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SOLUTION OF A BOUNDARY VALUE PROBLEM OF STATICS OF TWO-COMPONENT ELASTIC MIXTURES FOR A SPACE WITH TWO NONINTERSECTING SPHERICAL CAVITIES

Abstract. Using a general representation of solutions of a system of homogeneous differential equations of statics of two-component elastic mixtures which is expressed by six harmonic functions, we study boundary value problems of statics of two-component elastic mixtures for a space with two nonintersecting spherical cavities when different boundary conditions are given on the spherical surfaces. The uniqueness theorems are proved. The solution of the considered problems is reduced to the investigation of an infinite system of linear algebraic equations. It is proved that such systems are quasiregular. The question of regularity of the partial displacement vector is studied.

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[illegible]

Introduction

In recent years, considerable progress has been made in the investigation of physical and mechanical properties of composite materials, a stress-strained state of structural elements with regard for their structure. The results of the investigation in the composite material technology area have applications in various spheres of industry and engineering. The present work is dedicated to the effective solution of static problems of the linear theory of a mixture of two isotropic elastic materials. We use the fundamental equations derived in Green, Naghdi [6], Steel [14] and Green, Steel [7]. Some of the problems touched upon in the present work were considered in various aspects by other authors. For example, Atkin, Chadwick, Steel [1], Knops, Steel [10], Natroshvili, Jagmaidze, Svanadze [13] are dedicated to the uniqueness theorems of various linearized dynamic problems of the mixture theory. Effective solutions of problems of statics and steady-state oscillations of the elastic mixture theory are obtained for a ball in [4] and [13]. In the present work, a new approach is proposed to the solution of boundary value problems for concrete domains.

1. Some Notation, Auxiliary Formulas and Theorems

The three-dimensional Euclidean space is denoted by $\mathbb{R}^3$, and the points (vectors) of this space by x, y, z. The coordinates of these points in the basis $e_1 = (1,0,0)^\top$, $e_2 = (0,1,0)^\top$, $e_3 = (0,0,1)^\top$ are denoted by x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3. $\top$ is the transposition symbol.

We denote by Ω^+ a finite domain from $\mathbb{R}^3$ with the boundary $\partial\Omega$, and by Ω^- the complement of the set $\overline{\Omega}^+$ to $\mathbb{R}^3$ ($\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}^+$).

If a function $\Phi(x)$ defined in Ω^+ $[\Omega^-]$ is continuously extendable at a point $z \in \partial\Omega$, then we denote by $[\Phi(z)]^+$ ($[\Phi(z)]^-$) the limit

$$[\Phi(z)]^+ = \lim_{\Omega^+ \ni x \to z} \Phi(x) \quad \Big([\Phi(z)]^- = \lim_{\Omega^- \ni x \to z} \Phi(x) \Big).$$

We denote by r, ϑ, φ ($0 \le r < +\infty$, $0 \le \vartheta \le \pi$, $0 \le \varphi < 2\pi$) the spherical coordinates of a point $x \in \mathbb{R}^3$. Let us introduce the following vectors [12], [16]:

$$\begin{aligned}
&X_{mk}(\vartheta,\varphi) = e_r Y_k^{(m)}(\vartheta,\varphi), \quad k \ge 0,\\
&Y_{mk}(\vartheta,\varphi) = \frac{1}{\sqrt{k(k+1)}}\Big(e_\vartheta \frac{\partial}{\partial\vartheta} + \frac{e_\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\Big) Y_k^{(m)}(\vartheta,\varphi), \quad k \ge 1,\\
&Z_{mk}(\vartheta,\varphi) = \frac{1}{\sqrt{k(k+1)}}\Big(\frac{e_\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi} - e_\varphi \frac{\partial}{\partial\vartheta}\Big) Y_k^{(m)}(\vartheta,\varphi), \quad k \ge 1,
\end{aligned} \tag{1.1}$$

where $|m| \le k$, e_r, e_ϑ, e_φ are the unit orthogonal vectors

$$e_r = \big(\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta\big)^\top,$$
$$e_\vartheta = \big(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, -\sin\vartheta\big)^\top,$$

$$e_\varphi = (-\sin\varphi, \cos\varphi, 0)^\top,$$

$$Y_k^{(m)}(\vartheta,\varphi) = \sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\,P_k^{(m)}(\cos\vartheta)e^{im\varphi}, \tag{1.2}$$

$P_k^{(m)}(\cos\vartheta)$ is the first kind adjoint Legendre function of k-th degree and m-th order [15]. In the sequel, under a vector we mean a one-column matrix.

On the sphere of unit radius the set

$$\big\{X_{mk}(\vartheta,\varphi), Y_{mk}(\vartheta,\varphi), Z_{mk}(\vartheta,\varphi)\big\}_{|m|\le k,\, k=\overline{0,\infty}}$$

forms a complete orthonormalized system of vector functions in the space L_2.

The following identities are valid [2]:

$$e_r\cdot X_{mk}(\vartheta,\varphi)=Y_k^{(m)}(\vartheta,\varphi),\quad e_r\cdot Y_{mk}(\vartheta,\varphi)=0,\quad e_r\cdot Z_{mk}(\vartheta,\varphi)=0,$$
$$e_r\times X_{mk}(\vartheta,\varphi)=0,\quad e_r\times Y_{mk}(\vartheta,\varphi)=-Z_{mk}(\vartheta,\varphi),$$
$$e_r\times Z_{mk}(\vartheta,\varphi)=Y_{mk}(\vartheta,\varphi),$$
$$\operatorname{grad}\big[a(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\frac{da(r)}{dr}\,X_{mk}(\vartheta,\varphi)+\frac{\sqrt{k(k+1)}}{r}\,a(r)Y_{mk}(\vartheta,\varphi),$$
$$\operatorname{rot}\big[xa(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\sqrt{k(k+1)}\,a(r)Z_{mk}(\vartheta,\varphi),$$
$$\operatorname{rot}\operatorname{rot}\big[xa(r)Y_k^{(m)}(\vartheta,\varphi)\big]=$$
$$=\frac{k(k+1)}{r}\,a(r)X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi),$$
$$\operatorname{div}\big[a(r)X_{mk}(\vartheta,\varphi)\big]=\Big(\frac{d}{dr}+\frac{2}{r}\Big)a(r)Y_k^{(m)}(\vartheta,\varphi),$$
$$\operatorname{div}\big[a(r)Y_{mk}(\vartheta,\varphi)\big]=-\sqrt{k(k+1)}\,\frac{a(r)}{r}\,Y_k^{(m)}(\vartheta,\varphi),$$
$$\operatorname{div}\big[a(r)Z_{mk}(\vartheta,\varphi)\big]=0, \tag{1.3}$$
$$\operatorname{rot}\big[a(r)X_{mk}(\vartheta,\varphi)\big]=\sqrt{k(k+1)}\,\frac{a(r)}{r}\,Z_{mk}(\vartheta,\varphi),$$
$$\operatorname{rot}\big[a(r)Y_{mk}(\vartheta,\varphi)\big]=-\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Z_{mk}(\vartheta,\varphi),$$
$$\operatorname{rot}\big[a(r)Z_{mk}(\vartheta,\varphi)\big]=$$
$$=\sqrt{k(k+1)}\,\frac{a(r)}{r}\,X_{mk}(\vartheta,\varphi)+\Big(\frac{d}{dr}+\frac{1}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi),$$
$$\operatorname{grad}\operatorname{div}\big[a(r)X_{mk}(\vartheta,\varphi)\big]=\frac{d}{dr}\Big(\frac{d}{dr}+\frac{2}{r}\Big)a(r)X_{mk}(\vartheta,\varphi)+$$
$$+\frac{\sqrt{k(k+1)}}{r}\Big(\frac{d}{dr}+\frac{2}{r}\Big)a(r)Y_{mk}(\vartheta,\varphi),$$
$$\operatorname{grad}\operatorname{div}\big[a(r)Y_{mk}(\vartheta,\varphi)\big]=$$
$$=-\frac{\sqrt{k(k+1)}}{r}\Big(\frac{d}{dr}-\frac{1}{r}\Big)a(r)X_{mk}(\vartheta,\varphi)-\frac{k(k+1)}{r^2}\,a(r)Y_{mk}(\vartheta,\varphi),$$

where $a(r)$ is a function of r, $x=(x_1,x_2,x_3)^\top$, $a\cdot b$; $a\times b$ are the scalar and the vector product of the vectors a and b, respectively.

Taking into account the recurrent relations of Legendre polynomials and the orthogonality of the vectors (1.1), we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{m=-k}^{k} e_j a_k(r) Y_k^{(m)}(\vartheta,\varphi) = -a_1(r)\eta_{00}^{(j,1)}(A_{00})X_{00}(\vartheta,\varphi)+\\
&+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\Big\{\Big[a_{k-1}(r)\eta_{mk}^{(j,2)}(A_{mk}) - a_{k+1}(r)\eta_{mk}^{(j,1)}(A_{mk})\Big]X_{mk}(\vartheta,\varphi)+\\
&+\sqrt{k(k+1)}\Big[\frac{1}{k}\,a_{k-1}(r)\eta_{mk}^{(j,2)}(A_{mk})+\frac{1}{k+1}\,a_{k+1}(r)\eta_{mk}^{(j,1)}(A_{mk})\Big]Y_{mk}(\vartheta,\varphi)+\\
&\quad+(-1)^{j+1}\frac{a_k(r)}{\sqrt{k(k+1)}}\,\eta_{mk}^{(j,3)}(A_{mk})Z_{mk}(\vartheta,\varphi)\Big\},\quad j=1,2,\\
&\sum_{k=0}^{\infty}\sum_{m=-k}^{k} e_3 a_k(r) Y_k^{(m)}(\vartheta,\varphi)A_{mk} = \frac{1}{\sqrt{3}}\,a_1(r)A_{01}X_{00}(\vartheta,\varphi)+\\
&\quad+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\Big\{\Big[\sqrt{\frac{k^2-m^2}{4k^2-1}}\,a_{k-1}(r)A_{mk-1}+\\
&\quad+\sqrt{\frac{(k+1)^2-m^2}{(2k+1)(2k+3)}}\,a_{k+1}(r)A_{mk+1}\Big]X_{mk}(\vartheta,\varphi)+\\
&\quad+\sqrt{k(k+1)}\Big[\frac{1}{k}\sqrt{\frac{k^2-m^2}{4k^2-1}}\,a_{k-1}(r)A_{mk-1}-\\
&\quad-\frac{1}{k+1}\sqrt{\frac{(k+1)^2-m^2}{(2k+1)(2k+3)}}\,a_{k+1}(r)A_{mk+1}\Big]Y_{mk}(\vartheta,\varphi)+\\
&\quad+\frac{im}{\sqrt{k(k+1)}}\,a_k(r)A_{mk}Z_{mk}(\vartheta,\varphi)\Big\},
\end{aligned}\tag{1.4}$$

where A_{mk} is a constant and $a_k(r)$ is a function of r,

$$\begin{aligned}
\eta_{mk}^{(j,1)}(A_{mk}) &= \frac{\delta_{1j}+i\delta_{2j}}{2}\Big[\sqrt{\frac{(k+m+1)(k+m+2)}{(2k+1)(2k+3)}}\,A_{m+1k+1}+\\
&\quad+(-1)^j\sqrt{\frac{(k-m+1)(k-m+2)}{(2k+1)(2k+3)}}\,A_{m-1k+1}\Big],\\
\eta_{mk}^{(j,2)}(A_{mk}) &= \frac{\delta_{1j}+i\delta_{2j}}{2}\Big[\sqrt{\frac{(k-m)(k-m-1)}{4k^2-1}}\,A_{m+1k-1}+\\
&\quad+(-1)^j\sqrt{\frac{(k+m)(k+m-1)}{4k^2-1}}\,A_{m-1k-1}\Big],
\end{aligned}\tag{1.5}$$

$$\eta_{mk}^{(j,3)}(A_{mk}) = \frac{\delta_{2j}+i\delta_{1j}}{2}\Big[\sqrt{(k-m)(k+m+1)}\,A_{m+1k}- \\ -(-1)^j\sqrt{(k+m)(k-m+1)}\,A_{m-1k}\Big],\quad j=1,2,$$

$\delta_{\ell j}$ is the Kronecker symbol, $i=\sqrt{-1}$.

Using the formulas (1.3) and taking into account the identity

$$e_j\times\operatorname{grad}a_k(r)Y_k^{(m)}(\vartheta,\varphi) = -\operatorname{rot}\big[e_j a_k(r)Y_k^{(m)}(\vartheta,\varphi)\big],\quad j=1,2,3,$$

from (1.4) we obtain

$$\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\partial_j a_k(r)Y_k^{(m)}(\vartheta,\varphi)A_{mk} = \\ =\sum_{k=1}^{\infty}\sum_{m=-k}^{k}(-1)^j\bigg\{\frac{a_k(r)}{r}\,\eta_{mk}^{(j,3)}(A_{mk})X_{mk}(\vartheta,\varphi)+ \\ +\frac{1}{\sqrt{k(k+1)}}\Big(\frac{d}{dr}+\frac{1}{r}\Big)a_k(r)\eta_{mk}^{(j,3)}(A_{mk})Y_{mk}(\vartheta,\varphi)+ \\ +(-1)^j\sqrt{k(k+1)}\bigg[\frac{1}{k+1}\Big(\frac{d}{dr}+\frac{k+2}{r}\Big)a_{k+1}(r)\eta_{mk}^{(j,1)}(A_{mk})+ \\ +\frac{1}{k}\Big(\frac{d}{dr}-\frac{k-1}{r}\Big)a_{k-1}(r)\eta_{mk}^{(j,2)}(A_{mk})\bigg]Z_{mk}(\vartheta,\varphi)\bigg\},\quad j=1,2, \tag{1.6}$$

$$\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\partial_3 a_k(r)Y_k^{(m)}(\vartheta,\varphi)A_{mk} = \sum_{k=1}^{\infty}\sum_{m=-k}^{k}\bigg\{-\frac{im}{r}\,a_k(r)A_{mk}X_{mk}(\vartheta,\varphi)- \\ -\frac{im}{\sqrt{k(k+1)}}\Big(\frac{d}{dr}+\frac{1}{r}\Big)a_k(r)A_{mk}Y_{mk}(\vartheta,\varphi)+ \\ +\sqrt{k(k+1)}\bigg[\frac{1}{k}\sqrt{\frac{k^2-m^2}{4k^2-1}}\Big(\frac{d}{dr}-\frac{k-1}{r}\Big)a_{k-1}(r)A_{mk-1}- \\ -\frac{1}{k+1}\sqrt{\frac{(k+1)^2-m^2}{(2k+1)(2k+3)}}\Big(\frac{d}{dr}+\frac{k+2}{r}\Big)a_{k+1}(r)A_{mk+1}\bigg]Z_{mk}(\vartheta,\varphi)\bigg\},$$

where $\partial_j = e_j\times\operatorname{grad}$, $\eta_{mk}^{(j,\ell)}$, $j=1,2$, $\ell=1,2,3$, has form (1.5).

Using the formulas (1.3) and taking into account the identity

$$\operatorname{grad}\frac{\partial}{\partial x_j}\big[a_k(r)Y_k^{(m)}(\vartheta,\varphi)\big]=\operatorname{grad}\operatorname{div}\big[e_j a_k(r)Y_k^{(m)}(\vartheta,\varphi)\big],\quad j=1,2,3,$$

from (1.4) we obtain

$$\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\operatorname{grad}\frac{\partial}{\partial x_j}\,a_k(r)Y_k^{(m)}(\vartheta,\varphi)A_{mk} = \\ =-\frac{d}{dr}\Big(\frac{d}{dr}+\frac{2}{r}\Big)a_1(r)\eta_{00}^{(j,1)}(A_{00})X_{00}(\vartheta,\varphi)+$$

$$+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\bigg\{\Big[-\frac{d}{dr}\Big(\frac{d}{dr}+\frac{k+2}{r}\Big)a_{k+1}(r)\eta_{mk}^{(j,1)}(A_{mk})+$$
$$+\frac{d}{dr}\Big(\frac{d}{dr}-\frac{k-1}{r}\Big)a_{k-1}(r)\eta_{mk}^{(j,2)}(A_{mk})\Big]X_{mk}(\vartheta,\varphi)+$$
$$+\frac{\sqrt{k(k+1)}}{r}\Big[-\Big(\frac{d}{dr}+\frac{k+2}{r}\Big)a_{k+1}(r)\eta_{mk}^{(j,1)}(A_{mk})+$$
$$+\Big(\frac{d}{dr}-\frac{k-1}{r}\Big)a_{k-1}(r)\eta_{mk}^{(j,2)}(A_{mk})\Big]Y_{mk}(\vartheta,\varphi)\bigg\},\quad j=1,2, \tag{1.7}$$

$$\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\operatorname{grad}\frac{\partial}{\partial x_3}\,a_k(r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}=\frac{1}{\sqrt{3}}\,\frac{d}{dr}\Big(\frac{d}{dr}+\frac{2}{r}\Big)A_{01}X_{00}(\vartheta,\varphi)+$$
$$+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\bigg\{\Big[\sqrt{\frac{k^2-m^2}{4k^2-1}}\,\frac{d}{dr}\Big(\frac{d}{dr}-\frac{k-1}{r}\Big)a_{k-1}(r)A_{mk-1}+$$
$$+\sqrt{\frac{(k+1)^2-m^2}{(2k+1)(2k+3)}}\,\frac{d}{dr}\Big(\frac{d}{dr}+\frac{k+2}{r}\Big)a_{k+1}(r)A_{mk+1}\Big]X_{mk}(\vartheta,\varphi)+$$
$$+\sqrt{k(k+1)}\Big[\sqrt{\frac{k^2-m^2}{4k^2-1}}\,\frac{1}{r}\Big(\frac{d}{dr}-\frac{k-1}{r}\Big)a_{k-1}(r)A_{mk-1}+$$
$$+\sqrt{\frac{(k+1)^2-m^2}{(2k+1)(2k+3)}}\,\frac{1}{r}\Big(\frac{d}{dr}+\frac{k+2}{r}\Big)a_{k+1}(r)A_{mk+1}\Big]Y_{mk}(\vartheta,\varphi)\bigg\}.$$

Let the vector $f(\vartheta,\varphi)$ satisfy sufficient smoothness conditions under which it can be represented as a Fourier series

$$f(\vartheta,\varphi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{\alpha_{mk}X_{mk}(\vartheta,\varphi)+$$
$$+\sqrt{k(k+1)}\Big[\beta_{mk}Y_{mk}(\vartheta,\varphi)+\gamma_{mk}Z_{mk}(\vartheta,\varphi\Big]\Big\}, \tag{1.8}$$

where

$$\alpha_{mk}=\int_0^{2\pi}d\varphi\int_0^{\pi}f(\vartheta,\varphi)\overline{X}_{mk}(\vartheta,\varphi)\sin\vartheta\,d\vartheta,\quad k\geq 0,$$
$$\beta_{mk}=\frac{1}{\sqrt{k(k+1)}}\int_0^{2\pi}d\varphi\int_0^{\pi}f(\vartheta,\varphi)\overline{Y}_{mk}(\vartheta,\varphi)\sin\vartheta\,d\vartheta,\quad k\geq 1, \tag{1.9}$$
$$\gamma_{mk}=\frac{1}{\sqrt{k(k+1)}}\int_0^{2\pi}d\varphi\int_0^{\pi}f(\vartheta,\varphi)\overline{Z}_{mk}(\vartheta,\varphi)\sin\vartheta\,d\vartheta,\quad k\geq 1.$$

Note that in the sequel, in (1.8) and in analogous series the summation index k in the summands containing $Y_{mk}(\vartheta,\varphi)$ and $Z_{mk}(\vartheta,\varphi)$ varies from 1 to $+\infty$.

The following theorems are valid [2].

Theorem 1.1. *If $f(z)\in C^{(\ell)}(\partial\Omega)$, then the coefficients α_{mk}, β_{mk}, γ_{mk} defined by the formulas* (1.9) *admit the following estimates*

$$\alpha_{mk}=O(k^{-\ell}),\quad \beta_{mk}=O(k^{-\ell-1}),\quad \gamma_{mk}=O(k^{-\ell-1}),\ \ \ell\geq 1.$$

Theorem 1.2. *The vectors $X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ admit the following estimates for any $k\geq 0$*

$$\begin{aligned}
|X_{mk}(\vartheta,\varphi)|&\leq\sqrt{\frac{2k+1}{4\pi}},\ \ k\geq 0,\\
|Y_{mk}(\vartheta,\varphi)|&<\sqrt{\frac{2k(k+1)}{2k+1}},\ \ k\geq 1,\\
|Z_{mk}(\vartheta,\varphi)|&<\sqrt{\frac{2k(k+1)}{2k+1}},\ \ k\geq 1.
\end{aligned}\tag{1.10}$$

Moreover, as is known [4], [15],

$$|Y_k^{(m)}(\vartheta,\varphi)|\leq\sqrt{\frac{2k+1}{4\pi}},\ \ k\geq 0.\tag{1.11}$$

Definition 1.3. A vector u defined in the domain Ω will be called regular if $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$.

2. Statement of Boundary Value Problems. Uniqueness Theorems

The system of homogeneous differential equations of statics of the three-dimensional theory of mixture of two isotropic elastic materials is written in the form [6], [7], [14]

$$\begin{aligned}
&a_1\Delta u'+b_1\operatorname{grad}\operatorname{div}u'+c\Delta u''+d\operatorname{grad}\operatorname{div}u''=0,\\
&c\Delta u'+d\operatorname{grad}\operatorname{div}u'+a_2\Delta u''+b_2\operatorname{grad}\operatorname{div}u''=0,
\end{aligned}\tag{2.1}$$

where $u'=(u_1',u_2',u_3')^\top$, $u''=(u_1'',u_2'',u_3'')^\top$ are partial displacement vectors,

$$a_1=\mu_1-\lambda_5,\quad b_1=\mu_1+\lambda_5+\lambda_1-\frac{\rho_2}{\rho}\,\alpha_2',\quad a_2=\mu_2-\lambda_5,$$
$$b_2=\mu_2+\lambda_2+\lambda_5+\frac{\rho_1}{\rho}\,\alpha_2',\quad c=\mu_3+\lambda_5,\quad \alpha_2'=\lambda_3-\lambda_4,$$
$$d=\mu_3+\lambda_3-\lambda_5-\frac{\rho_1}{\rho}\,\alpha_2',\quad \rho=\rho_1+\rho_2,$$

ρ_1, ρ_2 are the partial densities of the mixture; $\lambda_1,\lambda_2,\dots,\lambda_5$, μ_1, μ_2, μ_3 are the elasticity moduli characterizing the mechanical properties of the mixture which satisfy the conditions [3]

$$\begin{gathered}\mu_1>0,\quad \mu_1\mu_2-\mu_3^2>0,\quad \lambda_5<0,\quad \lambda_1+\frac{2}{3}\,\mu_1-\frac{\rho_2}{\rho}\,\alpha_2'>0,\\ \Big(\lambda_1+\frac{2}{3}\,\mu_1-\frac{\rho_2}{\rho}\,\alpha_2'\Big)\Big(\lambda_2+\frac{2}{3}\,\mu_2+\frac{\rho_1}{\rho}\,\alpha_2'\Big)>\Big(\lambda_3+\frac{2}{3}\,\mu_3-\frac{\rho_1}{\rho}\,\alpha_2'\Big)^2.\end{gathered}\tag{2.2}$$

From these inequalities it follows that

$$a_1a_2-c^2>0,\quad (a_1+b_1)(a_2+b_2)-(c+d)^2>0.\tag{2.3}$$

Let us introduce the matrix differential operator

$$T(\partial x,n)=\begin{bmatrix}T^{(1)}(\partial x,n) & \vdots & T^{(2)}(\partial x,n)\\ \cdots & \cdots & \cdots\\ T^{(3)}(\partial x,n) & \vdots & T^{(4)}(\partial x,n)\end{bmatrix},\tag{2.4}$$

where

$$\begin{aligned}
&T^{(\ell)}(\partial x,n)=[T^{(\ell)}_{kj}(\partial x,n)]_{3\times 3},\quad \ell=1,2,3,4,\\
&T^{(1)}_{kj}(\partial x,n)=(\mu_1-\lambda_5)\delta_{kj}\,\frac{\partial}{\partial n}+(\mu_1+\lambda_5)n_j\,\frac{\partial}{\partial x_k}+\Big(\lambda_1-\frac{\rho_2}{\rho}\,\alpha_2'\Big)n_k\,\frac{\partial}{\partial x_j}\,,\\
&T^{(2)}_{kj}(\partial x,n)=(\mu_3+\lambda_5)\delta_{kj}\,\frac{\partial}{\partial n}+(\mu_3-\lambda_5)n_j\,\frac{\partial}{\partial x_k}+\Big(\lambda_3-\frac{\rho_1}{\rho}\,\alpha_2'\Big)n_k\,\frac{\partial}{\partial x_j}\,,\\
&T^{(3)}_{kj}(\partial x,n)=(\mu_3+\lambda_5)\delta_{kj}\,\frac{\partial}{\partial n}+(\mu_3-\lambda_5)n_j\,\frac{\partial}{\partial x_k}+\Big(\lambda_4+\frac{\rho_2}{\rho}\,\alpha_2'\Big)n_k\,\frac{\partial}{\partial x_j}\,,\\
&T^{(4)}_{kj}(\partial x,n)=(\mu_2-\lambda_5)\delta_{kj}\,\frac{\partial}{\partial n}+(\mu_2+\lambda_5)n_j\,\frac{\partial}{\partial x_k}+\Big(\lambda_2+\frac{\rho_1}{\rho}\,\alpha_2'\Big)n_k\,\frac{\partial}{\partial x_j}\,.
\end{aligned}$$

The operator $T(\partial x,n)$ defined by (2.4) is called the generalized stress operator, δ_{kj} is the Kronecker symbol.

The vector form of the notation for the expressions $T^{(\ell)}(\partial x,n)$, $\ell=1,2,3,4$, where u is a three-component vector, looks like

$$T^{(\ell)}(\partial x,n)u=\xi_\ell\,\frac{\partial u}{\partial n}+\eta_\ell n\,\mathrm{div}\,u+\zeta_\ell[n\times\mathrm{rot}\,u],\tag{2.5}$$

where

$$\begin{aligned}
&\xi_1=2\mu_1,\quad \eta_1=\lambda_1-\frac{\rho_2}{\rho}\,\alpha_2',\quad \zeta_1=\mu_1+\lambda_5,\\
&\xi_2=2\mu_3,\quad \eta_2=\lambda_3-\frac{\rho_1}{\rho}\,\alpha_2',\quad \zeta_2=\mu_3-\lambda_5,\\
&\xi_3=2\mu_3,\quad \eta_3=\lambda_4+\frac{\rho_2}{\rho}\,\alpha_2',\quad \zeta_3=\mu_3-\lambda_5,\\
&\xi_4=2\mu_2,\quad \eta_4=\lambda_2+\frac{\rho_1}{\rho}\,\alpha_2',\quad \zeta_4=\mu_2+\lambda_5.
\end{aligned}$$

Let $\partial\Omega_j$, $j=1,2$, be a spherical surface with center at the origin O_j and the radius R_j. Denote by Ω_j, $j=1,2$, the ball bounded by the surface $\partial\Omega_j$, while $\Omega^- = R^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$. Assume that a homogeneous medium occupies the domain Ω^- and $\overline{\Omega}_1 \bigcap \overline{\Omega}_2 = \varnothing$.

Problem. *Find in the domain Ω^- a regular solution $U(x)=(u'(x),u''(x))^\top$ of the system* (2.1) *which on the boundary $\partial\Omega_j$, $j=1,2$, satisfies one of the following conditions:*

$$[U(z)]^- = f^{(j)}(z), \quad j=1,2, \quad \textit{or} \tag{2.6}$$

$$\big[T(\partial z, n)U(z)\big]^- = f^{(j)}(z), \quad j=1,2, \quad \textit{or} \tag{2.7}$$

$$\begin{gathered}\big[n(z)\cdot u'(z)\big]^- = f_4^{(j,1)}(z), \quad \big[n(z)\times \operatorname{rot} u'(z)\big]^- = f^{(j,1)}(z),\\ \big[n(z)\cdot u''(z)\big]^- = f_4^{(j,2)}(z), \quad \big[n(z)\times \operatorname{rot} u''(z)\big]^- = f^{(j,2)}(z), \quad j=1,2,\end{gathered} \tag{2.8}$$

where $f^{(j)}(z) = (f^{(j,1)}(z), f^{(j,2)}(z))^\top$ is a six-component vector, $f^{(j,\ell)}(z) = (f_1^{(j,\ell)}(z),\ f_2^{(j,\ell)}(z), f_3^{(j,\ell)}(z))^\top$, $\ell, j = 1,2$, is a three-component vector, $f_k^{(j,\ell)}(z)$, $\ell, j = 1,2$, $k = 1,2,3,4$, are given functions on $\partial\Omega_j$, $n(z)$ is the outward normal vector with respect to Ω_j at the point $z \in \partial\Omega_j$.

In the neighborhood of infinity the vector $U(x)$ must satisfy the following conditions:

$$\begin{gathered}u_j'(x) = O(|x|^{-1}), \quad u_j''(x) = O(|x|^{-1}),\\ \frac{\partial u_j'(x)}{\partial x_k} = o(|x|^{-1}), \quad \frac{\partial u_j''(x)}{\partial x_k} = o(|x|^{-1}), \quad k,j = 1,2,3.\end{gathered} \tag{2.9}$$

Denote by $(I)^-$, $(II)^-$ and $(III)^-$ the problems containing the conditions (2.6), (2.7) and (2.8), respectively.

Theorem 2.1. *If $\partial\Omega_j \in \Lambda_1(\alpha)$, $0 < \alpha \le 1$, then Problems $(I)^-$, $(II)^-$ and $(III)^-$ admit at most one regular solution.*

Proof. The theorem will be proved if we show that the homogeneous problems $(I)_0^-$, $(II)_0^-$ and $(III)_0^-$ $(f^{(j)}(z) = 0,\ f_4^{(j,\ell)}(z) = 0,\ \ell, j = 1,2)$ have only the trivial solution.

Let us introduce the matrix differential operator $A(\partial x)$:

$$A(\partial x) = \begin{bmatrix} A^{(1)}(\partial x) & \vdots & A^{(2)}(\partial x) \\ \cdots\cdots & & \cdots\cdots \\ A^{(3)}(\partial x) & \vdots & A^{(4)}(\partial x)\end{bmatrix}_{6\times 6},$$

$$A^{(\ell)}(\partial x) = \big[A_{kj}^{(\ell)}(\partial x)\big]_{3\times 3}, \quad \ell = 1,2,3,4,$$

where

$$A_{kj}^{(1)}(\partial x) = a_1\delta_{kj}\Delta + b_1\,\frac{\partial^2}{\partial x_k \partial x_j}\,,$$

$$A_{kj}^{(\ell)}(\partial x) = c\delta_{kj}\Delta + d\,\frac{\partial^2}{\partial x_k \partial x_j}\,, \quad \ell = 2,3,$$

$$A_{kj}^{(4)}(\partial x) = a_2\delta_{kj}\Delta + b_2\,\frac{\partial^2}{\partial x_k \partial x_j}\,.$$

Using this notation, the system (2.1) can be written in the form $A(\partial x)U(x) = 0$.

Let us write Green's formula in the domain Ω^- for the system (2.1). We obtain [13]

$$\int\limits_{\Omega^-} \Big[U(x)\cdot A(\partial x)U(x) + E(U,U)\Big]\,dx = \\ = \int\limits_{\partial\Omega_1\cup\partial\Omega_2} [U(z)]^-\cdot\big[T(\partial z,n)U(z)\big]^-\,ds, \quad (2.10)$$

where

$$E(U,U) = \Big(\lambda_1 - \frac{\rho_2}{\rho}\,\alpha_2'\Big)(\operatorname{div} u')^2 + \Big(\lambda_2 + \frac{\rho_1}{\rho}\,\alpha_2'\Big)(\operatorname{div} u'')^2 + \\ + \frac{\mu_1}{2}\sum_{k,j=1}^{3}\Big(\frac{\partial u_k'}{\partial x_j} + \frac{\partial u_j'}{\partial x_k}\Big)^2 + \frac{\mu_2}{2}\sum_{k,j=1}^{3}\Big(\frac{\partial u_k''}{\partial x_j} + \frac{\partial u_j''}{\partial x_k}\Big)^2 + \\ + \mu_3\sum_{k,j=1}^{3}\Big(\frac{\partial u_k'}{\partial x_j} + \frac{\partial u_j'}{\partial x_k}\Big)\Big(\frac{\partial u_k''}{\partial x_j} + \frac{\partial u_j''}{\partial x_k}\Big) + \\ + 2\Big(\lambda_3 - \frac{\rho_1}{\rho}\,\alpha_2'\Big)\operatorname{div} u'\operatorname{div} u'' - \frac{\lambda_5}{2}\sum_{k,j=1}^{3}\Big(\frac{\partial u_k'}{\partial x_j} - \frac{\partial u_j'}{\partial x_k} - \frac{\partial u_k''}{\partial x_j} + \frac{\partial u_j''}{\partial x_k}\Big)^2.$$

The quadratic form of $E(U,U)$ is provided by the inequality (2.2).

If in (2.10) we take into account the boundary conditions of the homogeneous problems $(I)_0^-$, $(II)_0^-$ and $A(\partial x)U = 0$, then we obtain

$$\int\limits_{\Omega^-} E(U,U)\,dx = 0.$$

Hence it follows that $E(U,U) = 0$. A solution of this equation has the form [13]

$$u'(x) = b' + [a'\times x], \quad u''(x) = b'' + [a'\times x],$$

where a', b', b'' are arbitrary three-component constant vectors.

Taking into account the conditions at infinity (2.9), we conclude that $a' = b' = b'' = 0$, i.e. $u'(x) = 0$, $u''(x) = 0$, $x \in \Omega^-$.

In the case of Problem $(III)_0^-$ we need to consider the scalar derivative

$$U\cdot A(\partial x)U = (a_1u' + cu'')\Delta u' + (cu' + a_2u'')\Delta u'' + \\ + (b_1u' + du'')\operatorname{grad}\operatorname{div} u' + (du' + b_2u'')\operatorname{grad}\operatorname{div} u''. \quad (2.11)$$

Let $u=(u_1,u_2,u_3)^\top$ and $v=(v_1,v_2,v_3)^\top$ be three-component vectors. Then after some transformations we obtain

$$\begin{aligned} u\cdot\Delta v&=\operatorname{div}(u\operatorname{div}v)-\operatorname{div}u\operatorname{div}v+\operatorname{div}[u\times\operatorname{rot}v]-\operatorname{rot}u\cdot\operatorname{rot}v,\\ u\cdot\operatorname{grad}\operatorname{div}v&=\operatorname{div}(u\operatorname{div}v)-\operatorname{div}u\operatorname{div}v. \end{aligned}$$

Taking these equalities into account in (2.11), we obtain

$$\begin{aligned} U\cdot A(\partial x)U=\operatorname{div}\Big[&\big((a_1+b_1)u'+(c+d)u''\big)\operatorname{div}u'+\\ &+\big((c+d)u'+(a_2+b_2)u''\big)\operatorname{div}u''+a_1(u'\times\operatorname{rot}u')+\\ &+c(u''\times\operatorname{rot}u')+c(u'\times\operatorname{rot}u'')+a_2(u''\times\operatorname{rot}u'')\Big]-\widetilde{E}(U,U), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \widetilde{E}(U,U)&=\frac{1}{a_1+b_1}\Big[\big((a_1+b_1)\operatorname{div}u'+(c+d)\operatorname{div}u''\big)^2+d_1(\operatorname{div}u'')^2\Big]+\\ &+\frac{1}{a_1}\Big[\big(a_1\operatorname{rot}u'+c\operatorname{rot}u''\big)^2+d_2(\operatorname{rot}u'')^2\Big], \end{aligned} \tag{2.13}$$

$$d_1=(a_1+b_1)(a_2+b_2)-(c+d)^2>0,\quad d_2=a_1a_2-c^2>0.$$

Denote by $B(0,R)$ the ball bounded by the spherical surface $S(0,R)$ with the center at the origin and of the radius R. Let $\Omega_R^-=\Omega^-\cap B(0,R)$, where $R>0$ is sufficiently large so that $\partial\Omega_j\subset B(0,R)$, $j=1,2$.

Applying the Gauss–Ostrogradskiĭ theorem, from (2.12) we obtain

$$\begin{aligned} &\int\limits_{\Omega_R^-}\big[U\cdot A(\partial x)U+\widetilde{E}(U,U)\big]\,dx=\\ &=-\int\limits_{\partial\Omega_1\cup\partial\Omega_2}[U(z)]^-\cdot\big[P(\partial z,n)U(z)\big]^-\,ds+\int\limits_{S(0,R)}U(x)\cdot P(\partial x,n)U(x)\,ds, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} U\cdot P(\partial z,n)U=(n\cdot u')&\big[(a_1+b_1)\operatorname{div}u'+(c+d)\operatorname{div}u''\big]+\\ &+(n\cdot u'')\big[(c+d)\operatorname{div}u'+(a_2+b_2)\operatorname{div}u''\big]-\\ &-(a_1u'+cu'')\cdot[n\times\operatorname{rot}u']-(cu'+a_2u'')\cdot[n\times\operatorname{rot}u'']. \end{aligned} \tag{2.15}$$

Here we have used the identity $n\cdot[u\times\operatorname{rot}v]=-u\cdot[n\times\operatorname{rot}v]$.

If in (2.15) we use the estimates (2.9), then we have

$$U(x)\cdot P(\partial x\cdot n)U(x)=o(R^{-2}). \tag{2.16}$$

If in both parts of the equality (2.14) we pass to limit as $R\to\infty$ and take into account the estimate (2.16), then we obtain

$$\int\limits_{\Omega^-}\big[U\cdot A(\partial x)U+\widetilde{E}(U,U)\big]\,dx=-\int\limits_{\partial\Omega_1\cup\partial\Omega_2}[U(z)]^-\cdot\big[P(\partial z,n)U(z)\big]^-\,ds. \tag{2.17}$$

Taking into account the boundary conditions of the problem $(III)_0^-$ in the formula (2.15), we have $[U(z)]^- \cdot [P(\partial z, n)U(z)]^- = 0$, $z \in \partial\Omega_j$, $j = 1, 2$.

Since $A(\partial x)U = 0$, from the formula (2.17), with the latter equality taken into account, we obtain

$$\int\limits_{\Omega^-} \widetilde{E}(U, U)\, dx = 0. \tag{2.18}$$

Using (2.3), from (2.13) we obtain $\widetilde{E}(U, U) \geq 0$. Taking this inequality into account, from (2.18) we have $\widetilde{E}(U, U) = 0$, $x \in \Omega^-$.

Hence, with (2.13) taken into account, we obtain $\operatorname{div} u'(x) = 0$, $\operatorname{div} u''(x) = 0$, $\operatorname{rot} u'(x) = 0$, $\operatorname{rot} u''(x) = 0$, $x \in \Omega^-$.

A solution of this system has the form

$$u'(x) = \operatorname{grad} \Psi_1(x), \quad u''(x) = \operatorname{grad} \Psi_2(x), \quad x \in \Omega^-, \tag{2.19}$$

where $\Psi_j(x)$, $j = 1, 2$, is an arbitrary harmonic function.

Since $[n(z) \cdot u'(z)]^- = 0$, $[n(z) \cdot u''(z)]^- = 0$, the harmonic function $\Psi_j(x)$, $j = 1, 2$, on the boundary $\partial\Omega_j$ satisfies the Neumann condition

$$\left[\frac{\partial \Psi_j(z)}{\partial n(z)}\right]^- = 0, \quad z \in \partial\Omega_\ell, \quad \ell, j = 1, 2.$$

As is known, the Neumann homogeneous problem has the solution $\Psi_j(x) = c_j = \text{const}$, $j = 1, 2$, $x \in \Omega^-$. Inserting this value of the function $\Psi_j(x)$ into (2.19), we have that $u'(x) = 0$, $u''(x) = 0$, $x \in \Omega^-$. □

3. Solution of the Problem

Let us assume that the axes of the coordinate systems $O_j x_1^{(j)} x_2^{(j)} x_3^{(j)}$, $j = 1, 2$, are parallel and have the same orientation. We denote by $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)})$ and $(r_j, \vartheta_j, \varphi_j)$ the Cartesian and spherical coordinates of the point x with respect to the coordinate systems $O_j x_1^{(j)} x_2^{(j)} x_3^{(j)}$, $j = 1, 2$. The spherical coordinates of the point O_2 with respect to the system $O_1 x_1^{(1)} x_2^{(1)} x_3^{(1)}$ are denoted by $(h, \vartheta_0, \varphi_0)$.

Since the axes of the coordinate systems are parallel and have the same orientation, the following equality is true:

$$x^{(1)} = x^{(2)} + h e_0, \tag{3.1}$$

where $e_0 = (\cos\varphi_0 \sin\vartheta_0, \sin\varphi_0 \sin\vartheta_0, \cos\vartheta_0)^\top$.

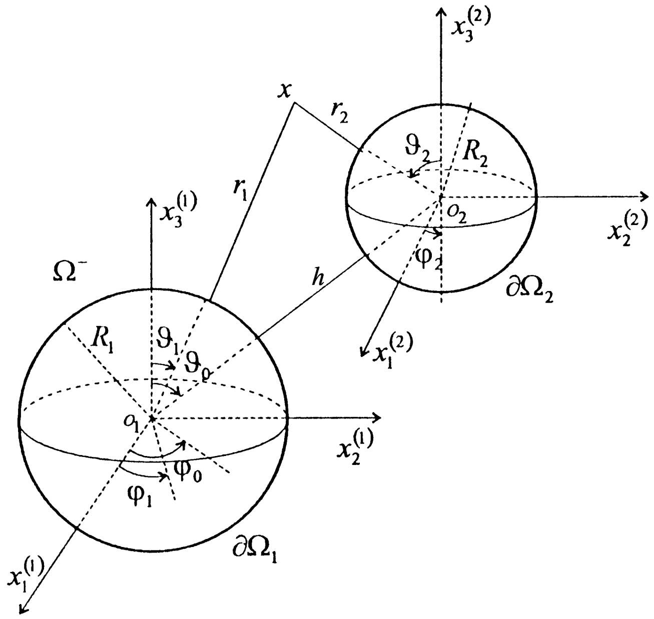
Passing to a new origin of the coordinates, we can write the transformation formula of the spherical coordinates in the form [5], [8]

$$r_q^{-k-1} Y_k^{(m)}(\vartheta_q, \varphi_q) = \\ = \sum_{p=0}^{\infty} \sum_{s=-p}^{p} G_{pk,q}^{(s,m)}(\vartheta_0, \varphi_0) r_j^p Y_p^{(s)}(\vartheta_j, \varphi_j), \quad q \neq j = 1, 2, \quad r_j < h, \tag{3.2}$$

where

$$G_{pk,q}^{(s,m)}(\vartheta_0,\varphi_0)=(-1)^s\big[(-1)^p\delta_{1q}+(-1)^k\delta_{2q}\big]\,\frac{1}{h^{k+p+1}}\times$$
$$\times\Big[\frac{4\pi(2k{+}1)(k{+}p{+}m{-}s)!(k{+}p{-}m{+}s)!}{(2p{+}1)(2k{+}2p{+}1)(k{+}m)!(k{-}m)!(p{+}s)!(p{-}s)!}\Big]^{1/2}Y_{k+p}^{(m-s)}(\vartheta_0,\varphi_0),$$

δ_{kj} is the Kronecker symbol, $Y_k^{(m)}(\vartheta_q,\varphi_q)$ has the form (1.2).



If in the formula (3.2) we take into account that

$$r_q^2=r_j^2+h^2-(-1)^q 2hr_j\cos\gamma_j,\quad q\neq j=1,2,$$

where

$$\cos\gamma_j=\sin\vartheta_j\sin\vartheta_0\cos(\varphi-\varphi_0)+\cos\vartheta_j\cos\vartheta_0,$$

and use the recurrent relations of Legendre polynomials [5], [15]

$$(k-m+1)\sin\vartheta P_k^{(m-1)}(\cos\vartheta)=\cos\vartheta P_k^{(m)}(\cos\vartheta)+P_{k-1}^{(m)}(\cos\vartheta),$$
$$(2k+1)\sin\vartheta P_k^{(m-1)}(\cos\vartheta)=P_{k-1}^{(m)}(\cos\vartheta)-P_{k+1}^{(m)}(\cos\vartheta),$$
$$\sin\vartheta P_k^{(m+1)}(\cos\vartheta)=(k-m)\cos\vartheta P_k^{(m)}(\cos\vartheta)-(k+m)P_{k-1}^{(m)}(\cos\vartheta),$$
$$\sin\vartheta\,\frac{d}{d\vartheta}\,P_k^{(m)}(\cos\vartheta)=\sin\vartheta P_k^{(m+1)}(\cos\vartheta)+m\cos\vartheta P_k^{(m)}(\cos\vartheta),$$

then we obtain

$$r_q^{-k+1} Y_k^{(m)}(\vartheta_q, \varphi_q) = \sum_{p=0}^{\infty} \sum_{s=-p}^{p} \Big[a_{pk,q}^{(s,m)}(\vartheta_0, \varphi_0) r_j^{p+2} + \\ + b_{pk,q}^{(s,m)}(\vartheta_0, \varphi_0) r_j^{p} \Big] Y_p^{(s)}(\vartheta_j, \varphi_j), \quad q \neq j = 1, 2, \quad (3.3)$$

where

$$a_{pk,q}^{(s,m)}(\vartheta_0, \varphi_0) = G_{pk,q}^{(s,m)}(\vartheta_0, \varphi_0) + (-1)^{q-1} h \zeta_{mskp}^{(1,q)}(\vartheta_0, \varphi_0),$$

$$b_{pk,q}^{(s,m)}(\vartheta_0, \varphi_0) = h^2 G_{pk,q}^{(s,m)}(\vartheta_0, \varphi_0) + (-1)^{q} h \zeta_{mskp}^{(2,q)}(\vartheta_0, \varphi_0),$$

$$\zeta_{mskp}^{(1,q)}(\vartheta_0, \varphi_0) = \sin\vartheta_0 \Bigg[e^{-i\varphi_0} \sqrt{\frac{(p-s+1)(p-s+2)}{(2p+1)(2p+3)}}\, G_{p+1k,q}^{(s-1,m)}(\vartheta_0, \varphi_0) - \\ - e^{i\varphi_0} \sqrt{\frac{(p+s+1)(p+s+2)}{(2p+1)(2p+3)}}\, G_{p+1k,q}^{(s+1,m)}(\vartheta_0, \varphi_0) \Bigg] + \\ + 2\cos\vartheta_0 \sqrt{\frac{(p+1)^2 - s^2}{(2p+1)(2p+3)}}\, G_{p+1k,q}^{(s,m)}(\vartheta_0, \varphi_0),$$

$$\zeta_{mskp}^{(2,q)}(\vartheta_0, \varphi_0) = \sin\vartheta_0 \Bigg[e^{-i\varphi_0} \sqrt{\frac{(p+s)(p+s-1)}{4p^2-1}}\, G_{p-1k,q}^{(s-1,m)}(\vartheta_0, \varphi_0) - \\ - e^{i\varphi_0} \sqrt{\frac{(p-s)(p-s-1)}{4p^2-1}}\, G_{p-1k,q}^{(s+1,m)}(\vartheta_0, \varphi_0) \Bigg] - \\ - 2\cos\vartheta_0 \sqrt{\frac{p^2 - s^2}{4p^2-1}}\, G_{p-1k,q}^{(s,m)}(\vartheta_0, \varphi_0).$$

A solution of the considered problem will be sought in the form [4]

$$\begin{aligned} u'(x) = \sum_{j=1}^{2} \Big\{ & \operatorname{grad} \Phi_1^{(j)}(x^{(j)}) + \operatorname{grad} r_j^2 \Big(r_j \frac{\partial}{\partial r_j} + 1 \Big) \big[\alpha_1 \Phi_2^{(j)}(x^{(j)}) + \\ & + \beta_1 \Phi_3^{(j)}(x^{(j)}) \big] + \operatorname{rot}\operatorname{rot} \big(x^{(j)} r_j^2 \Phi_2^{(j)}(x^{(j)}) \big) + \operatorname{rot} \big(x^{(j)} \Phi_5^{(j)}(x^{(j)}) \big) \Big\}, \\ u''(x) = \sum_{j=1}^{2} \Big\{ & \operatorname{grad} \Phi_4^{(j)}(x^{(j)}) + \operatorname{grad} r_j^2 \Big(r_j \frac{\partial}{\partial r_j} + 1 \Big) \big[\beta_2 \Phi_2^{(j)}(x^{(j)}) + \\ & + \alpha_2 \Phi_3^{(j)}(x^{(j)}) \big] + \operatorname{rot}\operatorname{rot} \big(x^{(j)} r_j^2 \Phi_3^{(j)}(x^{(j)}) \big) + \operatorname{rot} \big(x^{(j)} \Phi_6^{(j)}(x^{(j)}) \big) \Big\}, \end{aligned} \quad (3.4)$$

where $\Phi_\ell^{(j)}(x^{(j)})$, $j = 1, 2,$, $\ell = 1, 2, \dots, 6$ are scalar harmonic functions,

$$x^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)})^\top, \quad r_j = |x^{(j)}|, \quad r_j \frac{\partial}{\partial r_j} = x^{(j)} \cdot \operatorname{grad},$$

$$\alpha_1 = \frac{1}{d_1}\left[c(c+d) - a_1(a_2+b_2)\right], \quad \beta_1 = \frac{1}{d_1}(a_2 d - c b_2),$$
$$\alpha_2 = \frac{1}{d_1}\left[c(c+d) - a_2(a_1+b_1)\right], \quad \beta_2 = \frac{1}{d_1}(a_1 d - c b_1).$$

The harmonic functions $\Phi_\ell^{(j)}(x^{(j)})$, $j=1,2$, $\ell=1,2,\dots,6$ will be sought in the form

$$\Phi_\ell^{(j)}(x^{(j)}) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big(\frac{R_j}{r_j}\Big)^{k+1} Y_k^{(m)}(\vartheta_j,\varphi_j) A_{mk}^{(j,\ell)}, \tag{3.5}$$
$$j=1,2, \quad \ell=1,2,\dots,6,$$

where $A_{mk}^{(j,\ell)}$ are unknown constants.

We require of the $\Phi_\ell^{(j)}(x^{(j)})$, $j=1,2$, $\ell=2,3,5,6$, that

$$\int\limits_{\partial\Omega_j'} \Phi_\ell^{(j)}(x^{(j)})\,ds = 0, \quad j=1,2, \quad \ell=2,3,5,6, \tag{3.6}$$

where $\partial\Omega_j'$ is the sphere with center at the point O_j and of the radius R_j' ($R_q + h < R_j' < +\infty$, $q \neq j = 1,2$).

Inserting the value of the function $\Phi_\ell^{(j)}(x^{(j)})$, $j=1,2$, $\ell=2,3,5,6$, from (3.5) in (3.6) and taking into account the identity

$$\int\limits_0^{2\pi}\int\limits_0^{\pi} Y_k^{(m)}(\vartheta,\varphi)\sin\vartheta\,d\vartheta = \begin{cases} 2\sqrt{\pi}, & k=m=0,\\ 0, & \text{for others } k \text{ and } m,\end{cases}$$

we have $A_{00}^{(j,\ell)} = 0$, $j=1,2$, $\ell=2,3,5,6$.

Using the formulas (3.2), (3.3) and (3.5), we obtain:

$$\begin{gathered}
\Phi_\ell^{(j)}(x^{(j)}) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} r_q^k Y_k^{(m)}(\vartheta_q,\varphi_q) B_{mk}^{(q,\ell)}, \quad \ell=1,4,5,6,\\
r_j^2\Big(r_j\frac{\partial}{\partial r_j}+1\Big)\Phi_\ell^{(j)}(x^{(j)}) =\\
= -\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[r_q^{k+2}B_{mk}^{(q,\ell)} + r_q^k C_{mk}^{(q,\ell)}\Big] Y_k^{(m)}(\vartheta_q,\varphi_q),\\
r_j^2\Big(r_j\frac{\partial}{\partial r_j}+3\Big)\Phi_\ell^{(j)}(x^{(j)}) =\\
= -\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big[r_q^{k+2}D_{mk}^{(q,\ell)} + r_q^k E_{mk}^{(q,\ell)}\Big] Y_k^{(m)}(\vartheta_q,\varphi_q), \quad \ell=2,3,\\
\Big(2r_j\frac{\partial}{\partial r_j}+3\Big)\Phi_\ell^{(j)}(x^{(j)}) = -\sum_{k=0}^{\infty}\sum_{m=-k}^{k} r_q^k Y_k^{(m)}(\vartheta_q,\varphi_q) H_{mk}^{(q,\ell)}, \quad \ell=2,3,
\end{gathered} \tag{3.7}$$

where $q \neq j = 1,2$,

$$
\begin{aligned}
B_{mk}^{(q,\ell)} &= \sum_{p=0}^{\infty} \sum_{s=-p}^{p} R_j^{p+1} G_{kp,j}^{(m,s)}(\vartheta_0,\varphi_0) A_{sp}^{(j,\ell)}, \quad \ell = 1,4,\\
B_{mk}^{(q,\ell)} &= \sum_{p=1}^{\infty} \sum_{s=-p}^{p} R_j^{p+1} G_{kp,j}^{(m,s)}(\vartheta_0,\varphi_0) A_{sp}^{(j,\ell)}, \quad \ell = 5,6,\\
B_{mk}^{(q,\ell)} &= \sum_{p=1}^{\infty} \sum_{s=-p}^{p} pR_j^{p+1} a_{kp,j}^{(m,s)}(\vartheta_0,\varphi_0) A_{sp}^{(j,\ell)}, \quad \ell = 2,3,\\
C_{mk}^{(q,\ell)} &= \sum_{p=1}^{\infty} \sum_{s=-p}^{p} pR_j^{p+1} b_{kp,j}^{(m,s)}(\vartheta_0,\varphi_0) A_{sp}^{(j,\ell)}, \quad \ell = 2,3, \qquad (3.8)\\
D_{mk}^{(q,\ell)} &= \sum_{p=1}^{\infty} \sum_{s=-p}^{p} (p-2)R_j^{p+1} a_{kp,j}^{(m,s)}(\vartheta_0,\varphi_0) A_{sp}^{(j,\ell)}, \quad \ell = 2,3,\\
E_{mk}^{(q,\ell)} &= \sum_{p=1}^{\infty} \sum_{s=-p}^{p} (p-2)R_j^{p+1} b_{kp,j}^{(m,s)}(\vartheta_0,\varphi_0) A_{sp}^{(j,\ell)}, \quad \ell = 2,3,\\
H_{mk}^{(q,\ell)} &= \sum_{p=1}^{\infty} \sum_{s=-p}^{p} (2p-1)R_j^{p+1} G_{kp,j}^{(m,s)}(\vartheta_0,\varphi_0) A_{sp}^{(j,\ell)}, \quad \ell = 2,3.
\end{aligned}
$$

Taking the equalities (3.1) and (3.7) into account in (3.4) and using (1.5), we obtain

$$
\begin{aligned}
u'(x^{(j)}) = &\sum_{k=0}^{\infty} \sum_{m=-k}^{k} \bigg\{ \bigg[-\frac{k+1}{R_j}\Big(\frac{R_j}{r_j}\Big)^{k+2} A_{mk}^{(j,1)} + \\
&+ kR_j\big((k-1)(\alpha_1+1)+2\big)\Big(\frac{R_j}{r_j}\Big)^{k} A_{mk}^{(j,2)} + \beta_1 k(k-1)R_j\Big(\frac{R_j}{r_j}\Big)^{k} A_{mk}^{(j,3)} + \\
&+ kr_j^{k-1}\big(B_{mk}^{(j,1)} - \alpha_1 C_{mk}^{(j,2)} - \beta_1 C_{mk}^{(j,3)} - E_{mk}^{(j,2)}\big) - \\
&- r_j^{k+1}\Big(\alpha_1(k+2)B_{mk}^{(j,2)} + \beta_1(k+2)B_{mk}^{(j,3)} + \\
&+ (k+2)D_{mk}^{(j,2)} - 2H_{mk}^{(j,2)}\Big)\bigg] X_{mk}(\vartheta_j,\varphi_j) + \sqrt{k(k+1)}\times \\
&\times \bigg[\frac{1}{R_j}\Big(\frac{R_j}{r_j}\Big)^{k+2} A_{mk}^{(j,1)} - R_j\big(k(\alpha_1+1)-2\big)\Big(\frac{R_j}{r_j}\Big)^{k} A_{mk}^{(j,2)} - \\
&- \beta_1 kR_j\Big(\frac{R_j}{r_j}\Big)^{k} A_{mk}^{(j,3)} + r_j^{k-1}\big(B_{mk}^{(j,1)} - \alpha_1 C_{mk}^{(j,2)} - \beta_1 C_{mk}^{(j,3)}\big) - \\
&- r_j^{k+1}\Big(\alpha_1 B_{mk}^{(j,2)} + \beta_1 B_{mk}^{(j,3)} + D_{mk}^{(j,2)}\Big)\bigg] Y_{mk}(\vartheta_j,\varphi_j) + \\
&+ \sqrt{k(k+1)}\bigg[\Big(\frac{R_j}{r_j}\Big)^{k+1} A_{mk}^{(j,5)} + r_j^{k} B_{mk}^{(j,5)}\bigg] Z_{mk}(\vartheta_j,\varphi_j \bigg\} +
\end{aligned}
$$

$$+(-1)^j 2he_0\sum_{k=0}^{\infty}\sum_{m=-k}^{k} r_j^k Y_k^{(m)}(\vartheta_j,\varphi_j)H_{mk}^{(j,2)}+$$
$$+(-1)^j he_0\times\operatorname{grad}\sum_{k=0}^{\infty}\sum_{m=-k}^{k} r_j^k Y_k^{(m)}(\vartheta_j,\varphi_j)B_{mk}^{(j,5)}, \tag{3.9}$$

$$u''(x^{(j)})=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\bigg\{\bigg[-\frac{k+1}{R_j}\Big(\frac{R_j}{r_j}\Big)^{k+2}A_{mk}^{(j,4)}+$$
$$+\beta_2 k(k-1)R_j\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,2)}+kR_j\big((k-1)(\alpha_2+1)+2\big)\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,3)}+$$
$$+kr_j^{k-1}\big(B_{mk}^{(j,4)}-\beta_2C_{mk}^{(j,2)}-\alpha_2C_{mk}^{(j,3)}-E_{mk}^{(j,3)}\big)-$$
$$-r_j^{k+1}\Big(\beta_2(k+2)B_{mk}^{(j,2)}+\alpha_2(k+2)B_{mk}^{(j,3)}+$$
$$+(k+2)D_{mk}^{(j,3)}-2H_{mk}^{(j,3)}\Big)\bigg]X_{mk}(\vartheta_j,\varphi_j)+$$
$$+\sqrt{k(k+1)}\bigg[\frac{1}{R_j}\Big(\frac{R_j}{r_j}\Big)^{k+2}A_{mk}^{(j,4)}-\beta_2kR_j\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,2)}-$$
$$-R_j\Big(k(\alpha_2+1)-2\Big)\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,3)}+$$
$$+r_j^{k-1}\big(B_{mk}^{(j,4)}-\beta_2C_{mk}^{(j,2)}-\alpha_2C_{mk}^{(j,3)}-E_{mk}^{(j,3)}\big)-$$
$$-r_j^{k+1}\big(\beta_2B_{mk}^{(j,2)}+\alpha_2B_{mk}^{(j,3)}+D_{mk}^{(j,3)}\big)\bigg]Y_{mk}(\vartheta_j,\varphi_j)+$$
$$+\sqrt{k(k+1)}\Big[\Big(\frac{R_j}{r_j}\Big)^{k+1}A_{mk}^{(j,6)}+r_j^kB_{mk}^{(j,6)}\Big]Z_{mk}(\vartheta_j,\varphi_j\bigg\}+$$
$$+(-1)^j 2he_0\sum_{k=0}^{\infty}\sum_{m=-k}^{k} r_j^k Y_k^{(m)}(\vartheta_j,\varphi_j)H_{mk}^{(j,3)}+$$
$$+(-1)^j he_0\times\operatorname{grad}\sum_{k=0}^{\infty}\sum_{m=-k}^{k} r_j^k Y_k^{(m)}(\vartheta_j,\varphi_j)B_{mk}^{(j,6)},\quad j=1,2. \tag{3.10}$$

Using the formulas (1) and (1.8), we have

$$(-1)^j 2he_0\sum_{k=0}^{\infty}\sum_{m=-k}^{k} r_j^k Y_k^{(m)}(\vartheta_j,\varphi_j)H_{mk}^{(j,\ell)}=$$
$$=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\bigg\{\Big[-r_j^{k+1}\sigma_{mk}^{(j,1)}(H_{mk}^{(j,\ell)})+r_j^{k-1}\sigma_{mk}^{(j,2)}(H_{mk}^{(j,\ell)})\Big]X_{mk}(\vartheta_j,\varphi_j)+$$
$$+\sqrt{k(k+1)}\,\Big[\frac{1}{k+1}\,r_j^{k+1}\sigma_{mk}^{(j,1)}(H_{mk}^{(j,\ell)})+\frac{1}{k}\,r_j^{k-1}\sigma_{mk}^{(j,2)}(H_{mk}^{(j,\ell)})\Big]Y_{mk}(\vartheta_j,\varphi_j)+$$

$$+\sqrt{k(k+1)}\,r_j^k\sigma_{mk}^{(j,3)}(H_{mk}^{(j,\ell)})Z_{mk}(\vartheta_j,\varphi_j)\Big\},\quad j=1,2,\quad \ell=2,3, \tag{3.11}$$

$$(-1)^j he_0\times\operatorname{grad}\sum_{k=0}^{\infty}\sum_{m=-k}^{k} r_j^k Y_k^{(m)}(\vartheta_j,\varphi_j)B_{mk}^{(j,\ell)} =$$

$$=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{r_j^{k-1}\sigma_{mk}^{(j,4)}(B_{mk}^{(j,\ell)})\times\Big[X_{mk}(\vartheta_j,\varphi_j)+\frac{\sqrt{k(k+1)}}{k}\,Y_{mk}(\vartheta_j,\varphi_j)\Big]+$$

$$+\sqrt{k(k+1)}\,r_j^k\sigma_{mk}^{(j,5)}(B_{mk}^{(j,\ell)})Z_{mk}(\vartheta_j,\varphi_j)\Big\},\quad j=1,2,\quad \ell=5,6, \tag{3.12}$$

where

$$\sigma_{mk}^{(j,1)}(H_{mk}^{(j,\ell)})=(-1)^j 2h\Big[\cos\varphi_0\sin\vartheta_0\eta_{mk}^{(1,1)}(H_{mk}^{(j,\ell)})+$$

$$+\sin\varphi_0\sin\vartheta_0\eta_{mk}^{(2,1)}(H_{mk}^{(j,\ell)})-\cos\vartheta_0\sqrt{\frac{(k+1)^2-m^2}{(2k+1)(2k+3)}}\,H_{mk+1}^{(j,\ell)}\Big],$$

$$\sigma_{mk}^{(j,2)}(H_{mk}^{(j,\ell)})=(-1)^j 2h\Big[\cos\varphi_0\sin\vartheta_0\eta_{mk}^{(1,2)}(H_{mk}^{(j,\ell)})+$$

$$+\sin\varphi_0\sin\vartheta_0\eta_{mk}^{(2,2)}(H_{mk}^{(j,\ell)})+\cos\vartheta_0\sqrt{\frac{k^2-m^2}{4k^2-1}}\,H_{mk-1}^{(j,\ell)}\Big],$$

$$\sigma_{mk}^{(j,3)}(H_{mk}^{(j,\ell)})=\frac{2h(-1)^j}{k(k+1)}\Big[\cos\varphi_0\sin\vartheta_0\eta_{mk}^{(1,3)}(H_{mk}^{(j,\ell)})-\sin\varphi_0\sin\vartheta_0\times$$

$$\times\,\eta_{mk}^{(2,3)}(H_{mk}^{(j,\ell)})+im\cos\vartheta_0 H_{mk}^{(j,\ell)}\Big],\quad j=1,2,\ \ell=2,3, \tag{3.13}$$

$$\sigma_{mk}^{(j,4)}(H_{mk}^{(j,\ell)})=(-1)^j h\Big[-\cos\varphi_0\sin\vartheta_0\eta_{mk}^{(1,3)}(H_{mk}^{(j,\ell)})+$$

$$+\sin\vartheta_0\sin\varphi_0\eta_{mk}^{(2,3)}(B_{mk}^{(j,\ell)})-im\cos\vartheta_0 B_{mk}^{(j,\ell)}\Big],$$

$$\sigma_{mk}^{(j,5)}(H_{mk}^{(j,\ell)})=(-1)^j\,\frac{(2k+3)h}{k+1}\Big[\eta_{mk}^{(1,1)}(B_{mk}^{(j,\ell)})+\eta_{mk}^{(2,1)}(B_{mk}^{(j,\ell)})-$$

$$-\sqrt{\frac{(k+1)^2-m^2}{(2k+1)(2k+3)}}\,B_{mk+1}^{(j,\ell)}\Big],\quad j=1,2,\quad \ell=5,6.$$

Taking the equalities (3.11)and (3.12) into account in (3.9) and (3.10), we obtain

$$u'(x^{(j)})=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(1)}(r_j)X_{mk}(\vartheta_j,\varphi_j)+$$

$$+\sqrt{k(k+1)}\,\Big[v_{mk}^{(1)}(r_j)Y_{mk}(\vartheta_j,\varphi_j)+w_{mk}^{(1)}(r_j)Z_{mk}(\vartheta_j,\varphi_j)\Big]\Big\}, \tag{3.14}$$

$$u''(x^{(j)})=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}^{(2)}(r_j)X_{mk}(\vartheta_j,\varphi_j)+$$
$$+\sqrt{k(k+1)}\Big[v_{mk}^{(2)}(r_j)Y_{mk}(\vartheta_j,\varphi_j)+w_{mk}^{(2)}(r_j)Z_{mk}(\vartheta_j,\varphi_j)\Big]\Big\},\quad j=1,2,$$

where

$$\begin{aligned}
u_{mk}^{(1)}(r_j)&=-\frac{k+1}{R_j}\Big(\frac{R_j}{r_j}\Big)^{k+2}A_{mk}^{(j,1)}+\\
&+kR_j\big((k-1)(\alpha_1+1)+2\big)\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,2)}+\\
&+\beta_1k(k-1)R_j\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,3)}+r_j^{k+1}\ell_{mk}^{(j,1)}+r_j^{k-1}\ell_{mk}^{(j,2)},\\
v_{mk}^{(1)}(r_j)&=\frac{1}{R_j}\Big(\frac{R_j}{r_j}\Big)^{k+2}A_{mk}^{(j,1)}-R_j\big(k(\alpha_1+1)-2\big)\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,2)}-\\
&-\beta_1kR_j\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,3)}+r_j^{k+1}\ell_{mk}^{(j,3)}+\frac{1}{k}\,r_j^{k-1}\ell_{mk}^{(j,2)},\\
w_{mk}^{(1)}(r_j)&=\Big(\frac{R_j}{r_j}\Big)^{k+1}A_{mk}^{(j,5)}+r_j^{k}\ell_{mk}^{(j,4)}, \qquad (3.15)\\
u_{mk}^{(2)}(r_j)&=-\frac{k+1}{R_j}\Big(\frac{R_j}{r_j}\Big)^{k+2}A_{mk}^{(j,4)}+\beta_2k(k-1)R_j\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,2)}+\\
&+kR_j\Big((k-1)(\alpha_2+1)+2\Big)\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,3)}+\\
&+r_j^{k+1}\ell_{mk}^{(j,5)}+r_j^{k-1}\ell_{mk}^{(j,6)},\\
v_{mk}^{(2)}(r_j)&=\frac{1}{R_j}\Big(\frac{R_j}{r_j}\Big)^{k+2}A_{mk}^{(j,4)}-\beta_2kR_j\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,2)}-\\
&-R_j\big(k(\alpha_2+1)-2\big)\Big(\frac{R_j}{r_j}\Big)^{k}A_{mk}^{(j,3)}+r_j^{k+1}\ell_{mk}^{(j,7)}+\frac{1}{k}\,r_j^{k-1}\ell_{mk}^{(j,6)},\\
w_{mk}^{(2)}(r_j)&=\Big(\frac{R_j}{r_j}\Big)^{k+1}A_{mk}^{(j,6)}+r_j^{k}\ell_{mk}^{(j,8)};
\end{aligned}$$

$$\begin{aligned}
\ell_{mk}^{(j,1)}&=-\alpha_1(k+2)B_{mk}^{(j,2)}-\beta_1(k+2)B_{mk}^{(j,3)}-\\
&-(k+2)D_{mk}^{(j,2)}+2H_{mk}^{(j,2)}-\sigma_{mk}^{(j,1)}(H_{mk}^{(j,2)}),\\
\ell_{mk}^{(j,2)}&=kB_{mk}^{(j,1)}-\alpha_1kC_{mk}^{(j,2)}-\beta_1kC_{mk}^{(j,3)}-kE_{mk}^{(j,2)}+\\
&+\sigma_{mk}^{(j,2)}(H_{mk}^{(j,2)})+\sigma_{mk}^{(j,4)}(B_{mk}^{(j,5)}),\\
\ell_{mk}^{(j,3)}&=-\alpha_1B_{mk}^{(j,2)}-\beta_1B_{mk}^{(j,3)}-D_{mk}^{(j,2)}+\frac{1}{k+1}\,\sigma_{mk}^{(j,1)}(H_{mk}^{(j,2)}),\\
\ell_{mk}^{(j,4)}&=B_{mk}^{(j,5)}+\sigma_{mk}^{(j,3)}(H_{mk}^{(j,2)})+\sigma_{mk}^{(j,5)}(B_{mk}^{(j,5)}), \qquad (3.16)\\
\ell_{mk}^{(j,5)}&=2H_{mk}^{(j,3)}-\beta_2(k+2)B_{mk}^{(j,2)}-\alpha_2(k+2)B_{mk}^{(j,3)}-
\end{aligned}$$

$$
\begin{aligned}
&\quad - (k+2)D_{mk}^{(j,3)} - \sigma_{mk}^{(j,1)}(H_{mk}^{(j,3)}),\\
\ell_{mk}^{(j,6)} &= kB_{mk}^{(j,4)} - k\beta_2 C_{mk}^{(j,2)} - k\alpha_2 C_{mk}^{(j,3)} - kE_{mk}^{(j,3)} +\\
&\quad + \sigma_{mk}^{(j,2)}(H_{mk}^{(j,3)}) + \sigma_{mk}^{(j,4)}(B_{mk}^{(j,6)}),\\
\ell_{mk}^{(j,7)} &= -\beta_2 B_{mk}^{(j,2)} - \alpha_2 B_{mk}^{(j,3)} - D_{mk}^{(j,3)} + \frac{1}{k+1}\,\sigma_{mk}^{(j,1)}(H_{mk}^{(j,3)}),\\
\ell_{mk}^{(j,8)} &= B_{mk}^{(j,6)} + \sigma_{mk}^{(j,3)}(H_{mk}^{(j,3)}) + \sigma_{mk}^{(j,5)}(B_{mk}^{(j,6)}).
\end{aligned}
$$

Let us consider Problem $(I)^-$. It will be assumed that in the system (1.1) the vector function $f^{(j,\ell)}(z)$, $\ell, j = 1,2$, is expanded into a series as follows:

$$
\begin{aligned}
f^{(j,\ell)}(z) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} \Big\{ \alpha_{mk}^{(j,\ell)} X_{mk}(\vartheta_j,\varphi_j) + \\
+ \sqrt{k(k+1)}\Big[\beta_{mk}^{(j,\ell)} Y_{mk}(\vartheta_j,\varphi_j) + \gamma_{mk}^{(j,\ell)} Z_{mk}(\vartheta_j,\varphi_j)\Big]\Big\},\quad z\in\partial\Omega_j. \qquad (3.17)
\end{aligned}
$$

If in both parts of the equality (3) we pass to limit as $x \to z \in \partial\Omega_j$ $(r_j \to R_j)$ and take into account the boundary conditions of the problem $(I)^-$ and the formulas (3.15) and (3.17), then for the unknown constants we obtain the following system of algebraic equations:

$$
\begin{aligned}
&-\frac{1}{R_j}\,A_{00}^{(j,1)} + R_j\ell_{00}^{(j,1)} + R_j\ell_{00}^{(j,2)} = \alpha_{00}^{(j,1)},\\
&-\frac{1}{R_j}\,A_{00}^{(j,4)} + R_j\,\ell_{00}^{(j,5)} + \frac{1}{R_j}\,\ell_{00}^{(j,6)} = \alpha_{00}^{(j,2)},\quad j=1,2;
\end{aligned} \qquad (3.18)
$$

$$
\begin{aligned}
&-\frac{k+1}{R_j}\,A_{mk}^{(j,1)} + kR_j\big((k-1)(\alpha_1+1)+2\big)A_{mk}^{(j,2)} + \beta_1 k(k-1)R_j A_{mk}^{(j,3)} = x_{mk}^{(j,1)},\\
&\frac{1}{R_j}\,A_{mk}^{(j,1)} - R_j\big(k(\alpha_1+1)-2\big)A_{mk}^{(j,2)} - \beta_1 kR_j A_{mk}^{(j,3)} = x_{mk}^{(j,2)},\\
&-\frac{k+1}{R_j}\,A_{mk}^{(j,4)} + \beta_2 k(k-1)R_j A_{mk}^{(j,2)} +\\
&\quad + kR_j\big((k-1)(\alpha_2+1)+2\big)A_{mk}^{(j,3)} = x_{mk}^{(j,3)}, \qquad (3.19)\\
&\frac{1}{R_j}\,A_{mk}^{(j,4)} - \beta_2 kR_j A_{mk}^{(j,2)} - R_j\big(k(\alpha_2+1)-2\big)A_{mk}^{(j,3)} = x_{mk}^{(j,4)},\\
&A_{mk}^{(j,5)} = x_{mk}^{(j,5)},\quad A_{mk}^{(j,6)} = x_{mk}^{(j,6)},\quad k\ge 1,
\end{aligned}
$$

where

$$
\begin{aligned}
x_{mk}^{(j,1)} &= \alpha_{mk}^{(j,1)} - R_j^{k+1}\ell_{mk}^{(j,1)} - R_j^{k-1}\ell_{mk}^{(j,2)},\\
x_{mk}^{(j,2)} &= \beta_{mk}^{(j,1)} - R_j^{k+1}\ell_{mk}^{(j,3)} - \frac{1}{k}\,R_j^{k-1}\ell_{mk}^{(j,2)},\\
x_{mk}^{(j,3)} &= \alpha_{mk}^{(j,2)} - R_j^{k+1}\ell_{mk}^{(j,5)} - R_j^{k-1}\ell_{mk}^{(j,6)}, \qquad (3.20)
\end{aligned}
$$

$$x^{(j,4)}_{mk}=\beta^{(j,2)}_{mk}-R^{k+1}_j\ell^{(j,7)}_{mk}-\frac{1}{k}\,R^{k-1}_j\ell^{(j,6)}_{mk},$$
$$x^{(j,5)}_{mk}=\gamma^{(j,1)}_{mk}-R^k_j\ell^{(j,4)}_{mk},\quad x^{(j,6)}_{mk}=\gamma^{(j,2)}_{mk}-R^k_j\ell^{(j,8)}_{mk},\ \ k\geq 1,\ \ j=1,2.$$

If in the formulas (3) we use the notation of (3.8) and (3.13), then we have

$$\begin{aligned}
\ell^{(j,1)}_{mk}&=\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R^{p+1}_q\Big[C^{(j,1)}_{smpk}A^{(q,2)}_{sp}-\beta_1(k+2)pa^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)A^{(q,3)}_{sp}\Big],\\
\ell^{(j,2)}_{mk}&=\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R^{p+1}_q\Big[kG^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)A^{(q,1)}_{sp}+C^{(j,2)}_{smpk}A^{(q,2)}_{sp}-\\
&\quad-\beta_1kpb^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)A^{(q,3)}_{sp}+\sigma^{(j,4)}_{mk}(G^{(m,s)}_{kp,q})A^{(q,5)}_{sp}\Big]+\\
&\quad+kR_qG^{(m,0)}_{k0,q}(\vartheta_0,\varphi_0)A^{(q,1)}_{00},\\
\ell^{(j,3)}_{mk}&=\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R^{p+1}_q\Big[C^{(j,3)}_{smpk}A^{(q,2)}_{sp}-\beta_1pa^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)A^{(q,3)}_{sp}\Big],\\
\ell^{(j,4)}_{mk}&=\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R^{p+1}_q\Big[(2p-1)\sigma^{(j,3)}_{mk}(G^{(m,s)}_{kp,q})A^{(q,2)}_{sp}+C^{(j,4)}_{smpk}A^{(q,5)}_{sp}\Big],\\
\ell^{(j,5)}_{mk}&=\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R^{p+1}_q\Big[-\beta_2(k+2)pa^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)A^{(q,2)}_{sp}+C^{(j,5)}_{smpk}A^{(q,3)}_{sp}\Big],\quad (3.21)\\
\ell^{(j,6)}_{mk}&=\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R^{p+1}_q\Big[-\beta_2pkb^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)A^{(q,2)}_{sp}+C^{(j,6)}_{smpk}A^{(q,3)}_{sp}+\\
&\quad+kG^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)A^{(q,4)}_{sp}+\sigma^{(j,4)}_{mk}(G^{(m,s)}_{kp,q})A^{(q,6)}_{sp}\Big]+\\
&\quad+kR_qG^{(m,0)}_{k0,q}(\vartheta_0,\varphi_0)A^{(q,4)}_{00},\\
\ell^{(j,7)}_{mk}&=\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R^{p+1}_q\Big[-\beta_2pa^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)A^{(q,2)}_{sp}+C^{(j,7)}_{smpk}A^{(q,3)}_{sp}\Big],\\
\ell^{(j,8)}_{mk}&=\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R^{p+1}_q\Big[(2p-1)\sigma^{(j,3)}_{mk}(G^{(m,s)}_{kp,q})A^{(q,3)}_{sp}+C^{(j,8)}_{smpk}A^{(q,6)}_{sp}\Big],
\end{aligned}$$

where

$$\begin{aligned}
C^{(j,1)}_{smpk}&=-(k+2)\big[p(\alpha_1+1)-2\big]a^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)+\\
&\quad+2(2p-1)G^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)+(2p-1)\sigma^{(j,1)}_{mk}(G^{(m,s)}_{kp,q}),\\
C^{(j,2)}_{smpk}&=-k\big[p(\alpha_1+1)-2\big]b^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)+(2p-1)\sigma^{(j,2)}_{mk}(G^{(m,s)}_{kp,q}),\\
C^{(j,3)}_{smpk}&=-\big[p(\alpha_1+1)-2\big]a^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)+\frac{2p-1}{k+1}\,\sigma^{(j,1)}_{mk}(G^{(m,s)}_{kp,q}),\\
C^{(j,4)}_{smpk}&=G^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)+\sigma^{(j,5)}_{mk}(G^{(m,s)}_{kp,q}),
\end{aligned}$$

$$C_{smpk}^{(j,5)} = 2(2p-1)G_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)-$$
$$-(k+2)\big[p(\alpha_2+1)-2\big]a_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)-(2p-1)\sigma_{mk}^{(j,1)}(G_{kp,q}^{(m,s)}),$$
$$C_{smpk}^{(j,6)} = -k\big[p(\alpha_2+1)-2\big]b_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)+(2p-1)\sigma_{mk}^{(j,2)}(G_{kp,q}^{(m,s)}),$$
$$C_{smpk}^{(j,7)} = -\big[p(\alpha_2+1)-2\big]a_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)+\frac{2p-1}{k+1}\,\sigma_{mk}^{(j,1)}(G_{kp,q}^{(m,s)}),$$
$$C_{smpk}^{(j,8)} = G_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)+\sigma_{mk}^{(j,5)}(G_{kp,q}^{(m,s)}),\quad j\neq q=1,2.$$

With (3.22) taken into account, the system (3.18) implies

$$\begin{gathered}A_{00}^{(j,1)} = -R_j\alpha_{00}^{(j,1)}+\\ +\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R_j^2R_q^{p+1}\Big[C_{s0p0}^{(j,1)}A_{sp}^{(q,2)}-2\beta_1pa_{0p,q}^{(0,s)}(\vartheta_0,\varphi_0)A_{sp}^{(q,3)}\Big],\\ A_{00}^{(j,4)} = -R_j\alpha_{00}^{(j,2)}+\\ +\sum_{p=1}^{\infty}\sum_{s=-p}^{p}R_j^2R_q^{p+1}\Big[-2\beta_2pa_{0p,q}^{(0,s)}(\vartheta_0,\varphi_0)A_{sp}^{(q,2)}+C_{s0p0}^{(j,5)}A_{sp}^{(q,3)}\Big].\end{gathered}\tag{3.22}$$

A solution of the system (3) has the form

$$A_{mk} = H(k)x_{mk},\tag{3.23}$$

where

$$\begin{gathered}A_{mk} = \big[A_{mk}^{(1,1)},\dots,A_{mk}^{(1,6)},A_{mk}^{(2,1)},\dots,A_{mk}^{(2,6)}\big]^\top,\\ x_{mk} = \big[x_{mk}^{(1,1)},\dots,x_{mk}^{(1,6)},x_{mk}^{(2,1)},\dots,x_{mk}^{(2,6)}\big]^\top,\end{gathered}\tag{3.24}$$

$$H(k) = \begin{bmatrix} H^{(1)}(k) & \vdots & 0\\ \cdots & \cdots & \cdots\\ 0 & \vdots & H^{(2)}(k)\end{bmatrix}_{12\times 12},\quad H^{(j)}(k) = \big[h_{i\ell}^{(j)}(k)\big]_{6\times 6},\quad j=1,2,$$

$$h_{11}^{(j)}(k) = \frac{R_j}{2\Delta(k)}\big[-\Delta(k)+4(2k-1)(\alpha_2k-k-1)\big],$$
$$h_{12}^{(j)}(k) = \frac{2R_jk}{\Delta(k)}\Big[\frac{k(k-1)d_2}{d_1}+k(k+1)\alpha_2-(k^2-1)\alpha_1-(k+1)^2\Big],$$
$$h_{13}^{(j)}(k) = -\frac{2\beta_1R_jk(2k-1)}{\Delta(k)},\quad h_{14}^{(j)}(k) = -\frac{2\beta_1R_jk(2k^2+k-1)}{\Delta(k)},$$
$$h_{1\ell}^{(j)}(k) = 0,\quad \ell=5,6,$$
$$h_{21}^{(j)}(k) = \frac{2h_{11}^{(j)}(k)+R_j}{2(2k-1)R_j^2},\quad h_{22}^{(j)}(k) = \frac{2h_{12}^{(j)}(k)+R_j(k-1)}{2(2k-1)R_j^2},$$
$$h_{23}^{(j)}(k)=\frac{h_{13}^{(j)}(k)}{(2k-1)R_j^2},\quad h_{24}^{(j)}(k)=\frac{h_{14}^{(j)}(k)}{(2k-1)R_j^2},\quad h_{2\ell}^{(j)}(k)=0,\quad \ell=5,6,$$

$$h_{31}^{(j)}(k)=-\frac{1}{2\beta_1R_j^2k(2k-1)}\Big[2\big(k(\alpha_1+1)-2k-1\big)h_{11}^{(j)}(k)+R_j\big(k(\alpha_1+1)-2\big)\Big],$$

$$h_{32}^{(j)}(k)=-\frac{1}{2\beta_1R_j^2k(2k-1)}\Big[2\big(k(\alpha_1+1)-2k-1\big)h_{12}^{(j)}(k)+ +R_j\big((k-1)(\alpha_1+1)+2\big)\Big],$$

$$h_{3\ell}^{(j)}(k)=-\frac{k(\alpha_1+1)-2k-1}{\beta_1R_j^2k(2k-1)}\,h_{1\ell}^{(j)}(k),\ \ \ell=3,4,\quad h_{3\ell}^{(j)}(k)=0,\ \ \ell=5,6,$$

$$h_{41}^{(j)}(k)=-\frac{2\beta_2R_jk(2k-1)}{\Delta(k)},\quad h_{42}^{(j)}(k)=-\frac{2\beta_2R_jk(2k^2-k+1)}{\Delta(k)},$$

$$h_{43}^{(j)}(k)=\frac{R_j}{2\Delta(k)}\big[-\Delta(k)+4(2k-1)(k\alpha_1-k-1)\big],$$

$$h_{44}^{(j)}(k)=\frac{2R_jk}{\Delta(k)}\Big[\frac{k(k-1)d_2}{d_1}+k(k+1)\alpha_1-(k^2-1)\alpha_2-(k+1)^2\Big],$$

$$h_{4\ell}^{(j)}(k)=0,\ \ \ell=5,6,\quad h_{\ell\ell}^{(j)}(k)=1,\ \ \ell=5,6,$$

$$h_{5\ell}^{(j)}(k)=0,\ \ \ell=1,2,3,4,6,\quad h_{6\ell}^{(j)}(k)=0,\ \ \ell=1,2,\dots,5,$$

$$\Delta(k)=-4\Big[(\alpha_1\alpha_2-\beta_1\beta_2)k^2-(\alpha_1+\alpha_2)k(k+1)+(k+1)^2\Big].$$

Since $\alpha_1\alpha_2-\beta_1\beta_2=\frac{d_2}{d_1}>0$, $\alpha_1+\alpha_2<0$, we have $\Delta(k)<0$.

If the values $A_{00}^{(j,\ell)}$, $\ell=1,4$ from (3.22) are inserted into the expression $\ell_{mk}^{(j,\ell)}$, $\ell=2,6$, contained in (3.21), then we obtain

$$\begin{aligned}
\ell_{mk}^{(j,2)}=&-kR_q^2G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0)\alpha_{00}^{(q,1)}+\\
&+\sum_{p=1}^{\infty}\sum_{s=-p}^{p}\Big[kR_q^{p+1}G_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)A_{sp}^{(q,1)}+R_q^{p+1}C_{smpk}^{(j,2)}A_{sp}^{(q,2)}-\\
&-\beta_1kpR_q^{p+1}b_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)A_{sp}^{(q,3)}+R_q^{p+1}\sigma_{mk}^{(j,4)}(G_{kp,q}^{(m,s)})A_{sp}^{(q,5)}+\\
&+kR_q^3R_j^{p+1}G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0)C_{s0p0}^{(j,1)}A_{sp}^{(j,2)}-\\
&-2\beta_1kpR_q^3R_j^{p+1}G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0)a_{0p,j}^{(0,s)}(\vartheta_0,\varphi_0)A_{sp}^{(j,3)}\Big],\\
\ell_{mk}^{(j,6)}=&-kR_q^2G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0)\alpha_{00}^{(q,2)}+\\
&+\sum_{p=1}^{\infty}\sum_{s=-p}^{p}\Big[kR_q^{p+1}G_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)A_{sp}^{(q,4)}-\\
&-\beta_2pkR_q^{p+1}b_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)A_{sp}^{(q,2)}+R_q^{p+1}C_{smpk}^{(j,6)}A_{sp}^{(q,3)}+\\
&+R_q^{p+1}\sigma_{mk}^{(j,4)}(G_{kp,q}^{(m,s)})A_{sp}^{(q,6)}-\\
&-2\beta_2pkR_q^3R_j^{p+1}G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0)a_{0p,j}^{(0,s)}(\vartheta_0,\varphi_0)A_{sp}^{(j,2)}+\\
&+kR_q^3R_j^{p+1}G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0)C_{s0p0}^{(j,5)}A_{sp}^{(j,3)}\Big].
\end{aligned}\tag{3.25}$$

Substituting the values $\ell_{mk}^{(j,\ell)}$, $j = 1,2$, $\ell = 1,2,\dots,8$, from (3.21) and (3.24) into (3), we have

$$x_{mk} = \widetilde{\alpha}_{mk} + \sum_{p=1}^{\infty} \sum_{s=-p}^{p} N(s,m,p,k) A_{sp}, \tag{3.26}$$

where A_{sp}, x_{mk} have the form (3.24), and

$$N(s,m,p,k) = \begin{bmatrix} N^{(1)}(s,m,p,k) & \vdots & \widetilde{N}^{(2)}(s,m,p,k) \\ \cdots & \cdots & \cdots \\ \widetilde{N}^{(1)}(s,m,p,k) & \vdots & N^{(2)}(s,m,p,k) \end{bmatrix}_{12\times 12},$$

$$N^{(\ell)}(s,m,p,k) = \big[N_{ij}^{(\ell)}(s,m,p,k)\big]_{6\times 6}, \quad \ell = 1,2,$$

$$\widetilde{N}^{(\ell)}(s,m,p,k) = \big[\widetilde{N}_{ij}^{(\ell)}(s,m,p,k)\big]_{6\times 6}, \quad \ell = 1,2,$$

$$\widetilde{\alpha}_{mk} = \Big[\widetilde{\alpha}_{mk}^{(1,1)}, \dots, \widetilde{\alpha}_{mk}^{(1,6)}, \widetilde{\alpha}_{mk}^{(2,1)}, \dots, \widetilde{\alpha}_{mk}^{(1,6)}\Big]^{\top},$$

$$\begin{gathered} \widetilde{\alpha}_{mk}^{(j,2\ell-1)} = \alpha_{mk}^{(j,\ell)} + kR_q^2 R_j^{k-1} G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0)\alpha_{00}^{(q,\ell)}, \\ \widetilde{\alpha}_{mk}^{(j,2\ell)} = \beta_{mk}^{(j,\ell)} + R_q^2 R_j^{k-1} G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0)\alpha_{00}^{(q,\ell)}, \\ \widetilde{\alpha}_{mk}^{(j,5)} = \gamma_{mk}^{(j,1)}, \quad \widetilde{\alpha}_{mk}^{(j,6)} = \gamma_{mk}^{(j,2)}, \quad j \neq q = 1,2, \quad \ell = 1,2; \end{gathered} \tag{3.27}$$

$$\begin{aligned} N_{12}^{(j)}(s,m,p,k) &= -kR_q^3 R_j^{k+p} G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0) C_{s0p0}^{(j,1)}, \\ N_{13}^{(j)}(s,m,p,k) &= 2\beta_1 kpR_q^3 R_j^{k+p} G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0) a_{0p,j}^{(0,s)}(\vartheta_0,\varphi_0), \\ N_{32}^{(j)}(s,m,p,k) &= 2\beta_2 kpR_q^3 R_j^{k+p} G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0) a_{0p,j}^{(0,s)}(\vartheta_0,\varphi_0), \\ N_{33}^{(j)}(s,m,p,k) &= -kR_q^3 R_j^{k+p} G_{k0,q}^{(m,0)}(\vartheta_0,\varphi_0) C_{s0p0}^{(j,5)}, \\ N_{i\ell}^{(j)}(s,m,p,k) &= \frac{1}{k}\, N_{i-1\ell}^{(j)}(s,m,p,k), \quad i = 2,4, \quad \ell = 2,3, \\ N_{i\ell}^{(j)}(s,m,p,k) &= 0, \quad i = 1,2,3,4, \quad \ell = 1,4,5,6, \\ N_{i\ell}^{(j)}(s,m,p,k) &= 0, \quad i = 5,6, \quad \ell = 1,2,\dots,6, \quad j \neq q = 1,2; \end{aligned} \tag{3.28}$$

$$\begin{aligned} \widetilde{N}_{11}^{(j)}(s,m,p,k) &= -kR_j^{k-1} R_q^{p+1} G_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0), \\ \widetilde{N}_{12}^{(j)}(s,m,p,k) &= -R_j^{k-1} R_q^{p+1} \big[R_j^2 C_{smpk}^{(j,1)} + C_{smpk}^{(j,2)}\big], \\ \widetilde{N}_{13}^{(j)}(s,m,p,k) &= \beta_1 p R_j^{k-1} R_q^{p+1} \times \\ &\quad \times \big[R_j^2 (k+2) a_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0) + k b_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)\big], \\ \widetilde{N}_{15}^{(j)}(s,m,p,k) &= -R_j^{k-1} R_q^{p+1} \sigma_{mk}^{(j,4)}(G_{kp,q}^{(m,s)}), \\ \widetilde{N}_{1\ell}^{(j)}(s,m,p,k) &= 0, \quad \ell = 4,6, \\ \widetilde{N}_{21}^{(j)}(s,m,p,k) &= -R_j^{k-1} R_q^{p+1} G_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0), \end{aligned}$$

$$\widetilde{N}_{22}^{(j)}(s,m,p,k) = -R_j^{k-1}R_q^{p+1}\Big[R_j^2C_{smpk}^{(j,3)} + \frac{1}{k}\,C_{smpk}^{(j,2)}\Big],$$
$$\widetilde{N}_{23}^{(j)}(s,m,p,k) = \beta_1 pR_j^{k-1}R_q^{p+1}\big[R_j^2a_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0) + b_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)\big],$$
$$\widetilde{N}_{25}^{(j)}(s,m,p,k) = -\frac{1}{k}\,R_j^{k-1}R_q^{p+1}\sigma_{mk}^{(j,4)}(G_{kp,q}^{(m,s)}),$$
$$\widetilde{N}_{2\ell}^{(j)}(s,m,p,k) = 0, \quad \ell = 4,6,$$
$$\widetilde{N}_{32}^{(j)}(s,m,p,k) = \beta_2 pR_j^{k-1}R_q^{p+1}\times$$
$$\times\big[(k+2)R_j^2a_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0) + kb_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)\big],$$
$$\widetilde{N}_{33}^{(j)}(s,m,p,k) = -R_j^{k-1}R_q^{p+1}\big[R_j^2C_{smpk}^{(j,5)} + C_{smpk}^{(j,6)}\big],$$
$$\widetilde{N}_{34}^{(j)}(s,m,p,k) = -R_j^{k-1}R_q^{p+1}G_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0),$$
$$\widetilde{N}_{36}^{(j)}(s,m,p,k) = -R_j^{k-1}R_q^{p+1}\sigma_{mk}^{(j,4)}(G_{kp,q}^{(m,s)}),$$
$$\widetilde{N}_{3\ell}^{(j)}(s,m,p,k) = 0, \quad \ell = 1,5,$$
$$\widetilde{N}_{42}^{(j)}(s,m,p,k) = \beta_2 pR_j^{k-1}R_q^{p+1}\big[R_j^2a_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0) + b_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0)\big],$$
$$\widetilde{N}_{43}^{(j)}(s,m,p,k) = -R_j^{k-1}R_q^{p+1}\Big[R_j^2C_{smpk}^{(j,7)} + \frac{1}{k}\,C_{smpk}^{(j,6)}\Big],$$
$$\widetilde{N}_{44}^{(j)}(s,m,p,k) = -R_j^{k-1}R_q^{p+1}G_{kp,q}^{(m,s)}(\vartheta_0,\varphi_0),$$
$$\widetilde{N}_{46}^{(j)}(s,m,p,k) = -\frac{1}{k}\,R_j^{k-1}R_q^{p+1}\sigma_{mk}^{(j,4)}(G_{kp,q}^{(m,s)}),$$
$$\widetilde{N}_{4\ell}^{(j)}(s,m,p,k) = 0, \quad \ell = 1,5,$$
$$\widetilde{N}_{52}^{(j)}(s,m,p,k) = -(2p-1)R_j^kR_q^{p+1}\sigma_{mk}^{(j,3)}(G_{kp,q}^{(m,s)}), \tag{3.29}$$
$$\widetilde{N}_{55}^{(j)}(s,m,p,k) = -R_j^kR_q^{p+1}C_{smpk}^{(j,4)},$$
$$\widetilde{N}_{5\ell}^{(j)}(s,m,p,k) = 0, \quad \ell = 1,3,4,6,$$
$$\widetilde{N}_{63}^{(j)}(s,m,p,k) = -(2p-1)R_j^kR_q^{p+1}\sigma_{mk}^{(j,3)}(G_{kp,q}^{(m,s)}),$$
$$\widetilde{N}_{65}^{(j)}(s,m,p,k) = -R_j^kR_q^{p+1}C_{smpk}^{(j,4)},$$
$$\widetilde{N}_{6\ell}^{(j)}(s,m,p,k) = 0, \quad \ell = 1,2,4,6, \quad j \neq q = 1,2.$$

If the values of the vector x_{mk} from (3.26) are inserted into (3.23), then for the unknown constants we obtain the following infinite system of linear algebraic equations:

$$A_{mk} = \alpha_{mk} + \sum_{p=1}^{\infty}\sum_{s=-p}^{p} L_{smpk}A_{sp}, \quad k \geq 1, \tag{3.30}$$

where A_{mk} has the form (3.24),

$$\alpha_{mk} = H(k)\widetilde{\alpha}_{mk}, \quad L_{smpk} = H(k)N(s,m,p,k),$$

$$L_{smpk} = \begin{bmatrix} L^{(1)}_{smpk} & \vdots & L^{(2)}_{smpk} \\ \cdots & \cdots & \cdots \\ L^{(3)}_{smpk} & \vdots & L^{(4)}_{smpk} \end{bmatrix},$$

$$\begin{aligned} &L^{(1)}_{smpk} = H^{(1)}(k)N^{(1)}(s,m,p,k), \quad L^{(2)}_{smpk} = H^{(1)}(k)\widetilde{N}^{(2)}(s,m,p,k), \\ &L^{(3)}_{smpk} = H^{(2)}(k)\widetilde{N}^{(1)}(s,m,p,k), \quad L^{(4)}_{smpk} = H^{(2)}(k)N^{(2)}(s,m,p,k). \end{aligned} \tag{3.31}$$

Let us investigate the system (3.30). For this we need estimates of the elements of the matrix L_{smpk} with respect to p and k.

The following estimate is valid [8]:

$$\Big[\frac{(p+k+m-s)!(p+k-m+s)!}{(k+m)!(k-m)!(p+s)!(p-s)!}\Big]^{1/2} \le \frac{(p+k)!}{p!k!}, \quad k \ge 1, \quad p \ge 1. \tag{3.32}$$

Taking the inequalities (1.11) and (3.32) into account, we obtain

$$\begin{aligned} \big|G^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)\big| &\le \frac{1}{d^{k+p+1}}\sqrt{\frac{2p+1}{2k+1}}\,\frac{(p+k)!}{p!k!}, \\ \big|a^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)\big| &< \frac{3}{d^{k+p+1}}\sqrt{\frac{2p+1}{2k+1}}\,\frac{(p+k+1)!}{p!(k+1)!}, \\ \big|b^{(m,s)}_{kp,q}(\vartheta_0,\varphi_0)\big| &< \frac{3}{d^{k+p-1}}\sqrt{\frac{2p+1}{2k+1}}\,\frac{(p+k)!}{p!k!}. \end{aligned} \tag{3.33}$$

Using the inequalities (3.33), from (3.31), with the formulas (3.28) and (3.29) taken into account, we obtain the following estimates:

$$\begin{aligned} |L^{(1,\ell j)}_{smpk}| &\le \alpha k(p+1)^2\Big(\frac{R_1}{d}\Big)^{k+p}\sqrt{\frac{2p+1}{2k+1}}, \\ |L^{(2,\ell j)}_{smpk}| &\le \beta(p+1)(k+1)\Big(\frac{R_1}{d}\Big)^{k}\Big(\frac{R_2}{d}\Big)^{p+1}\times \\ &\quad\times\frac{(k+p+1)!}{p!(k+1)!}\sqrt{\frac{2p+1}{2k+1}}, \\ |L^{(3,\ell j)}_{smpk}| &\le \gamma(p+1)(k+1)\Big(\frac{R_2}{d}\Big)^{k}\Big(\frac{R_1}{d}\Big)^{p+1}\times \\ &\quad\times\frac{(k+p+1)!}{p!(k+1)!}\sqrt{\frac{2p+1}{2k+1}}, \\ |L^{(4,\ell j)}_{smpk}| &\le \delta k(p+1)^2\Big(\frac{R_2}{d}\Big)^{k+p}\sqrt{\frac{2p+1}{2k+1}}, \\ &\qquad \ell,j = 1,2,\dots,6, \quad k \ge 1, \quad p \ge 1, \end{aligned} \tag{3.34}$$

where α, β, γ, δ are positive constants not depending on p and k.

Let us show that the system (3.30) is quasiregular, which means that the regularity condition is fulfilled only in the rows, starting from a certain one, i.e., [9]

$$\sum_{p=1}^{\infty}\sum_{s=-p}^{p}\sum_{j=1}^{12}|L^{(\ell j)}_{smpk}|<1,\quad k=N+1,N+2,\dots,\quad \ell=1,2,\dots,12, \tag{3.35}$$

and, moreover,

$$\sum_{p=1}^{\infty}\sum_{s=-p}^{p}\sum_{j=1}^{12}|L^{(\ell j)}_{smpk}|<+\infty,\quad k=1,2,\dots,N,\quad \ell=1,2,\dots,12, \tag{3.36}$$

$$|\alpha_{mk,\ell}|\le M\Big(1-\sum_{p=N+1}^{\infty}\sum_{s=-p}^{p}\sum_{j=1}^{12}|L^{(\ell j)}_{smpk}|\Big), \tag{3.37}$$
$$k=N+1,N+2,\dots,\quad \ell=1,2,\dots,12,$$

where $M>0$ is a constant.

We first prove the existence of a natural number N such that for $k>N$ we have the following inequality:

$$\sum_{p=1}^{\infty}\sum_{s=-p}^{p}\sum_{j=1}^{12}|L^{(\ell j)}_{smpk}|<1-\rho, \tag{3.38}$$
$$k=N+1,N+2,\dots,\quad \ell=1,2,\dots,12,$$

where $0<\varepsilon'<\rho<\varepsilon<1$.

Taking into account the inequalities (3.34) and the identity [5]

$$\sum_{p=0}^{\infty}\frac{(k+p)!}{k!p!}\,t^p=\Big(\frac{1}{1-t}\Big)^{k+1},\quad 0<t<1,$$

we obtain

$$\sum_{p=1}^{\infty}\sum_{s=-p}^{p}\sum_{j=1}^{12}|L^{(\ell j)}_{smpk}|<\sigma_1k\Big(\frac{R_1}{d}\Big)^k+\sigma_2k\Big(\frac{R_2}{d}\Big)^k+$$
$$+\sigma_3(k+1)^4\Big(\frac{d}{d-R_2}\Big)^5\Big(\frac{R_1}{d-R_2}\Big)^k+\sigma_4(k+1)^4\Big(\frac{d}{d-R_1}\Big)^5\Big(\frac{R_2}{d-R_1}\Big)^k<$$
$$<\sigma k^4\Big(\frac{R_1}{d-R_2}\Big)^k,\quad k\ge 1, \tag{3.39}$$

where σ, σ_ℓ, $\ell=1,2,3,4$, are positive constants not depending on k.

The following lemma is true [3].

Lemma 3.1. *If* $0<\alpha=\mathrm{const}<1$, $c=\mathrm{const}>0$, m *and* k *are natural numbers, then the inequality* $k^ma^k<c$ *is valid for any*

$$k>N=\max\Big\{2,\big[4a^{\frac{2}{m}}c^{-\frac{1}{m}}(1-a^{1/m})^{-2}\big]\Big\},$$

where $[\circ]$ *is the integer part of the number enclosed in the brackets.*

By virtue of this lemma we conclude that the inequality

$$\sigma k^4\Big(\frac{R_1}{d-R_2}\Big)^k < 1-\rho$$

holds for any $k > N$ where

$$N=\max\Big\{2,\Big[4\sigma^{\frac{1}{4}}(1-\rho)^{-\frac{1}{4}}R_1^{\frac{1}{2}}\big(\sqrt[4]{d-R_2}-\sqrt[4]{R_1}\big)^{-2}\Big]\Big\}.$$

Thus we have proved the inequality (3.38) and, along with it, the inequality (3.35).

From the inequality (3.39) it follows that

$$\sum_{p=1}^{\infty}\sum_{s=-p}^{p}\sum_{j=1}^{12}|L_{smpk}^{(\ell j)}| < \sigma_1 k^4 \le \sigma N^4 < +\infty, \quad k=1,2,\dots,N, \quad \ell=1,2,\dots,12.$$

The inequality (3.36) is thereby proved.

Let us choose a constant M such that

$$M=\frac{1}{\varepsilon'}\max|\alpha_{mk,\ell}|, \quad k\ge 1, \quad \ell=1,2,\dots,12.$$

Hence, with (3.38) taken into account, it follows that

$$|\alpha_{mk,\ell}| \le \max_{k\ge 1}|\alpha_{mk,\ell}| = M\varepsilon' < M\rho <$$
$$< M\Big(1-\sum_{p=1}^{\infty}\sum_{s=-p}^{p}\sum_{j=1}^{12}|L_{smpk}^{(\ell j)}|\Big) < M\Big(1-\sum_{p=N+1}^{\infty}\sum_{s=-p}^{p}\sum_{j=1}^{12}|L_{smpk}^{(\ell j)}|\Big), \tag{3.40}$$
$$k=N+1,N+2,\dots, \quad \ell=1,2,\dots,12.$$

The inequality (3.37) is thereby proved.

Thus we have proved that the system (3.30) is quasiregular. As is known [9], the problem of the existence of a solution of such a system reduces to the problem of the existence of a solution of a finite system.

From the system (3) it follows that for $k\to\infty$

$$|x_{mk}^{(j,1)}| \sim |\alpha_{mk}^{(j,1)}|, \quad |x_{mk}^{(j,2)}| \sim |\beta_{mk}^{(j,1)}|,$$
$$|x_{mk}^{(j,3)}| \sim |\alpha_{mk}^{(j,2)}|, \quad |x_{mk}^{(j,4)}| \sim |\beta_{mk}^{(j,2)}|,$$
$$|x_{mk}^{(j,5)}| \sim |\gamma_{mk}^{(j,1)}|, \quad |x_{mk}^{(j,6)}| \sim |\gamma_{mk}^{(j,2)}|, \quad j=1,2.$$

The series (3) containing the constants $B_{mk}^{(j,\ell)}$, $\ell=1,2,\dots,6$, $C_{mk}^{(j,\ell)}$, $D_{mk}^{(j,\ell)}$, $E_{mk}^{(j,\ell)}$, $H_{mk}^{(j,\ell)}$, $\ell=2,3$, converge irrespective of the fact whether $x\in\Omega^-$ or $x\in\partial\Omega_j$, $j=1,2$. The regularity of the vector $U=(u',u'')$ depends on the convergence of the series (3) and their derivatives of first order containing the constants $A_{mk}^{(j,\ell)}$, $j=1,2$, $\ell=1,2,\dots,6$. If $x\in\Omega^-$, then $r_j>R_j$, $j=1,2$, and the above-mentioned series converge absolutely and uniformly in Ω^-. If $x\in\partial\Omega_j$, then, in view of the estimates (1.10), (1.11), for the convergence of the series (3) and their first degree derivatives it is sufficient that the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j=1,2$, admit the following estimates for $k\to\infty$

$$\alpha_{mk}^{(j)}=O(k^{-4}),\quad \beta_{mk}^{(j)}=O(k^{-5}),\quad \gamma_{mk}^{(j)}=O(k^{-5}),\quad j=1,2. \tag{3.41}$$

From Theorem 1.1 it follows that the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j=1,2$, admit the estimates (3.41) if the vector function $f^{(j)}(z)$ belongs to the class $f^{(j)}(z)\in C^4(\partial\Omega_j)$, $j=1,2$.

Note that since the coefficients $\alpha_{mk}^{(j)}$, $\beta_{mk}^{(j)}$, $\gamma_{mk}^{(j)}$, $j=1,2$, tend to zero for $k\to\infty$, we can always choose a constant $M>0$ such that the inequality (3.40) be fulfilled.

With (3.5) taken into account, from (3.4) it follows that for $|x|\to\infty$ we have

$$\begin{gathered} u'(x)=O(|x|^{-1}),\quad u''(x)=O(|x|^{-1}),\\ \frac{\partial u_j'(x)}{\partial x_k}=O(|x|^{-2}),\quad \frac{\partial u_j''(x)}{\partial x_k}=O(|x|^{-2}),\quad k,j=1,2,3. \end{gathered} \tag{3.42}$$

Problem $(I)^-$ is solved.

Problems $(II)^-$ and $(III)^-$ are solved analogously.

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