Memoirs on Differential Equations and Mathematical Physics
Volume 44, 2008, 89–104

Diabate Nabongo and Théodore K. Boni

NUMERICAL QUENCHING FOR A SEMILINEAR PARABOLIC EQUATION

Abstract. This paper concerns the study of the numerical approximation for the following boundary value problem:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = -u^{-p}(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = 1, \quad u(1,t) = 1, & t > 0, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \end{cases}$$

where $p > 0$. We obtain some conditions under which the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time and construct two discrete forms of the above problem which allow us to obtain some lower bounds of the numerical quenching time. Finally, we give some numerical experiments to illustrate our theoretical analysis.

2000 Mathematics Subject Classification. 5K55, 35B40, 65M06.
Key words and phrases. Semidiscretization, discretization, semilinear parabolic equation, semidiscrete quenching time, convergence.

რეზიუმე. ნაშრომში შესწავლილია

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = -u^{-p}(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = 1, \quad u(1,t) = 1, & t > 0, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \end{cases}$$

სასაზღვრო ამოცანის, სადაც $p > 0$, რიცხვითი მიახლოების საკითხი. მიღებულია პირობები, რომლებიც შესრულებისას ამ ამოცანის ნახევრადდისკრეტული ფორმის ამონახსნი სასრული დროის განმავლობაში ქრება და შეფასებულია მისი ნახევრადდისკრეტული ჩაქრობის დრო. დადგენილია ნახევრადდისკრეტული ჩაქრობის დროის კრებადობა და აგებულია ზემოხსენებული ამოცანის ორი დისკრეტული ფორმა, რომლებიც საშუალებას იძლევა მიღებული იქნეს რიცხვითი ჩაქრობის დროის ქვედა საზღვრები. თეორიული ანალიზის საილუსტრაციოდ მოყვანილია რამდენიმე რიცხვითი ექსპერიმენტი.

1. Introduction

Consider the following boundary value problem:

$$u_t(x,t) - u_{xx}(x,t) = -u^{-p}(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u(0,t) = 1, \quad u(1,t) = 1, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{3}$$

where $p > 0$, $u_0 \in C^0([0,1])$, $u_0(0) = 1$, $u_0(1) = 1$, $u_0(x) < 1$ for $x \in (0,1)$.

Definition 1.1. We say that the solution u of (1)–(3) quenches in a finite time if there exists a finite time T_q such that $\|u(x,t)\|_{\inf} > 0$ for $t \in [0, T_q)$ but

$$\lim_{t \to T_q} \|u(x,t)\|_{\inf} = 0,$$

where $\|u(x,t)\|_{\inf} = \min_{0 \le x \le 1} u(x,t)$. The time T_q is called the quenching time of the solution u.

The theoretical study of solutions for semilinear parabolic equations which quench in a finite time has been the subject of investigations of many authors (see [2], [4]–[7] and the references cited therein). Under some conditions, the authors have proved that the solution u of (1)–(3) quenches in a finite time and have given some estimations of the quenching time.

In this paper, we are interested in the numerical study of the phenomenon of quenching using a semidiscrete form of (1)–(3). We give some conditions under which the solution of a semidiscrete form of (1)–(3) quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero and construct two discrete forms of the problem (1)–(3) which allow us to obtain some lower bounds of the numerical quenching time. A similar study has been undertaken by some authors concerning the phenomenon of blow-up (see [1]). In [3], we may also find some results about numerical extinction.

This paper is organised as follows. In the next Section, we give some Lemmas which will be used later. In Section 3, under some conditions, we prove that the solution of a semidiscrete form of (1)–(3) quenches in a finite time and estimate its semidiscrete quenching time. In Section 4, we study the convergence of the semidiscrete quenching time. In Section 5, we study some results of Section 3 taking two discrete forms of (1)–(3). Finally, in the last section, we give some numerical results to illustrate our analysis.

2. The Semidiscrete Problem

In this section, we give some lemmas that will be used later.

Let I be a positive integer, and define the grid $x_i = ih$, $0 \le i \le I$, where $h = 1/I$. Let $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$. We approximate the solution u of the problem (1)–(3) by the solution $U_h(t)$ of the semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - U_i^{-p}(t), \;\; 1 \le i \le I-1, \;\; t \in (0, T_q^h), \tag{4}$$

$$U_0(t) = 1, \quad U_I(t) = 1, \;\; t \in (0, T_q^h), \tag{5}$$

$$U_i(0) = U_i^0 > 0, \;\; 0 \le i \le I, \tag{6}$$

where $U_i^0 < 1$ for $1 \le i \le I-1$,

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}.$$

Here, $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$, where $\|U_h(t)\|_{\inf} = \min_{0 \le i \le I} U_i(t)$. If T_q^h is finite, then we say that the solution $U_h(t)$ of (4)–(6) quenches in a finite time and the time T_q^h is called the semidiscrete quenching time of the solution $U_h(t)$.

The following lemma is a semidiscrete version of the maximum principle.

Lemma 2.1. *Let $\alpha_h \in C^0([0,T], \mathbb{R}^{I+1})$ and let $V_h \in C^1([0,T), \mathbb{R}^{I+1})$ be such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + \alpha_i(t) V_i(t) \ge 0, \;\; 1 \le i \le I-1, \;\; t \in (0,T), \tag{7}$$

$$V_0(t) \ge 0, \quad V_I(t) \ge 0, \;\; t \in (0,T), \tag{8}$$

$$V_i(0) \ge 0, \;\; 0 \le i \le I. \tag{9}$$

Then $V_i(t) \ge 0$, $0 \le i \le I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and introduce the vector $Z_h(t) = e^{\lambda t} V_h(t)$, where λ is such that $\alpha_i(t) - \lambda > 0$ for $t \in [0, T_0]$, $0 \le i \le I$. Let

$$m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t).$$

For $i = 0, \ldots, I$, the function $Z_i(t)$ is continuous on the compact $[0, T_0]$. Then there exist $i_0 \in \{0, 1, \ldots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$.

If $i_0 = 0$ or $i_0 = I$, then $m \ge 0$. If $i_0 \in \{0, 1, \ldots, I-1\}$, we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \tag{10}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0. \tag{11}$$

Due to (7), a straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (\alpha_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \ge 0. \tag{12}$$

It follows from (10)–(11) that $(\alpha_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \ge 0$, which implies that $Z_{i_0}(t_0) \ge 0$ because $\alpha_{i_0}(t_0) - \lambda > 0$. We deduce that $V_h(t) \ge 0$ for $t \in [0, T_0]$ and the proof is complete. $\square$

The lemma below shows a property of the semidiscrete solution.

Lemma 2.2. *Let $U_h(t)$ be the solution of* (4)–(6). *Then we have*

$$U_i(t) < 1, \ \ 1 \le i \le I-1, \ \ t \in (0, T_q^h). \tag{13}$$

Proof. Let $t_0 \in (0, T_q^h)$ be the first time $t > 0$ such that $U_i(t) < 1$ for $1 \le i \le I-1$, $t \in (0, t_0)$ but $U_j(t_0) = 1$ for a certain $j \in \{1, \dots, I-1\}$. We have

$$\frac{dU_j(t_0)}{dt} = \lim_{k \to 0} \frac{U_j(t_0) - U_j(t_0 - k)}{k} \ge 0,$$
$$\delta^2 U_j(t_0) = \frac{U_{j+1}(t_0) - 2U_j(t_0) + U_{j-1}(t_0)}{h^2} \le 0 \ \text{ if } \ 1 \le j \le I-1,$$

which implies that

$$\frac{dU_j(t_0)}{dt} - \delta^2 U_j(t_0) + U_j^{-p}(t_0) > 0.$$

But this contradicts (4) and the proof is complete. □

Another version of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.3. *Let $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. If V_h, $W_h \in C^1([0,T], \mathbb{R}^{I+1})$ are such that*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_i(t), t) <$$
$$< \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + f(W_i(t), t), \ \ 1 \le i \le I-1, \ \ t \in (0,T), \tag{14}$$
$$V_0(t) < W_0(t), \ \ V_I(t) < W_I(t), \ \ t \in (0,T), \tag{15}$$
$$V_i(0) < W_i(0), \ \ 0 \le i \le I, \ \ t \in (0,T), \tag{16}$$

then $V_i(t) < W_i(t)$ for $0 \le i \le I$, $t \in [0,T]$.

Proof. Let $Z_h(t) = W_h(t) - V_h(t)$ and let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $0 \le i \le I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I\}$. If $i_0 = 0$ or $i_0 = I$, we have a contradiction because of (15).

If $i_0 \in \{1, \dots, I-1\}$, we obtain

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0,$$

and

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + f\big(W_{i_0}(t_0), t_0\big) - f\big(V_{i_0}(t_0), t_0\big) \le 0.$$

This inequality contradicts (14) which ends the proof. □

The following results show some properties of the semidiscrete solution.

Lemma 2.4. *Let U_h be the solution of* (4)–(6) *such that*

$$U_i(0) = U_{I-i}(0),\ \ 0 \le i \le I, \quad U_{i+1}(0) < U_i(0),\ \ 0 \le i \le E\Big[\frac{I}{2}\Big] - 1. \tag{17}$$

Then we have for $t \in (0, T_q^h)$

$$U_{I-i}(t) = U_i(t),\ \ 0 \le i \le I, \quad U_{i+1}(t) < U_i(t),\ \ 0 \le i \le E\Big[\frac{I}{2}\Big] - 1, \tag{18}$$

where $E[\frac{I}{2}]$ is the integer part of the number $\frac{I}{2}$.

Proof. Let V_h be such that $V_i(t) = U_{I-i}(t)$ for $0 \le i \le I$ and let $W_h(t) = U_h(t) - V_h(t)$. It is not hard to see that there exists $\theta_i \in (U_i, W_i)$ such that

$$\frac{dW_i}{dt} - \delta^2 W_i + p\theta_i^{-p-1} W_i = 0,$$
$$W_0(t) = W_I(t) = 0,$$
$$W_i(0) = 0.$$

It follows from Lemma 2.1 that

$$W_i(t) = 0 \ \text{ for } \ t \in (0, T_q^h),\ \ 0 \le i \le I.$$

From Lemma 2.2, we have

$$U_i(t) < 1,\ \text{ for } \ 1 \le i \le I - 1,\ \ t \in (0, T_q^h). \tag{19}$$

Let t_1 be the first $t > 0$ such that $U_{i+1}(t) < U_i(t)$ for $t \in (0, t_1)$, $1 \le i \le E[\frac{I}{2}] - 1$, but

$$U_{k+1}(t_1) = U_k(t_1),\ \text{ for } a \text{ certain } k = 0, \dots, E\Big[\frac{I}{2}\Big] - 1. \tag{20}$$

Without loss of generality, we may suppose that k is the smallest integer which verifies (20).

If $k = 0$, then $U_1(t_1) = U_0(t_1) = 1$, which contradicts (19).

If $k = 1, \dots, E[\frac{I}{2}] - 2$, then we have

$$\frac{d}{dt}(U_{k+1} - U_k)(t_1) = \lim_{k \to 0} \frac{(U_{k+1} - U_k)(t_1) - (U_{k+1} - U_k)(t_1 - k)}{k} \le 0,$$

and

$$\delta^2(U_{k+1} - U_k)(t_1) =$$
$$= \frac{(U_{k+2} - U_k)(t_1) - 2(U_{k+1} - U_k)(t_1) + (U_k - U_{k-1})(t_1)}{h^2} > 0,$$

which implies that

$$\frac{d(U_{k+1} - U_k)(t_1)}{dt} - \delta^2(U_{k+1} - U_k)(t_1) + U_{k+1}^{-p}(t_1) - U_k^{-p}(t_1) < 0.$$

But this contradicts (4).

If $k = E[\frac{I}{2}] - 1$, then

$$U_{k+2}(t_1) = U_{E[\frac{I}{2}]+1}(t_1) = U_{I-E[\frac{I}{2}]-1}(t_1).$$

If I is even, then $U_{k+2}(t_1) = U_{E[\frac{I}{2}]-1} = U_k(t_1)$, which implies that

$$\delta^2(U_{k+1} - U_k)(t_1) = \frac{(U_k - U_{k-1})(t_1)}{h^2} > 0.$$

If I is odd, then

$$U_{k+2}(t_1) = U_{I-E[\frac{I-1}{2}]-1} = U_{E[\frac{I+1}{2}]-1}(t) = U_{k+1}(t_1),$$

which leads to $\delta^2(U_{k+1} - U_k)(t_1) = \frac{(U_k - U_{k-1})(t_1)}{h^2} > 0$. It is easy to see that

$$\frac{d(U_{k+1} - U_k)(t_1)}{dt} - \delta^2(U_{k+1} - U_k)(t_1) + U_{k+1}^{-p}(t_1) - U_k^{-p}(t_1) < 0,$$

which contradicts (4). This ends the proof. □

To end this section, let us give a property of the operator δ^2.

Lemma 2.5. *Let V_h and $U_h \in C^1([0,T], \mathbb{R}^{I+1})$. If*

$$\delta^+(U_i)\delta^+(V_i) \geq 0, \quad \textit{and} \quad \delta^-(U_i)\delta^-(V_i) \geq 0, \tag{21}$$

then

$$\delta^2(U_i V_i) \geq U_i \delta^2(V_i) + V_i \delta^2(U_i),$$

where $\delta^+(U_i) = \frac{U_{i+1} - U_i}{h}$ and $\delta^-(U_i) = \frac{U_{i-1} - U_i}{h}$.

Proof. A straightforward computation yields

$$\begin{aligned} h^2\delta^2(U_iV_i) &= U_{i+1}V_{i+1} - 2U_iV_i + U_{i-1}V_{i-1} = \\ &= (U_{i+1} - U_i)(V_{i+1} - V_i) + V_i(U_{i+1} - U_i) + U_i(V_{i+1} - V_i) + U_iV_i - 2U_iV_i + \\ &\quad + (U_{i-1} - U_i)(V_{i-1} - V_i) + (U_{i-1} - U_i)V_i + U_i(V_{i-1} - V_i) + U_iV_i, \end{aligned}$$

which implies that

$$\delta^2(U_iV_i) = \delta^+(U_i)\delta^+(V_i) + \delta^-(U_i)\delta^-(V_i) + U_i\delta^2(V_i) + V_i\delta^2(U_i).$$

Using (21), we obtain the desired result. □

3. Quenching in the Semidiscrete Problem

In this section, under some assumptions, we show that the solution U_h of (4)–(6) quenches in a finite time and estimate its semidiscrete quenching time. We need the following result.

Lemma 3.1. *Let $U_h \in \mathbb{R}^{I+1}$ be such that $U_h > 0$. Then we have*

$$\delta^2 U_i^{-p} \geq -pU_i^{-p-1}\delta^2 U_i.$$

Proof. Applying Taylor's expansion, we get

$$\begin{aligned} \delta^2 U_i^{-p} = -pU_i^{-p-1}\delta^2 U_i + (U_{i+1} - U_i)^2 \frac{p(p+1)}{2h^2}\theta_i^{-p-2} + \\ +(U_{i-1} - U_i)^2 \frac{p(p+1)}{2h^2}\eta_i^{-p-2}, \end{aligned}$$

where θ_i is an intermediate value between U_i and U_{i+1} and η_i the one between U_{i-1} and U_i. Use the fact that $U_h > 0$ to complete the rest of the proof. $\square$

The statement of our first result on the quenching time is the following.

Theorem 3.1. *Let U_h be the solution of* (4)–(6) *and assume that there exists a constant $A > 0$ such that*

$$\begin{gathered}\delta^2 U_i(0) - U_i^{-p}(0) \leq -Asin(ih\pi)U_i^{-p}(0),\\ 1 - \frac{4\pi^2}{A(p+1)}\|U_h(0)\|_{\inf}^{p+1} > 0.\end{gathered} \tag{22}$$

If (17) *holds, then U_h quenches in a finite time T_q^h with the following estimation*

$$T_q^h < -\frac{2}{\pi^2}\ln\Big(1 - \frac{4\pi^2}{A(p+1)}\,\|U_h(0)\|_{\inf}^{p+1}\Big).$$

Proof. Let $(0, T_q^h)$ be the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$. Our aim is to show that T_q^h is finite and satisfies the above inequality. From Lemma 2.4 we have

$$U_{I-i}(t) = U_i(t),\;\; 0 \leq i \leq I,\;\; U_{i+1}(t) < U_i(t),\;\; 0 \leq i \leq E\Big[\frac{I}{2}\Big] - 1. \tag{23}$$

Introduce the function $J_h(t)$ such that

$$J_i(t) = \frac{d}{dt}U_i(t) + C_i(t)U_i^{-p}(t),\;\; 0 \leq i \leq I,$$

where $C_i(t) = Ae^{-\lambda_h t}\sin(ih\pi)$ with $\lambda_h = \frac{2-2\cos(\pi h)}{h^2}$. It is not hard to see that

$$\frac{d}{dt}C_i(t) - \delta^2 C_i(t) = 0, \tag{24}$$

$$C_{I-i}(t)=C_i(t),\;\; 0\leq i\leq I,\quad C_{i+1}(t)>C_i(t),\;\; 0\leq i\leq E\Big[\frac{I}{2}\Big]-1. \tag{25}$$

From (23), (25) we get

$$\delta^+(U_i^{-p})\delta^+(C_i) \geq 0,\;\; \text{and}\;\; \delta^-(U_i^{-p})\delta^-(C_i) \geq 0. \tag{26}$$

A straightforward computation gives

$$\begin{aligned}\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) = \frac{d^2}{dt^2}U_i(t) + U_i^{-p}\,\frac{dC_i(t)}{dt} - pC_i(t)U_i^{-p-1}\,\frac{dU_i(t)}{dt} - \\ -\delta^2\Big(\frac{dU_i(t)}{dt}\Big) - \delta^2(C_i(t)U_i^{-p}(t)).\end{aligned}$$

It follows from (26), Lemma 2.5 and Lemma 3.1 that

$$\delta^2(C_i(t)U_i^{-p}(t)) \geq \delta^2(C_i(t))U_i^{-p}(t) - pC_i(t)U_i^{-p-1}(t)\delta^2 U_i(t). \tag{27}$$

We deduce that

$$\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \leq \frac{d}{dt}\Big(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)\Big) -$$

$$-pC_i(t)U_i^{-p-1}\Big(\frac{dU_i(t)}{dt}-\delta^2U_i(t)\Big)+U_i^{-p}(t)\Big(\frac{dC_i(t)}{dt}-\delta^2C_i(t)\Big).$$

From (4) and (24) we arrive at

$$\frac{dJ_i(t)}{dt}-\delta^2J_i(t)\le pU_i^{-p-1}(t)J_i(t),\ \ 1\le i\le I-1,\ \ t\in(0,T_q^h).$$

Obviously, $J_0(t)=0$ and $J_I(t)=0$. From the assumption (22), we get $J_h(0)\le 0$. It follows from Comparison Lemma 2.1 that $J_h(t)\le 0$, therefore we have

$$\frac{d}{dt}U_i(t)\le -Ae^{-\lambda_h t}\sin(ih\pi)U_i^{-p}(t),\ \ 0\le i\le I,$$

which implies that $U^p_{E[\frac{I}{2}]}(t)dU_{E[\frac{I}{2}]}(t)\le -Ae^{-\lambda_h t}sin(E[\frac{I}{2}]h\pi)dt$. We observe that $\frac{\pi^2}{2}\le\lambda_h\le 2\pi^2$, $sin(E[\frac{I}{2}]h\pi)\ge\frac{1}{2}$ for h small enough. Therefore, we get

$$U^p_{E[\frac{I}{2}]}(t)dU_{E[\frac{I}{2}]}(t)\le -\frac{A}{2}\,e^{-2\pi^2t}\,dt. \tag{28}$$

From Lemma 2.4, $U_{E[\frac{I}{2}]}(t)=\|U_h(t)\|_{\inf}$. Integrating the inequality (28) over $(0,T_q^h)$ and using the fact that $U_{E[\frac{I}{2}]}(0)=\|U_h(0)\|_{\inf}$, we arrive at

$$T_q^h\le-\frac{2}{\pi^2}\,\ln\Big(1-\frac{4\pi^2}{A(p+1)}\|U_h(0)\|_{\inf}^{p+1}\Big),$$

which implies that $T_q^h<\infty$ because of (22). This ends the proof. □

Remark 3.1. If there exists a constant $t_0\ge 0$ such that

$$1-\frac{4\pi^2}{A(p+1)}\,e^{-2\pi^2t_0}\|U_h(t_0)\|_{\inf}^{p+1}>0,$$

then integrating the inequality (28) over (t_0,T_q^h) we obtain

$$T_q^h-t_0\le-\frac{2}{\pi^2}\,\ln\Big(1-\frac{4\pi^2}{A(p+1)}\,e^{-2\pi^2t_0}\|U_h(t_0)\|_{\inf}^{p+1}\Big).$$

The theorem below gives a lower bound of the semidiscrete quenching time.

Theorem 3.2. *Let U_h be the solution of* (4)–(6). *Assume that U_h quenches at the time T_q^h. Then we have the following estimate*

$$T_q^h\ge\frac{\|U_h(0)\|_{\inf}^{p+1}}{p+1}.$$

Proof. Introduce the function $\alpha(t)$ defined by

$$\alpha(t)=\big(\|U_h(0)\|_{\inf}^{p+1}-(p+1)t\big)^{\frac{1}{p+1}}$$

and let $W_h(t)$ be a vector such that $W_i(t)=\alpha(t)$. A straightforward computation reveals that

$$\frac{d}{dt}W_i(t)-\delta^2W_i(t)+W_i^{-p}(t)=0,\ \ 1\le i\le I-1,$$

$$W_0(t) \le U_0(t),\ \ W_I(t) \le U_I(t),\quad W_i(0) \le U_i(0),\ \ 0 \le i \le I.$$

Lemma 2.1 implies that $W_i(t) \le U_i(t)$. We deduce that

$$\|U_h(t)\|_{\inf} \ge \left(\|U_h(0)\|_{\inf}^{p+1} - (p+1)t\right)^{\frac{1}{p+1}}.$$

This implies that if $t < \frac{\|U_h(0)\|_{\inf}^{p+1}}{p+1}$, then $\|U_h(t)\|_{\inf} > 0$. Therefore $T_q^h \ge \frac{\|U_h(0)\|_{\inf}^{p+1}}{p+1}$, and the proof is complete. □

4. Convergence of the Semidiscrete Quenching Time

In this section, under some assumptions, we prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero.

Firstly, we show that for each fixed time interval $[0, T]$ where the continuous solution u obeys $\|u(x,t)\|_{\inf} > 0$, the semidiscrete solution U_h approximates u as the mesh parameter h goes to zero.

Theorem 4.1. *Assume that* (1)–(3) *has a solution* $u \in C^{4,1}([0,1]\times[0,T])$ *such that* $\min_{t\in[0,T]} \|u(x,t)\|_{\inf} = \rho > 0$ *and the initial condition at (6) satisfies*

$$\|U_h^0 - u_h(0)\|_\infty = o(1) \ \ \textit{as} \ \ h \to 0, \tag{29}$$

where $u_h(t) = (u(x_0,t), \ldots, u(x_I,t))^T$. *Then, for h sufficiently small,* (4)–(6) *has a unique solution* $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ *such that*

$$\max_{0\le t\le T} \|U_h(t) - u_h(t)\|_\infty = O\big(\|U_h^0 - u_h(0)\|_\infty + h^2\big) \ \ \textit{as} \ \ h \to 0. \tag{30}$$

Proof. The problem (4)–(6) has for each h, a unique solution $U_h \in C^1([0,T_q^h), \mathbb{R}^{I+1})$. Let $t(h)$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < \frac{\rho}{2} \ \ \text{for} \ \ t \in (0, t(h)). \tag{31}$$

The relation (29) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T\}$. From the triangle inequality we get

$$\|U_h(t)\|_{\inf} \ge \|u_h(t)\|_{\inf} - \|U_h(t) - u_h(t)\|_\infty \ \ \text{for} \ \ t \in (0, t^*(h)),$$

which implies that

$$\|U_h(t)\|_{\inf} \ge \rho - \frac{\rho}{2} = \frac{\rho}{2} \ \ \text{for} \ \ t \in (0, t^*(h)).$$

Consider the error of discretization

$$e_h(t) = U_h(t) - u_h(t).$$

By a direct calculation, we have

$$\frac{d}{dt} e_i(t) - \delta^2 e_i(t) = p\Theta_i(t)^{-p-1} e_i(t) + \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t),$$

where Θ_i is an intermediate value between $U_i(t)$ and $u(x_i,t)$. Let $M > 0$ be such that

$$\frac{\|u_{xxxx}(x,t)\|_\infty}{12} \le M \ \ \text{for} \ \ t \in [0,T],\ \ p\Big(\frac{\rho}{2}\Big)^{-p-1} \le M.$$

It is not hard to see that

$$\frac{d}{dt}e_i(t) - \delta^2 e_i(t) \le M|e_i(t)| + Mh^2, \ \ t \in (0, t^*(h)).$$

Introduce the vector z_h such that

$$z_i(t) = e^{(M+1)t}\big(\|U_h^0 - u_h(0)\|_\infty + Mh^2\big), \ \ 0 \le i \le I.$$

A straightforward computation yields

$$\frac{d}{dt}z_i(t) - \delta^2 z_i(t) > M|z_i(t)| + Mh^2, \ \ 1 \le i \le I-1, \ \ t \in (0, t^*(h)),$$
$$z_0(t) > e_0(t), \ \ z_I(t) > e_I(t), \ \ t \in (0, t^*(h)),$$
$$z_i(0) > e_i(0), \ \ 0 \le i \le I.$$

It follows from Comparison Lemma 2.3 that

$$z_i(t) > e_i(t), \ \ t \in (0, t^*(h)), \ \ 0 \le i \le I.$$

By the same way, we also prove that

$$z_i(t) > -e_i(t), \ \ t \in (0, t^*(h)), \ \ 0 \le i \le I,$$

which implies that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(M+1)t}\big(\|U_h^0 - u_h(0)\|_\infty + Mh^2\big), \ \ t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (31) we obtain

$$\frac{\rho}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \le e^{(M+1)T}(\|U_h^0 - u_h(0)\|_\infty + Mh^2).$$

It is not hard to see that

$$e^{(M+1)T}(\|U_h^0 - u_h(0)\|_\infty + Mh^2) \to 0 \ \text{ when } \ h \to 0.$$

We deduce that $\frac{\rho}{2} \le 0$, which is impossible. Consequently, $t^*(h) = T$ and the proof is complete. □

Now we are in a position to prove the main result of this section.

Theorem 4.2. *Suppose that the solution u of* (1)–(3) *quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0,T_q))$ and the initial condition at* (6) *satisfies*

$$\|U_h^0 - u_h(0)\|_\infty = o(1) \ \ \textit{as} \ \ h \to 0.$$

Under the assumptions of Theorem 3.1, *the solution $U_h(t)$ of* (4)–(6) *quenches in a finite time T_q^h and we have*

$$\lim_{h \to 0} T_q^h = T_q.$$

Proof. Let $\varepsilon > 0$. There exists a constant $R > 0$ such that

$$-\frac{2}{\pi^2}\ln\Big(1 - \frac{4\pi^2}{A(p+1)}x^{p+1}\Big) < \frac{\varepsilon}{2} \ \ \text{for} \ \ x \in [0, R].$$

Since $\lim_{t \to T_q} \|u(x,t)\|_{\inf} = 0$, there exists $T_1 < T_q$ such that $|T_1 - T_q| < \frac{\varepsilon}{2}$ and $\|u(x,t)\|_{\inf} < \frac{R}{2}$ for $t \in (T_1, T_q)$. Let $T_2 = \frac{T_1 + T_q}{2}$. Obviously $\|u(x,t)\|_{\inf} <$

$\frac{R}{2}$ for $t \in [T_1, T_2]$. It follows from Theorem 4.1 that $\|U_h(t) - u_h(t)\|_\infty < \frac{R}{2}$ for $t \in [T_1, T_2]$. Using the triangle inequality, we obtain

$$\|U_h(t)\|_{\inf} \leq \|U_h(t) - u_h(t)\|_\infty + \|u_h(t)\|_{\inf} \leq$$
$$\leq \frac{R}{2} + \frac{R}{2} = R \text{ for } t \in [T_1, T_2].$$

From Theorem 3.1, $U_h(t)$ quenches in a finite time T_q^h. We deduce from Remark 3.1 that

$$|T_q^h - T_2| \leq -\frac{2}{\pi^2} \ln\Big(1 - \frac{4\pi^2}{A(p+1)}\, e^{-2\pi^2 T_2} \|U_h(T_2)\|^{p+1}\Big) < \frac{\varepsilon}{2},$$

which implies that $|T_q^h - T_q| \leq |T_q^h - T_2| + |T_2 - T_q| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and we have the desired result. $\square$

5. Full Discretizations

In this Section, we study the quenching phenomenon using full discrete schemes (explicit and implicit) of (1)–(3). At first, we approximate the solution $u(x,t)$ of (1)–(3) by the solution $U_h^{(n)} = (U_0^n, U_1^n, \ldots, U_I^n)^T$ of the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} - (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 1 \leq i \leq I-1, \tag{32}$$

$$U_0^{(n)} = 1, \quad U_I^{(n)} = 1, \tag{33}$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I, \tag{34}$$

where $n \geq 0$, $\Delta t_n = \min\{\frac{h^2}{2}, \tau \|U_h^{(n)}\|_{\inf}^{p+1}\}$ with $\tau = const \in (0,1)$.

We need the following definition.

Definition 5.1. We say that the solution $U_h^{(n)}$ of (32)–(34) quenches in a finite time if

$$\|U_h^{(n)}\|_{\inf} > 0 \text{ for } n \geq 0,$$
$$\lim_{n\to\infty} \|U_h^{(n)}\|_{\inf} = 0, \quad T(\infty) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \Delta t_j < \infty.$$

The value $T(\infty)$ is called the numerical quenching time of the solution $U_h^{(n)}$.

Theorem 5.1. *If the solution $U_h^{(n)}$ of (32)–(34) quenches in a finite time $T(\infty)$, then we have*

$$T(\infty) \geq \frac{Nh^2}{2} + \frac{\tau \|U_h^{(0)}\|_{\inf}^{1+p}}{(1+\tau)^{(p+1)N}((1+\tau)^{p+1} - 1)},$$

where N is an integer which satisfies $\frac{h^2}{2} \leq \frac{\tau \|U_h^{(0)}\|_{\inf}^{1+p}}{(1+\tau)^{(p+1)N}}$.

Proof. By a routine calculation, (32) gives

$$U_i^{(n+1)} = \frac{\frac{\Delta t_n}{h^2}(U_{i+1}^{(n)} + U_{i-1}^{(n)}) + (1 - 2\frac{\Delta t_n}{h^2})U_i^{(n)}}{1 + \Delta t_n (U_i^{(n)})^{-p-1}}, \quad 1 \leq i \leq I-1.$$

Let i_0 be such that $U_{i_0}^{(n)} = \|U_h^{(n)}\|_{\inf}$. Then we have

$$\|U_h^{(n+1)}\|_{\inf} = \frac{\frac{\Delta t_n}{h^2}(U_{i_0+1}^{(n)} + U_{i_0-1}^{(n)}) + (1 - 2\frac{\Delta t_n}{h^2})\|U_h^{(n)}\|_{\inf}}{1 + \Delta t_n \|U_h^{(n)}\|_{\inf}^{-p-1}},$$

which implies that

$$\|U_h^{(n+1)}\|_{\inf} \geq \frac{\|U_h^{(n)}\|_{\inf}}{1 + \Delta t_n \|U_h^{(n)}\|_{\inf}^{-p-1}}.$$

Since $\Delta t_n \leq \tau \|U_h^{\Delta t_n}\|_{\inf}^{p+1}$, we deduce that $\|U_h^{(n+1)}\|_{\inf} \geq \frac{\|U_h^{(n)}\|_{\inf}}{1+\tau}$, and by iteration we arrive at $\|U_h^{(n)}\|_{\inf} \geq \frac{\|U_h^{(0)}\|_{\inf}}{(1+\tau)^n}$. We deduce that

$$\sum_{n=0}^{\infty} \Delta t_n \geq \min\left\{\frac{h^2}{2}, \frac{\tau \|U_h^{(0)}\|_{\inf}^{p+1}}{((1+\tau)^{p+1})^n}\right\},$$

which implies that

$$\sum_{n=0}^{\infty} \Delta t_n \geq \frac{Nh^2}{2} + \sum_{n=N+1}^{\infty} \frac{\tau \|U_h^{(0)}\|_{\inf}^{p+1}}{[(1+\tau)^{p+1}]^n}.$$

Therefore, we have

$$T(\infty) = \sum_{n=0}^{\infty} \Delta t_n \geq \frac{Nh^2}{2} + \frac{\tau \|U_h^{(0)}\|_{\inf}^{p+1}}{(1+\tau)^{(N+1)(p+1)}} \frac{1}{(1 - \frac{1}{(1+\tau)^{p+1}})},$$

which leads us to the desired result. □

Now, approximate the solution u of (1)–(3) by the solution $U_h^{(n)} = (U_0^n, U_1^n, \ldots, U_I^n)^T$ of the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} - (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 1 \leq i \leq I-1, \tag{35}$$

$$U_0^{(n)} = 1, \quad U_I^{(n)} = 1, \quad n > 0, \tag{36}$$

$$U_i^{(0)} = \phi_i, \quad 0 \leq i \leq I, \tag{37}$$

where $n \geq 0$, $\Delta t_n = \tau \|U_h^{(n)}\|_{\inf}^{p+1}$, with $\tau = const \in (0,1)$.

We can write (35) in the following form

$$A_h^n V_h^{(n+1)} = W_h^{(n)}, \tag{38}$$

where

$$W_h^{(n)} = \left(U_1^{(n)} + \frac{\Delta t_n}{h^2}, U_2^{(n)}, \ldots, U_{I-2}^{(n)}, U_{I-1}^{(n)} + \frac{\Delta t_n}{h^2}\right)^T,$$

$$V_h^{(n+1)} = \left(U_1^{(n+1)}, U_2^{(n+1)}, \ldots, U_{I-2}^{(n+1)}, U_{I-1}^{(n+1)}\right)^T,$$

$$A_h^{(n)} = \begin{pmatrix} c_1 & -2\dfrac{\Delta t_n}{h^2} & 0 & \cdots & 0 \\ -\dfrac{\Delta t_n}{h^2} & c_2 & -\dfrac{\Delta t_n}{h^2} & 0 & \cdots \\ 0 & -\dfrac{\Delta t_n}{h^2} & c_3 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\dfrac{\Delta t_n}{h^2} \\ 0 & \cdots & 0 & -2\dfrac{\Delta t_n}{h^2} & c_{I-1} \end{pmatrix}$$

with $c_i = 1 + 2\frac{\Delta t_n}{h^2} + \Delta t_n (U_i^{(n)})^{-p-1}$.

Let us show that the problem (35)–(37) has a unique solution and, moreover, if $U_i^n \geq 0$, $\|U_h^{(n)}\|_\infty > 0$, then $U_i^{n+1} \geq 0$ and $\|U_h^{(n+1)}\|_\infty > 0$. Since $a_{ii} = c_i > 0$ and $a_{ij} \leq 0$ if $i \neq j$, we need only to prove that the spectral radius $\rho(Z_h^n) < 1$, where

$$Z_h^n = (X_h^n)^{-1} R_h^n, \quad X_h^n = diag(A_h^n), \quad A_h^n = X_h^n - R_h^n.$$

A direct calculation yields

$$Z_h^n = \begin{pmatrix} 0 & 2\dfrac{\Delta t_n}{c_2 h^2} & 0 & \cdots & 0 \\ \dfrac{\Delta t_n}{c_1 h^2} & 0 & \dfrac{\Delta t_n}{c_3 h^2} & 0 & \cdots \\ 0 & \dfrac{\Delta t_n}{c_2 h^2} & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \dfrac{\Delta t_n}{c_{I-1} h^2} \\ 0 & \cdots & 0 & 2\dfrac{\Delta t_n}{c_{I-2} h^2} & 0 \end{pmatrix}.$$

Using Gerschgorin's Theorem, we can obtain the following bound on the eigenvalues of Z_h^n : $|\nu_i| \leq \frac{2\Delta t_n}{c_i h^2} < 1$, where ν_i are eigenvalues of Z_h^n, which implies that $\rho(Z_h^n) < 1$.

Theorem 5.2. *If the solution $U_h^{(n)}$ of* (35)–(37) *quenches in a finite time, then its numerical quenching time $T(\infty)$ satisfies $T(\infty) \geq \frac{\tau \|U_h^{(0)}\|_{\inf}^{1+p} (1+\tau)^{1+p}}{(1+\tau)^{p+1} - 1}$.*

Proof. The equality (35) may be written in the following manner

$$\left(1 + 2\frac{\Delta t_n}{h^2} + \Delta t_n (U_i^{(n)})^{-p-1}\right) U_i^{(n+1)} = \frac{\Delta t_n}{h^2} U_{i+1}^{(n+1)} + \frac{\Delta t_n}{h^2} U_{i-1}^{(n+1)} + U_i^{(n)}.$$

Let i_0 be such that $U_{i_0}^{(n)} = \|U_h^{(n)}\|_{\inf}$. We obtain

$$\left(1 + 2\frac{\Delta t_n}{h^2} + \tau\right) \|U_h^{(n+1)}\|_{\inf} = \frac{\Delta t_n}{h^2} U_{i_0+1}^{(n+1)} + \frac{\Delta t_n}{h^2} U_{i_0-1}^{(n+1)} + \|U_h^{(n)}\|_{\inf},$$

which implies that

$$\Big(1+2\frac{\Delta t_n}{h^2}+\tau\Big)\|U_h^{(n+1)}\|_{\inf} \geq 2\frac{\Delta t_n}{h^2}\|U_h^{(n+1)}\|_{\inf} + \|U_h^{(n)}\|_{\inf}.$$

We deduce that $\|U_h^{(n+1)}\|_{\inf} \geq \frac{\|U_h^{(n)}\|_{\inf}}{(1+\tau)}$, and by iteration we arrive at $\|U_h^{(n)}\|_{\inf} \geq \frac{\|U_h^{(0)}\|_{\inf}}{(1+\tau)^n}$. Therefore, we get

$$T(\infty) \geq \sum_{n=0}^{\infty} \frac{\tau\|U_h^{(0)}\|_{\inf}^{p+1}}{(1+\tau)^{(p+1)n}},$$

which leads us to

$$T(\infty) \geq \frac{\tau\|U_h^{(0)}\|_{\inf}^{1+p}(1+\tau)^{1+p}}{(1+\tau)^{p+1}-1},$$

and we have the desired result. □

6. Numerical Experiments

In this section, we consider the problem (1)–(3) in the case where $p=1$ and $u_0(x) = 1-0.95\sin(\pi x)$. We use the explicit scheme (32)–(34) and the implicit scheme (35)–(37). In both cases, we take as initial condition $\varphi_i = (1-0.95*\sin(\pi i h))$ and $\tau = h^2$. In Tables 1 and 2, in the rows we present the numerical quenching times, the values of n, CPU times and the orders of the approximations corresponding to the meshes of 16, 32, 64, 128. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = |T^n - T^{n-1}| \leq 10^{-16}$. The order (s) of the method is computed from

$$s = \frac{\log((T_{4h}-T_{2h})/(T_{2h}-T_h))}{\log(2)}.$$

Table 1. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.002292	3454	0,8	-
32	0.002361	13535	03	-
64	0.002404	52517	20	0.68
128	0.002401	202298	687	3.85

Table 2. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.002287	3458	01	-
32	0.002374	13547	07	-
64	0.002407	52529	109	1.40
128	0.002403	202309	3300	3.05

References

1. L. Abia, J. C. López-Marcos and J. Martinez, On the blow-up time convergence of semidiscretizations of reaction-diffusion equations. Appl. Numer. Math. 26 (1998), no. 4, 399–414.
2. A. F. Acker and B. Kawohl, Remarks on quenching. *Nonlinear Anal.* **13** (1989), No. 1, 53–61.
3. T. K. Boni, Extinction for discretizations of some semilinear parabolic equations. *C. R. Acad. Sci. Paris Ser. I Math.* **333** (2001), No. 8, 795–800.
4. T. K. Boni, On quenching of solutions for some semilinear parabolic equations of second order. *Bull. Belg. Math. Soc. Simon Stevin* **7** (2000), No. 1, 73–95.
5. M. Fila, B. Kawohl and H. A. Levine, Quenching for quasilinear equations. *Comm. Partial Differential Equations* **17** (1992), No. 3-4, 593–614.
6. J.-S. Guo, On a quenching problem with the Robin boundary condition. *Nonlinear Anal.* **17** (1991), No. 9, 803–809.
7. H. A. Levine, Quenching, nonquenching, and beyond quenching for solution of some parabolic equations. *Ann. Mat. Pura Appl. (4)* **155** (1989), 243–260.

(Received 21.10.2007)

Authors' addresses:

D. Nabongo
Université d'Abobo-Adjamé, UFR-SFA
Département de Mathématiques et Informatiques
16 BP 372 Abidjan 16
Côte d'Ivoire
E-mail: nabongo_diabate@yahoo.fr

Th. K. Boni
Institut National Polytechnique Houphouët-Boigny de Yamoussoukro
BP 1093 Yamoussoukro
Côte d'Ivoire
E-mail: theokboni@yahoo.fr