limits_{\partial D_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\nu_0\,ds=\int\limits_{\Omega_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2 dx+\int\limits_{S_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\nu_0\,ds,$$

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\,(u_n^2)\,dx\,dt=\int\limits_{\partial D_\tau}u_n^2\nu_0\,ds=\int\limits_{\Omega_\tau}u_n^2\,dx,$$

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\,g(u_n)\,dx\,dt=\int\limits_{\partial D_\tau}g(u_n)\nu_0\,ds=\int\limits_{\Omega_\tau}g(u_n)\,dx,$$

$$\int\limits_{D_\tau}\frac{\partial^2 u_n}{\partial x_i^2}\,\frac{\partial u_n}{\partial t}\,dx\,dt=\int\limits_{\partial D_\tau}\frac{\partial u_n}{\partial x_i}\,\frac{\partial u_n}{\partial t}\,\nu_i\,ds-\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_n}{\partial x_i}\Big)^2 dx\,dt=$$

$$= \int\limits_{\partial D_\tau} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial t} \nu_i \, ds - \frac{1}{2} \int\limits_{\partial D_\tau} \Big(\frac{\partial u_n}{\partial x_i}\Big)^2 \nu_0 \, ds =$$

$$= \int\limits_{\partial D_\tau} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial t} \nu_i \, ds - \frac{1}{2} \int\limits_{S_\tau} \Big(\frac{\partial u_n}{\partial x_i}\Big)^2 \nu_0 \, ds - \frac{1}{2} \int\limits_{\Omega_\tau} \Big(\frac{\partial u_n}{\partial x_i}\Big)^2 dx, \quad i = 1, 2, 3,$$

whence by virtue of (6.9) we have

$$\int\limits_{D_\tau} F_n \frac{\partial u_n}{\partial t} \, dx \, dt =$$

$$= \int\limits_{S_\tau} \frac{1}{2\nu_0} \bigg[\sum_{i=1}^{3} \Big(\frac{\partial u_n}{\partial x_i} \nu_0 - \frac{\partial u_n}{\partial t} \nu_i \Big)^2 + \Big(\frac{\partial u_n}{\partial t}\Big)^2 \Big(\nu_0^2 - \sum_{j=1}^{3} \nu_j^2 \Big) \bigg] ds +$$

$$\frac{1}{2} \int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_n}{\partial t}\Big)^2 + \sum_{i=1}^{3} \Big(\frac{\partial u_n}{\partial x_i}\Big)^2 \bigg] dx + \int\limits_{\Omega_\tau} g(u_n) \, dx. \tag{6.10}$$

Since S_τ is a characteristic surface, we have

$$\Big(\nu_0^2 - \sum_{j=1}^{3} \nu_j^2 \Big) \Big|_{S_\tau} = 0. \tag{6.11}$$

Taking into account that $\big(\nu_0 \frac{\partial}{\partial x_i} - \nu_i \frac{\partial}{\partial t}\big)$, $i = 1, 2, 3$, is an inner differential operator on S_τ, by virtue of (6.7) we get

$$\Big(\frac{\partial u_n}{\partial x_i} \nu_0 - \frac{\partial u_n}{\partial t} \nu_i \Big) \Big|_{S_\tau} = 0, \quad i = 1, 2, 3. \tag{6.12}$$

Bearing in mind (6.11) and (6.12), we rewrite the equality (6.10) in the form

$$\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_n}{\partial t}\Big)^2 + \sum_{i=1}^{3} \Big(\frac{\partial u_n}{\partial x_i}\Big)^2 \bigg] dx + 2 \int\limits_{\Omega_\tau} g(u_n) \, dx =$$

$$= 2 \int\limits_{D_\tau} F_n \frac{\partial u_n}{\partial t} \, dx \, dt. \tag{6.13}$$

By (6.2), there exists a number $M_0 = \text{const} \geq 0$ such that

$$g(u) \geq -M_0, \quad u \in \mathbb{R}. \tag{6.14}$$

Using (6.14) and the Cauchy inequality $2F_n \frac{\partial u_n}{\partial t} \leq F_n^2 + \big(\frac{\partial u_n}{\partial t}\big)^2$, from (6.13) we find that

$$\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_n}{\partial t}\Big)^2 + \sum_{i=1}^{3} \Big(\frac{\partial u_n}{\partial x_i}\Big)^2 \bigg] dx \leq$$

$$\leq 2M_0 \operatorname{mes} \Omega_\tau + \int\limits_{D_\tau} \Big(\frac{\partial u_n}{\partial t}\Big)^2 dx \, dt + \int\limits_{D_\tau} F_n^2 \, dx \, dt. \tag{6.15}$$

From the equalities $u_n|_{S_\tau} = 0$ and $u_n(x,\tau) = \int\limits_{|x|}^{\tau} \frac{\partial u_n(x,t)}{\partial t}\,dt$, $x \in \Omega_\tau$, $0 < \tau \le T$, in a standard way we obtain the inequality [**49**, p. 63]

$$\int\limits_{\Omega_\tau} u_n^2\,dx \le T \int\limits_{D_\tau} \Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt, \quad 0 < \tau \le T. \tag{6.16}$$

Summing the inequalities (6.15) and (6.16), we get

$$\int\limits_{\Omega_\tau} \Big[u_n^2 + \Big(\frac{\partial u_n}{\partial t}\Big)^2 + \sum_{i=1}^{3}\Big(\frac{\partial u_n}{\partial x_i}\Big)^2\Big]\,dx \le$$

$$\le \frac{8}{3}\,\pi\tau^3 M_0 + (1+T)\int\limits_{D_\tau}\Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt + \int\limits_{D_\tau} F_n^2\,dx\,dt. \tag{6.17}$$

Introduce the notation

$$w(\delta) := \int\limits_{\Omega_\delta} \Big[u_n^2 + \Big(\frac{\partial u_n}{\partial t}\Big)^2 + \sum_{i=1}^{3}\Big(\frac{\partial u_n}{\partial x_i}\Big)^2\Big]\,dx.$$

Then by virtue of (6.17) we have

$$w(\delta) \le (1+T)\int\limits_{\Omega_\delta} \Big[u_n^2 + \Big(\frac{\partial u_n}{\partial t}\Big)^2 + \sum_{i=1}^{3}\Big(\frac{\partial u_n}{\partial x_i}\Big)^2\Big]\,dx\,dt +$$

$$+ \frac{8}{3}\,\pi T^3 M_0 + \int\limits_{\Omega_\delta} F_n^2\,dx\,dt =$$

$$= (1+T)\int\limits_0^\delta w(\sigma)\,d\sigma + \frac{8}{3}\,\pi T^3 M_0 + \|F_n\|^2_{L_2(D_\delta)}, \quad 0 < \delta \le T. \tag{6.18}$$

From (6.18), taking into account that $\|F_n\|^2_{L_2(D_\delta)}$ as a function of δ is nondecreasing, by Gronwall's lemma [**15**, p. 13] we obtain

$$w(\delta) \le \Big[\frac{8}{3}\,\pi\,T^3 M_0 + \|F_n\|^2_{L_2(D_\delta)}\Big]\exp(1+T)\delta \le c_1\big(1 + \|F_n\|^2_{L_2(D_T)}\big),$$

whence for $t = T$ it follows the inequality (6.8) with the constant

$$c_1 = \max\Big(\frac{8}{3}\pi\,T^3 M_0 \exp(1+T)T, \exp(1+T)T\Big).$$

2^0. By (6.4), we have $\operatorname{supp} F_n \cap S_T = \varnothing$. Therefore there exists a positive number $\delta_n < T$ such that

$$\operatorname{supp} F_n \subset D_{T,\delta_n} := \Big\{(x,t) \in D_T :\ t > |x| + \delta_n\Big\}. \tag{6.19}$$

At this step we will show that

$$u_n\big|_{D_T \setminus \overline{D}_{T,\delta_n}} = 0. \tag{6.20}$$

Indeed, let $(x^0,t^0)\in D_T\setminus\overline{D}_{T,\delta_n}$. Introduce into consideration the domain $D_{x^0,t^0}:=\big\{(x,t)\in\mathbb{R}^4:\ |x|<t<t^0-|x-x^0|\big\}$ which is bounded from below by the surface S_T and from above by the boundary $S^-_{x^0,t^0}:=\big\{(x,t)\in\mathbb{R}^4:\ t=t^0-|x-x^0|\big\}$ of the light cone of the past $G^-_{x^0,t^0}:=\big\{(x,t)\in\mathbb{R}^4:\ t<t^0-|x-x^0|\big\}$ with the vertex at the point (x^0,t^0). By (6.19), we have

$$F_n\big|_{D_{x^0,t^0}}=0,\ \ (x^0,t^0)\in D_T\setminus\overline{D}_{T,\delta_n}. \tag{6.21}$$

Let $D_{x^0,t^0,\tau}:=D_{x^0,t^0}\cap\{t<\tau\}$ and $\Omega_{x^0,t^0,\tau}:=D_{x^0,t^0}\cap\{t=\tau\}$, $0<\tau<t^0$. We have $\partial D_{x^0,t^0,\tau}=S_{1,\tau}\cup S_{2,\tau}\cup S_{3,\tau}$, where $S_{1,\tau}=\partial D_{x^0,t^0,\tau}\cap S_\infty$, $S_{2,\tau}=\partial D_{x^0,t^0,\tau}\cap S^-_{x^0,t^0}$, $S_{3,\tau}=\partial D_{x^0,t^0,\tau}\cap\overline{\Omega}_{x^0,t^0,\tau}$.

In the same way as in obtaining the equality (6.10), multiplying both parts of the equality (6.6) by $\frac{\partial u_n}{\partial t}$, integrating over the domain $D_{x^0,t^0,\tau}$, $0<\tau<t^0$ and taking into account (6.7) and (6.21), we obtain

$$0=\int\limits_{S_{1,\tau}\cup S_{2,\tau}}\frac{1}{2\nu_0}\bigg[\sum_{i=1}^3\Big(\frac{\partial u_n}{\partial x_i}\,\nu_0-\frac{\partial u_n}{\partial t}\,\nu_i\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big(\nu_0^2-\sum_{j=1}^3\nu_j^2\Big)\bigg]\,ds+$$
$$+\int\limits_{S_{2,\tau}\cup S_{3,\tau}}g(u_n)\nu_0\,ds+\frac12\int\limits_{S_{3,\tau}}\bigg[\Big(\frac{\partial u_n}{\partial t}\Big)^2+\sum_{j=1}^3\Big(\frac{\partial u_n}{\partial x_i}\Big)^2\bigg]\,dx. \tag{6.22}$$

By (6.7) and (6.11), bearing in mind that the surface $S_{2,\tau}$ is, just like $S_{1,\tau}$, a characteristic one and hence $\big(\nu_0^2-\sum_{j=1}^3\nu_j^2\big)\big|_{S_{1,\tau}\cup S_{2,\tau}}=0$, and

$$\nu_0\big|_{S_{1,\tau}}=-\frac{1}{\sqrt2}<0,\quad \nu_0\big|_{S_{2,\tau}}=\frac{1}{\sqrt2}>0,\quad \nu_0\big|_{S_{3,\tau}}=1,$$
$$\Big(\frac{\partial u_n}{\partial x_i}\,\nu_0-\frac{\partial u_n}{\partial t}\,\nu_i\Big)\bigg|_{S_{1,\tau}}=0,\quad \Big(\frac{\partial u_n}{\partial x_i}\,\nu_0-\frac{\partial u_n}{\partial t}\,\nu_i\Big)^2\bigg|_{S_{2,\tau}}\geq0,\ \ i=1,2,3,$$

we have

$$\int\limits_{S_{1,\tau}\cup S_{2,\tau}}\frac{1}{2\nu_0}\bigg[\sum_{i=1}^3\Big(\frac{\partial u_n}{\partial x_i}\,\nu_0-\frac{\partial u_n}{\partial t}\,\nu_i\Big)^2+\Big(\frac{\partial u_n}{\partial t}\Big)^2\Big(\nu_0^2-\sum_{j=1}^3\nu_j^2\Big)\bigg]\,ds\geq0. \tag{6.23}$$

Taking into account (6.2) and (6.23), the equality (6.11) yields

$$\int\limits_{S_{3,\tau}}\bigg[\Big(\frac{\partial u_n}{\partial t}\Big)^2+\sum_{i=1}^3\Big(\frac{\partial u_n}{\partial x_i}\Big)^2\bigg]\,dx\leq M_1\int\limits_{S_{2,\tau}\cup S_{3,\tau}}u_n^2\,ds,\ \ 0<\tau<t^0. \tag{6.24}$$

Since $u_n\in C^\infty(\overline{D}_T)$, $\nu_0|_{S_{2,\tau}\cup S_{3,\tau}}\geq0$, $|\nu_0|\leq1$, by virtue of (6.2) we can define a nonnegative constant M_1 independent of the parameter τ by the equality

$$M_1=2M_*=\text{const}>0. \tag{6.25}$$

Since $u_n|_{S_T} = 0$, where $S_T:\ t = |x|,\ t \le T$, we have

$$u_n(x,t) = \int\limits_{|x|}^{t} \frac{\partial u_n(x,\sigma)}{\partial t}\, d\sigma, \quad (x,t) \in S_{2,\tau} \cup S_{3,\tau}. \tag{6.26}$$

Reasoning in a standard way [**49**, p. 63], we get from (6.26) that

$$\int\limits_{S_{2,\tau}\cup S_{3,\tau}} u_n^2\, ds \le 2t^0 \int\limits_{D_{x^0,t^0,\tau}} \Big(\frac{\partial u_n}{\partial t}\Big)^2\, dx\, dt, \quad 0 < \tau \le t^0. \tag{6.27}$$

Putting $v(\tau) = \int\limits_{S_{3,\tau}} \big[\big(\frac{\partial u_n}{\partial t}\big)^2 + \sum\limits_{i=1}^{3} \big(\frac{\partial u_n}{\partial x_i}\big)^2\big]\, dx$, from (6.24) and (6.27) we easily obtain

$$v(\tau) \le 2t^0 M_1 \int\limits_0^{\tau} v(\delta)\, d\delta, \quad 0 < \tau \le t^0,$$

whence by (6.25) and Gronwall's lemma it immediately follows that $v(\tau) = 0$, $0 < \tau \le t^0$, and hence $\frac{\partial u_n}{\partial t} = \frac{\partial u_n}{\partial x_1} = \frac{\partial u_n}{\partial x_2} = \frac{\partial u_n}{\partial x_3} = 0$ in the domain D_{x^0,t^0}. Therefore $u_n|_{D_{x^0,t^0}} = \text{const}$, and taking into account the homogeneous boundary condition (6.7), we find that $u_n|_{D_{x^0,t^0}} = 0\ \forall\, (x^0,t^0) \in D_T \setminus \overline{D}_{T,\delta_n}$. Thus we have proved the equality (6.20).

3^0. We will now proceed directly to proving the a priori estimate (6.5). By (6.20), extending the values of the function u_n from the domain D_T into the layer $\Sigma_T := \big\{(x,t) \in \mathbb{R}^4:\ x \in \mathbb{R}^3,\ 0 < t < T\big\}$ by zero and preserving the notation, we obtain

$$u_n \in C^\infty(\overline{\Sigma}_T), \quad u_n\big|_{\overline{\Sigma}_T \setminus \overline{D}_{T,\delta_n}} = 0. \tag{6.28}$$

In particular, it follows from (6.28) that $u_n = 0$ for $|x| \ge T$.

Differentiating the equality (6.6) with respect to the variable x_i, we have

$$\square u_{n,x_i} = -f'(u_n)u_{n,x_i} + F_{n,x_i}, \quad i = 1,2,3, \tag{6.29}$$

where

$$u_{n,x_i} = \frac{\partial u_n}{\partial x_i}\,, \quad F_{n,x_i} = \frac{\partial F_n}{\partial x_i}\,, \quad \square := \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\,.$$

Let

$$E(\tau) := \frac{1}{2} \sum_{i=1}^{3} \int\limits_{\Omega_\tau} \Big(u_{n,x_i t}^2 + \sum_{k=1}^{3} u_{n,x_i x_k}^2\Big)\, dx, \quad \Omega_\tau = D_\infty \cap \{t = \tau\}. \tag{6.30}$$

By virtue of (6.28), in the right-hand side of (6.30) we can replace the domain Ω_τ by the three-dimensional ball $B_\tau(0,T):\ |x| < T$ in the plane $t = \tau$. Therefore, differentiating the equality (6.30) with respect to the variable τ and then integrating by parts, with regard for (6.6), (6.28) and (6.29) we obtain

$$\begin{aligned} E'(\tau) &= \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} \Big(u_{n,x_i t} u_{n,x_i tt} + \sum_{k=1}^{3} u_{n,x_i x_k} u_{n,x_i x_k t} \Big)\, dx = \\ &= \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} \Big(u_{n,x_i tt} u_{n,x_i t} - \sum_{k=1}^{3} u_{n,x_i x_k x_k} u_{n,x_i t} \Big)\, dx = \\ &= \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} (\square u_{n,x_i}) u_{n,x_i t}\, dx = \\ &= \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} \big[- f'(u_n) u_{n,x_i} + F_{n,x_i} \big] u_{n,x_i t}\, dx, \end{aligned} \tag{6.31}$$

where $B_\tau(0,T):\ |x| < T, t = \tau$.

By (6.1) and Gronwall's inequality [**22**, p. 134]

$$\Big| \int f_1 f_2 f_3\, dx \Big| \le \|f_1\|_{L_{p_1}} \|f_2\|_{L_{p_2}} \|f_3\|_{L_{p_3}}$$

for $p_1 = 3$, $p_2 = 6$, $p_3 = 2$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, as well as by the Cauchy inequality, for the right-hand side (6.31) we have the estimate

$$\begin{aligned} I &= \Big| \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} \big[- f'(u_n) u_{n,x_i} + F_{n,x_i} \big] u_{n,x_i t}\, dx \Big| \le \\ &\le \frac{1}{2} \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} F_{n,x_i}^2\, dx + \frac{1}{2} \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} u_{n,x_i}^2\, dx + \\ &\quad + \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} \big| f'(u_n) u_{n,x_i} u_{n,x_i t} \big|\, dx \le \\ &\le \frac{1}{2} \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} u_{n,x_i}^2\, dx + \frac{1}{2} \sum_{i=1}^{3} \int\limits_{B_\tau(0,T)} F_{n,x_i}^2\, dx + \\ &+ M \sum_{i=1}^{3} \big\| (1 + u_n^2) \big\|_{L_3(B_\tau(0,T))} \|u_{n,x_i}\|_{L_6(B_\tau(0,T))} \|u_{n,x_i}\|_{L_2(B_\tau(0,T))}. \end{aligned} \tag{6.32}$$

According to the theorem of embedding of the space $W_m^\ell(\Omega)$ into $L_p(\Omega)$, for $\dim \Omega = 3$, $m = 2$, $\ell = 1$, $p = 6$ [**49**, p. 84], [**48**, p. 111] there takes place the estimate

$$\|v\|_{L_6((|x|<T))} \le c_2 \|v\|_{\overset{\circ}{W}{}_2^1((|x|<T))} \quad \forall\, v \in \overset{\circ}{W}{}_2^1\big((|x| < T)\big) \tag{6.33}$$

with a positive constant c_2 not depending on v.

There also takes place [**49**, p. 117]

$$\sum_{i=1}^{3}\int\limits_{|x|<T} v_{x_i}^2\,dx \le c_3 \sum_{i,j=1}^{3}\int\limits_{|x|<T} v_{x_ix_j}^2\,dx \quad \forall\, v\in \overset{\circ}{W}{}_2^2((|x|<T)) \tag{6.34}$$

with a positive constant c_3 not depending on v.

Applying the inequality (6.33) to the functions u_n and u_{n,x_i} which, owing to (6.28), belong to the space $\overset{\circ}{W}{}_2^1((|x|<T))$ for fixed $t=\tau$, we obtain

$$\begin{aligned} \|u_n\|_{L_6(B_\tau(0,T))} &\le c_2\|u_n\|_{\overset{\circ}{W}{}_2^1(B_\tau(0,T))},\\ \|u_{n,x_i}\|_{L_6(B_\tau(0,T))} &\le c_2\|u_{n,x_i}\|_{\overset{\circ}{W}{}_2^1(B_\tau(0,T))}. \end{aligned} \tag{6.35}$$

By (6.8), (6.30) and (6.35), we have

$$\begin{aligned} &\big\|(1+u_n^2)\big\|_{L_3(B_\tau(0,T))}\|u_{n,x_i}\|_{L_6(B_\tau(0,T))}\|u_{n,x_it}\|_{L_2(B_\tau(0,T))} \le\\ &\le \Big(\sqrt[3]{\frac{4}{3}\,\pi}\,T+\|u_n\|^2_{L_6(B_\tau(0,T))}\Big)c_2\|u_n\|_{\overset{\circ}{W}{}_2^1(B_\tau(0,T))}[2E(\tau)]^{1/2}\le\\ &\le \Big[\sqrt[3]{\frac{4}{3}\,\pi}\,T+c_2^2c_1\big(1+\|F_n\|^2_{L_2(D_T)}\big)\Big]c_2[2E(\tau)]^{1/2}[2E(\tau)]^{1/2}\le\\ &\le c_4\big(1+\|F_n\|^2_{L_2(D_T)}\big)E(\tau), \end{aligned} \tag{6.36}$$

where

$$c_4 = 2c_2\sqrt[3]{\frac{4}{3}\,\pi}\,T+2c_2^3c_1.$$

It follows from (6.8), (6.32), (6.34) and (6.36) that

$$I \le c_3E(\tau)+\frac{1}{2}\|F_n\|^2_{W_2^1(B_\tau(0,T))}+3c_4M\big(1+\|F\|^2_{L_2(D_T)}\big)E(\tau). \tag{6.37}$$

By (6.31) and (6.37), we have

$$E'(\tau)\le \alpha(\tau)E(\tau)+\beta(\tau)\le \alpha(\tau)E(\tau)+\beta(\tau),\quad \tau\le T. \tag{6.38}$$

Here

$$\begin{aligned} \alpha(\tau) &= c_3+3c_4M\big(1+\|F_n\|^2_{L_2(D_T)}\big),\\ \beta(\tau) &= \frac{1}{2}\|F_n\|^2_{W_2^1(B_\tau(0,T))}. \end{aligned} \tag{6.39}$$

From (6.28) we have $E(0)=0$. Therefore, multiplying both parts of the inequality (6.38) by $\exp[-\alpha(T)\tau]$ and integrating, in a standard way we obtain

$$\begin{aligned} E(\tau) &\le e^{\alpha(T)\tau}\int\limits_0^\tau e^{-\alpha(T)\sigma}\beta(\sigma)\,d\sigma \le e^{\alpha(T)\tau}\int\limits_0^\tau \beta(\sigma)\,d\sigma =\\ &= \frac{1}{2}\,e^{\alpha(T)\tau}\int\limits_0^\tau \|F_n\|^2_{W_2^1(B_\tau(0,T))}\,d\sigma \le \frac{1}{2}\,e^{\alpha(T)\tau}\|F_n\|^2_{W_2^1(D_\tau)}\le \end{aligned}$$

$$\leq \frac{1}{2} e^{\alpha(T)T} \|F_n\|^2_{W_2^1(D_T)}, \quad 0 \leq \tau \leq T. \tag{6.40}$$

By virtue of (6.6), we have

$$u_{n,tt} = \Delta u_n - f(u_n) + F_n. \tag{6.41}$$

It follows from (6.1) that

$$|f(u_n)| = \left| \int\limits_0^{u_n} f'(\sigma)\, d\sigma \right| \leq M\Big(|u_n| + \frac{1}{3}\,|u_n|^3\Big). \tag{6.42}$$

Squaring both parts of the equality (6.41) and using (6.42) and the inequality $\big(\sum\limits_{i=1}^n a_i\big)^2 \leq n \sum\limits_{i=1}^n a_i^2$, we obtain

$$\int\limits_{\Omega_\tau} u_{n,tt}^2\, dx \leq \frac{8}{9}\, M^2 \int\limits_{B_\tau(0,T)} |u_n|^6\, dx + 4 \int\limits_{B_\tau(0,T)} \big[(\Delta u_n)^2 + 2M^2 u_n^2 + F_n^2\big]\, dx,$$

whence by virtue of (6.8), (6.35) and the facts that $(\Delta v_n)^2 \leq 3 \sum\limits_{i=1}^3 u^2_{n,x_i x_i}$ and $(a+b)^6 \leq 2^5(a^6+b^6)$, we find that

$$\int\limits_{\Omega_\tau} u_{n,tt}^2\, dx \leq$$
$$\leq M^2 c_2^6 \|u_k\|^6_{\overset{\circ}{W}{}_2^1(B_\tau(0,T))} + 24E(\tau) + 8M^2 c_1\big(1 + \|F_n\|^2_{L_2(D_\tau)}\big) +$$
$$+4\|F_n\|^2_{L_2(B_\tau(0,T))} \leq M^2 c_2^6 2^5 c_1^3 \big[1 + \|F_n\|^6_{L_2(D_\tau)}\big] +$$
$$+8c_1 M^2\big(1 + \|F_n\|^2_{L_2(D_\tau)}\big) + 4\|F_n\|^2_{L_2(B_\tau(0,T))} + 24E(\tau). \tag{6.43}$$

By (6.40), from (6.43) it follows that

$$\int\limits_{D_T} u_{n,tt}^2\; dx\, dt = \int\limits_0^T d\tau \int\limits_{\Omega_\tau} u_{n,tt}^2\, dx \leq$$
$$\leq M^2 c_2^6 2^5 c_1^3 T\big[1 + \|F_n\|^6_{L_2(D_T)}\big] + 8c_1 M^2 T\big(1 + \|F_n\|^2_{L_2(D_T)}\big) +$$
$$+4\|F_n\|^2_{L_2(D_T)} + 12Te^{\alpha(T)T}\|F_n\|^2_{W_2^1(D_T)} \leq$$
$$\leq c_5 + c_6\|F_n\|^2_{L_2(D_T)} + c_7\|F_n\|^2_{L_2(D_T)} + c_8\|F_n\|^2_{W_2^1(D_T)}. \tag{6.44}$$

Here

$$\begin{gathered} c_5 = M^2 c_2^6 2^5 c_1^3 T + 8c_1 M^2 T, \quad c_6 = 8c_1 M^2 T + 4, \\ c_7 = M^2 c_2^6 2^5 c_1^3 T, \quad c_8 = 12Te^{\alpha(T)T}. \end{gathered} \tag{6.45}$$

From (6.8), (6.30), (6.40) and (6.44), we have

$$\|u_n\|_{W_2^2(D_T)} = \int\limits_0^T d\tau \times$$

$$\times \int\limits_{\Omega_\tau} \Big[u_n^2 + \Big(\frac{\partial u_n}{\partial t}\Big)^2 + \sum_{i=1}^{3} \Big(\frac{\partial u_n}{\partial x_i}\Big)^2 + u_{n,tt}^2 + \sum_{i=1}^{3} u_{n,x_i t}^2 + \sum_{i,k=1}^{3} u_{n,x_i x_k}^2 \Big]\, dx \le$$

$$\le \int\limits_0^T c_1\big(1 + \|F_n\|^2_{L_2(D_\tau)}\big)\, d\tau + \int\limits_{D_\tau} u_{n,tt}^2\, dx\, dt + \int\limits_0^T 2E(\tau)\, d\tau \le$$

$$\le c_1 T + c_1 T \|F\|^2_{L_2(D_T)} + c_5 + c_6 \|F_n\|^2_{L_2(D_T)} +$$

$$+ c_7 \|F_n\|^6_{L_2(D_T)} + c_8 \|F_n\|^2_{W_2^1(D_T)} + T e^{\alpha(T)T} \|F_n\|^2_{W_2^1(D_T)} \le$$

$$\le c_9 + c_{10} \|F_n\|^2_{L_2(D_T)} + c_{11} \|F_n\|^6_{L_2(D_T)} + c_{12} \|F_n\|^2_{W_2^1(D_T)}. \qquad (6.46)$$

By (6.45) we obtain

$$c_9 = c_1 T + M^2 c_2^6 2^5 c_1^3 T + 8 c_1 M^2 T, \quad c_{10} = c_1 T + 8 c_1 M^2 T + 4,$$
$$c_{11} = M^2 c_2^6 2^5 c_1^3 T, \quad c_{12} = 13 T e^{\alpha(T)T}. \qquad (6.47)$$

Taking into account the obvious inequality $\big(\sum\limits_{i=1}^{n} |a_i|\big)^{1/2} \le \sum\limits_{i=1}^{n} |a_i|^{1/2}$ along with (6.39) and (6.47), from (6.46) we get

$$\|u_n\|_{W_2^2(D_T)} \le c\Big[1 + \|F_n\|_{L_2(D_T)} + \|F_n\|^3_{L_2(D_T)} +$$
$$+ \|F\|_{W_2^1(D_T)} \exp\big(c \|F_n\|^2_{L_2(D_T)}\big)\Big], \qquad (6.48)$$

where the positive constant c does not depend on u_n and F_n. By virtue of (6.3), passing in the inequality (6.48) to limit as $n \to \infty$, we obtain the a priori estimate (6.5).

Thus Lemma 6.1 is proved completely. □

Remark 6.2. Note that when deducing the a priori estimate (6.5), we have used essentially the fact that the spatial dimension of the equation (1.1) was assumed to be three (see, e.g., the equation (6.33)). Moreover, the same fact will be used below in proving the compactness of the corresponding to $f(u)$ nonlinear Nemytski operator.

Remark 6.3. Before we proceed to proving the global solvability of the nonlinear problem (1.1), (1.2) in the class W_2^2 on the basis of the a priori estimate (6.5), we will consider the same issue in the linear case, when $f = 0$, i.e., for the problem

$$L_0 u(x,t) = F(x,t), \;\; (x,t) \in D_T \;\; \big(L := \square\big), \qquad (6.49)$$

$$u(x,t) = 0, \;\; (x,t) \in S_T. \qquad (6.50)$$

In this case, for $F \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ we introduce the notion of a strong generalized solution $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ of the problem (6.49), (6.50) of the class W_2^2 in the domain D_T for which there exists a sequence of functions $u_n \in C^\infty(\overline{D}_T)$ satisfying the condition (6.50) and

$$\lim_{n\to\infty} \|u_n - u\|_{W_2^2(D_T)} = 0, \quad \lim_{n\to\infty} \|L_0 u_n - F\|_{W_2^1(D_T)} = 0. \tag{6.51}$$

Remark 6.4. Following the proof of the a priori estimate (6.5), it is not difficult to see that for $f = 0$, i.e., for a strong generalized solution of the linear problem (6.49), (6.50) of the class W_2^2 in the domain D_T the estimate

$$\|u\|_{W_2^2(D_T)} \le c_0 \|F\|_{W_2^1(D_T)} \tag{6.52}$$

is valid with a positive constant c_0 independent of u and F.

Since the space $C_0^\infty(\overline{D}_T, S_T) := \{F \in C^\infty(\overline{D}_T) : \ \operatorname{supp} F \cap S_T = \varnothing\}$ of infinitely differentiable in $\overline{D}_T$ functions vanishing in some neighborhood (its own for each such function) of the set S_T is dense in $\overset{\circ}{W}{}_2^1(D_T, S_T)$, for a function $F \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ there exists a sequence of functions $F_n \in C_0^\infty(\overline{D}_T, S_T)$ such that $\lim\limits_{n\to\infty} \|F_n - F\|_{W_2^1(D_T)} = 0$. For fixed n, extending the function F_n from the domain D_T into the layer $\Sigma_T := \{(x,t) \in \mathbb{R}^4 : 0 < t < T\}$ by zero and leaving the same notation, we have $F_n \in C^\infty(\overline{\Sigma}_T)$, for which the support $\operatorname{supp} F_n \subset D_\infty : \ t > |x|$. Denote by u_n a solution of the following linear Cauchy problem: $L_0 u_n = F_n$, $u_n|_{t=0} = 0$, $\frac{\partial u_n}{\partial t}\big|_{t=0} = 0$ in the layer Σ_T which, as is known, exists, is unique and belongs to the space $C^\infty(\overline{\Sigma}_T)$ [**17**, p. 192]. Note that since $\operatorname{supp} F_n \subset D_\infty$ and $u_n|_{t=0} = \frac{\partial u_n}{\partial t}\big|_{t=0} = 0$, taking into account the geometry of the domain of dependence of solution of the linear wave equation, we have $\operatorname{supp} u_n \subset D_\infty$ [**17**, p. 191]. Leaving for the restriction of the function u_n to the domain D_T the same notation, it can be easily seen that $u_n \in C^\infty(\overline{D}_T)$, $u_n|_{S_T} = 0$, and owing to the estimate (6.52) we have

$$\|u_n - u\|_{W_2^2(D_T)} \le c_0 \|F_n - F\|_{W_2^1(D_T)}. \tag{6.53}$$

Since the sequence $\{F_n\}$ is fundamental in $\overset{\circ}{W}{}_2^1(D_T, S_T)$, by virtue of (6.53) the sequence $\{u_n\}$ will be fundamental in the complete space

$$\overset{\circ}{W}{}_2^2(D_T, S_T) := \Big\{u \in W_2^2(D_T) : \ u\big|_{S_T} = 0\Big\}.$$

Therefore there exists a function $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ such that $\lim\limits_{n\to\infty} \|u_n - u\|_{W_2^2(D_T)} = 0$, and hence, due to the fact that $L_0 u_n = F_n \to F$ in the space $W_2^1(D_T)$, the function u will, by Remark 6.3, be a strong generalized solution of the problem (6.49), (6.50) of the class W_2^2 in the space D_T. According to what has been said, for the solution u of the problem (6.49), (6.50) we can write $u = L_0^{-1} F$, where $L_0^{-1} : \overset{\circ}{W}{}_2^1(D_T, S_T) \to \overset{\circ}{W}{}_2^2(D_T, S_T)$ is a linear continuous operator whose norm admits, by virtue of (6.52), the estimate

$$\|L_0^{-1}\|_{\overset{\circ}{W}{}_2^1(D_T,S_T) \to \overset{\circ}{W}{}_2^2(D_T,S_T)} \le c_0. \tag{6.54}$$

Remark 6.4. If the condition (6.1) is fulfilled, the Nemytski operator $\mathcal{N} : \overset{\circ}{W}{}_2^2(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ acting by the formula $\mathcal{N}u = -f(u)$ is continuous and compact. This assertion is a consequence of the following facts: (1) owing to $D_T \subset \mathbb{R}^4$ for $n = 4$, the embedding operator $I_1 : \overset{\circ}{W}{}_2^2(D_T, S_T) \to L_q(D_T)$ is continuous and compact for every $q \geq 1$ [**49**, p. 84]; (2) the embedding operator $I_2 : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_p(D_T)$ is continuous for $1 < p < 4$ [**49**, p. 83]; (3) the nonlinear Nemytski operator $\mathcal{H}$ acting by the formula $\mathcal{H}u = h(x, u)$, where the function $h = h(x, \xi)$ possesses the Carathéodory property, is continuous from the space $L_p(D_T)$ into $L_r(D_T)$, $p \geq 1$, $r \geq 1$, if and only if $|h(x, \xi)| \leq d(x) + \delta|\xi|^{p/r}$ $\forall\, \xi \in (-\infty, \infty)$, where $d \in L_r(D_T)$, and $\delta = \text{const} \geq 0$ [**48**, p. 66]; (4) according to the condition (6.1), the inequality

$$|f(u)| \leq M + 2M|u|^3, \quad u \in \mathbb{R},$$

holds, and hence according to the above-said, if $u_n \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, then $f(u_n) \to f(u)$ in the space $L_2(D_T)$ and $f'(u_n) \to f'(u)$ in the space $L_6(D_T)$; (5) if $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$, then $f'(u) \in L_q(D_T)$ for $q \geq 1$, and since $\frac{\partial u}{\partial x_i} \in W_2^1(D_T)$, therefore $\frac{\partial u}{\partial x_i} \in L_p(D_T)$ for $1 < p < 4$, and, in particular, $\frac{\partial u}{\partial x_i} \in L_3(D_T)$; (6) if $f_i \in L_{p_i}(D_T)$, $i = 1, 2$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r}$, $p_i > 1$, $r > 1$, then $f_1 f_2 \in L_r(D_T)$ [**58**, p. 45]; in particular, for $p_1 = 6$, $p_2 = 3$, $r = 2$ $(1/6 + 1/3 = 1/2)$, $f_1 = f'(u)$, $f_2 = \frac{\partial u}{\partial x_i}$, $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$, we obtain $\frac{\partial \mathcal{N}u}{x_i} = -f'(u)\,\frac{\partial u}{\partial x_i} \in L_2(D_T)$, $i = 1, 2, 3$; analogously, we have $\frac{\partial \mathcal{N}u}{\partial t} \in L_2(D_T)$, and hence $\mathcal{N}u \in W_2^1(D_T)$ if $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$. We will show below that in fact $\mathcal{N}u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$.

Indeed, let X be some bounded subset of the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$, and let $\{u_n\}$ be an arbitrary subset of elements from X. Since the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$ is compactly embedded into the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ [**48**, p. 183], there exist a subsequence $\{u_{n_k}\}$ and a function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ such that

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{L_2(D_T)} = \lim_{k\to\infty} \Big\| \frac{u_{n_k}}{\partial t} - \frac{\partial u}{\partial t} \Big\|_{L_2(D_T)} = \\ = \lim_{k\to\infty} \Big\| \frac{\partial u_{n_k}}{\partial x_i} - \frac{\partial u}{\partial x_i} \Big\|_{L_2(D_T)} = 0. \quad (6.55)$$

On the other hand, according to what has been said there exists a subsequence of the sequence $\{u_{n_k}\}$ (with the same notation) such that

$$\lim_{k\to\infty}\big\|f'(u_{n_k})-v_0\big\|_{L_6(D_T)}=0,\quad \lim_{k\to\infty}\Big\|\frac{\partial u_{n_k}}{\partial x_i}-v_i\Big\|_{L_2(D_T)}=0,\quad i=1,2,3, \tag{6.56}$$
$$\lim_{k\to\infty}\big\|f(u_{n_k})-v\big\|_{L_2(D_T)}=0,\quad \lim_{k\to\infty}\Big\|\frac{\partial u_{n_k}}{\partial t}-v_4\Big\|_{L_3(D_T)}=0,$$

where v_0, v, v_i, $i=1,\dots,4$, are some functions respectively from the spaces $L_6(D_T)$, $L_2(D_T)$ for v_0, v, and $L_3(D_T)$ for v_i. Using the definition of generalized derivatives due to Sobolev, from (6.55) and (6.56), reasoning in a standard way, we obtain

$$v_0=f'(u),\ \ v=f(u),\ \ v_i=\frac{\partial u}{\partial x_i}\,,\ \ i=1,2,3,\ \ v_4=\frac{\partial u}{\partial t}\,. \tag{6.57}$$

Let now show that

$$\begin{aligned}&\lim_{k\to\infty}\Big\|\frac{\partial \mathcal{N}u_{n_k}}{\partial x_i}-\frac{\partial \mathcal{N}u}{\partial x_i}\Big\|_{L_2(D_T)}=0,\ \ i=1,2,3,\\ &\lim_{k\to\infty}\Big\|\frac{\partial \mathcal{N}u_{n_k}}{\partial t}-\frac{\partial \mathcal{N}u}{\partial t}\Big\|_{L_2(D_T)}=0.\end{aligned} \tag{6.58}$$

Indeed, using Hölder's inequality for $p=3$, $q=3/2$ $(1/p+1/q=1)$, we will have

$$\begin{aligned}\Big\|\frac{\partial \mathcal{N}u_{n_k}}{\partial x_i}-\frac{\partial \mathcal{N}u}{\partial x_i}\Big\|_{L_2(D_T)}&=\\ =\int\limits_{D_T}\Big(f'(u_{n_k})\,\frac{\partial u_{n_k}}{\partial x_i}-f'(u)\,\frac{\partial u}{\partial x_i}\Big)^2\,dx\,dt&=\\ =\int\limits_{D_T}\Big[\big(f'(u_{n_k})-f'(u)\big)\,\frac{\partial u_{n_k}}{\partial x_i}+f'(u)\Big(\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\Big)\Big]^2\,dx\,dt&\leq\\ \leq 2\int\limits_{D_T}\big(f'(u_{n_k})-f'(u)\big)^2\Big(\frac{\partial u_{n_k}}{\partial x_i}\Big)^2\,dx\,dt&+\\ +2\int\limits_{D_T}\big(f'(u)\big)^2\Big(\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\Big)^2\,dx\,dt&\leq\\ \leq 2\Big\|\big(f'(u_{n_k})-f'(u)\big)^2\Big\|_{L_3(D_T)}\Big\|\Big(\frac{\partial u_{n_k}}{\partial x_i}\Big)^2\Big\|_{L_{3/2}(D_T)}&+\\ +2\big\|(f'(u))^2\big\|_{L_3(D_T)}\Big\|\Big(\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\Big)^2\Big\|_{L_{3/2}(D_T)}&=\\ =2\big\|f'(u_{n_k})-f'(u)\big\|^2_{L_6(D_T)}\Big\|\frac{\partial u_{n_k}}{\partial x_i}\Big\|^2_{L_3(D_T)}&+\\ +2\big\|(f'(u))\big\|^2_{L_6(D_T)}\Big\|\frac{\partial u_{n_k}}{\partial x_i}-\frac{\partial u}{\partial x_i}\Big\|^2_{L_3(D_T)}&.\end{aligned} \tag{6.59}$$

By virtue of (6.56), the sequence $\big\{\|\frac{\partial u_{n_k}}{\partial x_i}\|^2_{L_3(D_T)}\big\}$ is bounded. Therefore from (6.59), in view of (6.56) and (6.57), there follow the first three equalities from (6.58) for $i = 1, 2, 3$. The last equality from (6.58) is proved analogously. Thus the fact that $\mathcal{N}u_{n_k} \to \mathcal{N}u$ in the space $W_2^1(D_T)$ follows directly from (6.56), (6.57) and (6.58). So we have proved that the operator $\mathcal{N}$ from Remark 6.4 is compact, acting from the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$ to the space $W_2^1(D_T)$. This implies that this operator is also continuous since the above-mentioned spaces, being the Hilbert ones, are reflexive [**48**, p. 182]). Finally, the fact that the image $\mathcal{N}(\overset{\circ}{W}{}_2^2(D_T, S_T))$ is actually a subspace of the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$ follows from the following reasoning. If $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$, then there exists a sequence $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T) := \big\{u \in C^2(\overline{D}_T) : \ u|_{S_T} = 0\big\}$ such that $u_n \to u$ in the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$. But, according to the above-said, $\mathcal{N}u_n \to \mathcal{N}u$ in the space $W_2^1(D_T)$, and since $\mathcal{N}u_n = -f(u_n) \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T) \subset \overset{\circ}{W}{}_2^1(D_T, S_T)$ (recall that $f(0) = 0$ by the condition (6.1)), therefore taking into account that the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ is complete, we obtain $\mathcal{N}(\overset{\circ}{W}{}_2^2(D_T, S_T)) \subset \overset{\circ}{W}{}_2^1(D_T, S_T)$, and hence the operator $\mathcal{N} : \overset{\circ}{W}{}_2^2(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is continuous and compact.

Remark 6.5. As is mentioned in Remark 6.1, from the first equality (6.3) it follows that $\lim\limits_{n\to\infty} \|f(u_n) - f(u)\|_{L_2(D_T)} = 0$. The latter is a direct consequence of the assertion we formulated in Remark 6.4. From this reasoning it immediately follows that if $F \in \overset{\circ}{W}{}_2^2(D_T, S_T)$, then the function $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ is, by virtue of (6.54), a strong generalized solution of the problem (1.1), (1.2) of the class W_2^2 if and only if this function is a solution of the functional equation

$$u = L_0^{-1}\big(-f(u) + F\big) \tag{6.60}$$

in the space $\overset{\circ}{W}{}_2^2(D_T, S_T)$.

We rewrite the equation (6.60) in the form

$$u = Au := L_0^{-1}(\mathcal{N}u + F), \tag{6.61}$$

where the operator $\mathcal{N} : \overset{\circ}{W}{}_2^2(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is, by Remark 6.5, continuous and compact, and consequently, owing to (6.54), the operator $A : \overset{\circ}{W}{}_2^2(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is likewise continuous and compact. At the same time, by Lemma 6.1, for any parameter $\tau \in [0, 1]$ and every solution $u \in \overset{\circ}{W}{}_2^2(D_T, S_T)$ of the equation $u = \tau Au$ with the parameter τ the following a priori estimate is valid:

$$\|u\|_{W_2^2(D_T)} \le$$
$$\le c\Big[1+\tau\|F\|_{L_2(D_T)}+\tau^3\|F\|^3_{L_2(D_T)}+\tau\|F\|_{W_2^1(D_T)}\exp\big(c\tau^2\|F\|^2_{L_2(D_T)}\big)\Big] \le$$

$$\le c\Big[1+\|F\|_{L_2(D_T)}+\|F\|^3_{L_2(D_T)}+\|F\|_{W_2^1(D_T)}\exp\big(c\|F\|^2_{L_2(D_T)}\big)\Big]=$$
$$=C_0(c,F),$$

where $C_0 = C_0(c,F)$ is a positive constant not depending on u and the parameter τ.

Therefore, by the Leray–Schauder theorem [**66**, p. 375] the equation (6.61) and hence the problem (1.1), (1.2) has at least one strong generalized solution of the class W_2^2 in the domain D_T. Thus, by Remark 6.1 and Definitions 6.1, 6.2 and 6.3, the following theorem is valid.

Theorem 6.1. *Let* $n = 3$, $F \in L_{2,loc}(D_\infty)$ *and* $F \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ *for any* $T > 0$. *Then the problem* (1.1), (1.2) *is globally solvable in the class* W_2^2, *i.e., for any* $T > 0$ *this problem has a solution of the class* W_2^2 *in the domain* D_T *in the sense of Definition* 6.1.

Assume

$$\overset{\circ}{W}{}^k_{2,loc}(D_\infty,S_\infty)=\Big\{v\in L_{2,loc}(D_\infty):\ v\big|_{D_T}\in \overset{\circ}{W}{}^k_2(D_T,S_T)\ \forall\, T>0\Big\}.$$

In the next section we will prove the uniqueness of solution of the problem (1.1), (1.2) of the class W_2^2 in the sense of Definition 6.1. This circumstance along with Theorem 6.1 allows us to conclude that the theorem below is valid.

Theorem 6.2. *Let* $n = 3$, $F \in \overset{\circ}{W}{}^1_{2,loc}(D_\infty,S_\infty)$. *Then the problem* (1.1), (1.2) *has in the light cone* D_∞ *of future a global solution* u *from the space* $\overset{\circ}{W}{}^1_{2,loc}(D_T,S_T)$ *which satisfies the equation* (1.1) *almost everywhere in the domain* D_∞ *as well as the boundary condition* (1.2) *in the sense of the trace theory.*

7. The Uniqueness of a Solution of the Problem (1.1), (1.2) in the Class W_2^2

Lemma 7.1. *Let* $n = 3$ *and the condition* (6.1) *be fulfilled. Then the problem* (1.1), (1.2) *cannot have more than one solution of the class* W_2^2 *in the domain* D_T *in the sense of Definition* 1.1.

Proof. Let u_1 and u_2 be two solutions of the problem (1.1), (1.2) of the class W_2^2 in the domain D_T in the sense of Definition 6.1. Then for the difference $u = u_2 - u_1$ we have

$$\frac{\partial^2 u}{\partial t^2}-\Delta u=-\big(f(u_2)-f(u_1)\big), \tag{7.1}$$

$$u,\ u_1,\ u_2\in \overset{\circ}{W}{}^2_2(D_T,S_T). \tag{7.2}$$

Multiplying both parts of the equality (7.1) by u_t and integrating over the domain D_τ, just as in obtaining (6.13) we have

$$\int\limits_{\Omega_\tau}\Big[u_t^2+\sum_{i=1}^3 u_{x_i}^2\Big]\,dx=-2\int\limits_{D_\tau}\big(f(u_2)-f(u_1)\big)u_t\,dx\,dt,\quad 0<\tau\le T. \tag{7.3}$$

We estimate the right-hand side of the equality (7.3). By (6.1) we have

$$\bigg|-2\int\limits_{D_\tau}\big(f(u_2)-f(u_1)\big)u_t\,dx\,dt\bigg|=$$

$$=2\bigg|\int\limits_{D_\tau}\Big[(u_2-u_1)\int\limits_0^1 f'\big(u_1+s(u_2-u_1)\big)\,ds\Big]u_t\,dx\,dt\bigg|\le$$

$$\le 2M\int\limits_{D_\tau}|u_2-u_1|\big(1+2|u_1|^2+2|u_2|^2\big)|u_t|\,dx\,dt=$$

$$=4M\int\limits_{D_\tau}\big(|u_1|^2+|u_2|^2\big)|u|\,|u_t|\,dx\,dt+2M\int\limits_{D_\tau}|u|\,|u_t|\,dx\,dt=$$

$$=4M\int\limits_0^\tau d\sigma\int\limits_{\Omega_\sigma}\big(|u_1|^2+|u_2|^2\big)|u|\,|u_t|\,dx+2M\int\limits_0^\tau d\sigma\int\limits_{\Omega_\sigma}|u|\,|u_t|\,dx. \tag{7.4}$$

Using Hölder's inequality for $p_1=3$, $p_2=6$, $p_3=2$ ($1/3+1/6+1/2=1$) and the Schwarz inequality, we obtain

$$\int\limits_{\Omega_\sigma}\big(|u_1|^2+|u_2|^2\big)|u|\,|u_t|\,dx\le$$

$$\le\big(\|u_1^2\|_{L_3(\Omega_\sigma)}+\|u_2^2\|_{L_3(\Omega_\sigma)}\big)\|u\|_{L_6(\Omega_\sigma)}\|u_t\|_{L_2(\Omega_\sigma)}=$$

$$=\big(\|u_1\|^2_{L_6(\Omega_\sigma)}+\|u_2\|^2_{L_6(\Omega_\sigma)}\big)\|u\|_{L_6(\Omega_\sigma)}\|u_t\|_{L_2(\Omega_\sigma)},\quad 0<\sigma\le T, \tag{7.5}$$

$$\int\limits_{\Omega_\sigma}|u|\,|u_t|\,dx\le\|u\|_{L_2(\Omega_\sigma)}+\|u_t\|_{L_2(\Omega_\sigma)}. \tag{7.6}$$

By the embedding theorems [**49**, pp. 69, 78], we have

$$\begin{aligned}
&\big\|v\big|_{\Omega_\sigma}\big\|_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)}\le C(T)\|v\|_{W_2^2(D_T)}\quad\big(\dim\Omega_\sigma=3,\ \dim D_T=4\big),\\
&\big\|v\big|_{\Omega_\sigma}\big\|_{L_6(\Omega_\sigma)}\le\beta\big\|v\big|_{\Omega_\sigma}\big\|_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)}\le\beta C(T)\|v\|_{W_2^2(D_T)},\\
&\big\|v_t\big|\Omega_\sigma\big\|_{L_2(\Omega_\sigma)}\le C_1(T)\|v\|_{W_2^2(D_T)},
\end{aligned} \tag{7.7}$$

where the positive constants $C(T)$, $C_1(T)$ and β do not depend on the parameter $\sigma\in(0,T]$ and the function v.

Due to (7.2), from (7.5), (7.6) and (7.7) it follows that

$$\int\limits_{\Omega_\sigma} \big(|u_1|^2+|u_2|^2\big)|u|\,|u_t|\,dx \le 2M_4\|u\|_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)}\|u_t\|_{L_2(\Omega_\sigma)} \le$$

$$\le M_4\Big(\|u\|^2_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)}+\|u_t\|^2_{L_2(\Omega_\sigma)}\Big) = M_4\int\limits_{\Omega_\sigma}\Big[u_t^2+\sum_{i=1}^3 u_{x_i}^2\Big]\,dx, \tag{7.8}$$

$$\int\limits_{\Omega_\sigma}|u|\,|u_t|\,dx \le \frac12\Big(\|u\|^2_{L_2(\Omega_\sigma)}+\|u_t\|^2_{L_2(\Omega_\sigma)}\Big)\le$$

$$\le \frac12\int\limits_{\Omega_\sigma}[u^2+u_t^2]\,dx \le M_5\int\limits_{\Omega_\sigma}\Big[u_t^2+\sum_{i=1}^3 u_{x_i}^2\Big]\,dx, \tag{7.9}$$

where

$$M_4=\beta^3C(T)\max\Big(\|u_1\|^2_{W_2^2(D_T)},\|u_2\|^2_{W_2^2(D_T)}\Big)<+\infty,\ \ M_5=\text{const}>0;$$

here we have used the fact that in the space $\overset{\circ}{W}{}_2^1(\Omega_\sigma)$ the norm

$$\|u\|_{\overset{\circ}{W}{}_2^1(\Omega_\sigma)}=\bigg\{\int\limits_{\Omega_\sigma}\Big[u^2+\sum_{i=1}^3 u_{x_i}^2\Big]\,dx\bigg\}^{1/2}$$

is equivalent to the norm [**49**, p. 62]

$$\|u\|=\bigg\{\int\limits_{\Omega_\sigma}\Big[\sum_{i=1}^3 u_{x_i}^2\Big]\,dx\bigg\}^{1/2}.$$

Assuming $w(\tau)=\int\limits_{\Omega_\sigma}\big[u_t^2+\sum_{i=1}^3 u_{x_i}^2\big]\,dx$ and taking into account (7.3), (7.4), (7.8) and (7.9), we obtain

$$w(\tau)\le M_6\int\limits_0^\tau w(\sigma)\,d\sigma,\ \ M_6=\text{const}>0,$$

whence by Gronwall's lemma we find that $w=0$, i.e., $u_t=u_{x_i}=0$, $i=1,2,3$. Consequently, $u=\text{const}$, and since $u|_{S_T}=0$, therefore $u=0$, i.e., $u_2=u_1$, which proves our lemma. □

CHAPTER 3

Sobolev's Problem for Multi-Dimensional Nonlinear Wave Equations in a Conic Domain of Time Type

1. Statement of the Problem

Consider the nonlinear wave equation of the type

$$L_\lambda u := \frac{\partial^2 u}{\partial t^2} - \Delta u + \lambda |u|^p u = F, \tag{1.1}$$

where $\lambda \neq 0$ and $p > 0$ are given real numbers, $F = F(x,t)$ is a given and u is an unknown real function, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$.

Let D be a conic domain in the space $\mathbb{R}^{n+1}$ of the variables $x = (x_1, \dots, x_n)$ and t, i.e., D contains, along with the point $(x,t) \in D$, the whole ray $\ell :\ (\tau x, \tau t)$, $0 < \tau < \infty$. By S we denote the conic surface ∂D. D is assumed to be homeomorphic to the conic domain $\omega :\ t > |x|$, and $S \setminus O$ is a connected n-dimensional manifold of the class C^∞, where $O = (0, \dots, 0, 0)$ is the vertex of S. Assume also that D lies in the half-space $t > 0$, and $D_T := \big\{(x,t) \in D :\ t < T\big\}$, $S_T := \big\{(x,t) \in S :\ t \leq T\big\}$, $T > 0$. In case $T = \infty$, it is obvious that $D_\infty = D$ and $S_\infty = S$.

For the equation (1.1), we consider the problem: find in the domain D_T a solution $u(x,t)$ of that equation according to the boundary condition

$$u\big|_{S_T} = g, \tag{1.2}$$

where g is a given real function on S_T.

In case the conic manifold $S = \partial D$ is time-oriented, i.e.,

$$\Big(\nu_0^2 - \sum_{i=1}^{n} \nu_i^2\Big)\Big|_S < 0, \ \ \nu_0\big|_S < 0, \tag{1.3}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ is the unit vector of the outer normal to $S \setminus O$, and the equation is linear, i.e., for $\lambda = 0$, the problem (1.1), (1.2) has been formulated and investigated by S. L. Sobolev in [**63**]. Note that in the case (1.3), the problem (1.1), (1.2) can be considered as a multi-dimensional version of the second Darboux problem [**2**, pp. 228, 233] for the nonlinear equation (1.1).

Below, the condition (1.3) will be assumed to be fulfilled.

Remark 1.1. The embedding operator $I : W_2^1(D_T) \to L_q(D_T)$ is linear, continuous and compact for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [**49**, p. 81]. At the same time, the Nemytski operator $K : L_q(D_T) \to L_2(Q_T)$ acting by the formula $Ku := \lambda |u|^p u$ is continuous and bounded if $q \geq 2(p+1)$ [**47**, pp. 349], [**48**, pp. 66, 67]. Thus if $p < \frac{2}{n-1}$, i.e., $2(p+1) < \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < 2(p+1) \leq q < \frac{2(n+1)}{n-1}$ and hence the operator

$$K_0 = KI : W_2^1(D_T) \to L_2(D_T) \tag{1.4}$$

is continuous and compact. In addition, from $u \in W_2^1(D_T)$ it follows that $u \in L_{p+1}(D_T)$.

As mentioned above, it is assumed that here and in the sequel $p > 0$.

Definition 1.1. Let $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ and $0 < p < \frac{2}{n-1}$. The function $u \in W_2^1(D_T)$ is said to be a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T if there exists a sequence of functions $u_k \in C^2(\overline{D}_T)$ such that $u_k \to u$ in the space $W_2^1(D_T)$, $L_\lambda u_k \to F$ in the space $L_2(D_T)$, and $u_k|_{S_T} \to g$ in the space $W_2^1(S_T)$. Besides, the convergence of the sequence $\{\lambda |u_k|^p u_k\}$ to the function $\lambda |u|^p u$ in the space $L_2(D_T)$ as $u_k \to u$ in the space $W_2^1(D_T)$ follows from Remark 1.1. Note that since $|u|^{p+1} \in L_2(D_T)$ and the domain D_T is bounded, the function $u \in L_{p+1}(D_T)$.

Definition 1.2. Let $0 < p < \frac{2}{n-1}$, $F \in L_{2,loc}(D)$, $g \in W_{2,loc}^1(S)$, and $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ for any $T > 0$. We say that the problem (1.1), (1.2) is globally solvable in the class W_2^1 if for any $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T.

2. A Priori Estimate of a Solution of the Problem (1.1), (1.2) in the Class W_2^1

Lemma 2.1. *Let $\lambda > 0$, $0 < p < \frac{2}{n-1}$, $F \in L_2(D_T)$, and $g \in W_2^1(S_T)$. Then for every strong generalized solution u of the problem* (1.1), (1.2) *of the class W_2^1 in the domain D_T the a priori estimate*

$$\|u\|_{W_2^1(D_T)} \leq c\Big(\|F\|_{L_2(D_T)} + \|g\|_{W_2^1(S_T)}\Big) \tag{2.1}$$

is valid with a positive constant c not depending on u and F.

Proof. Let $u \in W_2^1(D_T)$ be a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T. Then by Definition 1.1 there exists a sequence of function $u_k \in C^2(\overline{D}_T)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{W_2^1(D_T)} = 0, \quad \lim_{k\to\infty} \|L_\lambda u_k - F\|_{L_2(D_T)} = 0, \tag{2.2}$$

$$\lim_{k\to\infty} \big\|u_k\big|_{S_T} - g\|_{W_2^1(S_T)} = 0. \tag{2.3}$$

Consider the function $u_k \in C^2(\overline{D}_T)$ as a solution of the problem

$$L_\lambda u_k = F_k, \tag{2.4}$$

$$u_k\big|_{S_T} = g_k. \tag{2.5}$$

Here

$$F_k := L_\lambda u_k, \quad g_k := u_k\big|_{S_T}. \tag{2.6}$$

Multiplying both parts of the equation (2.7) by $\frac{\partial u_k}{\partial t}$ and integrating over the domain $D_\tau := \big\{(x,t) \in D: \ t < \tau\big\}$, $0 < \tau \leq T$, we obtain

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_k}{\partial t}\Big)^2 \, dx\, dt - \int\limits_{D_\tau} \Delta u_k \, \frac{\partial u_k}{\partial t} \, dx\, dt + \\ + \frac{\lambda}{p+2}\int\limits_{D_\tau} \frac{\partial}{\partial t} \, |u_k|^{p+2} \, dx\, dt = \int\limits_{D_\tau} F_k \, \frac{\partial u_k}{\partial t} \, dx\, dt. \tag{2.7}$$

Assume $\Omega_\tau := D \cap \{t = \tau\}$. Clearly, $\Omega_\tau = D_\tau \cap \{t = \tau\}$ for $0 < \tau < T$. Then taking into account the equality (2.5) and our reasoning in Chapter II for (2.8), we integrate the left-hand side (2.7) by parts and obtain

$$\int\limits_{D_\tau} F_k \, \frac{\partial u_k}{\partial t} \, dx\, dt = \\ = \int\limits_{S_\tau} \frac{1}{2\nu_0}\bigg[\sum_{i=1}^n \Big(\frac{\partial u_k}{\partial x_i}\,\nu_0 - \frac{\partial u_k}{\partial t}\,\nu_i\Big)^2 + \Big(\frac{\partial u_k}{\partial t}\Big)^2\Big(\nu_0^2 - \sum_{j=1}^n \nu_j^2\Big)\bigg] ds + \\ + \frac{1}{2}\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_k}{\partial x_i}\Big)^2\bigg] dx + \\ + \frac{\lambda}{p+2}\int\limits_{S_T} |g_k|^{p+2}\nu_0 \, ds + \frac{\lambda}{p+2}\int\limits_{\Omega_\tau} |u_k|^{p+2} \, dx, \tag{2.8}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ is the unit vector of the outer normal to ∂D_τ.

By virtue of $\lambda > 0$ and (1.3), it follows from (2.8) that

$$\frac{1}{2}\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_k}{\partial x_i}\Big)^2\bigg] dx \leq \\ \leq \int\limits_{S_\tau} \frac{1}{2|\nu_0|}\bigg[\sum_{i=1}^n \Big(\frac{\partial u_k}{\partial x_i}\,\nu_0 - \frac{\partial u_k}{\partial t}\,\nu_i\Big)^2\bigg] ds + \\ + \frac{\lambda}{p+2}\int\limits_{S_T} |g_k|^{p+2}\nu_0 \, ds + \int\limits_{D_\tau} F_k \, \frac{\partial u_k}{\partial t} \, dx\, dt. \tag{2.9}$$

Since S is a conic manifold, $\sup\limits_{S\setminus O} |\nu_0|^{-1} = \sup\limits_{S\cap\{t=1\}} |\nu_0|^{-1}$. At the same time, $S \setminus O$ is a smooth manifold and $S \cap \{t = 1\} = \partial\Omega_{\tau=1}$ is compact.

Therefore, taking into account that ν_0 is a continuous function on $S \setminus O$, we have

$$M_0 := \sup_{S\setminus O} |\nu_0|^{-1} = \sup_{S\cap\{t=1\}} |\nu_0|^{-1} < +\infty, \quad |\nu_0| \leq |\nu| = 1. \tag{2.10}$$

Noticing that $\big(\nu_0 \frac{\partial}{\partial x_i} - \nu_i \frac{\partial}{\partial t}\big)$, $i = 1, \dots, n$, is an inner differential operator on S_T, by virtue of (2.5) we can easily see that

$$\int\limits_{S_\tau} \Big[\sum_{i=1}^n \Big(\frac{\partial u_k}{\partial x_i}\, \nu_0 - \frac{\partial u_k}{\partial t}\, \nu_i \Big)^2 \Big]\, ds \leq \big\| u_k \big|_{S_T} \big\|^2_{W_2^1(S_T)} = \|g_k\|^2_{W_2^1(S_T)}. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\int\limits_{S_\tau} \frac{1}{2|\nu_0|} \Big[\sum_{i=1}^n \Big(\frac{\partial u_k}{\partial x_i}\, \nu_0 - \frac{\partial u_k}{\partial t}\, \nu_i \Big)^2 \Big]\, ds \leq \frac{1}{2}\, M_0 \|g_k\|^2_{W_2^1(S_T)}. \tag{2.12}$$

Taking into account the Cauchy inequality $2F_k \frac{\partial u_k}{\partial t} \leq |F_k|^2 + \big(\frac{\partial u_k}{\partial t}\big)^2$, by virtue of (2.12) from (2.9) we find that

$$\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial u_k}{\partial t} \Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_k}{\partial x_i} \Big)^2 \Big]\, dx \leq$$

$$\leq M_0 \|g_k\|^2_{W_2^1(S_T)} + \frac{2}{p+2} \int\limits_{S_T} |g_k|^{p+2}\, ds + \int\limits_{D_\tau} \Big(\frac{\partial u_k}{\partial t} \Big)^2\, dx\, dt + \int\limits_{D_\tau} F_k^2\, dx\, dt. \tag{2.13}$$

If $t = \gamma(x)$ is the equation of the conic surface S, then by (2.5) we have

$$u_k(x,\tau) = u_k(x,\gamma(x)) + \int\limits_{\gamma(x)}^{\tau} \frac{\partial}{\partial t}\, u_k(x,s)\, ds =$$

$$= g_k(x) + \int\limits_{\gamma(x)}^{\tau} \frac{\partial}{\partial t}\, u_k(x,s)\, ds, \quad (x,\tau) \in \Omega_\tau.$$

Squaring both parts of the obtained equality, integrating over the domain Ω_τ and using the Schwarz inequality, we obtain

$$\int\limits_{\Omega_\tau} u_k^2\, dx \leq 2 \int\limits_{\Omega_\tau} g_k^2(x,\gamma(x))\, dx + 2 \int\limits_{\Omega_\tau} \Big(\int\limits_{\gamma(x)}^{\tau} \frac{\partial}{\partial t}\, u_k(x,s)\, ds \Big)^2\, dx \leq$$

$$\leq 2 \int\limits_{S_\tau} g_k^2\, ds + 2 \int\limits_{\Omega_\tau} (\tau - \gamma(x)) \Big[\int\limits_{\gamma(x)}^{\tau} \Big(\frac{\partial u_k}{\partial t} \Big)^2\, ds \Big]\, dx \leq$$

$$\leq 2 \int\limits_{S_\tau} g_k^2\, ds + 2T \int\limits_{\Omega_\tau} \Big[\int\limits_{\gamma(x)}^{\tau} \Big(\frac{\partial u_k}{\partial t} \Big)^2\, ds \Big]\, dx =$$

$$= 2\int\limits_{S_\tau} g_k^2\,ds + 2T\int\limits_{D_\tau}\Big(\frac{\partial u_k}{\partial t}\Big)^2\,dx\,dt. \tag{2.14}$$

Adding the inequalities (2.13) and (2.14), we get

$$\int\limits_{\Omega_\tau}\Big[u_k^2+\Big(\frac{\partial u_k}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u_k}{\partial x_i}\Big)^2\Big]\,dx \le$$
$$\le (2T+1)\int\limits_{D_\tau}\Big(\frac{\partial u_k}{\partial t}\Big)^2\,dx\,dt+\frac{\lambda}{p+2}\int\limits_{S_T}|g_k|^{p+2}\,ds+$$
$$+\int\limits_{D_\tau}F_k^2\;dx\,dt+(M_0+2)\|g_k\|^2_{W_2^1(S_T)}\le$$
$$\le (2T+1)\int\limits_{D_\tau}\Big[u_k^2+\Big(\frac{\partial u_k}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u_k}{\partial x_i}\Big)^2\Big]\,dx\,dt+$$
$$+\frac{\lambda}{p+2}\int\limits_{S_T}|g_k|^{p+2}\,ds+(M_0+2)\Big[\int\limits_{D_\tau}F_k^2\;dx\,dt+\|g_k\|^2_{W_2^1(S_T)}\Big]. \tag{2.15}$$

It follows from (2.3), (2.6) and our reasoning in Remark 1.1 that

$$\lim_{k\to\infty}|g_k|^{p+2}\,ds=\int\limits_{S_T}|g|^{p+2}\,ds,$$

and also $\int\limits_{S_T}|g|^{p+2}\,ds\le C_1\|g\|^2_{W_2^1(S_T)}$ with a positive constant C_1 not depending on $g\in W_2^1(S_T)$. Therefore, putting

$$w(\tau):=\int\limits_{\Omega_\tau}\Big[u_k^2+\Big(\frac{\partial u_k}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u_k}{\partial x_i}\Big)^2\Big]\,dx \tag{2.16}$$

from (2.15) we find

$$w(\tau)\le(2T+1)\int\limits_0^\tau w(s)\,ds+\Big(M_0+\frac{\lambda}{p+2}\,C_1+2\Big)\Big[\|F_k\|^2_{L_2(D_T)}+\|g_k\|^2_{W_2^1(S_T)}\Big],$$

whence by Gronwall's lemma it follows that

$$w(\tau)\le\Big(M_0+\frac{\lambda}{p+2}\,C_1+2\Big)\Big[\|F_k\|^2_{L_2(D_T)}+\|g_k\|^2_{W_2^1(S_T)}\Big]\exp(2T+1)\tau. \tag{2.17}$$

Owing to (2.16) and (2.17), we have

$$\|u_k\|^2_{W_2^1(S_T)}=\int\limits_0^T w(\tau)\,d\tau\le$$

$$\leq \Big(M_0 + \frac{\lambda}{p+2}\,C_1 + 2\Big)T\big(\exp(2T+1)T\big)\Big[\|F_k\|^2_{L_2(D_T)} + \|g_k\|^2_{W_2^1(S_T)}\Big]. \quad (2.18)$$

From (2.18) we get

$$\|u_k\|_{W_2^1(D_T)} \leq c\Big(\|F_k\|_{L_2(D_T)} + \|g_k\|_{W_2^1(S_T)}\Big). \quad (2.19)$$

Here

$$c = \sqrt{\Big(M_0 + \frac{\lambda}{p+2}\,C_1 + 2\Big)T}\ \exp\frac{1}{2}\,(2T+T)T. \quad (2.20)$$

By (2.2) and (2.3), passing in (2.19) to limit as $k \to \infty$, we obtain the estimate (2.1) with the constant c defined from (2.20) which by virtue of (2.10) does not depend on u, g and F. □

3. The Global Solvability of the Problem (1.1), (1.2) in the Class W_2^1

First of all, let us consider the issue of the solvability of the corresponding to (1.1), (1.2) linear problem, when in the equation the parameter $\lambda = 0$, i.e., for the problem

$$L_0 u(x,t) = F(x,t), \quad (x,t) \in D_T, \quad (3.1)$$

$$u(x,t) = g(x,t), \quad (x,t) \in S_T. \quad (3.2)$$

In this case, for $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ we introduce analogously the notion of a strong generalized solution $u \in W_2^1(D_T)$ of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T for which there exists a sequence of functions $u_k \in C^2(\overline{D}_T)$ such that $u_k \to u$ in the space $W_2^1(D_T)$, $L_0 u_k \to F$ in the space $L_2(D_T)$ and $u_k|_{S_T} \to g$ in the space $W_2^1(S_T)$. Note here that as is seen from the proof of Lemma 2.1, the a priori estimate (2.1) is likewise valid for a strong generalized solution of the problem (3.1), (3.2).

Introduce into consideration the weighted Sobolev space $W^k_{2,\alpha}(D)$, $0 < \alpha < \infty$, $k = 1, 2, \ldots$, consisting of the functions belonging to the class $W^k_{2,loc}(D)$ and for which the norm ([**46**])

$$\|u\|^2_{W^k_{2,\alpha}(D)} = \sum_{i=0}^{k} \int\limits_D r^{-2\alpha-2(k-1)} \Big|\frac{\partial^i u}{\partial x^{i'}\partial t^{i_0}}\Big|^2\,dx\,dt,$$

where

$$r = \Big(\sum_{j=1}^n x_j^2 + t^2\Big)^{1/2}, \quad \frac{\partial^2 u}{\partial x^{i'}\partial t^{i_0}} = \frac{\partial^i u}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}\partial t^{i_0}}\,, \quad i = i_1 + \cdots + i_n + i_0,$$

is finite.

Analogously, we introduce the space $W^k_{2,\alpha}(S)$, $S = \partial D$.

Along with the problem (3.1), (3.2), we consider an analogous problem in the infinite cone D. The problem is posed as follows:

$$L_0 u(x,t) = F(x,t), \ \ (x,t) \in D, \tag{3.3}$$

$$u(x,t) = g(x,t), \ \ (x,t) \in S. \tag{3.4}$$

By (1.3), according to a result obtained in [**24**, p. 114], there exists a sequence $\alpha_0 = \alpha_0(k) > 1$ such that for $\alpha \geq \alpha_0$ the problem (3.3), (3.4) has a unique solution $u \in W^k_{2,\alpha}(D)$ for every $F \in W^{k-1}_{2,\alpha-1}(D)$ and $g \in W^k_{2,\alpha-\frac{1}{2}}(S)$.

Since the space $C^\infty_0(D_T)$ of finitary, infinitely differentiable in D_T functions is dense in $L_2(D_T)$, for a given $F \in L_2(D_T)$ there exists a sequence of functions $F_\ell \in C^\infty_0(D_T)$ such that $\lim\limits_{\ell\to\infty} \|F_\ell - F\|_{L_2(D_T)} = 0$. For the fixed ℓ, extending the function F_ℓ by zero beyond the domain D_T and leaving the same as above notation, we have $F_\ell \in C^\infty_0(D)$. Obviously, $F_\ell \in W^{k-1}_{2,\alpha-1}(D)$ for any $k \geq 1$ and $\alpha > 1$, and hence for $\alpha \geq \alpha_0 = \alpha_0(k)$. If $g \in W^1_2(S_T)$, then, as is known, there exists a function $\widetilde{g} \in W^1_2(S)$ such that $g = \widetilde{g}|_{S_T}$ and $\operatorname{diam}\operatorname{supp}\widetilde{g} < +\infty$. At the same time, the space $C^\infty_*(S) := \big\{g \in C^\infty(S) : \ \operatorname{diam}\operatorname{supp} g < +\infty, \ 0 \notin \operatorname{supp} g\big\}$ is dense in $W^1_2(S)$. Therefore there exists a sequence of functions $g_\ell \in C^\infty_*(S)$ such that $\lim\limits_{\ell\to\infty} \|g_\ell - \widetilde{g}\|_{W^1_2(S)} = 0$. It can be easily seen that $g_\ell \in W^k_{2,\alpha-\frac{1}{2}}(S)$ for any $k \geq 2$ and $\alpha > 1$, and hence for $\alpha \geq \alpha_0 = \alpha_0(k)$, as well. According to what has been said, there exists a solution $\widetilde{u}_\ell \in W^k_{2,\alpha}(D)$ of the problem (3.3), (3.4) for $F = F_\ell$ and $g = g_\ell$. Assume $u_\ell = \widetilde{u}_\ell|_{D_T}$. Since $u_\ell \in W^k_2(D_T)$, when the number k is sufficiently large, namely, $k > \frac{n+1}{2} + 2$, by the embedding theorem [**49**, p. 84] the function $u_\ell \in C^2(\overline{D}_T)$. As far as the a priori estimate (2.1) is likewise valid for a strong generalized solution of the problem (3.1), (3.2) of the class W^1_2 in the domain D_T, we have

$$\|u_\ell - u_{\ell'}\|_{W^1_2(D_T)} \leq c\Big(\|F_\ell - F_{\ell'}\|_{L_2(D_T)} + \|g_\ell - g_{\ell'}\|_{W^1_2(S_T)}\Big). \tag{3.5}$$

Since the sequences $\{F_\ell\}$ and $\{g_\ell\}$ are fundamental respectively in the spaces $L_2(D_T)$ and $W^1_2(D_T)$, owing to (3.5) the sequence $\{u_\ell\}$ will be fundamental in the space $W^1_2(D_T)$. Therefore because the space $W^1_2(D_T)$ is complete, there exists a function $u \in W^1_2(D_T)$ such that $\lim\limits_{\ell\to\infty} \|u_\ell - u\|_{W^1_2(D_T)} = 0$, and since $L_0 u_\ell = F_\ell \to F$ in the space $L_2(D_T)$ and $g_\ell = u_\ell|_{S_T} \to g$ in the space $W^1_2(S_T)$, this function is a strong generalized solution of the problem (3.1), (3.2) of the class W^1_2 in the domain D_T. The uniqueness of the solution of the problem (3.1), (3.2) of the class W^1_2 in the domain D_T follows from the a priori estimate (2.1). Consequently, for a solution u of the problem (3.1), (3.2) we can write $u = L_0^{-1}(F,g)$, where $L_0^{-1} : L_2(D_T) \times W^1_2(S_T) \to W^1_2(D_T)$ is a linear continuous operator whose norm, by virtue of (2.1), admits the estimate

$$\|L_0^{-1}\|_{L_2(D_T)\times W^1_2(S_T)\to W^1_2(D_T)} \leq c, \tag{3.6}$$

where the constant c is defined from (2.20).

Since the operator $L_0^{-1} : L_2(D_T) \times W_2^1(S_T) \to W_2^1(D_T)$ is linear, there takes place the representation

$$L_0^{-1}(F,g) = L_{01}^{-1}(F) + L_{02}^{-1}(g), \tag{3.7}$$

where $L_{01}^{-1} : L_2(D_T) \to W_2^1(D_T)$ and $L_{02}^{-1} : W_2^1(S_T) \to W_2^1(D_T)$ are linear continuous operators, and by (3.6)

$$\|L_{01}^{-1}\|_{L_2(D_T)\to W_2^1(D_T)} \le c, \quad \|L_{02}^{-1}\|_{W_2^1(S_T)\to W_2^1(D_T)} \le c. \tag{3.8}$$

Remark 3.1. Note that for $F \in L_2(D_T)$, $g \in W_2^1(S_T)$, $0 < p < \frac{2}{n-1}$, by virtue of (3.6), (3.7), (3.8) and Remark 1.1 the function $u \in W_2^1(D_T)$ is a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T if and only if u is a solution of the functional equation

$$u = L_{01}^{-1}(-\lambda |u|^p u) + L_{01}^{-1}(F) + L_{02}^{-1}(g) \tag{3.9}$$

in the space $W_2^1(D_T)$.

We rewrite the equation (3.9) in the form

$$u := Au = -L_{01}^{-1}(K_0 u) + L_{01}^{-1}(F) + L_{02}^{-1}(g), \tag{3.10}$$

where the operator $K_0 : W_2^1(D_T) \to L_2(D_T)$ from (1.4) is, by Remark 1.1, continuous and compact. Consequently, by (3.8) the operator $A : W_2^1(D_T) \to W_2^1(D_T)$ is continuous and compact, as well. At the same time, by Lemma 2.1 and (2.10), (2.20) for any parameter $\tau \in [0,1]$ and every solution of the equation $u = \tau A u$ with the parameter τ the same a priori estimate (2.1) is valid with a positive constant c not depending on u, F, g and τ. Therefore by the Leray–Schauder theorem [**66**, p. 375] the equation (3.10) and hence by Remark 3.1 the problem (1.1), (1.2) has at least one solution $u \in W_2^1(D_T)$.

Thus we have proved the following theorem.

Theorem 3.1. *Let $\lambda > 0$, $0 < p < \frac{2}{n-1}$, $F \in L_{2,loc}(D)$, $g \in W_{2,loc}^1(S)$ and $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ for any $T > 0$. Then the problem* (1.1), (1.2) *is globally solvable in the class W_2^1, i.e., for any $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T.*

4. The Non-Existence of the Global Solvability of the Problem (1.1), (1.2)

Below we will restrict ourselves to the case where the boundary condition (1.2) is homogeneous, i.e.,

$$u\big|_{S_T} = 0. \tag{4.1}$$

For $(x^0, t^0) \in D_T$, we introduce into consideration the domain D_{x^0,t^0} which is bounded from below by the conic surface S and from above by the light cone of the past $S^-_{x^0,t^0} :\ t = t^0 - |x - x^0|$ with the vertex at the point (x^0, t^0).

Lemma 4.1. *Let $F \in C(\overline{D}_T)$ and $u \in C^2(\overline{D}_T)$ be a classical solution of the problem* (1.1), (4.1). *Then if for some point $(x^0, t^0) \in D_T$ the function $F|_{D_{x^0,t^0}} = 0$, then $u|_{D_{x^0,t^0}} = 0$ as well.*

Proof. Since the proof of the above lemma is, to a certain extent, analogous to that of Lemma 2.1, we cite only the main points of that proof.

Assume $D_{x^0,t^0,\tau} := D_{x^0,t^0} \cap \{t < \tau\}$, $\Omega_{x^0,t^0,\tau} := D_{x^0,t^0} \cap \{t = \tau\}$, $0 < t < \tau$. Then $\partial D_{x^0,t^0,\tau} = S_{1,\tau} \cup S_{2,\tau} \cup S_{3,\tau}$, where $S_{1,\tau} = \partial D_{x^0,t^0,\tau} \cap S$, $S_{2,\tau} = \partial D_{x^0,t^0,\tau} \cap S^-_{x^0,t^0}$, $S_{3,\tau} = \partial D_{x^0,t^0,\tau} \cap \overline{\Omega}_{x^0,t^0,\tau}$. Just in the same way as in obtaining the equality (2.8), multiplying both parts of the equality (1.1) by $\frac{\partial u}{\partial t}$, integrating over the domain $D_{x^0,t^0,\tau}$, $0 < \tau < t^0$, and taking into account (1.1) and the fact that $F|_{D_{x^0,t^0}} = 0$, we obtain

$$0 = \int\limits_{S_{1,\tau} \cup S_{2,\tau}} \frac{1}{2\nu_0} \left[\sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\,\nu_0 - \frac{\partial u}{\partial t}\,\nu_i \Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 \Big(\nu_0^2 - \sum_{j=1}^n \nu_j^2\Big) \right] ds +$$
$$+ \int\limits_{S_{2,\tau} \cup S_{3,\tau}} \frac{\lambda}{p+2}\, |u|^{p+2} \nu_0 \, ds + \int\limits_{S_{3,\tau}} \left[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2 \right] dx. \qquad (4.2)$$

By (1.3) and (4.1), bearing in mind that

$$\Big(\nu_0^2 - \sum_{i=1}^n \nu_i^2\Big)\Big|_{S_{1,\tau}} < 0, \quad \nu_0\big|_{S_{1,\tau}} < 0,$$
$$\Big(\nu_0^2 - \sum_{i=1}^n \nu_i^2\Big)\Big|_{S_{2,\tau}} = 0, \quad \nu_0\big|_{S_{2,\tau}} = \frac{1}{\sqrt{2}} > 0,$$
$$\Big(\frac{\partial u}{\partial x_i}\,\nu_0 - \frac{\partial u}{\partial t}\,\nu_i \Big)\Big|_{S_{1,\tau}} = 0, \quad \Big(\frac{\partial u}{\partial x_i}\,\nu_0 - \frac{\partial u}{\partial t}\,\nu_i \Big)^2\Big|_{S_{2,\tau}} \geq 0, \quad i = 1, \dots, n,$$

we find that

$$\int\limits_{S_{1,\tau} \cup S_{2,\tau}} \frac{1}{2\nu_0} \left[\sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\,\nu_0 - \frac{\partial u}{\partial t}\,\nu_i \Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 \Big(\nu_0^2 - \sum_{j=1}^n \nu_j^2\Big) \right] ds \geq 0. \quad (4.3)$$

In view of (4.3), from (4.2) we get

$$\int\limits_{S_{3,\tau}} \left[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2 \right] dx \leq M \int\limits_{S_{2,\tau} \cup S_{3,\tau}} u^2 \, ds, \quad 0 < \tau < t^0. \qquad (4.4)$$

Here, since $u \in C^2(\overline{D}_T)$ and $|\nu_0| \leq 1$, in the capacity of the nonnegative constant M independent of the parameter τ we can take

$$M = \frac{|\lambda|}{p+2}\, \|u\|^p_{C(\overline{D}_T)} < +\infty. \qquad (4.5)$$

By (4.1), reasoning as in proving the inequality (2.14) we obtain

$$\int\limits_{S_{2,\tau}\cup S_{3,\tau}} u^2\,ds \le 2t^0 \int\limits_{D_{x^0,t^0,\tau}} \Big(\frac{\partial u}{\partial t}\Big)^2\,dx,\ \ 0<\tau<t^0. \tag{4.6}$$

Putting

$$w(\tau) := \int\limits_{S_{3,\tau}} \bigg[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}\Big)^2\bigg]\,dx,$$

from (4.4) and (4.6) we easily find that

$$w(\tau) \le 2t^0 M \int\limits_0^\tau w(\delta)\,d\delta,\ \ 0<\tau<t^0,$$

whence by (4.5) and Gronwall's lemma it immediately follows that $w(\tau)=0$, $0<\tau<t^0$, and hence $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x_1}=\cdots=\frac{\partial u}{\partial x_n}=0$ in the domain D_{x^0,t^0}. Therefore $u|_{D_{x^0,t^0}}=\text{const}$, and taking into account the homogeneous boundary condition (4.1), we finally obtain that $u|_{D_{x^0,t^0}}=0$. Thus the lemma is proved. □

Below we will restrict ourselves to the consideration of the case where the equation (1.1) involves a parameter $\lambda<0$ and the spatial dimension $n=2$. For simplicity of our exposition, we assume that

$$S:\ t=k_0|x|,\ \ k_0=\text{const}>1. \tag{4.7}$$

Obviously, for the conic surface S given by the equality (4.7) the condition (1.3) is fulfilled. In this case, $D_T=\big\{(x,t)\in\mathbb{R}^3:\ k_0|x|<t<T\big\}$.

Let $G_a:\ t>|x|+a$ be the light cone of future with the vertex at the point $(0,0,a)$, where $a=\text{const}>0$. By (4.7), it is evident that $D\setminus G_a=\big\{(x,t)\in\mathbb{R}^3:\ k_0|x|<t<|x|+a,\ |x|<\frac{a}{k_0-1}\big\}$ and

$$D\setminus\overline{G}_a\subset\big\{(x,t)\in\mathbb{R}^3:\ 0<t<b\big\},\ \ b=\frac{ak_0}{k_0-1}\,. \tag{4.8}$$

Lemma 4.2. *Let $n=2$, $\lambda<0$, $F\in C(\overline{D}_T)$, $T\ge b=\frac{ak_0}{k_0-1}$, $\operatorname{supp}F\subset\overline{G}_a$ and $F\ge 0$. Then if $u\in C^2(\overline{D}_T)$ is a classical solution of the problem* (1.1), (4.1), *then $u|_{D_b}\ge 0$.*

Proof. First, let us show that $u|_{D\setminus\overline{G}_a}=0$. Indeed, let $(x^0,t^0)\in D\setminus\overline{G}_a$. Then since $\operatorname{supp}F\subset\overline{G}_a$, we have that $F|_{D_{x^0,t^0}}=0$, and according to Lemma 4.1 the equality $u|_{D_{x^0,t^0}}=0$ holds. Therefore taking into account (4.8), extending the functions u and F by zero beyound D_b into the strip $\Sigma_b:=\big\{(x,t)\in\mathbb{R}^3:\ 0<t<b\big\}$, and leaving the same as above notation,

we obtain that $u \in C^2(\overline{\Sigma}_b)$ is a classical solution of the Cauchy problem

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} - \Delta u = -\lambda |u|^p u + F, \\ u\big|_{t=0} = 0, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \end{gathered} \tag{4.9}$$

in the strip Σ_b. As is known, for the solution $u \in C^2(\overline{\Sigma}_b)$ of the problem (4.9) the integral representation [**69**, pp. 213–216]

$$u(x,t) = -\frac{\lambda}{2\pi} \int\limits_{\Omega_{x,t}} \frac{|u|^p u}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau + F_0(x,t), \quad (x,t) \in \Sigma_b, \tag{4.10}$$

is valid.

Here

$$F_0(x,t) = \frac{1}{2\pi} \int\limits_{\Omega_{x,t}} \frac{F(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau, \tag{4.11}$$

where $\Omega_{x,t} := \big\{(\xi,\tau) \in \mathbb{R}^3 : |\xi - x| < t,\ 0 < \tau < t - |\xi - x|\big\}$ is a circular cone with the vertex at the point (x,t); its base is the circle $d: \ |\xi - x| < t$, $\tau = 0$ in the plane $\tau = 0$ of the variables ξ_1 and ξ_2, $\xi = (\xi_1, \xi_2)$.

Let $(x^0, t^0) \in D_b$ and $\widetilde{\psi}_0 = \widetilde{\psi}_0(x,t) \in C(\overline{\Omega}_{x^0,t^0})$. Then the linear operator $\Psi : C(\overline{\Omega}_{x^0,t^0}) \to C(\overline{\Omega}_{x^0,t^0})$ acting by the formula

$$\Psi v(x,t) = \frac{1}{2\pi} \int\limits_{\Omega_{x,t}} \frac{\widetilde{\psi}_0(\xi,\tau) v(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau, \quad (x,t) \in \overline{\Omega}_{x^0,t^0},$$

is continuous, and for its norm the estimate [**69**, p. 215]

$$\|\Psi\|_{C(\overline{\Omega}_{x^0,t^0}) \to C(\overline{\Omega}_{x^0,t^0})} \le \frac{(t_0)^2}{2}\, \|\widetilde{\psi}_0\|_{C(\overline{\Omega}_{x^0,t^0})} \le \frac{T^2}{2}\, \|\widetilde{\psi}_0\|_{C(\overline{\Omega}_{x^0,t^0})}$$

is valid.

Consider the integral equation

$$v(x,t) = \int\limits_{\Omega_{x,t}} \frac{\psi_0(\xi,\tau) v(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau + F_0(x,t), \quad (x,t) \in \overline{\Omega}_{x^0,t^0}, \tag{4.12}$$

with respect to the unknown function v. Here

$$\psi_0(\xi,\tau) = -\frac{\lambda}{2\pi}\, |u(\xi,\tau)|^p \in C(\overline{\Omega}_{x^0,t^0}), \tag{4.13}$$

where u is the classical solution of the problem (1.1), (4.1) appearing in Lemma 4.2. Since ψ_0, $F_0 \in C(\overline{\Omega}_{x^0,t^0})$, and the operator in the right-hand side (4.12) is an integral operator of Volterra type (with respect to the variable t) with a weak singularity, the equation (4.12) is uniquely solvable in the space $C(\overline{\Omega}_{x^0,t^0})$. In addition, the solution v of the equation (4.12) can be obtained by the Picard method of successive approximations:

$$v_0 = 0, \quad v_{k+1}(x,t) = \int\limits_{\Omega_{x,t}} \frac{\psi_0(\xi,\tau) v_k(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau + F_0(x,t), \qquad (4.14)$$

$$k = 1, 2, \dots .$$

Indeed, let $\omega_\tau = \Omega_{x^0,t^0} \cap \{t = \tau\}$, $w_m|_{\overline{\Omega}_{x^0,t^0}} = v_{m+1} - v_m$ $(w_0|_{\overline{\Omega}_{x^0,t^0}} = F_0)$, $\lambda_m(t) = \max\limits_{x \omega \overline{\omega}_t} |w_m(x,t)|$, $m = 0, 1, \dots$; $\delta = \int\limits_{|\eta|<1} \frac{d\eta_1\, d\eta_2}{\sqrt{1-|\eta|^2}} \|\psi_0\|_{C(\overline{\Omega}_{x^0,t^0})} = 2\pi \|\psi_0\|_{C(\overline{\Omega}_{x^0,t^0})}$. Then denoting $B_\beta \varphi(t) = \delta \int\limits_0^t (t-\tau)^{\beta-1} \varphi(\tau)\, d\tau$, $\beta > 0$, and taking onto account the equality [**15**, p. 206]

$$B_\beta^m \varphi(t) = \frac{1}{\Gamma(m\beta)} \int\limits_0^t (\delta \Gamma(\beta))^m (t-\tau)^{m\beta-1} \varphi(\tau)\, d\tau,$$

owing to (4.14) we have

$$|w_m(x,t)| = \Bigg| \int\limits_{\Omega_{x,t}} \frac{\psi_0 w_{m-1}}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau \Bigg| \le$$

$$\le \int\limits_0^t d\tau \int\limits_{|x-\xi|<t-\tau} \frac{|\psi_0|\, |w_{m-1}|}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi \le$$

$$\le \|\psi_0\|_{C(\overline{\Omega}_{x^0,t^0})} \int\limits_0^t d\tau \int\limits_{|x-\xi|<t-\tau} \frac{\lambda_{m-1}(\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi =$$

$$= \|\psi_0\|_{C(\overline{\Omega}_{x^0,t^0})} \int\limits_0^t (t-\tau) \lambda_{m-1}(\tau)\, d\tau \int\limits_{|\eta|<1} \frac{d\eta_1\, d\eta_2}{\sqrt{1-|\eta|^2}} =$$

$$= B_2 \lambda_{m-1}(t), \quad (x,t) \in \Omega_{x^0,t^0},$$

whence

$$\lambda_m(t) \le B_2 \lambda_{m-1}(t) \le \dots \le B_2^m \lambda_0(t) =$$

$$= \frac{1}{\Gamma(2m)} \int\limits_0^t (\delta \Gamma(2))^m (t-\tau)^{2m-1} \lambda_0(\tau)\, d\tau \le$$

$$\le \frac{\delta^m}{\Gamma(2m)} \int\limits_0^t (t-\tau)^{2m-1} \|w_0\|_{C(\overline{\Omega}_{x^0,t^0})}\, d\tau =$$

$$= \frac{(\delta T^2)^m}{\Gamma(2m) 2m} \|F\|_{C(\overline{\Omega}_{x^0,t^0})} = \frac{(\delta T^2)^m}{(2m)!} \|F_0\|_{C(\overline{\Omega}_{x^0,t^0})},$$

and consequently,

$$\|w_m\|_{C(\overline{\Omega}_{x^0,t^0})} = \|\lambda_m\|_{C([0,t^0])} \leq \frac{(\delta T^2)^m}{(2m)!} \|F_0\|_{C(\overline{\Omega}_{x^0,t^0})}.$$

Therefore the series $v = \lim\limits_{m\to\infty} v_m = v_0 + \sum\limits_{m=0}^{\infty} w_m$ converges in the class $C(\overline{\Omega}_{x^0,t^0})$, and its sum is a solution of the equation (4.12). The uniqueness of solution of the equation (4.12) in the space $C(\overline{\Omega}_{x^0,t^0})$ is proved analogously.

Since $\lambda < 0$, by virtue of (4.13) the function $\psi_0(\xi,\tau) = -\frac{\lambda}{2\pi}\,|u(\xi,\tau)|^p \geq 0$, and according to the equality (4.11) the function $F_0(x,t) \geq 0$, as well, because, by the condition, $F(x,t) \geq 0$. Therefore the successive approximations from (4.14) are nonnegative, and since $\lim\limits_{k\to\infty} \|v_k - v\|_{C(\overline{\Omega}_{x^0,t^0})} = 0$, therefore the solution $v \geq 0$ in the closed domain $\overline{\Omega}_{x^0,t^0}$. It remains only to note that due to (4.10), (4.12) and (4.13), the function u is likewise a solution of the equation (4.12), and owing to the unique solvability of the equation, we have $u = v \geq 0$ in $\overline{\Omega}_{x^0,t^0}$. Thus $u(x^0,t^0) \geq 0$ for any point $(x^0,t^0) \in D_b$, which was to be demonstrated. □

Let c_R and φ_R be, respectively, the first characteristic value and eigenfunction of the Dirichlet problem in the circle $\omega_R:\ x_1^2 + x_2^2 < R^2$. Consequently,

$$(\Delta\varphi_R + c_R\varphi_R)\big|_{\omega_R} = 0, \quad \varphi_R\big|_{\partial\omega_R} = 0. \tag{4.15}$$

As is known, $c_R > 0$, and changing the sign and performing the corresponding normalization, we may assume [**59**, p. 25] that

$$\varphi_R\big|_{\omega_R} > 0, \quad \int\limits_{\omega_R} \varphi_R\, dx = 1. \tag{4.16}$$

Below, the conditions of Lemma 4.2 will be assumed to be fulfilled. As is shown in proving this lemma, extending the functions u and F by zero beyond D_b into the strip $\Sigma_b = \big\{(x,t) \in \mathbb{R}^3 :\ 0 < t < b\big\}$ and leaving the same notation, we find that $u \in C^2(\overline{\Sigma}_b)$ is a classical solution of the Cauchy problem (4.9) in the strip Σ_b.

Remark 4.1. In the equation (1.1), without restriction of generality we may assume that $\lambda = -1$, since the case $\lambda < 0$, $\lambda \neq -1$, by virtue of $p > 0$ reduces to the case $\lambda = -1$ after we introduce a new unknown function $v = |\lambda|^{1/p}u$. Therefore the function v will satisfy the equation

$$v_{tt} - \Delta u = v^{p+1} + |\lambda|^{1/p}F(x,t), \ \ (x,t) \in \Sigma_b.$$

According to the above remark, instead of (4.9) we consider the following Cauchy problem:

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} - \Delta u = u^{p+1} + F(x,t), \quad (x,t) \in \Sigma_b, \\ u\big|_{t=0} = 0, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0,\end{gathered} \tag{4.17}$$

where $u|_{\Sigma_b} \geq 0$ and $u \in C^2(\overline{\Sigma}_b)$. In addition, as is shown in proving Lemma 4.2, we have

$$u\big|_{\Sigma_b \setminus \overline{G}_a} = 0. \tag{4.18}$$

Take $R \geq b > \frac{a}{k_0-1}$, where the number $\frac{a}{k_0-1}$ is the radius of the circle obtained by intersection of the domain $D: \ t > k_0|x|$ and the plane $t = b$. Introduce into consideration the functions

$$E(t) = \int\limits_{\omega_R} u(x,t)\varphi_R(x)\,dx, \quad f_R(x) = \int\limits_{\omega_R} F(x,t)\varphi_R(x)\,dx, \ \ 0 \leq t \leq b. \tag{4.19}$$

Since $U|_{\Sigma_b} \geq 0$, $u \in C^2(\overline{\Sigma}_b)$ and $F \in C(\overline{\Sigma}_b)$, we have $E \geq 0$, $E \in C^2([0,b])$ and $f_R \in C([0,R])$.

By (4.15), (4.18) and (4.19), the integration by parts results in

$$\int\limits_{\omega_R} \Delta u \varphi_R\,dx = \int\limits_{\omega_R} u \Delta\varphi_R\,dx = -c_R \int\limits_{\omega_R} u\varphi_R\,dx = -c_R E. \tag{4.20}$$

By virtue of (4.16) and the fact that $p > 0$ and $u|_{\Sigma_b} \geq 0$, using Jensen's [**59**, p. 26] inequality we obtain

$$\int\limits_{\omega_R} u^{p+1}\varphi_R\,dx \geq \bigg(\int\limits_{\omega_R} u\varphi_R\,dx\bigg)^{p+1} = E^{p+1}. \tag{4.21}$$

It immediately follows from (4.17)–(4.21) that

$$E'' + c_R E \geq E^{p+1} + f_R, \ \ 0 \leq t \leq b, \tag{4.22}$$

$$E(0) = 0, \quad E'(0) = 0. \tag{4.23}$$

To investigate the problem (4.22), (4.23), we make use of the method of test functions [**53**, pp. 10–12]. Towards this end, we take b_1, $0 < b_1 < b_2$, and consider a nonnegative test function $\psi \in C^2([0,b])$ such that

$$0 \leq \psi \leq 1, \ \ \psi(t) = 1, \ \ 0 \leq t \leq b; \quad \psi^{(i)}(b) = 0, \ \ i = 0,1,2. \tag{4.24}$$

It follows from (4.22)–(4.24) that

$$\int\limits_0^b E^{p+1}(t)\psi(t)\,dt \leq \int\limits_0^b E(t)\big[\psi''(t) + c_R\psi(t)\big]\,dt - \int\limits_0^b f_R(t)\psi(t)\,dt. \tag{4.25}$$

If in the Young inequality with the parameter $\varepsilon > 0$

$$yz \leq \frac{\varepsilon}{\alpha}\,y^\alpha + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\,z^{\alpha'}, \ \ y,z \geq 0, \ \ \alpha' = \frac{\alpha}{\alpha-1}\,,$$

we take $\alpha = p+1$, $\alpha' = \frac{p+1}{p}$, $y = E\psi^{\frac{1}{p+1}}$, $z = \frac{|\psi''+c_R\psi|}{\psi^{\frac{1}{p+1}}}$ and bear in mind that $\frac{\alpha'}{\alpha} = \frac{1}{\alpha-1} = \alpha'-1$, then we will obtain

$$E|\psi''+c_R\psi| = E\psi^{1/\alpha}\,\frac{|\psi''+c_R\psi|}{\psi^{1/\alpha}} \le$$
$$\le \frac{\varepsilon}{\alpha}\,E^{\alpha}\psi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\,\frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}\,. \tag{4.26}$$

By (4.26), from (4.25) we have

$$\Big(1-\frac{\varepsilon}{\alpha}\Big)\int\limits_0^b E^{\alpha}\psi\,dt \le \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int\limits_0^b \frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}\,dt - \int\limits_0^b f_R(t)\psi(t)\,dt. \tag{4.27}$$

Taking into consideration that $\inf\limits_{0<\varepsilon<\alpha}\big[\frac{\alpha-1}{\alpha-\varepsilon}\,\frac{1}{\varepsilon^{\alpha'-1}}\big] = 1$ which is achieved for $\varepsilon = 1$, from (4.27) with regard for (4.24) we obtain

$$\int\limits_0^{b_1} E^{\alpha}\psi\,dt \le \int\limits_0^b \frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}\,dt - \alpha'\int\limits_0^b f_R(t)\psi(t)\,dt. \tag{4.28}$$

We take now in the capacity of the test function ψ the function of the type

$$\psi(t) = \psi_0(\tau), \quad \tau = \frac{t}{b_1}\,, \quad 0 \le \tau \le \tau_1 = \frac{b}{b_1}\,. \tag{4.29}$$

Here

$$\begin{gathered}\psi_0 \in C^2([0,\tau_1]), \ \ 0 \le \psi_0 \le 1, \quad \psi_0(\tau) = 1, \ \ 0 \le \tau \le 1,\\ \psi_0^{(i)}(\tau_1) = 0, \ \ i = 0,1,2.\end{gathered} \tag{4.30}$$

It is not difficult to see that

$$c_R = \frac{c_1}{R^2} \le \frac{c_1}{b^2} \le \frac{c_1}{b_1^2}\,, \quad \varphi_R(x) = \frac{1}{R^2}\,\varphi_1\Big(\frac{x}{R}\Big). \tag{4.31}$$

In view of (4.29), (4.30) and (4.31), taking into account that $\psi''(t) = 0$ for $0 \le t \le b_1$ and $f_R \ge 0$ because $F \ge 0$ as well as the known inequality $|y+z|^{\alpha'} \le 2^{\alpha'-1}\big(|y|^{\alpha'}+|z|^{\alpha'}\big)$, from (4.28) we get

$$\int\limits_0^{b_1} E^{\alpha}\,dt \le$$
$$\le \int\limits_0^{b_1} \frac{c_R^{\alpha'}\psi^{\alpha'}}{\psi^{\alpha'-1}}\,dt + \int\limits_{b_1}^b \frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}\,dt - \alpha'\int\limits_0^b f_R(t)\psi(t)\,dt \le$$
$$\le c_R^{\alpha'}\int\limits_0^{b_1}\psi\,dt + b_1\int\limits_1^{r_1} \frac{\big|\frac{1}{b_1^2}\,\psi_0''(\tau)+c_R\psi_0(\tau)\big|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\,d\tau - \alpha'\int\limits_0^{b_1} f_R(t)\,dt \le$$

$$\leq c_R^{\alpha'} b_1 + \frac{2^{\alpha'-1}}{b_1^{2\alpha'-1}} \int\limits_1^{r_1} \frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\, d\tau +$$

$$+ b_1 2^{\alpha'-1} c_R^{\alpha'} \int\limits_1^{r_1} \psi_0(\tau)\, d\tau - \alpha' \int\limits_0^{b_1} f_R(t)\, dt \leq$$

$$\leq \frac{c_1^{\alpha'}}{b_1^{2\alpha'-1}} + \frac{2^{\alpha'-1}}{b_1^{2\alpha'-1}} \int\limits_1^{r_1} \frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\, d\tau +$$

$$+ \frac{2^{\alpha'-1} c_1^{\alpha'}}{b_1^{2\alpha'-1}} (\tau_1 - 1) - \alpha' \int\limits_0^{b_1} f_R(t)\, dt. \tag{4.32}$$

Assuming now that $R = b = \frac{ak_0}{k_0-1}$ and the number $\tau_1 > 1$ is such that

$$b_1 = \frac{b}{\tau_1} = a + 2\,\frac{b-a}{3} = \frac{a+2b}{3} = \frac{a}{3}\Big(\frac{3k_0-1}{k_0-1}\Big), \tag{4.33}$$

from (4.32) we find

$$\int\limits_0^{b_1} E^\alpha\, dt \leq b_1^{1-2\alpha'} \bigg[c_1^{\alpha'}\big(1 + 2^{\alpha'-1}(\tau_1-1)\big) + 2^{\alpha'-1} \int\limits_1^{r_1} \frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\, d\tau -$$

$$- \alpha' b_1^{2\alpha'-1} \int\limits_0^{b_1} f_b(t)\, dt \bigg], \quad 2\alpha' - 1 = \frac{p+2}{p}\,. \tag{4.34}$$

As is known, the function ψ_0 with the property (4.30) for which the integral

$$d(\psi_0) = \int\limits_1^{r_1} \frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\, d\tau < +\infty \tag{4.35}$$

is finite does exist [**53**, p. 11].

Bearing in mind (4.19) and (4.31), we have

$$J(b) = \int\limits_0^{b_1} f_b(t)\, dt = \int\limits_0^{b_1} dt \int\limits_{\omega_b} F(x,t)\varphi_b(x)\, dx =$$

$$= \int\limits_0^{b_1} dt \int\limits_{\omega_b} F(x,t)\, \frac{1}{b^2}\, \varphi_1\Big(\frac{x}{b}\Big)\, dx = \int\limits_0^{b_1} dt \int\limits_{\omega_1} F(b\xi, t)\varphi_1(\xi)\, d\xi. \tag{4.36}$$

By virtue of (4.35), the value

$$\varkappa_0 = \varkappa_0(c_1, \alpha', \psi_0) = \frac{\tau_1^{2\alpha'-1}}{\alpha'} \Big[c_1^{\alpha'}\big(1 + 2^{\alpha'-1}(\tau_1 - 1)\big) + 2^{\alpha'-1} d(\psi_0) \Big] \tag{4.37}$$

is likewise finite.

From the above reasoning we have the following

Theorem 4.1. *Let $n=2$, $\lambda=-1$, $F\in C(\overline{D})$, $F\geq 0$ and* $\operatorname{supp} F\subset \overline{G}_a:\ t\geq |x|+a$, $a=\mathrm{const}>0$. *If the condition*

$$b^{\frac{p+2}{p}}\int\limits_0^{\frac{1}{\tau_1}b}dt\int\limits_{\omega_1}F(b\xi,t)\varphi_1(\xi)\,d\xi>\varkappa_0,\quad b=\frac{ak_0}{k_0-1}\,,\quad \tau_1=\frac{3k_0}{3k_0-1}\,, \tag{4.38}$$

is fulfilled, then for $T\geq b$ the problem (1.1), (4.1) *fails to have a classical solution $u\in C^2(\overline{D}_T)$ in the domain D_T.*

Proof. Indeed, by (4.33) and (4.36)–(4.38) the right-hand side of the inequality (4.34) is negative, but this is impossible because the left-hand side of this inequality is nonnegative. Therefore for $T\geq b$ the problem (1.1), (4.1) cannot have a classical solution $u\in C^2(\overline{D}_T)$ in the domain D_T. Thus the theorem is proved. □

Remark 4.2. As we can see from the proof, if the conditions of Theorem 4.1 are fulfilled and a solution $u\in C^2(\overline{D}_T)$ of the problem (1.1), (4.1) exists in the domain D_T, then T is contained in the interval $(0,b)$, i.e., $0<T<b=\frac{ak_0}{k_0-1}$.

Remark 4.3. In Theorem 4.1, it is assumed that $\lambda=-1$. Taking into account Remark 4.1, we can conclude that Theorem 4.1 remains also valid in case $\lambda<0$, provided in the right-hand side of the inequality (4.38) instead of $\varkappa_0$ we write $|\lambda|^{-\frac{1}{p}}\varkappa_0$.

Corollary 4.1. *Let $n=2$, $\lambda<0$, $F=\mu F_0$, where $\mu=\mathrm{const}>0$, $F_0\in C(\overline{D})$, $F_0\geq 0$, $\operatorname{supp} F_0\subset\overline{G}_a$ and $F_0|_{D_b}\not\equiv 0$. Then there exists a positive number μ_0 such that if $\mu>\mu_0$, then the problem* (1.1), (4.1) *cannot have a classical solution $u\in C^2(\overline{D}_T)$ for $T\geq b$.*

CHAPTER 4

Some Multi-Dimensional Versions of the First Darboux Problem for Nonlinear Wave Equations

1. Statement of the Problems

In the Euclidean space $\mathbb{R}^{n+1}$ of the variables t, $x_1,\dots,x_n$, $n\geq 2$, we consider the nonlinear wave equation of the type

$$L_\lambda u := \frac{\partial^2 u}{\partial t^2} - \Delta u + \lambda f(u) = F, \tag{1.1}$$

where f and F are given real functions, $f\in C(\mathbb{R})$ is a nonlinear function, $f(0)=0$, and u is an unknown real function, $\Delta=\sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$, $\lambda\neq 0$ is a given real number.

By $D:\ t>|x|,\ x_n>0$ we denote one half of the light cone of future which is bounded by the part $S^0=D\cap\{x_n=0\}$ of the hyperplane $x_n=0$ and by the half $S:\ t=|x|,\ x_n\geq 0$ of the characteristic conoid $C:\ t=|x|$ of the equation (1.1). Assume $D_T:=\big\{(x,t)\in D:\ t<T\big\}$, $S_T^0:=\big\{(x,t)\in S^0:\ t\leq T\big\}$, $S_T:=\big\{(x,t)\in S:\ t\leq T\big\}$, $T>0$. In case $T=\infty$, it is obvious that $D_\infty=D$, $S_\infty^0=S^0$ and $S_\infty=S$.

For the equation (1.1), we consider the following problem: find in the domain D_T a solution $u(x,t)$ of that equation satisfying one of the following boundary conditions:

$$\frac{\partial u}{\partial x_n}\Big|_{S_T^0}=0,\quad u\big|_{S_T}=0 \tag{1.2}$$

or

$$u\big|_{S_T^0}=0,\quad u\big|_{S_T}=0. \tag{1.3}$$

The problems (1.1), (1.2) and (1.1), (1.3) are multi-dimensional versions of the first Darboux problem for the equation (1.1), when one part of the data support is a characteristic manifold and another part is of time type [**2**, pp. 228, 233].

Let $f\in C(\mathbb{R})$. If $u\in C^2(\overline{D}_T)$ is a classical solution of the problem (1.1), (1.2), then multiplying both parts of the equation (1.1) by an arbitrary function $\varphi\in C^2(\overline{D}_T)$ satisfying the condition $\varphi|_{t=T}=0$, after integration by parts we obtain

$$\int\limits_{S_T^0\cup S_T} \frac{\partial u}{\partial N}\,\varphi\,ds - \int\limits_{D_T} u_t\varphi_t\,dx\,dt + \int\limits_{D_T} \nabla_x u\nabla_x\varphi\,dx\,dt + \lambda\int\limits_{D_T} f(u)\varphi\,dx\,dt = \int\limits_{D_T} F\varphi\,dx\,dt, \quad (1.4)$$

where $\frac{\partial}{\partial N} = \nu_0\,\frac{\partial}{\partial t} - \sum\limits_{i=1}^{n}\nu_i\,\frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal, $\nu = (\nu_1,\dots,\nu_n,\nu_0)$ is the unit vector of the outer normal to ∂D_T, $\nabla_x = \big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\big)$. Taking into account that $\frac{\partial u}{\partial N}\big|_{S_T^0} = \frac{\partial u}{\partial x_n}$ and S_T is a characteristic manifold in which $\frac{\partial}{\partial N}$ is an inner differential operator, by virtue of (1.2) we have $\frac{\partial u}{\partial N}\big|_{S_T^0\cup S_T} = 0$. Therefore the equality (1.4) takes the form

$$-\int\limits_{D_T} u_t\varphi_t\,dx\,dt + \int\limits_{D_T} \nabla_x u\nabla_x\varphi\,dx\,dt + \lambda\int\limits_{D_T} f(u)\varphi\,dx\,dt = \int\limits_{D_T} F\varphi\,dx\,dt. \quad (1.5)$$

The equality (1.5) can be considered as a basis of the definition of a weak generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T.

Suppose $\overset{\circ}{W}{}_2^1(D_T,S_T) := \big\{u\in W_2^1(D_T):\ u|_{S_T}=0\big\}$, where $W_2^1(D_T)$ is the well-known Sobolev's space, and the equality $u|_{S_T}=0$ is understood in the sense of the trace theory [**49**, p. 70].

Definition 1.1. Let $F\in L_2(D_T)$. The function $u\in \overset{\circ}{W}{}_2^1(D_T,S_T)$ is said to be a weak generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T, if $f(u)\in L_2(D_T)$ and for every function $\varphi\in W_2^1(D_T)$ such that $\varphi|_{t=T}=0$ the equality (1.5) is fulfilled.

Remark 1.1. In a standard way [**49**, p. 113] it is proved that if a weak solution u of the problem (1.1), (1.2) belongs to the space $W_2^1(D_T)$, then for that solution the homogeneous boundary conditions (1.2) will be fulfilled in the sense of the trace theory.

Assume $\overset{\circ}{C}{}^2(\overline{D}_T,S_T^0,S_T) := \big\{u\in C^2(\overline{D}_T):\ \frac{\partial u}{\partial x_n}\big|_{S_T^0}=0,\ u|_{S_T}=0\big\}$.

Definition 1.2. Let $F\in L_2(D_T)$. The function $u\in \overset{\circ}{W}{}_2^1(D_T,S_T)$ is said to be a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T, if there exists a sequence of functions

$u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ such that $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ and $L_\lambda u_k \to F$ in the space $L_2(D_T)$.

Remark 1.2. It can be easily verified that if $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1, then this solution will automatically be a weak generalized solution of that problem if the nonlinear Nemytski operator $K : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T)$ acting by the formula $Ku = f(u)$ is continuous. Therefore, if it is additionally known that $u \in W_2^2(D_T)$, then the boundary conditions (1.2) for that solution will be fulfilled in the sense of the trace theory. Below we will distinguish the cases where the operator K is continuous from the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ to $L_2(D_T)$.

Definition 1.3. Let $F \in L_{2,loc}(D)$ and $F \in L_2(D_T)$ for any $T > 0$. We say that the problem (1.1), (1.2) is globally solvable in the class W_2^1 if for every $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T.

Remark 1.3. We can define analogously a weak generalized solution of the problem (1.1), (1.3) of the class W_2^1 in the domain D_T as a function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T) := \big\{v \in W_2^1(D_T) : \ v|_{S_T^0 \cup S_T} = 0\big\}$ for which $f(u) \in L_2(D_T)$ and the integral equality (1.5) is valid for every function $\varphi \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$, where $F \in L_2(D_T)$. The function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ is said to be a strong generalized solution of the problem (1.1), (1.3) of the class W_2^1 in the domain D_T if there exists a sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0 \cup S_T) := \big\{v \in C^2(\overline{D}_T) : \ v|_{S_T^0 \cup S_T} = 0\big\}$ such that $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T^0 \cup S_T)$ and $L_\lambda u_k \to F$ in the space $L_2(D_T)$. Analogously, we say that the problem (1.1), (1.3) is globally solvable in the class W_2^1 if for every $T > 0$ this problem has a strong generalized solution of the class W_2^1 in the domain D_T.

Below we will distinguish particular cases for the nonlinear function $f = f(u)$, when the problem (1.1), (1.3) is globally solvable in the class W_2^1 in one case, and such solvability does not take place in the other case.

2. A Priori Estimates

Lemma 2.1. *Let $\lambda \geq 0$, $f(u) = |u|^p u$, $p > 0$ and $F \in L_2(D_T)$. Then for every strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T the a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{\frac{e}{2}}\, T \|F\|_{L_2(D_T)} \tag{2.1}$$

is valid.

Proof. Let $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ be a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T. By Definition 1.2, there exists a sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0, \quad \lim_{k\to\infty} \|L_\lambda u_k - F\|_{L_2(D_T)} = 0. \tag{2.2}$$

Consider the function $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$ as a solution of the problem

$$L_\lambda u_k = F_k, \tag{2.3}$$

$$\frac{\partial u_k}{\partial x_n}\Big|_{S_T^0} = 0, \quad u_k\big|_{S_k} = 0. \tag{2.4}$$

Here

$$F_k := L_\lambda u_k. \tag{2.5}$$

Multiplying both parts of the equation (2.3) by $\frac{\partial u_k}{\partial t}$ and integrating over the domain D_τ, $0 < \tau \le T$, we get

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_k}{\partial t}\Big)^2\,dx\,dt - \int\limits_{D_\tau} \Delta u_k \,\frac{\partial u_k}{\partial t}\,dx\,dt + \frac{\lambda}{p+2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\,|u_k|^{p+2}\,dx\,dt =$$
$$= \int\limits_{D_\tau} F_k\,\frac{\partial u_k}{\partial t}\,dx\,dt. \tag{2.6}$$

Assume $\Omega_\tau := D_T \cap \{t = \tau\}$, $0 < \tau < T$. Obviously, $\partial D_\tau = S_\tau^0 \cup S_\tau \cup \Omega_\tau$. Taking into account (2.4) and the equalities $\nu|_{\Omega_\tau} = (0, \dots, 0, 1)$, $\nu|_{S_T^0} = (0, \dots, -1, 0)$, by integration by parts we obtain

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_k}{\partial t}\Big)^2\,dx\,dt = \int\limits_{\partial D_\tau}\Big(\frac{\partial u_k}{\partial t}\Big)^2\nu_0\,dx\,dt =$$
$$= \int\limits_{\Omega_\tau}\Big(\frac{\partial u_k}{\partial t}\Big)^2\,dx + \int\limits_{S_\tau}\Big(\frac{\partial u_k}{\partial t}\Big)^2\nu_0\,ds,$$

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\,(u_k)^2\,dx\,dt = \int\limits_{\partial D_\tau} u_k^2\nu_0\,ds = \int\limits_{\Omega_\tau} u_k^2\,dx,$$

$$\int\limits_{D_\tau}\frac{\partial}{\partial t}\,|u_k|^{p+2}\,dx\,dt = \int\limits_{\partial D_\tau}|u_k|^{p+2}\nu_0\,ds = \int\limits_{\Omega_\tau}|u_k|^{p+2}\,dx,$$

$$\int\limits_{D_\tau}\frac{\partial^2 u_k}{\partial x_i^2}\,\frac{\partial u_k}{\partial t}\,dx\,dt = \int\limits_{\partial D_\tau}\frac{\partial u_k}{\partial x_i}\,\frac{\partial u_k}{\partial t}\,\nu_i\,ds - \frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u_k}{\partial x_i}\Big)^2\,dx\,dt =$$
$$= \int\limits_{\partial D_\tau}\frac{\partial u_k}{\partial x_i}\,\frac{\partial u_k}{\partial t}\,\nu_i\,ds - \frac{1}{2}\int\limits_{\partial D_\tau}\Big(\frac{\partial u_k}{\partial x_i}\Big)^2\nu_0\,ds =$$

$$= \int\limits_{S_\tau} \frac{\partial u_k}{\partial x_i}\, \frac{\partial u_k}{\partial t}\, \nu_i\, ds - \frac{1}{2} \int\limits_{S_\tau} \Big(\frac{\partial u_k}{\partial x_i}\Big)^2 \nu_0\, ds - \frac{1}{2} \int\limits_{\Omega_\tau} \Big(\frac{\partial u_k}{\partial x_i}\Big)^2 dx,$$

whence by virtue of (2.6) we get

$$\int\limits_{D_\tau} F_k\, \frac{\partial u_k}{\partial t}\, dx\, dt =$$

$$= \int\limits_{S_\tau} \frac{1}{2\nu_0} \bigg[\sum_{i=1}^{n} \Big(\frac{\partial u_k}{\partial x_i}\, \nu_0 - \frac{\partial u_k}{\partial t}\, \nu_i\Big)^2 + \Big(\frac{\partial u_k}{\partial t}\Big)^2 \Big(\nu_0^2 - \sum_{j=1}^{n} \nu_j^2\Big) \bigg] ds +$$

$$+ \frac{1}{2} \int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u_k}{\partial x_i}\Big)^2 \bigg] dx + \frac{\lambda}{p+2} \int\limits_{\Omega_\tau} |u_k|^{p+2}\, dx. \tag{2.7}$$

Since S_τ is a characteristic manifold, we have

$$\Big(\nu_0^2 - \sum_{j=1}^{n} \nu_j^2\Big)\Big|_{S_\tau} = 0. \tag{2.8}$$

Taking into account that $\big(\nu_0^2\, \frac{\partial}{\partial x_i} - \nu_i\, \frac{\partial}{\partial t}\big)$, $i = 1, \dots, n$, is an inner differential operator on S_τ, by (2.4) we find that

$$\Big(\frac{\partial u_k}{\partial x_i}\, \nu_0 - \frac{\partial u_k}{\partial t}\, \nu_i\Big)\Big|_{S_\tau} = 0, \quad i = 1, \dots, n. \tag{2.9}$$

Owing to (2.8), (2.9), from (2.7) it follows

$$\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u_k}{\partial x_i}\Big)^2 \bigg] dx + \frac{2\lambda}{p+2} \int\limits_{\Omega_\tau} |u_k|^{p+2}\, dx =$$

$$= 2 \int\limits_{D_\tau} F_k\, \frac{\partial u_k}{\partial t}\, dx\, dt,$$

whence in view of $\lambda \geq 0$, it follows that

$$\int\limits_{\Omega_\tau} \bigg[\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u_k}{\partial x_i}\Big)^2 \bigg] dx \leq 2 \int\limits_{D_\tau} F_k\, \frac{\partial u_k}{\partial t}\, dx\, dt. \tag{2.10}$$

Putting

$$w(\delta) := \int\limits_{\Omega_\delta} \bigg[\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u_k}{\partial x_i}\Big)^2 \bigg] dx$$

and taking into account the inequality $2F_k\, \frac{\partial u_k}{\partial t} \leq \varepsilon \big(\frac{\partial u_k}{\partial t}\big)^2 + \frac{1}{\varepsilon}\, F_k^2$ which is valid for any $\varepsilon = \text{const} > 0$, from (2.10) we obtain

$$w(\delta) \leq \varepsilon \int\limits_0^\delta w(\sigma)\, d\sigma + \frac{1}{\varepsilon}\, \|F_k\|^2_{L_2(D_\delta)}, \quad 0 < \delta \leq T. \tag{2.11}$$

From (2.11), bearing in mind that $\|F_k\|^2_{L_2(D_\delta)}$ as a function of δ is nondecreasing, by Gronwall's lemma we get

$$w(\delta) \leq \frac{1}{\varepsilon}\,\|F_k\|^2_{L_2(D_\delta)} \exp\delta\varepsilon,$$

whence with regard for the fact that $\inf\limits_{\varepsilon>0} \frac{\exp\delta\varepsilon}{\varepsilon} = e\delta$ which is achieved for $\varepsilon = 1/\delta$, we obtain

$$w(\delta) \leq e\delta\|F_k\|^2_{L_2(D_\delta)}, \ \ 0 < \delta \leq T. \tag{2.12}$$

From (2.12) it in its turn follows that

$$\|u_k\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = \int\limits_{D_T} \left[\Big(\frac{\partial u_k}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_k}{\partial x_i}\Big)^2\right] dx\,dt = \\ = \int\limits_0^T w(\delta)\,d\delta \leq \frac{e}{2}\,T^2\|F_k\|^2_{L_2(D_T)}. \tag{2.13}$$

Here we have used the fact that in the space $\overset{\circ}{W}{}^1_2(D_T,S_T)$ one of the equivalent norms is given by means of the expression

$$\left\{\int\limits_{D_T} \left[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\right] dx\,dt\right\}^{1/2}.$$

Indeed, from the equalities $u|_{S_T} = 0$ and $u(x,t) = \int\limits_{\psi(x)}^{t} \frac{\partial u(x,\tau)}{\partial t}\,d\tau$, $(x,t) \in \overline{D}_T$, where $t-\psi(x)=0$ is the equation of the conic manifold S_T, standard reasoning results in the inequality

$$\int\limits_{D_T} u^2\,dx\,dt \leq T^2 \int\limits_{D_T} \Big(\frac{\partial u}{\partial t}\Big)^2\,dx\,dt.$$

Now, due to (2.2) and (2.5), passing in the inequality (2.13) to limit as $k\to\infty$, we obtain (2.1), which proves the above lemma. □

An a priori estimate for the solution of the problem (1.1), (1.3) is proved analogously.

Lemma 2.2. *Let $\lambda \geq 0$, $f(u) = |u|^p u$, $p > 0$ and $F \in L_2(D_T)$. Then for any strong generalized solution $u \in \overset{\circ}{W}{}^1_2(D_T, S^0_T \cup S_T)$ of the problem (1.1), (1.3) of the class W^1_2 in the domain D_T the a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}^1_2(D_T,S^0_T\cup S_T)} \leq \sqrt{\frac{e}{2}}\,T\|F\|_{L_2(D_T)} \tag{2.14}$$

holds.

3. The Global Solvability

First, let us consider the issue of the solvability of the corresponding to (1.1), (1.2) linear problem, when in the equation (1.2) the parameter $\lambda = 0$, i.e., for the problem

$$L_0 u(x,t) = F(x,t), \quad (x,t) \in D_T, \tag{3.1}$$

$$\frac{\partial u}{\partial x_n}\Big|_{S_T^0} = 0, \quad u\big|_{S_T} = 0. \tag{3.2}$$

In this case, for $F \in L_2(D_T)$ we introduce analogously the notion of a strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T for which there exists a sequence of functions $u_k \in \overset{\circ}{C}{}_2^1(\overline{D}_T, S_T^0, S_T)$ such that $\lim\limits_{k\to\infty} \|u_k - u\|_{W_2^1(D_T,S_T)} = 0$, $\lim\limits_{k\to\infty} \|L_0 u_k - F\|_{L_2(D_T)} = 0$. It should be noted that in view of Lemma 2.1, for $\lambda = 0$, the a priori estimate (2.1) is likewise valid for a strong generalized solution of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T.

Since the space $C_0^\infty(D_T)$ of finitary infinitely differentiable in D_T functions is dense in $L_2(D_T)$, for a given $F \in L_2(D_T)$ there exists a sequence of functions $F_k \in C_0^\infty(D_T)$ such that $\lim\limits_{k\to\infty} \|F_k - F\|_{L_2(D_T)} = 0$. For a fixed k, extending the function F_k evenly with respect to the variable x_n into the domain $D_T^- := \big\{(x,t) \in \mathbb{R}^{n+1} : \ x_n < 0, \ |x| < t < T\big\}$ and then by zero beyond the domain $D_T \cup D_T^-$, and leaving the same as above notation, we have $F_k \in C^\infty(\mathbb{R}^{n+1})$ with the support $\operatorname{supp} F_k \subset D_\infty \cup D_\infty^-$, where $\mathbb{R}_+^{n+1} := \mathbb{R}^{n+1} \cap \{t \geq 0\}$. Denote by u_k the solution of the Cauchy problem

$$L_0 u_k = F_k, \quad u_k\big|_{t=0} = 0, \quad \frac{\partial u_k}{\partial t}\Big|_{t=0} = 0, \tag{3.3}$$

which, as is known, exists, is unique and belongs to the space $C^\infty(\mathbb{R}_+^{n+1})$ [**17**, p. 192]. In addition, since $\operatorname{supp} F_k \subset D_\infty \cup D_\infty^- \subset \big\{(x,t) \in \mathbb{R}^{n+1} : t > |x|\big\}$ and $u_k|_{t=0} = 0$, $\frac{\partial u_k}{\partial t}\big|_{t=0} = 0$, taking into account the geometry of the domain of dependence of a solution of the linear wave equation $L_0 u = F$, we have $\operatorname{supp} u_k \subset \big\{(x,t) \in \mathbb{R}^{n+1} : \ t > |x|\big\}$ [**17**, p. 191], and, in particular, $u_k|_{S_T} = 0$. On the other hand, the function $\widetilde{u}_k(x_1, \dots, x_n, t) = u_k(x_1, \dots, -x_n, t)$ is likewise a solution of the same Cauchy problem (3.3), since F_k is an even function with respect to the variable x_n. Therefore, owing to the uniqueness of the solution of the Cauchy problem, we have $\widetilde{u}_k = u_k$, i.e., $u_k(x_1, \dots, -x_n, t) = u_k(x_1, \dots, x_n, t)$, and hence the function u_k is likewise even with respect to the variable x_n. This, in turn, implies that $\frac{\partial u_k}{\partial x_n}\big|_{x_n=0} = 0$, which along with the condition $u_k|_{S_T} = 0$ means that if we leave for the restriction of the function u_k to the domain D_T the same notation, then $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T^0, S_T)$. Further, by (2.1) and (3.3) there takes place the inequality

$$\|u_k - u_l\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{\frac{e}{2}}\, T \|F_k - F_l\|_{L_2(D_T)} \tag{3.4}$$

since the a priori estimate (2.1) is valid for a strong generalized solution of the problem (3.1), (3.2) of the class W_2^1 in the domain D_T, as well.

Since the sequence $\{F_k\}$ is fundamental in $L_2(D_T)$, therefore by virtue of (3.4) the sequence $\{u_k\}$ is also fundamental in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ which is complete. Therefore, there exists a function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ such that $\lim\limits_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0$, and since $L_0 u_k = F_k \to F$ in the space $L_2(D_T)$, this function will, by the definition, be a strong generalized solution of the problem (3.1), (3.2). The uniqueness of solution from the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ follows from the a priori estimate (2.1). Consequently, for the solution u of the problem (3.1), (3.2) we can write $u = L_0^{-1} F$, where $L_0^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a linear continuous operator whose norm, owing to (2.1), admits the estimate

$$\|L_0^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{\frac{e}{2}}\, T. \tag{3.5}$$

Remark 3.1. The embedding operator $I : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_q(D_T)$ is linear, continuous and compact for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$ [**49**, p. 81]. At the same time, Nemytski's operator $K : L_q(D_T) \to L_2(D_T)$ acting by the formula $Ku := -\lambda |u|^p u$ is continuous and bounded if $q \geq 2(p+1)$ [**47**, p. 349], [**48**, pp. 66, 67]. Thus if $p < \frac{2}{n-1}$, i.e., $2(p+1) \leq \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < 2(p+1) \leq q < \frac{2(n+1)}{n-1}$, and hence the operator

$$K_0 = KI : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T) \tag{3.6}$$

is continuous and compact. In addition, from $u_k \to u$ in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ it follows that $K_0 u_k \to K_0 u$ in the space $L_2(D_T)$. Therefore, according to Remark 1.2, a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T will also be a weak generalized solution of that problem of the class W_2^1 in the domain D_T.

Remark 3.2. For $F \in L_2(D_T)$, $0 < p < \frac{2}{n-1}$, by virtue of (3.5) and Remark 3.1 a function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T if and only if u is a solution of the functional equation

$$u = L_0^{-1}\big(-\lambda |u|^p u + F\big) \tag{3.7}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

We rewrite the equation (3.7) as follows:

$$u = Au := L_0^{-1}(K_0 u + F), \tag{3.8}$$

where the operator $K_0 : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T)$ from (3.6) is, by Remark 3.1, continuous and compact. Consequently, by (3.5) the operator $A : \overset{\circ}{W}{}_2^1(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is likewise continuous and compact. At the same time, by Lemma 2.1 for any parameter $\tau \in [0,1]$ and every solution of the equation $u = \tau Au$ with the parameter τ the a priori estimate $\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq c\|F\|_{L_2(D_T)}$ is valid with a positive constant c independent of u, F and τ. Therefore, according to the Leray–Schauder theorem [**66**, p. 375] the equation (3.8), and hence the problem (1.1), (1.2) has at least one strong generalized solution of the class W_2^1 in the domain D_T. Thus we have proved the following

Theorem 3.1. *Let $\lambda > 0$, $f(u) = |u|^p u$, $0 < p < \frac{2}{n-1}$, $F \in L_{2,loc}(D)$ and $F \in L_2(D_T)$ for any $T > 0$. Then the problem* (1.1), (1.2) *is globally solvable in the class W_2^1, i.e., for any $T > 0$ this problem has a weak generalized solution of the class W_2^1 in the domain D_T.*

Reasoning analogously, we can prove that the statement of Theorem 3.1 is likewise valid for the problem (1.1), (1.3).

4. The Non-Existence of the Global Solvability

Below we will consider the case where in the equation (1.1) the function $f(u) = -|u|^{p+1}$, $p > 0$, i.e., the equation

$$L_\lambda u := \frac{\partial^2 u}{\partial t^2} - \Delta u - \lambda |u|^{p+1} = F, \tag{4.1}$$

as well as the more general than (1.2) boundary condition

$$\frac{\partial u}{\partial x_n}\Big|_{S_T^0} = 0, \quad u\big|_{S_T} = g, \tag{4.2}$$

where g is a given real function on S_T.

Remark 4.1. Under the assumption that $F \in L_2(D_T)$, $g \in W_2^1(S_T)$ and $0 < p < \frac{2}{n-1}$, similarly to Definitions 1.1 and 1.2 concerning a weak and a strong generalized solution of the problem (1.1), (1.2) of the class W_2^1 in the domain D_T, with regard for Remark 3.1 we introduce the notions of a weak and a strong generalized solution of the problem (4.1), (4.2) of the class W_2^1 in the domain D_T:

(i) a function $u \in W_2^1(D_T)$ is said to be a weak generalized solution of the problem (4.1), (4.2) of the class W_2^1 in the domain D_T if for every function $\varphi \in W_2^1(D_T)$ such that $\varphi|_{t=T} = 0$ the integral equation

$$-\int\limits_{D_T} u_t\varphi_t\,dx\,dt+\int\limits_{D_T}\nabla_x u\nabla_x\varphi\,dx\,dt=$$
$$=\lambda\int\limits_{D_T}|u|^{p+1}\varphi\,dx\,dt+\int\limits_{D_T}F\varphi\,dx\,dt-\int\limits_{S_T}\frac{\partial g}{\partial N}\,\varphi\,ds \tag{4.3}$$

holds, where $\frac{\partial}{\partial N}=\nu_0\,\frac{\partial}{\partial t}-\sum\limits_{i=1}^{n}\nu_i\,\frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal being an inner differential operator on S_T since the conic manifold S_T is characteristic, and $\nu=(\nu_1,\dots,\nu_n,\nu_0)$ is the unit vector of the outer normal to ∂D_T, $\nabla_x:=\big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\big)$;

(ii) a function $u\in W_2^1(D_T)$ is said to be a strong generalized solution of the problem (4.1), (4.2) of the class W_2^1 in the domain D_T, if there exists a sequence of functions $u_k\in \overset{\circ}{C}{}_*^2(D_T,S_T):=\big\{u\in C^2(\overline{D}_T):\ \frac{\partial u}{\partial x_n}\big|_{S_T^0}=0\big\}$ such that $u_k\to u$ in the space $W_2^1(D_T)$, $L_\lambda u_k\to F$ in the space $L_2(D_T)$ and $u_k|_{S_T}\to g$ in the space $W_2^1(S_T)$.

Note also that according to Remarks 1.2 and 3.1 a strong generalized solution of the problem (4.1), (4.2) of the class W_2^1 in the domain D_T is likewise a weak generalized solution of that problem of the class W_2^1 in the domain D_T.

Analogously, we introduce the notion of the global solvability of the problem (4.1), (4.2) of the class W_2^1.

Remark 4.2. Below we will use the fact that the derivative with respect to the conormal $\frac{\partial}{\partial N}$, being an inner differential operator on the characteristic conic manifold S, coincides with the derivative $\frac{\partial}{\partial r}$ with respect to the spherical variable $r=(t^2+|x|^2)^{1/2}$ with minus sign.

We have the following theorem on the non-existence of the global solvability of the problem (4.1), (4.2).

Theorem 4.1. *Let $F\in L_{2,loc}(D)$, $g\in W_{2,loc}^1(S)$ and $F\in L_2(D_T)$, $g\in W_2^1(S_T)$ for any $T>0$. Then if $\lambda>0$, $0<p<\frac{2}{n-1}$ and*

$$F\big|_D\geq 0,\quad g\big|_S\geq 0,\quad \frac{\partial g}{\partial r}\bigg|_S\geq 0, \tag{4.4}$$

then there exists a positive number $T_0=T_0(F,g)$ such that for $T>T_0$ the problem (4.1), (4.2) cannot have a weak generalized solution of the class W_2^1 (for $F=0$ and $g=0$, nontrivial) in the domain D_T.

Proof. Let $G_T:\ |x|<t<T$, $G_T^-:=G_T\cap\{x_n<0\}$, $S_T^-:\ t=|x|$, $x_n\leq 0$, $t\leq T$. Obviously, $D_T=G_T^+:=G_T\cap\{x_n>0\}$ and $G_T=G_T^-\cup S_T^0\cup D_T$, where $S_T^0=\partial D_T\cap\{x_n=0\}$. We extend the functions u, F and g evenly with respect to the variable x_n into G_T^- and S_T^-, respectively. For the sake of simplicity, for the extended functions defined in G_T and $S_T^- \cup S_T$ we leave the same notation u, F and g. Then if $u \in W_2^1(D_T)$ is a weak generalized solution of the problem (4.1), (4.2) of the class W_2^1 in the domain D_T, then for every function $\psi \in W_2^1(G_T)$ such that $\psi|_{t=T} = 0$ the equality

$$-\int\limits_{G_T} u_t \psi_t \, dx \, dt + \int\limits_{G_T} \nabla_x u \nabla_x \psi \, dx \, dt =$$

$$= \lambda \int\limits_{G_T} |u|^{p+1} \psi \, dx \, dt + \int\limits_{G_T} F\psi \, dx \, dt - \int\limits_{S_T^- \cup S_T} \frac{\partial g}{\partial N} \, \psi \, ds \tag{4.5}$$

holds.

Indeed, if $\psi \in W_2^1(G_T)$ and $\psi|_{t=T} = 0$, then, obviously, $\psi|_{D_T} \in W_2^1(D_T)$ and $\widetilde{\psi} \in W_2^1(D_T)$, where, by definition, $\widetilde{\psi}(x_1, \dots, x_n, t) = \psi(x_1, \dots, -x_n, t)$, $(x_1, \dots, x_n, t) \in D_T$, and $\widetilde{\psi}|_{t=T} = 0$. Therefore, by the equality (4.3) we have

$$-\int\limits_{D_T} u_t \psi_t \, dx \, dt + \int\limits_{D_T} \nabla_x u \nabla_x \psi \, dx \, dt =$$

$$= \lambda \int\limits_{D_T} |u|^{p+1} \psi \, dx \, dt + \int\limits_{D_T} F\psi \, dx \, dt - \int\limits_{S_T} \frac{\partial g}{\partial N} \, \psi \, ds, \tag{4.6}$$

$$-\int\limits_{D_T} u_t \widetilde{\psi}_t \, dx \, dt + \int\limits_{D_T} \nabla_x u \nabla_x \widetilde{\psi} \, dx \, dt =$$

$$= \lambda \int\limits_{D_T} |u|^{p+1} \widetilde{\psi} \, dx \, dt + \int\limits_{D_T} F\widetilde{\psi} \, dx \, dt - \int\limits_{S_T} \frac{\partial g}{\partial N} \, \widetilde{\psi} \, ds. \tag{4.7}$$

Taking now into account that u, F and g are even functions with respect to the variable x_n, as well as the equality

$$\widetilde{\psi}(x_1, \dots, x_n, t) = \psi(x_1, \dots, -x_n, t), \ \ (x_1, \dots, x_n, t) \in D_T,$$

we find that

$$-\int\limits_{D_T} u_t \widetilde{\psi}_t \, dx \, dt + \int\limits_{D_T} \nabla_x u \nabla_x \widetilde{\psi} \, dx \, dt =$$

$$= -\int\limits_{G_T^-} u_t \psi_t \, dx \, dt + \int\limits_{G_T^-} \nabla_x u \nabla_x \psi \, dx \, dt, \tag{4.8}$$

$$\lambda \int\limits_{D_T} |u|^{p+1} \widetilde{\psi} \, dx \, dt + \int\limits_{D_T} F\widetilde{\psi} \, dx \, dt - \int\limits_{S_T} \frac{\partial g}{\partial N} \, \widetilde{\psi} \, ds =$$

$$= \lambda \int\limits_{D_T} |u|^{p+1} \psi \, dx \, dt + \int\limits_{G_T^-} F\psi \, dx \, dt - \int\limits_{S_T} \frac{\partial g}{\partial N} \, \psi \, ds. \tag{4.9}$$

From (4.7), (4.8) and (4.9) it follows that

$$-\int\limits_{G_T^-} u_t\psi_t\,dx\,dt+\int\limits_{G_T^-}\nabla_x u\nabla_x\psi\,dx\,dt=$$
$$=\lambda\int\limits_{G_T^-}|u|^{p+1}\psi\,dx\,dt+\int\limits_{G_T^-}F\psi\,dx\,dt-\int\limits_{S_T^-}\frac{\partial g}{\partial N}\,\psi\,ds. \qquad (4.10)$$

Finally, adding the equalities (4.6) and (4.10) we obtain (4.5).

Note that the inequality $\frac{\partial g}{\partial r}\big|_S\geq 0$ in the condition (4.4) is understood in the generalized sense, i.e., by the assumption $g\in W^1_{2,loc}(S)$ there exists the generalized derivative $\frac{\partial g}{\partial r}\in L_{2,loc}(S)$ which is nonnegative, and hence, for every function $\beta\in C(S)$ finitary with respect to the variable r, $\beta\geq 0$, the inequality

$$\int\limits_S\frac{\partial g}{\partial r}\,\beta\,ds\geq 0 \qquad (4.11)$$

holds.

Here we will use the method of test functions [**53**, pp. 10–12]. In the capacity of a test function in the equality (4.5) we take $\psi(x,t)=\psi_0\big[\frac{2}{T^2}\,(t^2+|x|^2)\big]$, where $\psi_0\in C^2((-\infty,+\infty))$, $\psi_0\geq 0$, $\psi_0'\leq 0$, $\psi_0(\sigma)=1$ for $0\leq\sigma\leq 1$ and $\psi_0(\sigma)=0$ for $\sigma\geq 2$ [**53**, p. 22]. Obviously, $\psi|_{t=T}=0$ and $\psi\in C^2(\overline{G}_T)$, and all the more, $\psi\in W_2^1(G_T)$.

Integrating the left-hand side (4.5) by parts, we obtain

$$\int\limits_{G_T}u\Box\psi\,dx\,dt=\lambda\int\limits_{G_T}|u|^{p+1}\psi\,dx\,dt+$$
$$+\int\limits_{G_T}F\psi\,dx\,dt+\int\limits_{S_T^-\cup S_T}g\,\frac{\partial\psi}{\partial N}\,ds-\int\limits_{S_T^-\cup S_T}\frac{\partial g}{\partial N}\,\psi\,ds. \qquad (4.12)$$

Taking into account Remark 4.1, (4.4) and (4.11), we have

$$\int\limits_{D_T}F\psi\,dx\,dt\geq 0,\qquad \int\limits_{S_T^-\cup S_T}g\,\frac{\partial\psi}{\partial N}\,ds\geq 0,\qquad \int\limits_{S_T^-\cup S_T}\frac{\partial g}{\partial N}\,\psi\,ds\leq 0, \qquad (4.13)$$

where ψ is the above-introduced test function.

Assuming that the functions F, g and ψ are fixed, we introduce into consideration the function of one variable T,

$$\gamma(T)=\int\limits_{G_T}F\psi\,dx\,dt+\int\limits_{S_T^-\cup S_T}g\,\frac{\partial\psi}{\partial N}\,ds-\int\limits_{S_T^-\cup S_T}\frac{\partial g}{\partial N}\,\psi\,ds,\ \ T>0. \qquad (4.14)$$

Owing to the absolute continuity of the integral and the inequalities (4.9), the function $\gamma(T)$ from (4.10) is nonnegative, continuous and nondecreasing. Note that $\lim\limits_{T\to\infty}\gamma(T)=0$.

Taking into account (4.10), we rewrite the equality (4.8) in the form

$$\lambda \int\limits_{G_T} |u|^{p+1}\psi \, dx\, dt = \int\limits_{G_T} u\Box\psi \, dx\, dt - \gamma(T).$$

The rest of our reasoning allowing for proving Theorem 4.1 word by word repeats that of Section 5 in Chapter II for $\alpha = p+1$. $\square$

Remark 4.3. The conclusion of Theorem 4.1 remains valid for the limiting case $p = \frac{2}{n-1}$ as well, if we take advantage of the reasoning presented in [**53**, p. 23]. The conclusion of that theorem ceases to be valid if the condition $p > \frac{2}{n-1}$ and the second condition of (4.4), i.e., the condition $g|_S \geq 0$, are violated simultaneously. Indeed, the function $u(x,t) = -\varepsilon(1+t^2-|x|^2)^{-1/p}$, $\varepsilon = \text{const} > 0$, is a global classical, and hence, generalized solution of the problem (4.1), (4.2) for $g = -\varepsilon$ $\big(\frac{\partial g}{\partial r}|_S = 0\big)$ and $F = \big[2\varepsilon\,\frac{n+1}{p} - 4\varepsilon\,\frac{p+1}{p^2}\,\frac{t^2-|x|^2}{1+t^2-|x|^2} - \lambda\varepsilon^{p+1}\big](1+t^2-|x|^2)^{\frac{p+1}{p}}$; in addition, as it can be easily verified, $F|_D \geq 0$ if $p > \frac{2}{n-1}$ and $0 < \varepsilon \leq \big\{\frac{2}{\lambda}\big[\frac{n+1-\frac{2(p+1)}{p}}{p}\big]\big\}^{1/p}$. Note that the inequality $n+1-\frac{2(p+1)}{p} > 0$ is equivalent to $p > \frac{2}{n-1}$.

Remark 4.4. The conclusion of Theorem 4.1 also ceases to be valid if only the third condition of (4.4) is violated, i.e., the condition $\frac{\partial g}{\partial r}\big|_S \geq 0$. Indeed, the function $u(x,t) = c_0\big[(t_0+1)^2 - |x|^2\big]^{-1/p}$, where $c_0 = \lambda^{-1/p}\big[\frac{4(p+1)}{p^2} - \frac{2(n+1)}{p}\big]^{-1/p}$, is a global classical solution of the problem (4.1), (4.2) for $F=0$ and $g = u|_S = c_0\big[(t+1)^2 - t^2\big]^{-1/p} > 0$.

Remark 4.5. In case $-1 < p < 0$, the problem (4.1), (4.2) may have more than one global solution. For example, for $F=0$ and $g=0$, the conditions (4.4) are fulfilled, but the problem (4.1), (4.2) has, besides the trivial solution, an infinite set of global linearly independent solutions $u_\alpha(x,t)$ depending on the parameter $\alpha \geq 0$ and given by the formula

$$u_\alpha(x,t) = \begin{cases} c_0\big[(t-\alpha)^2 - |x|^2\big]^{-1/p}, & t > \alpha + |x|, \\ 0, & |x| \leq t \leq \alpha + |x|, \end{cases}$$

where $c_0 = \lambda^{-1/p}\big[\frac{4(p+1)}{p^2} - \frac{2(n+1)}{p}\big]^{-1/p}$. It is not difficult to see that $u_\alpha \in C^1(\overline{D})$ for $p<0$, while for $-1/2 < p < 0$ the function $u_\alpha \in C^2(\overline{D})$.

5. The Local Solvability

Remark 5.1. Just as is mentioned in Remarks 3.1 and 3.2, for $0 < p < \frac{2}{n-1}$ the operator

$$K_1 : \overset{\circ}{W}{}^1_2(D_T, S_T) \to L_2(D_T) \quad \big(K_1 u = \lambda |u|^{p+1}\big) \tag{5.1}$$

is continuous and compact, and the problem (4.1), (4.2) for $g = 0$ is equivalent to the functional equation

$$u = A_1 u + u_0 \tag{5.2}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$, where

$$A_1 = L_0^{-1} K_1, \quad u_0 = L_0^{-1} F \in \overset{\circ}{W}{}_2^1(D_T, S_T) \tag{5.3}$$

with regard for (5.1). Here $L_0^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a linear continuous operator whose norm admits the estimate (3.5).

Remark 5.2. Let $B(0,d) := \big\{u \in \overset{\circ}{W}{}_2^1(D_T, S_T) : \ \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le d\big\}$ be the closed (convex) sphere in the Hilbert space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ of radius $d > 0$ with the center at the zero element. Since by the above Remark 5.1 the operator $A_1 : \overset{\circ}{W}{}_2^1(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ for $0 < p < \frac{2}{n-1}$ is continuous and compact, according to the Schauder principle for showing the solvability of the equation (5.2) it suffices to prove that the operator A_2 acting by the formula $A_2 u = A_1 u + u_0$ transforms the ball $B(0,d)$ into itself for some $d > 0$ [**66**, p. 379]. Towards this end, we will give here the needed estimate for $\|Au\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}$.

If $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$, we denote by $\widetilde{u}$ the function which is, in fact, the extension of the function u evenly through the planes $x_n = 0$ and $t = T$. Obviously $\widetilde{u} \in \overset{\circ}{W}{}_2^1(D_T^*)$, where $D_T^* : \ |x| < t < 2T - |x|$.

Using the inequality [**72**, p. 258]

$$\int\limits_{\Omega} |v| \, d\Omega \le (\operatorname{mes} \Omega)^{1-\frac{1}{q}} \|v\|_{q,\Omega}, \quad q \ge 1,$$

and taking into account the equalities

$$\|\widetilde{u}\|_{L_q(D_T^*)}^q = 4\|u\|_{L_q(D_T)}^q, \quad \|\widetilde{u}\|_{\overset{\circ}{W}{}_2^1(D_T^*)}^2 = 4\|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}^2,$$

from the well-known multiplicative inequality [**49**, p. 78]

$$\|v\|_{q,\Omega} \le \beta \|\nabla v\|_{m,\Omega}^{\widetilde{\alpha}} \|v\|_{r,\Omega}^{1-\widetilde{\alpha}} \ \ \forall\, v \in \overset{\circ}{W}{}_2^1(\Omega), \ \ \Omega \subset \mathbb{R}^{n+1},$$
$$\widetilde{\alpha} = \Big(\frac{1}{r} - \frac{1}{q}\Big)\Big(\frac{1}{r} - \frac{1}{\widetilde{m}}\Big)^{-1}, \ \ \widetilde{m} = \frac{(n+1)m}{n+1-m}\,,$$

for $\Omega = D_T^* \subset \mathbb{R}^{n+1}$, $v = \widetilde{u}$, $r = 1$, $m = 2$ and $1 < q \le \frac{2(n+1)}{n-1}$, where $\beta = \text{const} > 0$ does not depend on v and T, we obtain the following inequality:

$$\|u\|_{L_q(D_T)} \le c_0 (\operatorname{mes} D_T)^{\frac{1}{q}+\frac{1}{n+1}-\frac{1}{2}} \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \ \ \forall\, u \in \overset{\circ}{W}{}_2^1(D_T, S_T), \tag{5.4}$$

where $c_0 = \text{const} > 0$ does not depend on u.

Taking into account that $\operatorname{mes} D_T = \frac{\omega_n}{2(n+1)}\, T^{n+1}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, for $q = 2(p+1)$ (5.4) yields

$$\|u\|_{L_{2(p+1)}(D_T)} \le$$
$$\le c_0 \widetilde{\ell}_{p,n} T^{(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)} \|u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \quad \forall\, u \in \overset{\circ}{W}{}_2^1(D_T,S_T), \qquad (5.5)$$

where $\widetilde{\ell}_{p,n} = \left(\frac{\omega_n}{2(n+1)}\right)^{\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)}$.

For the value $\|K_1 u\|_{L_2(D_T)}$, where $u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$ and the operator K_1 acts by the equality from (5.1), by virtue of (5.5) the estimate

$$\|K_1 u\|_{L_2(D_T)} \le \lambda \Bigg[\int\limits_{D_T} |u|^{2(p+1)}\, dx\, dt\Bigg]^{1/2} = \lambda \|u\|^{p+1}_{L_{2(p+1)}(D_T)} \le$$
$$\le \lambda \ell_{p,n} T^{(p+1)(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)} \|u\|^{p+1}_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \qquad (5.6)$$

holds, where $\ell_{p,n} = [c_0 \widetilde{\ell}_{p,n}]^{p+1}$.

Now from (3.5) and (5.6), for $\|A_1 u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)}$, where by virtue of (5.3) $A_1 u = L_0^{-1} K_1 u$, the estimate

$$\|A_1 u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \le \|L_0^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}_2^1(D_T,S_T)} \|K_1 u\|_{L_2(D_T)} \le$$
$$\le \sqrt{\frac{e}{2}}\, \lambda \ell_{p,n} T^{1+(p+1)(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)} \|u\|^{p+1}_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \qquad (5.7)$$
$$\forall\, u \in \overset{\circ}{W}{}_2^1(D_T,S_T)$$

is valid.

Note that $\frac{1}{2(p+1)} + \frac{1}{n+1} - \frac{1}{2} > 0$ for $p < \frac{2}{n-1}$.

Consider the equation

$$a z^{p+1} + b = z \qquad (5.8)$$

with respect to the unknown function z, where

$$a = \sqrt{\frac{e}{2}}\, \lambda \ell_{p,n} T^{1+(p+1)(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)}, \quad b = \sqrt{\frac{e}{2}}\, T \|F\|_{L_2(D_T)}. \qquad (5.9)$$

For $T > 0$, it is evident that $a > 0$ and $b \ge 0$. A simple analysis similar to that which for $p = 2$ is performed in [**66**, pp. 373, 374] shows that:

(1) if $b = 0$, the equation (5.8) along with the zero root $z_1 = 0$ has the unique positive root $z_2 = a^{-1/p}$;
(2) if $b > 0$, then for $0 < b < b_0$, where

$$b_0 = \big[(p+1)^{-\frac{1}{p}} - (p+1)^{-\frac{p+1}{p}}\big] a^{-\frac{1}{p}}, \qquad (5.10)$$

the equation (5.8) has two positive roots z_1 and z_2, $0 < z_1 < z_2$, which for $b = b_0$ merge into one positive root

$$z_1 = z_2 = z_0 = \big[(p+1)a\big]^{-\frac{1}{p}};$$

(3) if $b > b_0$, then the equation (5.8) has no nonnegative root.

Note that for $0 < b < b_0$ there take place the inequalities

$$z_1 < z_0 = \big[(p+1)a\big]^{-\frac{1}{p}} < z_2.$$

Owing to (5.9) and (5.10), the condition $b \le b_0$ is equivalent to the condition

$$\sqrt{\frac{e}{2}}\,T\|F\|_{L_2(D_T)} \le$$

$$\le \bigg[\sqrt{\frac{e}{2}}\,\lambda\ell_{p,n}T^{1+(p+1)(n+1)\left(\frac{1}{2(p+1)}+\frac{1}{n+1}-\frac{1}{2}\right)}\bigg]^{-\frac{1}{p}}\big[(p+1)^{-\frac{1}{p}}-(p+1)^{-\frac{p+1}{p}}\big]$$

or

$$\|F\|_{L_2(D_T)} \le \gamma_{n,\lambda,p}T^{-\alpha_n}, \quad \alpha_n > 0, \tag{5.11}$$

where

$$\gamma_{n,\lambda,p} = \big[(p+1)^{-\frac{1}{p}} - (p+1)^{-\frac{p+1}{p}}\big](\lambda\ell_{p,n})^{-\frac{1}{p}} \exp\Big[-\frac{1}{2}\Big(1+\frac{1}{p}\Big)\Big],$$

$$\alpha_n = 1 + \frac{1}{p}\bigg[1 + (p+1)(n+1)\Big(\frac{1}{2(p+1)} + \frac{1}{n+1} - \frac{1}{2}\Big)\bigg].$$

Bearing in mind that the Lebesgue integral is absolutely continuous, we have $\lim\limits_{T\to 0}\|F\|_{L_2(D_T)} = 0$. At the same time, $\lim\limits_{T\to 0} T^{-\alpha_n} = +\infty$. Therefore, there exists a number $T_1 = T_1(F)$, $0 < T_1 < +\infty$ such that inequality (5.11) holds for

$$0 < T \le T_1(F). \tag{5.12}$$

Let us now show that if the condition (5.12) is fulfilled, then the operator $A_2 : \overset{\circ}{W}{}^1_2(D_T,S_T) \to \overset{\circ}{W}{}^1_2(D_T,S_T)$ acting by the formula $A_2 = A_1u + u_0$ transforms the ball $B(0,z_2)$ mentioned in Remark 5.2 into itself, where z_2 is the maximal positive root of the equation (3.8). Indeed, if $u \in B(0,z_2)$, then by (5.7), (5.8) and (5.9) we have

$$\|A_2u\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \le a\|u\|^{p+1}_{\overset{\circ}{W}{}^1_2(D_T,S_T)} + b \le az_2^{p+1} + b = z_2.$$

Therefore, according to Remarks 5.1 and 5.2 the following theorem is valid.

Theorem 5.1. *Let $0 < p < \frac{2}{n-1}$, $g = 0$, $F \in L_{2,loc}(D)$ and $F \in L_2(D_T)$ for any $T > 0$. Then the problem* (4.1), (4.2) *in the domain D_T has at least one strong generalized solution of the class W_2^1 if T satisfies the inequality* (5.12).

Note that analogous results are valid for the problem (4.1), (4.3) as well.

CHAPTER 5

Characteristic Boundary Value Problems for Nonlinear Equations with the Iterated Wave Operator in the Principal Part

1. Statement of the First Characteristic Boundary Value Problem

In the Euclidean space $\mathbb{R}^{n+1}$ of the variables $x_1, \dots, x_n, t$, we consider the nonlinear equation of the type

$$L_\lambda u := \square^2 u + \lambda f(u) = F, \tag{1.1}$$

where λ is a given real constant, $f : \mathbb{R} \to \mathbb{R}$ is a given continuous nonlinear function, $f(0) = 0$, F is a given and u is an unknown real function, $\square := \frac{\partial^2}{\partial t^2} - \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$.

By $D_T : \ |x| < t < T - |x|$ we denote the domain which is the intersection of the light cone of future $K_0^+ : \ \ t > |x|$ with the vertex at the origin $O(0, \dots, 0)$ and the light cone of past $K_A^- : \ t < T - |x|$ with the vertex at the point $A(0, \dots, 0, T)$, $T = \text{const} > 0$.

For the equation (1.1), we consider the characteristic boundary value problem: find in the domain D_T a solution $u(x_1, \dots, x_n, t)$ of that equation according to the boundary condition

$$u\big|_{\partial D_T} = 0. \tag{1.2}$$

Assume $\overset{\circ}{C}{}^k(\overline{D}_T, \partial D_T) := \big\{ u \in C^k(\overline{D}_T) : \ u|_{\partial D_T} = 0 \big\}$, $k \geq 1$. Let $u \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T)$ be a classical solution of the problem (1.1), (1.2). Multiplying both parts of the equation (1.1) by an arbitrary function $\varphi \in \overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$ and integrating the obtained equality by parts over the domain D_T, we obtain

$$\int\limits_{D_T} \square u \square \varphi \, dx \, dt + \lambda \int\limits_{D_T} f(u) \varphi \, dx \, dt = \int\limits_{D_T} F \varphi \, dx \, dt. \tag{1.3}$$

When deducing (1.3), we have used the equality

$$\int\limits_{D_T} \square u \square \varphi \, dx \, dt = \int\limits_{\partial D_T} \frac{\partial \varphi}{\partial N} \square \varphi \, ds - \int\limits_{\partial D_T} \varphi \, \frac{\partial}{\partial N} \square u \, ds + \int\limits_{D_T} \varphi \square^2 u \, dx \, dt$$

and the fact that since ∂D_T is a characteristic manifold, the derivative with respect to the conormal $\frac{\partial}{\partial N}=\nu_{n+1}\frac{\partial}{\partial t}-\sum\limits_{i=1}^{n}\nu_i\frac{\partial}{\partial x_i}$, where $\nu=(\nu_1,\dots,\nu_n,\nu_{n+1})$ is the unit vector of the outer normal to ∂D_T, is an inner differential operator on the characteristic manifold ∂D_T, and hence if $v\in \overset{\circ}{C}{}^1(\overline{D}_T,\partial D_T)$, then $\frac{\partial v}{\partial N}\big|_{\partial D_T}=0$.

Introduce the Hilbert space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ as the completion with respect to the norm

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)}=\int\limits_{D_T}\left[u^2+\Big(\frac{\partial u}{\partial t}\Big)^2+\sum_{i=1}^{n}\Big(\frac{\partial u}{\partial x_i}\Big)^2+(\Box u)^2\right]dx\,dt \tag{1.4}$$

of the classical space $\overset{\circ}{C}{}^2(\overline{D}_T,\partial D_T)$. It follows from (1.4) that if $u\in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$, then $u\in \overset{\circ}{W}{}^1_2(D_T)$ and $\Box u\in L_2(D_T)$. Here $W_2^1(D_T)$ is the well-known Sobolev space [**49**, p. 56] consisting of the elements $L_2(D_T)$ having the first order generalized derivatives from $L_2(D_T)$, and $\overset{\circ}{W}{}^1_2(D_T)=\big\{u\in W_2^1(D_T):\ u|_{\partial D_T}=0\big\}$, where the equality $u|_{\partial D_T}=0$ is understood in the sense of the trace theory [**30**, p. 70].

We take the equality (1.3) as a basis for our definition of the generalized solution of the problem (1.1), (1.2).

Definition 1.1. Let $F\in L_2(D_T)$. The function $u\in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ is said to be a weak generalized solution of the problem (1.1), (1.2), if $f(u)\in L_2(D_T)$ and for any function $\varphi\in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ the integral equality (1.3) is valid, i.e.,

$$\int\limits_{D_T}\Box u\Box\varphi\,dx\,dt+\lambda\int\limits_{D_T}f(u)\varphi\,dx\,dt=\int\limits_{D_T}F\varphi\,dx\,dt\ \ \forall\,\varphi\in \overset{\circ}{W}{}^1_{2,\Box}(D_T). \tag{1.5}$$

It is not difficult to verify that if a solution u of the problem (1.1), (1.2) belongs, in the sense of Definition 1.1, to the class $C^4(\overline{D}_T)$, then it will also be a classical solution of that problem.

2. The Solvability of the Problem (1.1), (1.2) in Case of the Nonlinearity of the Type $f(u)=|u|^\alpha\operatorname{sgn}u$

Let a nonlinear function f in the equation (1.1) be of the form

$$f(u)=|u|^\alpha\operatorname{sgn}u,\ \ \alpha=\text{const}>0,\ \ \alpha\neq 1. \tag{2.1}$$

Then according to (2.1) the equation (1.1) and the integral equality (1.5) take the form

$$L_\lambda u:=\Box^2u+\lambda|u|^\alpha\operatorname{sgn}u=F \tag{2.2}$$

and

$$\int\limits_{D_T} \square u \square \varphi \, dx\, dt + \lambda \int\limits_{D_T} \varphi |u|^\alpha \operatorname{sgn} u \, dx\, dt =$$
$$= \int\limits_{D_T} F\varphi \, dx\, dt \ \ \forall\, \varphi \in \overset{\circ}{W}{}^1_{2,\square}(D_T). \tag{2.3}$$

Lemma 2.1. *The inequality*

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\square}(D_T)} \le c\|\square u\|_{L_2(D_T)} \ \ \forall\, u \in \overset{\circ}{W}{}^1_{2,\square}(D_T) \tag{2.4}$$

holds, where the norm of the space $\overset{\circ}{W}{}^1_{2,\square}(D_T)$ *is given by the equality* (1.4) *and the positive constant* c *does not depend on* u.

Proof. Since the space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$ is a dense subspace of the space $\overset{\circ}{W}{}^1_{2,\square}(D_T)$, it suffices to prove that $\forall\, u \in \overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\square}(D^+_{T/2})} \le c^2 \|\square u\|^2_{L_2(D^+_{T/2})}, \ \ \|u\|^2_{\overset{\circ}{W}{}^1_{2,\square}(D^-_{T/2})} \le c^2 \|\square u\|^2_{L_2(D^-_{T/2})}, \tag{2.5}$$

where $D^+_{T/2} = D_T \cap \{t < T/2\}$, $D^-_{T/2} = D_T \cap \{t > T/2\}$ and the norm $\|\cdot\|_{\overset{\circ}{W}{}^1_{2,\square}(D^\pm_{T/2})}$ is given by the equality (1.4) in which instead of D_T we have to take $D^\pm_{T/2}$.

We restrict ourselves to the proof of the first inequality (2.5) since the second one is word by word proved analogously.

Assume $\Omega_\tau := \overline{D}{}^\pm_{T/2} \cap \{t = \tau\}$, $D^+_\tau := D^+_{T/2} \cap \{t < \tau\}$, $S^+_\tau := \big\{(x,t) \in \partial D^+_\tau : \ t = |x|\big\}$, $0 < \tau \le T/2$, and let $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ be the unit vector of the outer normal to ∂D^+_τ. For $u \in \overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$, in view of the equalities $u|^+_{S_\tau} = 0$, $\Omega_\tau = \partial D^+_\tau \cap \{t = \tau\}$ and $\nu|_{\Omega_\tau} = (0, \dots, 0, 1)$, the integration by parts provides us with

$$\int\limits_{D^+_\tau} \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} \, dx\, dt = \frac{1}{2} \int\limits_{D^+_\tau} \frac{\partial}{\partial t} \Big(\frac{\partial u}{\partial t}\Big)^2 dx\, dt = \frac{1}{2} \int\limits_{D^+_\tau} \Big(\frac{\partial u}{\partial t}\Big)^2 \nu_{n+1}\, ds =$$
$$= \frac{1}{2} \int\limits_{\Omega_\tau} \Big(\frac{\partial u}{\partial t}\Big)^2 dx + \frac{1}{2} \int\limits_{S^+_\tau} \Big(\frac{\partial u}{\partial t}\Big)^2 \nu_{n+1}\, ds, \ \ \tau \le \frac{T}{2}\,, \tag{2.6}$$

$$\int\limits_{D^+_\tau} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial u}{\partial t} \, dx\, dt = \int\limits_{\partial D^+_\tau} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial t} \nu_i \, ds - \frac{1}{2} \int\limits_{D^+_\tau} \frac{\partial}{\partial t} \Big(\frac{\partial u}{\partial x_i}\Big)^2 dx\, dt =$$
$$= \int\limits_{\partial D^+_\tau} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial t} \nu_i \, ds - \frac{1}{2} \int\limits_{\partial D^+_\tau} \Big(\frac{\partial u}{\partial x_i}\Big)^2 \nu_{n+1}\, ds =$$

$$= \int\limits_{\partial D_\tau^+} \frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial t}\,\nu_i\,ds - \frac{1}{2}\int\limits_{S_\tau^+}\Big(\frac{\partial u}{\partial x_i}\Big)^2 \nu_{n+1}\,ds - \frac{1}{2}\int\limits_{\Omega_\tau}\Big(\frac{\partial u}{\partial x_i}\Big)^2 dx,\quad \tau \le \frac{T}{2}\,. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\int\limits_{D_\tau^+} \Box u\,\frac{\partial u}{\partial t}\,dx\,dt =$$

$$= \int\limits_{S_\tau^+} \frac{1}{2\nu_{n+1}}\Big[\sum_{i=1}^n\Big(\frac{\partial u}{\partial x_i}\,\nu_{n+1} - \frac{\partial u}{\partial t}\,\nu_i\Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2\Big(\nu_{n+1}^2 - \sum_{j=1}^n \nu_j^2\Big)\Big]\,ds +$$

$$+\frac{1}{2}\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n\Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\,dx,\quad \tau \le \frac{T}{2}\,. \tag{2.8}$$

Since $u|_{S_\tau^+} = 0$ and $\big(\nu_{n+1}\,\frac{\partial}{\partial x_i} - \nu_i\,\frac{\partial}{\partial t}\big)$, $1 \le i \le n$, is an inner differential operator on S_τ^+, there take place the following equalities:

$$\Big(\frac{\partial u}{\partial x_i}\,\nu_{n+1} - \frac{\partial u}{\partial t}\,\nu_i\Big)\Big|_{S_\tau^+} = 0,\quad i = 1,\dots,n. \tag{2.9}$$

Therefore, taking into account that $\nu_{n+1}^2 - \sum\limits_{j=1}^n \nu_j^2 = 0$ on the characteristic manifold S_τ^+, by virtue of (2.8) and (2.9) we have

$$\int\limits_{\Omega_\tau}\Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n\Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\,dx = 2\int\limits_{D_\tau^+} \Box u\,\frac{\partial u}{\partial t}\,dx\,dt,\quad \tau \le \frac{T}{2}\,. \tag{2.10}$$

Putting

$$w(\delta) := \int\limits_{\Omega_\delta}\Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n\Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\,dx$$

and using the equality

$$2\Box u\,\frac{\partial u}{\partial t} \le \varepsilon\Big(\frac{\partial u}{\partial t}\Big)^2 + \frac{1}{\varepsilon}\,|\Box u|^2$$

which is valid for every $\varepsilon = \text{const} > 0$, from (2.10) we obtain

$$w(\delta) \le \varepsilon\int\limits_0^\delta w(\sigma)\,d\sigma + \frac{1}{\varepsilon}\,\|\Box u\|^2_{L_2(D_\delta^+)},\quad 0 < \delta \le \frac{T}{2}\,. \tag{2.11}$$

From (2.11), taking into account that $\|\Box u\|^2_{L_2(D_\delta^+)}$ as a function of δ is nondecreasing, by Gronwall's lemma [**15**, p. 13] we find that

$$w(\delta) \le \frac{1}{\varepsilon}\,\|\Box u\|^2_{L_2(D_\delta^+)}\exp \delta\varepsilon,$$

whence bearing in mind that $\inf\limits_{\varepsilon>0} \frac{1}{\varepsilon}\exp\delta\varepsilon = e\delta$ is achieved for $\varepsilon = 1/\delta$, we obtain

$$w(\delta) \le e\delta\|\Box u\|^2_{L_2(D^+_\delta)}, \quad 0<\delta\le\frac{T}{2}\,. \tag{2.12}$$

In its turn, from (2.12) it follows that

$$\int\limits_{D^+_{T/2}} \bigg[\Big(\frac{\partial u}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u}{\partial x_i}\Big)^2\bigg]\,dx\,dt=\int\limits_0^{T/2} w(\delta)\,d\delta\le\frac{e}{8}\,T^2\|\Box u\|^2_{L_2(D^+_{T/2})}. \tag{2.13}$$

Using the equalities $u|_{S_{T/2}}=0$ and $u(x,t)=\int\limits_{|x|}^{t}\frac{\partial u(x,\tau)}{\partial t}\,d\tau$, $(x,t)\in\overline{D}^+_{T/2}$, which are valid for every function $u\in\overset{\circ}{C}{}^2(\overline{D}_T,\partial D_T)$, and reasoning in a standard way [**49**, p. 69], it is not difficult to get the inequality

$$\int\limits_{D^+_{T/2}} u^2(x,t)\,dx\,dt\le\frac{T^2}{4}\int\limits_{D^+_{T/2}}\Big(\frac{\partial u}{\partial t}\Big)^2\,dx\,dt. \tag{2.14}$$

Owing to (2.13) and (2.14), we have

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D^+_{T/2})}=\int\limits_{D^+_{T/2}}\bigg[u^2+\Big(\frac{\partial u}{\partial t}\Big)^2+\sum_{i=1}^n\Big(\frac{\partial u}{\partial x_i}\Big)^2+(\Box u)^2\bigg]\,dx\,dt\le$$
$$\le\Big(1+\frac{e}{8}\,T^2+\frac{e}{32}\,T^4\Big)\|\Box u\|^2_{L_2(D^+_{T/2})},$$

whence we obtain the first inequality from (2.5) with the constant $c^2=1+\frac{e}{8}\,T^2+\frac{e}{32}\,T^4$. Thus we have proved the lemma. □

Lemma 2.2. *Let $F\in L_2(D_T)$, $0<\alpha<1$, and in the case $\alpha>1$ we additionally require that $\lambda>0$. Then for a weak generalized solution $u\in\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ of the problem (1.1), (1.2) with nonlinearity of the type (2.1), i.e., of the problem (2.2), (1.2) in the sense of the integral equality (2.3) for $|u|^\alpha\in L_2(D_T)$, the a priori estimate*

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)}\le c_1\|F\|_{L_2(D_T)}+c_2 \tag{2.15}$$

is valid with nonnegative constants $c_i(T,\alpha,\lambda)$, $i=1,2$, independent of u and F and $c_1>0$.

Proof. First, let $\alpha>1$ and $\lambda>0$. Putting $\varphi=u\in\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ in the equality (2.3) and taking into account (1.4) for any $\varepsilon>0$, we obtain

$$\|\Box u\|^2_{L_2(D_T)}=\int\limits_{D_T}(\Box u)^2\,dx\,dt=-\lambda\int\limits_{D_T}|u|^{\alpha+1}\,dx\,dt+\int\limits_{D_T}Fu\,dx\,dt\le$$

$$\leq \int\limits_{D_T} Fu\,dx\,dt \leq \frac{1}{4\varepsilon}\int\limits_{D_T} F^2\,dx\,dt + \varepsilon\|u\|^2_{L_2(D_T)} \leq$$

$$\leq \frac{1}{4\varepsilon}\,\|F\|^2_{L_2(D_T)} + \varepsilon\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)}. \tag{2.16}$$

By virtue of (2.4) and (2.16), we have

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} \leq c^2\|\Box u\|^2_{L_2(D_T)} \leq \frac{c^2}{4\varepsilon}\,\|F\|^2_{L_2(D_T)} + c^2\varepsilon\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)},$$

whence for $\varepsilon = \frac{1}{2c^2} < \frac{1}{c^2}$ we obtain

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} \leq \frac{c^2}{4\varepsilon(1-\varepsilon c^2)}\,\|F\|^2_{L_2(D_T)} = c^4\|F\|^2_{L_2(D_T)}. \tag{2.17}$$

In case $\alpha > 1$ and $\lambda > 0$, from (2.17) it follows the inequality (2.15) with $c_1 = c^2$ and $c_2 = 0$.

Let now $0 < \alpha < 1$. Using the well-known inequality $ab \leq \frac{\varepsilon a^p}{p} + \frac{b^q}{q\varepsilon^{q-1}}$ with the parameter $\varepsilon > 0$ for $a = |u|^{\alpha+1}$, $b = 1$, $p = \frac{2}{\alpha+1} > 1$, $q = \frac{2}{a-\alpha}$, $\frac{1}{p} + \frac{1}{q} = 1$, analogously as when deducing the inequality (2.16) we have

$$\|\Box u\|^2_{L_2(D_T)} = \int\limits_{D_T} (\Box u)^2\,dx\,dt = -\lambda\int\limits_{D_T} |u|^{\alpha+1}\,dx\,dt + \int\limits_{D_T} Fu\,dx\,dt \leq$$

$$\leq |\lambda|\int\limits_{D_T}\Big[\varepsilon\,\frac{1+\alpha}{2}\,|u|^2 + \frac{1-\alpha}{2\varepsilon^{q-1}}\Big]\,dx\,dt + \frac{1}{4\varepsilon}\int\limits_{D_T} F^2\,dx\,dt + \varepsilon\int\limits_{D_T} u^2\,dx\,dt =$$

$$\leq \frac{1}{4\varepsilon}\,\|F\|^2_{L_2(D_T)} + \varepsilon\Big(|\lambda|\,\frac{1+\alpha}{2} + 1\Big)\|u\|^2_{L_2(D_T)} + |\lambda|\,\frac{1-\alpha}{2\varepsilon^{q-1}}\,\operatorname{mes} D_T. \tag{2.18}$$

By virtue of (1.4) and (2.4), it follows from (2.18) that

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} \leq c^2\|\Box u\|^2_{L_2(D_T)} \leq$$

$$\leq \frac{c^2}{4\varepsilon}\,\|F\|^2_{L_2(D_T)} + \varepsilon c^2\Big(|\lambda|\,\frac{1+\alpha}{2} + 1\Big)\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} + c^2|\lambda|\,\frac{1-\alpha}{2\varepsilon^{q-1}}\,\operatorname{mes} D_T,$$

$$q = \frac{2}{1-\alpha},$$

whence for $\varepsilon = \frac{1}{2}\,c^{-2}\big(|\lambda|\,\frac{1+\alpha}{2} + 1\big)^{-1}$ we obtain

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} \leq$$

$$\leq \Big[1 - \varepsilon c^2\Big(|\lambda|\,\frac{1+\alpha}{2} + 1\Big)\Big]^{-1}\Big(\frac{c^2}{4\varepsilon}\,\|F\|_{L_2(D_T)} + c^2|\lambda|\,\frac{1-\alpha}{2\varepsilon^{q-1}}\,\operatorname{mes} D_T\Big) =$$

$$= c^4\Big(|\lambda|\,\frac{1+\alpha}{2} + 1\Big)\|F\|^2_{L_2(D_T)} + 2c^2|\lambda|\,\frac{1-\alpha}{2\varepsilon^{q-1}}\,\operatorname{mes} D_T. \tag{2.19}$$

From (2.19), in case $0 < \alpha < 1$ it follows the inequality (2.15) with $c_1 = c^2\big(|\lambda|\,\frac{1+\alpha}{2} + 1\big)^{1/2}$ and $c_2 = c\big(2|\lambda|\,\frac{1-\alpha}{2\varepsilon^{q-1}}\,\mathrm{mes}\,D_T\big)^{1/2}$, where $q = \frac{1}{1-\alpha}$. Thus the lemma is proved completely. □

Remark 2.1. It follows from the proof of Lemma 2.2 that the constants c_1 and c_2 in the estimate (2.15) are equal to:
(1) $\alpha > 1$, $\lambda > 0$: $c_1 = c^2$, $c_2 = 0$; (2.20)
(2) $0 < \alpha < 1$, $-\infty < \lambda < +\infty$:

$$c_1 = c^2\Big(|\lambda|\,\frac{1+\alpha}{2} + 1\Big)^{1/2}, \quad c_2 = c\Big(2|\lambda|\,\frac{1-\alpha}{2\varepsilon^{q-1}}\,\mathrm{mes}\,D_T\Big)^{1/2}, \tag{2.21}$$

where the constant $c = \big(1 + \frac{e}{8}T^2 + \frac{e}{32}T^4\big)^{1/2}$ is taken from the estimate (2.4) and $q = \frac{1}{1-\alpha}$.

Remark 2.2. Below we will first consider the linear problem corresponding to (1.1), (1.2), i.e., the case where $\lambda = 0$. In this case, for $F \in L_2(D_T)$ we introduce analogously the notion of a weak generalized solution $u \in \overset{\circ}{W}{}^1_{2,\square}(D_T)$ of that problem when the integral equality

$$(u,\varphi)_\square := \int\limits_{D_T} \square u \square\varphi\, dx\, dt = \int\limits_{D_T} F\varphi\, dx\, dt \ \ \forall\, \varphi \in \overset{\circ}{W}{}^1_{2,\square}(D_T) \tag{2.22}$$

holds.

Remark 2.3. By (1.4) and (2.4), taking into account that

$$\big|(\square u, \square\varphi)_{L_2(D_T)}\big| = \Bigg|\int\limits_{D_T} \square u \square\varphi\, dx\, dt\Bigg| \leq$$
$$\leq \|\square u\|_{L_2(D_T)}\|\square\varphi\|_{L_2(D_T)} \leq \|\square u\|_{\overset{\circ}{W}{}^1_{2,\square}(D_T)}\|\square\varphi\|_{\overset{\circ}{W}{}^1_{2,\square}(D_T)},$$

we can take the bilinear form $(u,\varphi)_\square := \int\limits_{D_T} \square u \square\varphi\, dx\, dt$ from (2.22) as a scalar product in the Hilbert space $\overset{\circ}{W}{}^1_{2,\square}(D_T)$. Therefore, for $F \in L_2(D_T)$

$$\Bigg|\int\limits_{D_T} F\varphi\, dx\, dt\Bigg| \leq \|F\|_{L_2(D_T)}\|\varphi\|_{L_2(D_T)} \leq \|F\|_{L_2(D_T)}\|\varphi\|_{\overset{\circ}{W}{}^1_{2,\square}(D_T)},$$

and by the Riesz theorem [**10**, p. 83] there exists a unique function u from the space $\overset{\circ}{W}{}^1_{2,\square}(D_T)$ which satisfies the equality (2.22) for every $\varphi \in \overset{\circ}{W}{}^1_{2,\square}(D_T)$ and for its norm the estimate

$$\|u\|_{\overset{\circ}{W}{}^1_{2,\square}(D_T)} \leq \|F\|_{L_2(D_T)} \tag{2.23}$$

is valid. Thus introducing the notation $u = L_0^{-1}F$, we find that to the linear problem corresponding to (1.1), (1.2), i.e., for $\lambda = 0$, there corresponds the linear bounded operator $L_0^{-1}: L_2(D_T) \to \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ and for its norm the estimate

$$\|L_0^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}^1_{2,\Box}(D_T)} \leq \|F\|_{L_2(D_T)} \tag{2.24}$$

holds by virtue of (2.23).

Taking into account Definition 1.1 and Remark 2.3, we can rewrite the equality (2.3), equivalent to the problem (2.2), (1.2), in the form

$$u = L_0^{-1}\big[-\lambda|u|^\alpha \operatorname{sgn} u + F\big] \tag{2.25}$$

in the Hilbert space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$.

Remark 2.4. The embedding operator $I : \overset{\circ}{W}{}^1_2(D_T) \to L_q(D_T)$ is linear, continuous and compact for $1 < q < \frac{2(n+1)}{n-1}$, when $n \geq 2$ [**49**, p. 81]. At the same time, the Nemytski operator $N : L_q(D_T) \to L_2(D_T)$ acting by the formula $Nu = -\lambda|u|^\alpha \operatorname{sgn} u$, $\alpha > 1$, is continuous and bounded if $q \geq 2\alpha$ [**47**, p. 349], [**48**, pp. 66, 67]. Thus if $1 < \alpha < \frac{n+1}{n-1}$, then there exists a number q such that $1 < 2\alpha \leq q < \frac{2(n+1)}{n-1}$ and hence the operator

$$N_1 = NI : \overset{\circ}{W}{}^1_2(D_T) \to L_2(D_T) \tag{2.26}$$

is continuous and compact. In addition, from $u \in \overset{\circ}{W}{}^1_2(D_T)$ there follows $f(u) = |u|^\alpha \operatorname{sgn} u \in L_2(D_T)$. Next, since due to (1.4) the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ is continuously embedded into the space $\overset{\circ}{W}{}^1_2(D_T)$, bearing in mind (2.26) we will see that the operator

$$N_2 = NII_1 : \overset{\circ}{W}{}^1_{2,\Box}(D_T) \to L_2(D_T), \tag{2.27}$$

where $I_1 : \overset{\circ}{W}{}^1_{2,\Box}(D_T) \to \overset{\circ}{W}{}^1_2(D_T)$ is the embedding operator, is likewise continuous and compact for $1 < \alpha < \frac{n+1}{n-1}$. For $0 < \alpha < 1$ the operator (2.27) is also continuous and compact since, by Rellikh's theorem [**49**, p. 64], the space $\overset{\circ}{W}{}^1_2(D_T)$ is continuously and compactly embedded into $L_2(D_T)$, and the space $L_2(D_T)$ is, in its turn, continuously embedded into $L_p(D_T)$ for $0 < p < 2$.

We rewrite the equation (2.25) as follows:

$$u = Au := L_0^{-1}(N_2 u + F), \tag{2.28}$$

where the operator $N_2 : \overset{\circ}{W}{}^1_{2,\Box}(D_T) \to L_2(D_T)$, by Remark 2.4, for $0 < \alpha < \frac{n+1}{n-1}$, $\alpha \neq 1$, is continuous and compact. Then, taking into account (2.24) we conclude that the operator $A : \overset{\circ}{W}{}^1_{2,\Box}(D_T) \to \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ from (2.28) is likewise continuous and compact. At the same time, according to the a priori estimate (2.15) of Lemma 2.2 in which the constants c_1 and c_2

are given by the equalities (2.20) and (2.21), for any parameter $\tau \in [0,1]$ and for every solution $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ of the equation $u = \tau A u$ with the above-mentioned parameter the a priori estimate (2.15) is valid with positive constants $c_1 > 0$ and $c_2 \geq 0$ independent of u, F and τ. Therefore, by the Leray–Schauder theorem [**66**, p. 375] the equation (2.28) and hence the problem (2.2), (1.2) has at least one weak generalized solution u from the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$. Thus the following theorem is valid.

Theorem 2.1. *Let $0 < \alpha < \frac{n+1}{n-1}$, $\alpha \neq 1$, $\lambda \neq 0$ and $\lambda > 0$ for $\alpha > 1$. Then for any $F \in L_2(D_T)$ the problem (2.2), (1.2) has at least one weak generalized solution $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$.*

3. The Uniqueness of a Solution of the Problem (1.1), (1.2) in Case of the Nonlinearity of the Type $f(u) = |u|^\alpha \operatorname{sgn} u$

Let $F \in L_2(D_T)$, and moreover, let u_1 and u_2 be two weak generalized solutions of the problem (2.2), (1.2) from the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$, i.e., according to (2.3) the equalities

$$\int\limits_{D_T} \Box u_i \Box \varphi \, dx \, dt = \\ = -\lambda \int\limits_{D_T} \varphi |u_i|^\alpha \operatorname{sgn} u_i \, dx \, dt + \int\limits_{D_T} F \varphi \, dx \, dt \ \ \forall \varphi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T) \tag{3.1}$$

are valid and $|u_i|^\alpha \in L_2(D_T)$, $i = 1, 2$.

From (3.1), for the difference $v = u_2 - u_1$ we have

$$\int\limits_{D_T} \Box v \Box \varphi \, dx \, dt = \\ = -\lambda \int\limits_{D_T} \varphi \big(|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1 \big) \, dx \, dt \ \ \forall \varphi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T). \tag{3.2}$$

Putting $\varphi = v \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ in the equality (3.2), we obtain

$$\int\limits_{D_T} (\Box v)^2 \, dx \, dt = -\lambda \int\limits_{D_T} \big(|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1 \big)(u_2 - u_1) \, dx \, dt. \tag{3.3}$$

Note that for the finite values u_1 and u_2, for $\alpha > 0$ the inequality

$$\big(|u_2|^\alpha \operatorname{sgn} u_2 - |u_1|^\alpha \operatorname{sgn} u_1 \big)(u_2 - u_1) \geq 0 \tag{3.4}$$

holds.

From (3.3) and the equality (3.4) which is fulfilled for almost all points $(x,t) \in D_T$ for $u_i \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$, $i = 1, 2$, in case $\alpha > 0$ and $\lambda > 0$ it follows that

$$\int\limits_{D_T} (\Box v)^2\, dx\, dt \le 0$$

whence, owing to (2.4), we find that $v = 0$, i.e. $u_2 = u_1$.

Thus the following theorem is valid.

Theorem 3.1. *Let $\alpha > 0$, $\alpha \ne 1$ and $\lambda > 0$. Then for any $F \in L_2(D_T)$ the problem (2.2), (1.2) cannot have more than one generalized solution in the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$.*

From Theorems 2.1 and 3.1 it in its turn follows

Theorem 3.2. *Let $0 < \alpha < \frac{n+1}{n-1}$, $\alpha \ne 1$ and $\lambda > 0$. Then for any $F \in L_2(D_T)$ the problem (2.2), (1.2) has a unique weak generalized solution in the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$.*

4. The Non-Existence of a Solution of the Problem (1.1), (1.2) in the Case of the Nonlinearity of the Type $f(u) = |u|^\alpha$

Let now in the equation (1.1), and hence in the integral equality (1.2), the function $f(u) = |u|^\alpha$, $\alpha > 1$.

Theorem 4.1. *Let $F^0 \in L_2(D_T)$, $\|F^0\|_{L_2(D_T)}) \ne 0$, $F^0 \ge 0$, and $F = \mu F^0$, $\mu = \mathrm{const} > 0$. Then in case $f(u) = |u|^\alpha$, $\alpha > 1$, for $\lambda < 0$ there exists a number $\mu_0 = \mu_0(F^0, \mu, \alpha) > 0$ such that for $\mu > \mu_0$ the problem (1.1), (1.2) cannot have a weak generalized solution from the space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$.*

Proof. Assume that the conditions of the theorem are fulfilled and the solution $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ of the problem (1.1), (1.2) does exist for any fixed $\mu > 0$. Then the equality (1.5) takes the form

$$\int\limits_{D_T} \Box u \Box \varphi\, dx\, dt =$$

$$= -\lambda \int\limits_{D_T} |u|^\alpha \varphi\, dx\, dt + \mu \int\limits_{D_T} F^0 \varphi\, dx\, dt \ \ \forall\, \varphi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T). \tag{4.1}$$

It can be easily verified that

$$\int\limits_{D_T} \Box u \Box \varphi\, dx\, dt = \int\limits_{D_T} u \Box^2 \varphi\, dx\, dt \ \ \forall\, \varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T), \tag{4.2}$$

where $\overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T) := \big\{ u \in C^4(\overline{D}_T) :\ u|_{\partial D_T} = 0 \big\} \subset \overset{\circ}{W}{}^1_{2,\Box}(D_T)$. Indeed, since $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ and the space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$ is dense in $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$, there exists a sequence $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} = 0. \tag{4.3}$$

Taking into account that

$$\int\limits_{D_T} \Box u_k \Box\varphi \, dx\, dt = \\ = \int\limits_{\partial D_T} \frac{\partial u_k}{\partial N} \Box\varphi \, ds - \int\limits_{\partial D_T} u_k \frac{\partial}{\partial N} \Box\varphi \, ds + \int\limits_{D_T} u_k \Box^2\varphi \, dx\, dt, \tag{4.4}$$

where the derivative with respect to the conormal $\frac{\partial}{\partial N} = \nu_{n+1} \frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}$ is an inner differential operator on the characteristic manifold ∂D_T, and hence $\frac{\partial u_k}{\partial N}\big|_{\partial D_T} = 0$ since $u_k|_{\partial D_T} = 0$, from (4.4) we obtain

$$\int\limits_{D_T} \Box u_k \Box\varphi \, dx\, dt = \int\limits_{D_T} u_k \Box^2\varphi \, dx\, dt, \tag{4.5}$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the unit vector of the outer normal to ∂D_T. Passing in (4.5) to limit as $k \to \infty$, by virtue of (1.4) and (4.3) we obtain (4.2).

In view of (4.2), we rewrite the equality (4.1) as follows:

$$-\lambda \int\limits_{D_T} |u|^\alpha \varphi \, dx\, dt = \\ = \int\limits_{D_T} u \Box^2\varphi \, dx\, dt - \mu \int\limits_{D_T} F^0 \varphi \, dx\, dt \;\; \forall\, \varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T). \tag{4.6}$$

Below we will use the method of test functions [**53**, pp. 10–12]. As a test function we take $\varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T)$ such that $\varphi|_{D_T} > 0$. If in Young's inequality with the parameter $\varepsilon > 0$

$$ab \le \frac{\varepsilon}{\alpha} a^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} b^{\alpha'}; \;\; a, b \ge 0, \;\; \alpha' = \frac{\alpha}{\alpha - 1}$$

we take $a = |u|\varphi^{1/\alpha}$, $b = |\Box^2\varphi|/\varphi^{1/\alpha}$, then taking into account that $\alpha'/\alpha = \alpha' - 1$ we will have

$$|u\Box^2\varphi| = |u|\varphi^{1/\alpha} \frac{|\Box^2\varphi|}{\varphi^{1/\alpha}} \le \frac{\varepsilon}{\alpha} |u|^\alpha \varphi + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,. \tag{4.7}$$

By virtue of (4.7) and the fact that $-\lambda = |\lambda|$, from (4.6) there follows the inequality

$$\Big(|\lambda| - \frac{\varepsilon}{\alpha}\Big) \int\limits_{D_T} |u|^\alpha \varphi \, dx\, dt \le \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx\, dt - \mu \int\limits_{D_T} F^0 \varphi \, dx\, dt,$$

whence for $\varepsilon < |\lambda|\alpha$ we get

$$\int\limits_{D_T} |u|^\alpha \varphi\, dx\, dt \leq$$

$$\leq \frac{\alpha}{(|\lambda|\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} \int\limits_{D_T} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt - \frac{\alpha\mu}{|\lambda|\alpha-\varepsilon} \int\limits_{D_T} F^0\varphi\, dx\, dt. \tag{4.8}$$

Taking into account the equalities $\alpha' = \frac{\alpha}{\alpha-1}$, $\alpha = \frac{\alpha'}{\alpha'-1}$ and $\min\limits_{0<\varepsilon<|\lambda|\alpha} \frac{\alpha}{(|\lambda|\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = \frac{1}{|\lambda|^{\alpha'}}$ which is achieved for $\varepsilon = |\lambda|$, from (4.8) we find that

$$\int\limits_{D_T} |u|^\alpha \varphi\, dx\, dt \leq \frac{1}{|\lambda|^{\alpha'}} \int\limits_{D_T} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt - \frac{\alpha'\mu}{|\lambda|} \int\limits_{D_T} F^0\varphi\, dx\, dt. \tag{4.9}$$

Note that it is not difficult to show the existence of a test function φ such that

$$\varphi \in \overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T), \quad \varphi\big|_{D_T} > 0, \quad \varkappa_0 = \int\limits_{D_T} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt < +\infty. \tag{4.10}$$

Indeed, it can be easily verified that the function

$$\varphi(x,t) = \Big[(t^2-|x|^2)\big((T-t)^2-|x|^2\big)\Big]^m$$

for a sufficiently large positive m satisfies the conditions (4.10).

Since by the condition of the theorem $F^0 \in L_2(D_T)$, $\|F^0\|_{L_2(D_T)} \neq 0$, $F^0 \geq 0$, and $\operatorname{mes} D_T < +\infty$, due to the fact that $\varphi|_{D_T} > 0$ we will have

$$0 < \varkappa_1 = \int\limits_{D_T} F^0\varphi\, dx\, dt < +\infty. \tag{4.11}$$

Denote by $g(\mu)$ the left-hand side of the inequality (4.9) which is a linear function with respect to μ, and by (4.10) and (4.11) we will have

$$g(\mu) < 0 \ \text{ for } \ \mu > \mu_0 \ \text{ and } \ g(\mu) > 0 \ \text{ for } \ \mu < \mu_0, \tag{4.12}$$

where

$$g(\mu) = \frac{\varkappa_0}{|\lambda|^{\alpha'}} - \frac{\alpha'\mu}{|\lambda|}\varkappa_1, \quad \mu_0 = \frac{|\lambda|}{\alpha'|\lambda|^{\alpha'}} \cdot \frac{\varkappa_0}{\varkappa_1} > 0.$$

Owing to (4.12) for $\mu > \mu_0$, the right-hand side of the inequality (4.9) is negative, whereas the left-hand side of that inequality is nonnegative. The obtained contradiction proves the theorem. $\Box$

5. The Characteristic Cauchy Problem

For the nonlinear equation (1.1) with $f(u) = |u|^\alpha$, $\alpha = \text{const} > 0$, i.e., for the equation

$$L_\lambda := \square^2 u + \lambda |u|^\alpha = F, \quad \lambda = \text{const} < 0, \tag{5.1}$$

we consider the characteristic Cauchy problem: find in the frustrum of the cone of future $D_T^+ : \ |x| < t < T$ a solution $u(x,t)$ of that equation according the boundary conditions

$$u\Big|_{S_T} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{S_T} = 0, \tag{5.2}$$

where $S_T : \ t = |x|, t \leq T$ is the characteristic manifold being a conic portion of the boundary D_T^+, and $\frac{\partial}{\partial \nu}$ is the derivative with respect to the outer normal to ∂D_T^+. Considering the case $T = +\infty$, we assume $D_\infty^+ : \ t > |x|$ and $S_\infty = \partial D_\infty^+ : \ t = |x|$.

Below it will be shown that under certain conditions imposed on the nonlinearity exponent α and on the function F, the problem (5.1), (5.2) has no global solution, although, as it will be proved, this problem is locally solvable.

Let $\overset{\circ}{W}{}_2^2(D_T^+, S_T) := \{u \in W_2^2(D_T^+) : \ u|_{S_T} = 0, \ \frac{\partial u}{\partial \nu}\big|_{S_T} = 0\}$, where $W_2^2(D_T^+)$ is the well-known Sobolev's space [**49**, p. 56] consisting of the elements $L_2(D_T^+)$ having generalized derivatives up to the second order, inclusive, from $L_2(D_T^+)$, and the conditions (5.2) are understood in the sense of the trace theory [**49**, p. 70].

Definition 5.1. Let $F \in L_2(D_T^+)$. The function u is said to be a weak generalized solution of the problem (5.1), (5.2) of the class W_2^2 in the domain D_T^+ if $u \in \overset{\circ}{W}{}_2^2(D_T^+, S_T)$, $|u|^\alpha \in L_2(D_T^+)$, and for every function $\varphi \in W_2^2(D_T^+)$ such that $\varphi|_{t=T} = 0$, $\frac{\partial \varphi}{\partial t}\big|_{t=T} = 0$, the integral equality

$$\int\limits_{D_T^+} \square u \square \varphi \, dx\, dt + \lambda \int\limits_{D_T^+} |u|^\alpha \varphi \, dx\, dt = \int\limits_{D_T^+} F\varphi \, dx\, dt \tag{5.3}$$

is valid.

The integration by parts allows us to verify that the classical solution $u \in \overset{\circ}{C}{}^4(\overline{D}_T^+, S_T) := \{u \in C^4(\overline{D}_T) : \ u|_{S_T} = 0, \ \frac{\partial u}{\partial \nu}\big|_{S_T} = 0\}$ of the problem (5.1), (5.2) is also a weak generalized solution of that problem of the class W_2^2 in the sense of Definition 5.1. Conversely, if a weak generalized solution of the problem (5.1), (5.2) of the class W_2^2 belongs to the space $C^4(\overline{D}_T^+)$, then this solution will also be classical. Here we have used the fact that if $u \in C^4(\overline{D}_T^+)$ and the conditions (5.2) are fulfilled, then as far as S_T is a characteristic manifold, the equality $\square u|_{S_T} = 0$ is true. In addition, since the derivative with respect to the conormal $\frac{\partial}{\partial N} = \nu_{n+1} \frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}$

$(\nu=(\nu_1,\dots,\nu_n,\nu_{n+1}))$ is an inner differential operator on the characteristic manifold S_T, therefore $\frac{\partial}{\partial N}\,\Box u\big|_{S_T}=0$, and also $\frac{\partial u}{\partial N}\big|_{S_T}=0$ because $u|_{S_T}=0$.

Definition 5.2. Let $F\in L_2(D_T^+)$. The function u is said to be a strong generalized solution of the problem (5.1), (5.2) of the class W_2^2 in the domain D_T^+ if $u\in \overset{\circ}{W}{}_2^2(D_T^+,S_T)$, $|u|^\alpha\in L_2(D_T)$ and there exists a sequence of functions $u_m\in \overset{\circ}{C}{}^4(\overline{D}_T^{\,+},S_T)$ such that $u_m\to u$ in the space $\overset{\circ}{W}{}_2^2(D_T^+,S_T)$ and $|u_m|^\alpha\to|u|^\alpha$, $L_\lambda u_m\to F$ in the space $L_2(D_T^+)$.

Obviously, the classical solution of the problem (5.1), (5.2) from the space $\overset{\circ}{C}{}^4(\overline{D}_T^{\,+},S_T)$ is a strong generalized solution of that problem of the class W_2^2. In its turn, a strong generalized solution of the problem (5.1), (5.2) of the class W_2^2 is a weak generalized solution of that problem of the class W_2^2.

Definition 5.3. Let $F\in L_{2,loc}(D_\infty^+)$ and $F\in L_2(D_T^+)$ for any $T>0$. We say that the problem (5.1), (5.2) is globally solvable in the weak (strong) sense in the class W_2^2 if for any $T>0$ this problem has a weak (strong) generalized solution of the class W_2^2 in the domain D_T^+.

Remark 5.1. It can be easily seen that if the problem (5.1), (5.2) is not globally solvable in the weak sense, then it will not be globally solvable in the strong sense in the class W_2^2. Obviously, the global solvability of the problem (5.1), (5.2) in the strong sense implies the global solvability of that problem in the weak sense in the class W_2^2.

Theorem 5.1. *Let $F\in L_{2,loc}(D_\infty^+)$, $F\ge 0$, $F\not\equiv 0$ and $F\in L_2(D_T^+)$ for any $T>0$. Then if the nonlinearity exponent α in the equation* (5.1) *satisfies the inequalities*

$$\begin{cases}1<\alpha<\dfrac{n+1}{n-2}\,, & n>3,\\ 1<\alpha<\infty, & n=2,3,\end{cases}\tag{5.4}$$

and in the limiting case $\alpha=\frac{n+1}{n-3}$ for $n>3$ the function F satisfies the condition

$$\lim_{T\to\infty}\int\limits_{D_T}F\,dx\,dt=\infty,\tag{5.5}$$

then the problem (5.1), (5.2) *is not globally solvable in the weak sense in the class W_2^2, i.e., there exists a number $T_0=T_0(F)>0$, such that for $T>T_0$ the problem* (5.1), (5.2) *fails to have a weak generalized solution of the class W_2^2 in the domain D_T^+.*

Proof. Assume that u is a weak generalized solution of the problem (5.1), (5.2) of the class W_2^2 in the domain D_T^+, i.e., the integral equality (5.3) is valid for any function $\varphi\in W_2^2(D_T^+)$ such that $\varphi|_{t=T}=0$, $\frac{\partial\varphi}{\partial t}\big|_{t=T}=0$.

Integrating the left-hand side of the equality (5.3) by parts, we obtain

$$\int\limits_{D_T^+} \Box u \Box\varphi \, dx\, dt =$$

$$\int\limits_{\partial D_T^+} \frac{\partial u}{\partial N} \Box\varphi \, ds - \int\limits_{\partial D_T^+} \frac{\partial u}{\partial t} \frac{\partial}{\partial t} \Box\varphi \, dx\, dt + \int\limits_{D_T^+} \nabla_x u \nabla_x(\Box\varphi)\, dx\, dt =$$

$$= \int\limits_{\partial D_T^+} \frac{\partial u}{\partial N} \Box\varphi \, ds - \int\limits_{\partial D_T^+} u \frac{\partial}{\partial N} \Box\varphi \, ds + \int\limits_{D_T^+} u\Box^2\varphi \, dx\, dt, \tag{5.6}$$

where $\frac{\partial}{\partial N}$ is the derivative with respect to the conormal.

Let the function $\varphi_0 = \varphi_0(\sigma)$ of one real variable σ be such that

$$\varphi_0 \in C^4((-\infty,+\infty)), \ \ \varphi_0 \geq 0, \ \ \varphi_0' \leq 0, \ \ \varphi_0(\sigma) = \begin{cases} 1, & 0 \leq \sigma \leq 1, \\ 0, & \sigma \geq 2. \end{cases} \tag{5.7}$$

We use here the method of test functions [**53**, pp. 10–12]. In the capacity of the test function in the equality (5.3) we take the function $\varphi(x,t) = \varphi_0\big[\frac{2}{T^2}\,(t^2+|x|^2)\big]$. Taking into account that $u|_{S_T} = 0$ and hence $\frac{\partial u}{\partial N}\big|_{S_T} = 0$, since $\frac{\partial}{\partial N} = \nu_{n+1}\,\frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \nu_i\, \frac{\partial}{\partial x_i}$ is an inner differential operator on S_T as well as by virtue of (5.7) and the equalities $\frac{\partial^i\varphi}{\partial t^i}\big|_{t=T} = 0$, $0 \leq i \leq 4$, $\Box\varphi|_{t=T} = \frac{\partial}{\partial N}\,\Box\varphi\big|_{t=T} = 0$, it follows from (5.6) that $\int\limits_{D_T^+} \Box u \Box\varphi\, dx\, dt = \int\limits_{D_T^+} u\Box^2\varphi\, dx\, dt$.

Thus we can rewrite the equality (5.3) in the form

$$-\lambda \int\limits_{D_T^+} |u|^\alpha \varphi \, dx\, dt = \int\limits_{D_T^+} u\Box^2\varphi\, dx\, dt - \int\limits_{D_T^+} F\varphi\, dx\, dt. \tag{5.8}$$

If in Young's inequality with the parameter $\varepsilon > 0$

$$ab \leq \frac{\varepsilon}{\alpha}\, a^\alpha + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\, b^{\alpha'}, \ \ a,b \geq 0, \ \ \alpha' = \frac{\alpha}{\alpha-1}$$

we take $a = |u|\varphi^{1/\alpha}$, $b = |\Box^2\varphi|/\varphi^{1/\alpha}$, then in view of the fact that $\alpha'/\alpha = \alpha' - 1$ we will have

$$|u\Box^2\varphi| = |u|\varphi^{1/\alpha}\, \frac{|\Box^2\varphi|}{\varphi^{1/\alpha}} \leq \frac{\varepsilon}{\alpha}\, |u|^\alpha\varphi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\, \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\,. \tag{5.9}$$

Owing to (5.9) and $|\lambda| = -\lambda$, from (5.8) it follows the inequality

$$\Big(|\lambda| - \frac{\varepsilon}{\alpha}\Big) \int\limits_{D_T^+} |u|^\alpha\varphi\, dx\, dt \leq \frac{1}{\alpha'\varepsilon^{\alpha'-1}} \int\limits_{D_T^+} \frac{|\Box^2\varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt - \int\limits_{D_T} F\varphi\, dx\, dt,$$

whence for $\varepsilon < |\lambda|\alpha$ we get

$$\int\limits_{D_T^+} |u|^\alpha \varphi\, dx\, dt \leq$$

$$\leq \frac{\alpha}{(|\lambda|\alpha - \varepsilon)\alpha' \varepsilon^{\alpha'-1}} \int\limits_{D_T^+} \frac{|\Box^2 \varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt - \frac{\alpha}{|\lambda|\alpha - \varepsilon} \int\limits_{D_T^+} F\varphi\, dx\, dt. \qquad (5.10)$$

Bearing in mind the equalities $\alpha' = \frac{\alpha}{\alpha-1}$, $\alpha = \frac{\alpha'}{\alpha'-1}$ and $\min\limits_{0<\varepsilon<|\lambda|\alpha} \frac{\alpha}{(|\lambda|\alpha-\varepsilon)\alpha'\varepsilon^{\alpha'-1}} = \frac{1}{\lambda^{\alpha'}}$ which is achieved for $\varepsilon = |\lambda|$, it follows from (5.10) that

$$\int\limits_{D_T^+} |u|^\alpha \varphi\, dx\, dt \leq \frac{1}{|\lambda|^{\alpha'}} \int\limits_{D_T^+} \frac{|\Box^2 \varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt - \frac{\alpha'}{|\lambda|} \int\limits_{D_T^+} F\varphi\, dx\, dt. \qquad (5.11)$$

According to the properties (5.7) of the function φ_0, the test function $\varphi(x,t) = \varphi_0\big[\frac{2}{T^2}\,(t^2 + |x|^2)\big] = 0$ for $r = (t^2 + |x|^2)^{1/2} \geq T$. Therefore, after the change of variables $t = T\xi_0$ and $x = T\xi$ we have

$$\int\limits_{D_T^+} \frac{|\Box^2 \varphi|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt =$$

$$= \int\limits_{r=(t^2+|x|^2)^{1/2}<T} \frac{\big|c_1 T^{-4}\varphi_0'' + (c_2 t^2 + c_3 |x|^2) T^{-6} \varphi_0''' + c_4 T^{-8} (t^2 - |x|^2)^2 \varphi_0''''\big|^{\alpha'}}{\varphi^{\alpha'-1}}\, dx\, dt =$$

$$= T^{n+1-4\alpha'} \int\limits_{\substack{1<2(\xi_0^2+|\xi|^2)<2,\\ \xi_0>|\xi|}} \frac{\big|c_1 \varphi_0'' + (c_2 \xi_0^2 + c_3 |\xi|^2) \varphi_0''' + c_4 (\xi_0^2 - |\xi|^2)^2 \varphi_0''''\big|^{\alpha'}}{\varphi_0^{\alpha'-1}}\, dx\, dt, \quad (5.12)$$

where $c_i = c_i(n)$, $i = 1, \dots, 4$, are certain integers.

As is known, the test function $\varphi(x,t) = \varphi_0\big[\frac{2}{T^2}\,(t^2 + |x|^2)\big]$ with the above-mentioned properties for which the integrals in the right-hand sides of (5.11) and (5.12) are finite does exist [**53**, p. 28].

Due to (5.12), from the inequality (5.11) and the fact that $\varphi_0(\sigma) = 1$ for $0 \leq \sigma \leq 1$ we obtain the inequality

$$\int\limits_{\substack{r=(t^2+|x|^2)^{1/2}<\frac{T}{\sqrt{2}},\\ t>|x|}} |u|^\alpha\, dx\, dt \leq$$

$$\leq \int\limits_{D_T^+} |u|^\alpha \varphi\, dx\, dt \leq \frac{T^{n+1-4\alpha'}}{|\lambda|^{\alpha'}}\, \varkappa_0 - \frac{\alpha'}{|\lambda|}\, \gamma(T), \quad (5.13)$$

where

$$\gamma(T)=\int\limits_{D_T^+} F\varphi\,dx\,dt,$$

$$\varkappa_0=\int\limits_{\substack{1<2(\xi_0^2+|\xi|^2)<2,\\ \xi_0>|\xi|}} \frac{\big|c_1\varphi_0''+(c_2\xi_0^2+c_3|\xi|^2)\varphi_0'''+c_4(\xi_0^2-|\xi|^2)^2\varphi_0''''\big|^{\alpha'}}{\varphi_0^{\alpha'-1}}\,d\xi_0\,d\xi<+\infty.$$

Consider first the case $q=n+1-4\alpha'<0$ which according to the condition (5.4) implies that $\alpha<\frac{n+1}{n-3}$ for $n>3$ and $\alpha<\infty$ for $n=2,3$. In this case, the equation

$$g(T)=\frac{T^{n+1-4\alpha'}}{|\lambda|^{\alpha'}}\,\varkappa_0-\frac{\alpha'}{|\lambda|}\,\gamma(T)=0 \tag{5.14}$$

has a unique positive root $T=T_0>0$ since the function $g_1(T)=\frac{T^{n+1-4\alpha'}}{|\lambda|^{\alpha'}}\,\varkappa_0$ is positive, continuous, strictly decreasing on the interval $(0,+\infty)$ with $\lim\limits_{T\to 0} g_1(T)=+\infty$ and $\lim\limits_{T\to+\infty} g_1(T)=0$, and the function $\gamma(T)=\int\limits_{D_T^+} F\varphi\,dx\,dt$ is, by virtue of $F\geq 0$ and (5.7), nonnegative and nondecreasing and is, because of the absolute continuity of the integral, also continuous. Moreover, $\lim\limits_{T\to+\infty}\gamma(T)>0$, since $F\geq 0$ and $F\not\equiv 0$, i.e., $F\neq 0$ on some set of the positive Lebesgue measure. Thus $g(T)<0$ for $T>T_0$ and $g(T)>0$ for $0<T<T_0$. Consequently, for $T>T_0$ the right-hand side of the inequality (5.13) is negative, but this is impossible.

Consider now the limiting case $q=n+1-4\alpha'=0$, i.e., when $\alpha=\frac{n+1}{n-3}$ for $n>3$. In this case, the equation (5.14) takes the form $\frac{1}{|\lambda|^{\alpha'}}\,\varkappa_0-\frac{\alpha'}{|\lambda|}\,\gamma(T)=0$ and likewise has, owing to the obvious equality $\lim\limits_{T\to 0}\gamma(T)=0$ and the conditions (5.5) and (5.7), a unique positive root $T=T_0>0$. For $T>T_0$, the right-hand side of the inequality (5.13) is negative, and this again leads to a contradiction. Thus the theorem is proved completely. □

Remark 5.2. It follows from the proof of Theorem 5.2 that if the conditions of the theorem are fulfilled and there exists a weak generalized solution of the problem (5.1), (5.2) of the class W_2^2 in the domain D_T^+, then the estimate

$$T\leq T_0 \tag{5.15}$$

is valid, where T_0 is a unique positive root of the equation (5.14).

Below we will prove the local solvability of the problem (5.1), (5.2). First we will consider the linear case when in the equation (5.1) the parameter $\lambda=0$, i.e., we consider the problem

$$L_0u(x,t)=F(x,t),\quad (x,t)\in D_T^+, \tag{5.16}$$

$$u|_{S_T} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{S_T} = 0, \tag{5.17}$$

where $L_0 = \Box^2$.

Definition 5.4. Let $F \in L_2(D_T^+)$. The function u is said to be a strong generalized solution of the problem (5.16), (5.17) of the class W_2^2 in the domain D_T^+ if $u \in \overset{\circ}{W}{}_2^2(D_T^+, S_T)$ and there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^4(\overline{D}_T^+, S_T)$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}_2^2(D_T^+, S_T)$ and $L_0 u_m \to F$ in the space $L_2(D_T^+)$.

Obviously, the classical solution $u \in \overset{\circ}{C}{}^4(\overline{D}_T^+, S_T)$ of the problem (5.16), (5.17) is a strong generalized solution of the problem of the class W_2^2 in the domain D_T.

Lemma 5.1. *For a strong generalized solution u of the problem* (5.16), (5.17) *of the class W_2^2 in the domain D_T^+ the estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^2(D_T^+, S_T)} \le c_n T^2 \|F\|_{L_2(D_T^+)} \tag{5.18}$$

holds, where the positive constant c_n does not depend on u, F and T.

Proof. The same reasoning as when deducing the inequality (2.13) allows us to prove the inequality

$$\|v\|_{\overset{\circ}{W}{}_2^1(D_T^+, S_T)} \le \sqrt{\frac{e}{2}}\, T \|\Box v\|_{L_2(D_T^+)} \quad \forall\, v \in \overset{\circ}{C}{}^2(\overline{D}_T^+, S_T), \tag{5.19}$$

where $\overset{\circ}{C}{}^2(\overline{D}_T^+, S_T) := \{v \in C^2(\overline{D}_T^+) : \ v|_{S_T} = 0\}$ and in the space $\overset{\circ}{W}{}_2^1(D_T^+, S_T) := \{v \in W_2^1(D_T^+) : \ v|_{S_T} = 0\}$ we take, by virtue of (2.14), the norm

$$\|v\|^2_{\overset{\circ}{W}{}_2^1(D_T^+, S_T)} = \int\limits_{D_T^+} \Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\Big]\, dx\, dt.$$

By the definition, if u is a strong generalized solution of the problem (5.16), (5.17) of the class W_2^2 in the domain D_T^+, then there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^4(\overline{D}_T^+, S_T) := \{u \in C^4(\overline{D}_T^+) : \ u|_{S_T} = 0, \ \frac{\partial u}{\partial \nu}\big|_{S_T} = 0\}$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^2(D_T^+, S_T)} = 0, \quad \lim_{m\to\infty} \|\Box^2 u_m - F\|_{L_2(D_T^+)} = 0. \tag{5.20}$$

Since $u_m \in \overset{\circ}{C}{}^4(\overline{D}_T^+, S_T)$ satisfies the homogeneous boundary conditions (5.17) and S_T is a characteristic manifold corresponding to the operator $\Box$, therefore, as is known [**8**, p. 546],

$$\Box u_m\big|_{S_T} = 0. \tag{5.21}$$

Owing to (5.11), the function $v = \Box u_m \in \overset{\circ}{C}{}^2(\overline{D}_T^{\,+}, S_T)$, due to (5.19), satisfies the inequalities

$$\begin{aligned} \|\Box u_m\|^2_{L_2(D_T^+)} &\le \frac{e}{2}\, T^2 \|\Box^2 u_m\|^2_{L_2(D_T^+)}, \\ \Big\|\Box \frac{\partial u_m}{\partial t}\Big\|^2_{L_2(D_T^+)} &\le \frac{e}{2}\, T^2 \|\Box^2 u_m\|^2_{L_2(D_T^+)}, \\ \Big\|\Box \frac{\partial u_m}{\partial x_i}\Big\|^2_{L_2(D_T^+)} &\le \frac{e}{2}\, T^2 \|\Box^2 u_m\|^2_{L_2(D_T^+)}, \quad i = 1, \dots, n. \end{aligned} \tag{5.22}$$

Since $\frac{\partial u_m}{\partial t}, \frac{\partial u_m}{\partial x_i} \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$, by (5.19) and (5.22) we have

$$\begin{aligned} &\|u_m\|_{\overset{\circ}{W}{}^2_2(D_T^+, S_T)} = \\ &= \int\limits_{D_T^+} \bigg[\Big(\frac{\partial u_m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 + \Big(\frac{\partial^2 u_m}{\partial t^2}\Big)^2 + \\ &\quad + \sum_{i=1}^n \Big(\frac{\partial^2 u_m}{\partial t \partial x_i}\Big)^2 + \sum_{i,j=1}^n \Big(\frac{\partial^2 u_m}{\partial x_i \partial x_j}\Big)^2 \bigg]\, dx\, dt \le \\ &\le \Big\|\frac{\partial u_m}{\partial t}\Big\|^2_{\overset{\circ}{W}{}^1_2(D_T^+, S_T)} + \sum_{i=1}^n \Big\|\frac{\partial u_m}{\partial x_i}\Big\|^2_{\overset{\circ}{W}{}^1_2(D_T^+, S_T)} \le \\ &\le \frac{e}{2}\, T^2 \Big\|\Box \frac{\partial u_m}{\partial t}\Big\|^2_{L_2(D_T^+)} + \frac{e}{2}\, T^2 \sum_{i=1}^n \Big\|\Box \frac{\partial u_m}{\partial x_i}\Big\|^2_{L_2(D_T^+)} \le \\ &\le \Big(\frac{e}{2}\Big)^2 (n+1) T^4 \|\Box^2 u_m\|^2_{L_2(D_T^+)}, \end{aligned}$$

whence

$$\|u_m\|_{\overset{\circ}{W}{}^2_2(D_T^+, S_T)} \le c_n T^2 \|\Box^2 u_m\|_{L_2(D_T^+)}, \quad c_n = \sqrt{n+1}\, \frac{e}{2}\,. \tag{5.23}$$

By virtue of (5.20), passing in the inequality (5.23) to limit as $m \to \infty$, we obtain (5.18), which proves our lemma. □

Lemma 5.2. *For any $F \in L_2(D_T)$ there exists a unique strong generalized solution u of the problem* (5.16), (5.17) *of the class W_2^2 in the domain D_T^+ for which the estimate* (5.18) *is valid.*

Proof. Since the space $C_0^\infty(D_T^+)$ of finitary infinitely differentiable in D_T^+ functions is dense in $L_2(D_T^+)$, for a given $F \in L_2(D_T)$ there exists a sequence of functions $F_m \in C_0^\infty(D_T^+)$ such that $\lim\limits_{m\to\infty} \|F_m - F\|_{L_2(D_T^+)} = 0$. For the fixed m, extending the function F_m by zero beyond the domain D_T^+ and leaving the same notation, we have $F_m \in C^\infty(\mathbb{R}^{n+1}_+)$ for which the support $\operatorname{supp} F_m \subset D_\infty^+$, where $\mathbb{R}^{n+1}_+ := \mathbb{R}^{n+1} \cap \{t \ge 0\}$. Denote by u_m the solution of the Cauchy problem $L_0 u_m = F_m$, $\frac{\partial^i u}{\partial t^i}\big|_{t=0} = 0$, $0 \le i \le 3$, which, as is known, exists, is unique and belongs to the space $C^\infty(\mathbb{R}^{n+1}_+)$ [**17**, p. 192]. In addition, since $\operatorname{supp} F_m \subset D^+_\infty$, $\frac{\partial^i u}{\partial t^i}\big|_{t=0} = 0$, $0 \le i \le 3$, taking into account the geometry of the domain of dependence of a solution of the linear equation $L_0 u_m = F_m$ of hyperbolic type we find that $\operatorname{supp} u_m \subset D^+_\infty$ [**17**, p. 191]. Leaving for the restriction of the function u_m to the domain D_T the same notation, we can easily see that $u_m \in \overset{\circ}{C}{}^4(\overline{D}{}^+_T, S_T)$, and by (5.18), the inequality

$$\|u_m - u_k\|_{\overset{\circ}{W}{}^2_2(D^+_T, S_T)} \le c_n T^2 \|F_m - F_k\|_{L_2(D^+_T)} \tag{5.24}$$

is valid.

Since the sequence $\{F_m\}$ is fundamental in $L_2(D^+_T)$, owing to (5.24) the sequence $\{u_m\}$ is fundamental in the complete space $\overset{\circ}{W}{}^2_2(D^+_T, S_T)$. Therefore, there exists a function $u \in \overset{\circ}{W}{}^2_2(D^+_T, S_T)$ such that $\lim\limits_{m\to\infty} \|u_m - u_k\|_{\overset{\circ}{W}{}^2_2(D^+_T, S_T)} = 0$, and since $L_0 u_m = F_m \to F$ in the space $L_2(D^+_T)$, this function u will, by Definition 5.4, be a strong generalized solution of the problem (5.16), (5.17) of the class W^2_2 in the domain D^+_T, for which the estimate (5.18) is valid. The uniqueness of the solution follows from the estimate (5.18). Thus the lemma is proved completely. □

Remark 5.3. By Lemma 5.2, for a strong generalized solution u of the problem (5.16), (5.17) of the class W^2_2 in the domain D^+_T we can write $u = L_0^{-1} F$, where $L_0^{-1} : L_2(D^+_T) \to \overset{\circ}{W}{}^2_2(D^+_T, S_T)$ is a linear continuous operator whose norm, by virtue of (5.18), admits the estimate

$$\|L_0^{-1}\|_{L_2(D^+_T) \to \overset{\circ}{W}{}^2_2(D^+_T, S_T)} \le c_n T^2. \tag{5.25}$$

Remark 5.4. The embedding operator $I : \overset{\circ}{W}{}^2_2(D^+_T, S_T) \to L_q(D^+_T)$ is linear, continuous and compact for $1 < q < \frac{2(n+1)}{n-3}$, when $n > 3$, and $1 < q < \infty$ when $n = 2, 3$ [**49**, p. 84]. At the same time, the Nemytski operator $N : L_q(D^+_T) \to L_2(D^+_T)$ acting by the formula $Nu = -\lambda |u|^\alpha$ is continuous and bounded if $q \ge 2\alpha$ [**47**, p. 349], [**48**, pp. 66, 67]. Thus if the nonlinearity exponent α in the equation (5.1) satisfies the inequalities (5.4), then putting $q = 2\alpha$ we find that the operator

$$N_0 = NI : \overset{\circ}{W}{}^2_2(D^+_T, S_T) \to L_2(D^+_T) \tag{5.26}$$

is continuous and compact. Moreover, from $u \in \overset{\circ}{W}{}^2_2(D^+_T, S_T)$ it follows that $|u|^\alpha \in L_2(D^+_T)$, and taking in Definition 5.2 into account the fact that $u_m \to u$ in the space $\overset{\circ}{W}{}^2_2(D^+_T, S_T)$, it automatically follows that $|u_m|^\alpha \to |u|^\alpha$ in the space $L_2(D^+_T)$, as well.

Remark 5.5. If $F \in L_2(D_T^+)$ and the nonlinearity exponent α satisfies the inequalities (5.4), then according to Definition 5.2 and Remarks 5.3 and 5.4 the function $u \in \overset{\circ}{W}{}_2^2(D_T^+, S_T)$ is a strong generalized solution of the problem (5.1), (5.2) of the class W_2^2 in the domain D_T^+ if and only if u is a solution of the functional equation

$$u = L_0^{-1}\big(-\lambda |u|^\alpha + F\big) \tag{5.27}$$

in the space $\overset{\circ}{W}{}_2^2(D_T^+, S_T)$.

We rewrite the equation (5.27) in the form

$$u = Ku + u_0, \tag{5.28}$$

where the the operator $K := L_0^{-1} N_0 : \overset{\circ}{W}{}_2^2(D_T^+, S_T) \to \overset{\circ}{W}{}_2^2(D_T^+, S_T)$ is, by virtue of (5.25), (5.26) and Remark 5.4, continuous, compact and acting in the space $\overset{\circ}{W}{}_2^2(D_T^+, S_T)$, while $u_0 := L_0^{-1} F \in \overset{\circ}{W}{}_2^2(D_T^+, S_T)$.

Remark 5.6. Let $B(0, R_0) := \big\{ u \in \overset{\circ}{W}{}_2^2(D_T^+, S_T) : \ \|u\|_{\overset{\circ}{W}{}_2^2(D_T^+, S_T)} \leq R_0 \big\}$ be the closed (convex) ball in the Hilbert space $\overset{\circ}{W}{}_2^2(D_T^+, S_T)$ of radius $R_0 > 0$ with the center at the zero element. Since the operator $K : \overset{\circ}{W}{}_2^2(D_T^+, S_T) \to \overset{\circ}{W}{}_2^2(D_T^+, S_T)$ is continuous and compact (provided the inequalities (5.4) are fulfilled), by the Schauder principle for showing the solvability of the equation (5.28) it suffices to show that the operator K_1 acting by the formula $K_1 u = Ku + u_0$ transforms the ball $B(0, R_0)$ into itself for some $R_0 > 0$ [**66**, p. 370]. By (5.25), analogously as in proving Theorem 5.1 of Chapter IV, one can prove that for sufficiently small T such a ball $B(0, R_0)$ does exist. Thus we have the following theorem on the local solvability of the problem (5.1), (5.2).

Theorem 5.2. *Let $F \in L_{2,loc}(D_\infty^+)$ and $F \in L_2(D_T^+)$ for any $T > 0$. Then if the nonlinearity exponent α in the equation* (5.1) *satisfies the inequalities* (5.4)*, then there exists a number $T_1 = T_1(F) > 0$ such that for $T \leq T_1$ the problem* (5.1), (5.2) *has at least one strong generalized solution of the class W_2^2 in the domain D_T^+ in the sense of Definition* 5.2*, which is also a weak generalized solution of that problem of the class W_2^2 in the domain D_T^+ in the sense of Definition* 5.1.

Remark 5.7. It follows from Theorems 5.1 and 5.2 that if $F \in L_{2,loc}(D_\infty^+)$, $F \geq 0$, $F \not\equiv 0$, $F \in L_2(D_T^+)$ for any $T > 0$ and the nonlinearity exponent α satisfies the inequalities (5.4), then there exists a number $T_* = T_*(F) > 0$ such that for $T < T_*$ there exists a strong (weak) generalized solution of the problem (5.1), (5.2) of the class W_2^2 in the domain D_T, while for $T > T_*$ such a solution does not exist, and in view of the estimate (5.15) we have $T_* \in [T_1, T_0]$.

Remark 5.8. In case $0 < \alpha < 1$, the problem (5.1), (5.2) may have more than one global solution. For example, for $F = 0$ the problem (5.1), (5.2) in the domain D_∞ has, besides the trivial solution, an infinite set of global linearly independent solutions $u_\sigma \in \overset{\circ}{C}{}^2(\overline{D}{}^+_\infty, S_\infty)$ depending on the parameter $\sigma \geq 0$ and given by the formula

$$u_\sigma(x,t) = \begin{cases} \beta\big[(t-\sigma)^2 - |x|^2\big]^{\frac{2}{1-\alpha}}, & t > \sigma + |x|, \\ 0 & |x| \leq t \leq \sigma + |x|, \end{cases}$$

where $\beta = |\lambda|^{\frac{1}{1-\alpha}}\big[4k(k-1)(n+2k-1)(n+2k-3)\big]^{-\frac{1}{1-\alpha}}$, $k = \frac{2}{1-\alpha}$, $\lambda < 0$, and for $1/2 < \alpha < 1$ the function $u_\sigma \in C^4(\overline{D}_\infty)$.

Remark 5.9. Note that for $n = 2$ and $n = 3$, according to the well-known properties [**8**, p. 745], [**2**, p. 84] of solutions of the linear characteristic problem $\square v = g$ in D_∞, $v|_{S_\infty} = 0$, if $g \geq 0$, then $v \geq 0$ as well. Therefore, for $n = 2, 3$, if $F \geq 0$, then the classical solution u of the nonlinear problem (5.1), (5.2), analogously to (5.21) satisfying also the condition $\square u|_{S_\infty} = 0$, will likewise be nonnegative. But in this case, for $\alpha = 1$, this solution will satisfy the following linear problem:

$$\square^2 u + \lambda u = F,$$

$$u\Big|_{S_\infty} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{S_\infty} = 0$$

which is globally solvable in the corresponding functional spaces.

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Author's address:
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: kharibegashvili@yahoo.com