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EXISTENCE RESULTS FOR DIFFERENTIAL EQUATIONS WITH FRACTIONAL ORDER AND IMPULSES

Abstract. In this paper we establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential equations involving the Caputo fractional derivative of order $\alpha \in (1, 2]$.

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1. Introduction

This paper deals with the existence and uniqueness of solutions for the initial value problems (IVP for short) for fractional order differential equations

$$\begin{gathered} {}^cD^\alpha y(t) = f(t,y), \quad \text{for each } \ t \in J = [0,T], \\ t \neq t_k, \quad k = 1,\dots,m, \quad 1 < \alpha \leq 2, \end{gathered} \tag{1}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{2}$$

$$\Delta y'\big|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,\dots,m, \tag{3}$$

$$y(0) = y_0, \quad y'(0) = y_1, \tag{4}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, I_k, $\overline{I}_k : \mathbb{R} \to \mathbb{R}$, $k = 1,\dots,m$, and y_0, $y_1 \in \mathbb{R}$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1,\dots,m$. Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc. (see [16], [22], [23], [26], [34], [35], [39]). There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas *et al.* [29], Miller and Ross [36], Podlubny [40], Samko *et al.* [43] and the papers of Agarwal *et al.* [1], [2], Benchohra and Hamani [5], Benchohra *et al.* [8], [9], Delbosco and Rodino [15], Diethelm *et al.* [16], [17], [18], El-Sayed [19], [20], [21], Kaufmann and Mboumi [27], Kilbas and Marzan [28], Mainardi [34], Momani and Hadid [37], Momani *et al.* [38], Podlubny *et al.* [42], Yu and Gao [45] and Zhang [46], and the references therein. Very recently some basic theory for the initial value problems of fractional differential equations involving Riemann–Liouville differential operator of order $0 < \alpha \leq 1$ has been discussed by Lakshmikantham and Vatsala [31], [32], [33]. In [4] the authors considered a class of perturbed fractional differential equations and in [6] a boundary value problem for differential equations involving the Caputo fractional derivative.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions containing $y(0)$, $y'(0)$, etc. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville and Caputo types see [25], [41].

Impulsive differential equations (for $\alpha \in \mathbb{N}$) have become important in recent years as mathematical models of phenomena in both physical and social sciences. There has been a significant development in impulsive theory, especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Bainov and Simeonov [3], Benchohra *et al.* [7], Lakshmikantham *et al* [30], and Samoilenko and Perestyuk [44] and the references therein. In [10], Benchohra and Slimani have initiated the study of fractional differential equations with impulses in which they considered a class of initial value problems for differential equations involving the Caputo fractional derivative of order $\alpha \in (0,1]$ and the impulsive effect. The aim of this paper is to continue this study by giving several existence and uniqueness results for the initial value problem (1)–(4). This paper is organized as follows. In Section 2, we present some preliminary results about fractional derivation and integration needed in the following sections. Section 3 will be concerned with existence and uniqueness results for the IVP (1)–(4). We give three results: the first one is based on the Banach fixed point theorem (Theorem 3.5), the second one is based on Schaefer's fixed point theorem (Theorem 3.6), the third one on the nonlinear alternative of the Leray–Schauder type (Theorem 3.7) and the fourth one (Theorem 3.9) on the Burton–Kirk fixed point theorem for the sum of contraction and completely continuous operators. In Section 4, we indicate some generalizations to nonlocal initial value problems. The last section is devoted to an example illustrating the applicability of the imposed conditions. These results can be considered as a contribution to this emerging field.

2. Preliminaries

In this section, we introduce notation, definitions and preliminary facts that are used throughout this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\big\{|y(t)| : \ t \in J\big\}.$$

Definition 2.1 ([29], [40])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, h(s)\, ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2 ([29], [40])**.** For a function h given on the interval $[a,b]$, the αth Riemann–Liouville fractional-order derivative of h is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)\, ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([28])**.** For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h is defined by

$$({}^cD^\alpha_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds.$$

3. Existence of Solutions

Consider the following space

$$PC(J,\mathbb{R}) = \Big\{ y : J \to \mathbb{R} : \ \ y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \dots, m+1$$
$$\text{and there exist } y(t_k^-) \text{ and } y(t_k^+),\ k = 1, \dots, m,\ \text{with } y(t_k^-) = y(t_k) \Big\}.$$

$PC(J,\mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} \ |y(t)|.$$

Set $J' := [0,T] \setminus \{t_1, \dots, t_m\}$.

Definition 3.1. A function $y \in PC(J,\mathbb{R})$ with its α-derivative existing on J' is said to be a solution of (1)–(4) if y satisfies the equation ${}^cD^\alpha y(t) = f(t, y(t))$ on J', and the conditions

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \ \ k = 1, \dots, m,$$
$$\Delta y'\big|_{t=t_k} = \overline{I}_k(y(t_k^-)), \ \ k = 1, \dots, m,$$

and

$$y(0) = y_0, \quad y'0) = y_1$$

are satisfied.

For proving the existence of solutions for the problem (1)–(4), we need the following auxiliary lemmas:

Lemma 3.2 ([46])**.** *Let $\alpha > 0$. Then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n$, $n = [\alpha] + 1$.

Lemma 3.3 ([46])**.** *Let $\alpha > 0$. Then*

$$I^\alpha {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n$, $n = [\alpha] + 1$.

As a consequence of Lemma 3.2 and Lemma 3.3, we have the following result which is useful in what follows.

Lemma 3.4. *Let $1 < \alpha \leq 2$ and let $h : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} h(s)\, ds \quad \textit{if } \ t \in [0, t_1], \\ y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum\limits_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)\, ds + \\ \quad + \frac{1}{\Gamma(\alpha-1)} \sum\limits_{i=1}^{k} (t - t_i) \int\limits_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-2} h(s)\, ds + \\ \quad + \frac{1}{\Gamma(\alpha)} \int\limits_{t_k}^{t} (t-s)^{\alpha-1} h(s)\, ds + \\ \quad + \sum\limits_{i=1}^{k} I_i(y(t_i^-)) + \sum\limits_{i=1}^{k} (t - t_i) \overline{I}_i(y(t_i^-)), \\ \qquad\qquad \textit{if } \ t \in (t_k, t_{k+1}], \quad k = 1, \dots, m, \end{cases} \tag{5}$$

if and only if y is a solution of the fractional IVP

$$^cD^{\alpha} y(t) = h(t), \quad \textit{for eacht } t \in J', \tag{6}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{7}$$

$$\Delta y'\big|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{8}$$

$$y(0) = y_0, \quad y'(0) = y_1. \tag{9}$$

Proof. Assume y satisfies (6)–(9). If $t \in [0, t_1]$, then

$$^cD^{\alpha} y(t) = h(t).$$

Lemma 3.3 implies

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} h(s)\, ds.$$

Hence $c_0 = y_0$ and $c_1 = y_1$. Thus

$$y(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} h(s)\, ds.$$

If $t \in (t_1, t_2]$, then Lemma 3.3 implies

$$y(t) = c_0 + c_1 (t - t_1) + \frac{1}{\Gamma(\alpha)} \int\limits_{t_1}^t (t-s)^{\alpha-1} h(s)\, ds, \tag{10}$$

$$\Delta y\big|_{t=t_1} = y(t_1^+) - y(t_1^-) =$$

$$= c_0 - \left(y_0 + y_1 t_1 + \frac{1}{\Gamma(\alpha)} \int\limits_0^{t_1} (t_1 - s)^{\alpha-1} h(s)\, ds \right) =$$
$$= I_1(y(t_1^-)).$$

Hence

$$c_0 = y_0 + y_1 t_1 + \frac{1}{\Gamma(\alpha)} \int\limits_0^{t_1} (t_1 - s)^{\alpha-1} h(s)\, ds + I_1(y(t_1^-)), \tag{11}$$

$$\Delta y'\big|_{t=t_1} = y'(t_1^+) - y'(t_1^-) =$$
$$= c_1 - \left(y_1 + \frac{1}{\Gamma(\alpha-1)} \int\limits_0^{t_1} (t_1 - s)^{\alpha-2} h(s)\, ds \right) =$$
$$= \overline{I}_1(y(t_1^-)),$$

and

$$c_1 = y_1 + \frac{1}{\Gamma(\alpha-1)} \int\limits_0^{t_1} (t_1 - s)^{\alpha-2} h(s)\, ds + \overline{I}_1(y(t_1^-)). \tag{12}$$

Then by (10)–(12) we have

$$y(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int\limits_0^{t_1} (t_1 - s)^{\alpha-1} h(s)\, ds +$$
$$+ \frac{(t-t_1)}{\Gamma(\alpha-1)} \int\limits_0^{t_1} (t_1 - s)^{\alpha-2} h(s)\, ds + I_1(y(t_1^-)) +$$
$$+ (t-t_1)\overline{I}_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int\limits_{t_1}^{t} (t - s)^{\alpha-1} h(s)\, ds.$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 3.3 we get (5).

Conversely, assume that y satisfies the impulsive fractional integral equation (5). If $t \in [0, t_1]$, then $y(0) = y_0$, $y'(0) = y_1$, and using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha y(t) = h(t), \quad \text{for each } \ t \in [0, t_1].$$

If $t \in [t_k, t_{k+1})$, $k = 1, \dots, m$, then using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$${}^cD^\alpha y(t) = h(t), \quad \text{for each } \ t \in [t_k, t_{k+1}).$$

Also, we can easily show that

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m,$$

and

$$\Delta y'\big|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m.$$

Our first result is based on the Banach fixed point theorem. $\square$

Theorem 3.5. *Assume that:*

(H1) *There exists a constant $l > 0$ such that*

$$\big|f(t,u) - f(t,\overline{u})\big| \le l|u - \overline{u}| \ \text{ for each } \ t \in J \ \text{ and all } \ u, \overline{u} \in \mathbb{R}.$$

(H2) *There exist constants $l^*, \overline{l}^* > 0$ such that*

$$\big|I_k(u) - I_k(\overline{u})\big| \le l^*|u - \overline{u}| \ \text{ for each } \ u, \overline{u} \in \mathbb{R} \ \text{ and } \ k = 1, \dots, m,$$

and

$$\big|\overline{I}_k(u) - \overline{I}_k(\overline{u})\big| \le \overline{l}^*|u - \overline{u}| \ \text{ for each } \ u, \overline{u} \in \mathbb{R} \ \text{ and } \ k = 1, \dots, m.$$

If

$$\left[\frac{mlT^\alpha}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{lT^\alpha}{\Gamma(\alpha+1)} + m(l^* + T\overline{l}^*)\right] < 1, \tag{13}$$

then the IVP (1)–(4) *has a unique solution on J.*

Proof. Transform the problem (1)–(4) into a fixed point problem. Consider the operator

$$F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$$

defined by

$$\begin{aligned} F(y)(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, y(s))\, ds + \\ + \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_i} (t_k - s)^{\alpha-2} f(s, y(s))\, ds + \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, y(s))\, ds + \\ + \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t - t_k)\overline{I}_k(y(t_k^-)). \end{aligned}$$

Clearly, the fixed points of the operator F are solutions of the problem (1)–(4). We will use the Banach contraction principle to prove that F has a fixed point. We will show that F *is a contraction.*

Let $x,\, y \in PC(J,\mathbb{R})$. Then for each $t \in J$ we have

$$\big|F(x)(t) - F(y)(t)\big| \le$$

$$\le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_i} (t_k - s)^{\alpha-1} \big|f(s,x(s)) - f(s,y(s))\big|\, ds +$$

$$+\frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} \big|f(s,x(s)) - f(s,y(s))\big|\, ds +$$

$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \big|f(s,x(s)) - f(s,y(s))\big|\, ds +$$

$$+\sum_{0<t_k<t} \big|I_k(x(t_k^-)) - I_k(y(t_k^-))\big| +$$

$$+\sum_{0<t_k<t} (t-t_k) \big|\overline{I}_k(x(t_k^-)) - \overline{I}_k(y(t_k^-))\big| \le$$

$$\le \frac{l}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |x(s) - y(s)|\, ds +$$

$$+\frac{l}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_i} (t_k - s)^{\alpha-2} |x(s) - y(s)|\, ds +$$

$$+\frac{l}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |x(s) - y(s)|\, ds +$$

$$+l^* \sum_{0<t_k<t} \big|x(t_k^-) - y(t_k^-)\big| + \overline{l}^* \sum_{0<t_k<t} (t-t_i) \big|x(t_k^-) - y(t_k^-)\big| \le$$

$$\le \frac{mlT^\alpha}{\Gamma(\alpha+1)} \|x-y\|_\infty + \frac{mlT^\alpha}{\Gamma(\alpha)} \|x-y\|_\infty + \frac{lT^\alpha}{\Gamma(\alpha+1)} \|x-y\|_\infty +$$

$$+ml^* \|x-y\|_\infty + m\overline{l}^* T \|x-y\|_\infty =$$

$$= \left[\frac{mlT^\alpha}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{lT^\alpha}{\Gamma(\alpha+1)} + m(l^* + T\overline{l}^*)\right] \|x-y\|_\infty .$$

Thus

$$\|F(x) - F(y)\|_\infty \le \left[\frac{mlT^\alpha}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{lT^\alpha}{\Gamma(\alpha+1)} + m(l^* + T\overline{l}^*)\right] \|x-y\|_\infty .$$

Consequently, by (13) F is a contraction. As a consequence of the Banach fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1)–(4). □

The second result is based on Schaefer's fixed point theorem.

Theorem 3.6. *Assume that:*

(H3) *The function* $f : J \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(H4) *There exists a constant* $M > 0$ *such that*

$$|f(t,u)| \leq M \ \text{ for each } \ t \in J \ \text{ and all } \ u \in \mathbb{R}.$$

(H5) *The functions* I_k, $\overline{I}_k : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* M^*, $\overline{M}^* > 0$ *such that*

$$|I_k(u)| \leq M^*, \quad |\overline{I}_k(u)| \leq \overline{M}^* \ \text{ for each } \ u \in \mathbb{R}, \ \text{ and } \ k = 1, \ldots, m.$$

Then the IVP (1)–(4) *has at least one solution on* J.

Proof. We will use Schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given in several steps.

Step 1: *F is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $PC(J,\mathbb{R})$. Then for each $t \in J$

$$\big|F(y_n)(t) - F(y)(t)\big| \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \big|f(s,y_n(s)) - f(s,y(s))\big|\, ds +$$

$$+\frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_i} (t_k - s)^{\alpha-2} \big|f(s,y_n(s)) - f(s,y(s))\big|\, ds +$$

$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \big|f(s,y_n(s)) - f(s,y(s))\big|\, ds +$$

$$+ \sum_{0<t_k<t} \big|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\big| + \sum_{0<t_k<t} (t-t_k) \big|\overline{I}_k(y_n(t_k^-)) - \overline{I}_k(y(t_k^-))\big|.$$

Since f, I_k and $\overline{I}_k$, $k = 1, \ldots, m$, are continuous functions, we have

$$\big\|F(y_n) - F(y)\big\|_\infty \to 0 \ \text{ as } \ n \to \infty.$$

Step 2: *F maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.*

Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $y \in B_{\eta^*} = \big\{y \in PC(J,\mathbb{R}) : \ \|y\|_\infty \leq \eta^*\big\}$ we have $\|F(y)\|_\infty \leq \ell$. By (H4) and (H5), we have for each $t \in J$

$$|F(y)(t)| \leq |y_0| + T|y_1| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |f(s,y(s))|\, ds +$$

$$+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int\limits_{t_{k-1}}^{t_i}(t_k-s)^{\alpha-2}|f(s,y(s))|\,ds+$$
$$+\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y(s))|\,ds+$$
$$+\sum_{0<t_k<t}\left|I_k(y(t_k^-))\right|+\sum_{0<t_k<t}(t-t_k)\left|\overline{I}_k(y(t_k^-))\right|\leq$$
$$\leq|y_0|+T|y_1|+\frac{mMT^\alpha}{\Gamma(\alpha+1)}+\frac{mMT^\alpha}{\Gamma(\alpha)}+$$
$$+\frac{MT^\alpha}{\Gamma(\alpha+1)}+mM^*+mT\overline{M}^*.$$

Thus

$$\|F(y)\|_\infty\leq|y_0|+T|y_1|+$$
$$+\frac{mMT^\alpha}{\Gamma(\alpha+1)}+\frac{mMT^\alpha}{\Gamma(\alpha)}+\frac{MT^\alpha}{\Gamma(\alpha+1)}+mM^*+mT\overline{M}^*:=\ell.$$

Step 3: *F maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.*

Let τ_1, $\tau_2\in J$, $\tau_1<\tau_2$, B_{η^*} be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, and let $y\in B_{\eta^*}$. Then

$$\left|F(y)(\tau_2)-F(y)(\tau_1)\right|=$$
$$=\frac{1}{\Gamma(\alpha)}\int\limits_0^{\tau_1}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right||f(s,y(s))|\,ds+$$
$$+\frac{1}{\Gamma(\alpha)}\int\limits_{\tau_1}^{\tau_2}\left|(\tau_2-s)^{\alpha-1}\right||f(s,y(s))|\,ds+$$
$$+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<\tau_2-\tau_1}(\tau_2-t_k)\int\limits_{t_{k-1}}^{t_i}(t_k-s)^{\alpha-2}|f(s,y(s))|\,ds+$$
$$+\frac{1}{\Gamma(\alpha-1)}\sum_{0<t_k<\tau_1}(\tau_2-\tau_1)\int\limits_{t_{k-1}}^{t_i}(t_k-s)^{\alpha-2}|f(s,y(s))|\,ds+$$
$$+\int\limits_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}|f(s,y(s))|\,ds+$$
$$+\int\limits_{t_k}^{\tau_1}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right||f(s,y(s))|\,ds+$$

$$+\sum_{0<t_k<\tau_2-\tau_1}\left|I_k(y(t_k^-))\right|+\sum_{0<t_k<\tau_2-\tau_1}(\tau_2-t_k)\left|\overline{I}_k(y(t_k^-))\right|+$$
$$+(\tau_2-\tau_1)\sum_{0<t_k<\tau_1}\left|\overline{I}_k(y(t_k^-))\right|.$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $F : PC(J,\mathbb{R}) \longrightarrow PC(J,\mathbb{R})$ is continuous and completely continuous.

Step 4: *A priori bounds.*

Now it remains to show that the set

$$\mathcal{E}=\left\{y\in PC(J,\mathbb{R}):\ y=\lambda F(y)\ \text{ for some }\ 0<\lambda<1\right\}$$

is bounded.

Let $y \in \mathcal{E}$. Then $y = \lambda F(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$ we have

$$y(t)=\lambda y_0+T|y_1|+\frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,y(s))\,ds+$$
$$+\frac{\lambda}{\Gamma(\alpha-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_i}(t_k-s)^{\alpha-2}f(s,y(s))\,ds+$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,y(s))\,ds+$$
$$+\lambda\sum_{0<t_k<t}I_k(y(t_k^-))+\lambda\sum_{0<t_k<t}(t-t_k)\overline{I}_k(y(t_k^-)).$$

This implies by (H4) and (H5) (as in Step 2) that for each $t \in J$ we have

$$y(t)\le|y_0|+T|y_1|+\frac{mMT^\alpha}{\Gamma(\alpha+1)}+\frac{mMT^\alpha}{\Gamma(\alpha)}+\frac{MT^\alpha}{\Gamma(\alpha+1)}+mM^*+mT\overline{M}^*.$$

Thus for every $t \in J$ we have

$$\|y\|_\infty\le|y_0|+T|y_1|+$$
$$+\frac{mMT^\alpha}{\Gamma(\alpha+1)}+\frac{mMT^\alpha}{\Gamma(\alpha)}+\frac{MT^\alpha}{\Gamma(\alpha+1)}+mM^*+mT\overline{M}^*:=R.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1)–(4). □

In the following theorem we will give an existence result for the problem (1)–(4) by means of an application of the nonlinear alternative of Leray–Schauder type with the conditions (H4) and (H5) weakened.

Theorem 3.7. *Assume that* (H2) *and the following conditions hold.*

(H6) *There exist* $\phi_f \in C(J, \mathbb{R}^+)$ *and* $\psi : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$|f(t, u)| \le \phi_f(t)\psi(|u|) \ \text{for each} \ t \in J \ \text{and all} \ u \in \mathbb{R}.$$

(H7) *There exist* $\psi^*, \overline{\psi}^* : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$|I_k(u)| \le \psi^*(|u|) \ \text{for each} \ u \in \mathbb{R},$$

and

$$|\overline{I}_k(u)| \le \overline{\psi}^*(|u|) \ \text{for each} \ u \in \mathbb{R}.$$

(H8) *There exists a number* $\overline{M} > 0$ *such that*

$$\frac{\overline{M}}{|y_0| + T|y_1| + a\psi(\overline{M}) + m\psi^*(\overline{M}) + mT\overline{\psi}^*(\overline{M})} > 1, \tag{14}$$

where $\phi_f^0 = \sup\{\phi_f(t) : \ t \in J\}$ *and*

$$a = \frac{mT^\alpha \phi_f^0}{\Gamma(\alpha+1)} + \frac{mT^\alpha \phi_f^0}{\Gamma(\alpha)} + \frac{T^\alpha \phi_f^0}{\Gamma(\alpha+1)}.$$

Then the IVP (1)–(4) *has at least one solution on* J.

Proof. Consider the operator F defined in Theorems 3.5 and 3.6. It can be easily shown that F is continuous and completely continuous. For $\lambda \in [0, 1]$, let y be such that for each $t \in J$ we have $y(t) = \lambda(Fy)(t)$. Then from (H6)–(H7) we have for each $t \in J$

$$\begin{aligned}
|y(t)| \le |y_0| + T|y_1| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \phi_f(s)\psi(|y(s)|)\, ds +& \\
+ \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-2} \phi_f(s)\psi(|y(s)|)\, ds +& \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \phi_f(s)\psi(|y(s)|)\, ds +& \\
+ \sum_{0<t_k<t} \psi^*(|y(t_k)|) + \sum_{0<t_k<t} (t - t_k)\overline{\psi}^*(|y(t_k)|) \le& \\
\le |y_0| + T|y_1| + \psi(\|y\|_\infty) \frac{mT^\alpha \phi_f^0}{\Gamma(\alpha+1)} + \psi(\|y\|_\infty) \frac{mT^\alpha \phi_f^0}{\Gamma(\alpha)} +& \\
+ \psi(\|y\|_\infty) \frac{T^\alpha \phi_f^0}{\Gamma(\alpha+1)} + m\psi^*(\|y\|_\infty) + mT\overline{\psi}^*(\|y\|_\infty).&
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{|y_0| + T|y_1| + a\psi(\|y\|_\infty) + m\psi^*(\|y\|_\infty) + mT\overline{\psi}^*(\|y\|_\infty)} \le 1.$$

Then by the condition (14) there exists $\overline{M}$ such that $\|y\|_\infty \neq \overline{M}$.

Let

$$U = \left\{y \in PC(J,\mathbb{R}) : \ \|y\|_\infty < \overline{M}\right\}.$$

The operator $F : \overline{U} \to PC(J,\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda F(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [24], we deduce that F has a fixed point y in $\overline{U}$ which is a solution of the problem (1)–(4). This completes the proof. □

We now present another existence result for the IVP (1)–(4) relying on the following Burton and Kirk's fixed point theorem [11].

Theorem 3.8. *Let X be a Banach space, and A, $B : X \to X$ two operators satisfying*

(i) *A is a contraction, and*
(ii) *B is completely continuous.*

Then either

(a) *the operator equation $y = A(y) + B(y)$ has a solution, or*
(b) *the set $\mathbb{E} = \left\{u \in X : \ u = \lambda A\left(\frac{u}{\lambda}\right) + \lambda B(u)\right\}$ is unbounded for $\lambda \in (0,1)$.*

Theorem 3.9. *Assume that* (H1) *and* (H7) *hold. Furthermore, if*

$$\frac{mT^\alpha l}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{T^\alpha l}{\Gamma(\alpha+1)} < 1 \tag{15}$$

and

$$\limsup_{u\to+\infty} \frac{\left(1 - \frac{mT^\alpha l}{\Gamma(\alpha+1)} - \frac{mlT^\alpha}{\Gamma(\alpha)} - \frac{T^\alpha l}{\Gamma(\alpha+1)}\right)u}{|y_0|+|y_1|T+\frac{mT^\alpha f^*}{\Gamma(\alpha+1)}+\frac{mT^\alpha f^*}{\Gamma(\alpha)}+\frac{T^\alpha f^*}{\Gamma(\alpha+1)}+m\psi^*(u)+mT\overline{\psi}^*(u)} > 1, \tag{16}$$

then the IVP (1)–(4) *has at least one solution on J.*

Proof. Consider the operators A, $B : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$(Ay)(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, y(s))\, ds +$$

$$+\frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_i} (t_k - s)^{\alpha-2} f(s, y(s))\, ds$$

$$+\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, y(s))\, ds,$$

and

$$(By)(t) = \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t-t_k)\overline{I}_k(y(t_k^-)).$$

From (H1) and (15) we can easily show that A is a contraction. By (H7) it is clear that the operator B is continuous and completely continuous. To conclude for the existence of the fixed point of the operator $A+B$, it suffices to prove that the set $\mathbb{E}$ in Theorem 3.8 is bounded.

Let $y \in \mathbb{E}$. Then for each $t \in J$

$$y(t) = \lambda A\Big(\frac{u}{\lambda}\Big)(t) + \lambda B(u)(t).$$

From (H1) and (H7) we have

$$\begin{aligned}
|y(t)| &\le \lambda|y_0| + \lambda|y_1|T + \frac{\lambda}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1}\Big|f\Big(s,\frac{y(s)}{\lambda}\Big)\Big|\,ds + \\
&+\frac{\lambda}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_i} (t_k-s)^{\alpha-2}\Big|f\Big(s,\frac{y(s)}{\lambda}\Big)\Big|\,ds + \\
&+\frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\Big|f\Big(s,\frac{y(s)}{\lambda}\Big)\Big|\,ds + \\
&+\lambda \sum_{0<t_k<t} \big|I_k(y(t_k^-))\big| + \lambda \sum_{0<t_k<t} (t-t_k)\big|\overline{I}_k(y(t_k^-))\big| \le \\
&\le |y_0| + |y_1|T + \frac{l}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1}|y(s)|\,ds + \\
&+\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1}|f(s,0)|\,ds + \\
&+\frac{l}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_i} (t_k-s)^{\alpha-2}|y(s)|\,ds + \\
&+\frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_i} (t_k-s)^{\alpha-2}|f(s,0)|\,ds + \\
&+\frac{l}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}|y(s)|\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}|f(s,0)|\,ds + \\
&+\sum_{0<t_k<t} \psi^*\big(|y(t_k^-)|\big) + \sum_{0<t_k<t} (t-t_k)\overline{\psi}^*\big(|y(t_k^-)|\big) \le
\end{aligned}$$

$$\leq |y_0| + |y_1|T + \frac{mT^\alpha l}{\Gamma(\alpha+1)}\|y\|_\infty + \frac{mT^\alpha f^*}{\Gamma(\alpha+1)} + \\ + \frac{mlT^\alpha}{\Gamma(\alpha)}\|y\|_\infty + \frac{mT^\alpha f^*}{\Gamma(\alpha)} + \frac{T^\alpha l}{\Gamma(\alpha+1)}\|y\|_\infty + \frac{T^\alpha f^*}{\Gamma(\alpha+1)} + \\ + m\psi^*(\|y\|_\infty) + mT\overline{\psi}^*(\|y|_\infty).$$

Thus

$$\frac{\left(1 - \frac{mT^\alpha l}{\Gamma(\alpha+1)} - \frac{mlT^\alpha}{\Gamma(\alpha)} - \frac{T^\alpha l}{\Gamma(\alpha+1)}\right)\|y\|_\infty}{|y_0| + |y_1|T + \frac{mT^\alpha f^*}{\Gamma(\alpha+1)} + \frac{mT^\alpha f^*}{\Gamma(\alpha)} + \frac{T^\alpha f^*}{\Gamma(\alpha+1)} + m\psi^*(\|y\|_\infty) + mT\overline{\psi}^*(\|y|_\infty)} \leq 1. \tag{17}$$

From (16) it follows that there exists a constant $R > 0$ such that for each $y \in \mathbb{E}$ with $\|y\|_\infty > R$ the condition (17) is violated. Hence $\|y\|_\infty \leq R$ for each $y \in \mathbb{E}$, which means that the set $\mathbb{E}$ is bounded. □

4. Nonlocal Impulsive Differential Equations

This section is concerned with a generalization of the results presented in the previous section to nonlocal impulsive fractional differential equations. More precisely, we will present some existence and uniqueness results for the following nonlocal problem

$$\begin{gathered} {}^cD^\alpha y(t) = f(t,y), \ \text{ for each } \ t \in J = [0,T], \\ t \neq t_k, \ \ k = 1, \dots, m, \ \ 1 < \alpha \leq 2, \end{gathered} \tag{18}$$

$$\Delta y\big|_{t=t_k} = I_k(y(t_k^-)), \ \ k = 1, \dots, m, \tag{19}$$

$$\Delta y'\big|_{t=t_k} = \overline{I}_k(y(t_k^-)), \ \ k = 1, \dots, m, \tag{20}$$

$$y(0) + g(y) = y_0, \ \ y'(0) = y_1, \tag{21}$$

where f, I_k, $\overline{I}_k$, $k = 1, \dots, m$, are as in Section 3 and $g : PC(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function. Nonlocal conditions were initiated by Byszewski [14] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [12], [13], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(y)$ may be given by

$$g(y) = \sum_{i=1}^{p} c_i y(\tau_i),$$

where c_i, $i = 1, \dots, p$, are given constants and $0 < \tau_1 < \dots < \tau_p \leq T$. Let us introduce the following set of conditions.

(H9) There exists a constant $M^{**} > 0$ such that

$$|g(u)| \leq M^{**} \ \text{ for each } \ u \in PC(J, \mathbb{R}).$$

(H10) There exists a constant $l^{**} > 0$ such that

$$\big|g(u) - g(\overline{u})\big| \leq l^{**}|u - \overline{u}| \ \text{ for each } \ u, \overline{u} \in PC(J, \mathbb{R}).$$

(H11) There exists $\psi^{**} : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$|g(u)| \leq \psi^{**}(|u|) \text{ for each } u \in PC(J, \mathbb{R}).$$

(H12) There exists a number $\overline{M}^* > 0$ such that

$$\frac{\overline{M}^*}{|y_0| + T|y_1| + \psi^{**}(\overline{M}^*) + a\psi(\overline{M}^*) + m\psi^*(\overline{M}^*) + mT\overline{\psi}^*(\overline{M}^*)} > 1. \quad (22)$$

Theorem 4.1. *Assume that* (H1)*,* (H2)*,* (H10) *hold. If*

$$\left[\frac{mlT^\alpha}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{lT^\alpha}{\Gamma(\alpha+1)} + m(l^* + T\overline{l}^*) + l^{**}\right] < 1, \quad (23)$$

then the nonlocal problem (18)–(21) *has a unique solution on* J*.*

Proof. Transform the problem (18)–(21) into a fixed point problem. Consider the operator

$$\widetilde{F} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$$

defined by

$$\widetilde{F}(y)(t) = y_0 - g(y) + y_1 t + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, y(s))\, ds +$$

$$+ \frac{1}{\Gamma(\alpha-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_i} (t_k - s)^{\alpha-2} f(s, y(s))\, ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s, y(s))\, ds +$$

$$+ \sum_{0<t_k<t} I_k(y(t_k^-)) + \sum_{0<t_k<t} (t - t_k)\overline{I}_k(y(t_k^-)).$$

Clearly, the fixed points of the operator $\widetilde{F}$ are solutions of the problem (18)–(21). Using (H1), (H2), (H10) and (23) we can easily show the $\widetilde{F}$ is a contraction. □

Theorem 4.2. *Assume that* (H3)–(H5)*,* (H9) *hold. Then the nonlocal problem* (18)–(21) *has at least one solution on* J*.*

Theorem 4.3. *Assume that* (H6)–(H7)*,* (H11)–(H12) *hold. Then the nonlocal problem* (18)–(21) *has at least one solution on* J*.*

5. An Example

In this section, we give an example to illustrate the usefulness of our main results. Let us consider the following impulsive fractional initial value problem

$$ {}^cD^\alpha y(t) = \frac{e^{-t}|y(t)|}{(9+e^t)(1+|y(t)|)}\,, \quad t \in J := [0,1], \quad t \neq \frac{1}{2}\,, \quad 1 < \alpha \leq 2, \tag{24} $$

$$ \Delta y\big|_{t=\frac{1}{2}} = \frac{\left|y(\frac{1}{2}^-)\right|}{3+\left|y(\frac{1}{2}^-)\right|}\,, \tag{25} $$

$$ \Delta y'\big|_{t=\frac{1}{2}} = \frac{\left|y(\frac{1}{2}^-)\right|}{5+\left|y(\frac{1}{2}^-)\right|}\,, \tag{26} $$

$$ y(0) = 0, \quad y'(0) = 0. \tag{27} $$

Set

$$ f(t,x) = \frac{e^{-t}x}{(9+e^t)(1+x)}, \quad (t,x) \in J \times [0,\infty), $$
$$ I_k(x) = \frac{x}{3+x}\,, \quad x \in [0,\infty), $$

and

$$ \overline{I}_k(x) = \frac{x}{5+x}\,, \quad x \in [0,\infty). $$

Let $x, y \in [0,\infty)$ and $t \in J$. Then we have

$$ |f(t,x) - f(t,y)| = \frac{e^{-t}}{(9+e^t)}\left|\frac{x}{1+x} - \frac{y}{1+y}\right| = $$
$$ = \frac{e^{-t}|x-y|}{(9+e^t)(1+x)(1+y)} \leq \frac{e^{-t}}{(9+e^t)}\,|x-y| \leq \frac{1}{10}\,|x-y|. $$

Hence the condition $(H1)$ holds with $l = \frac{1}{10}$.

Let x, $y \in [0,\infty)$. Then we have

$$ |I_k(x) - I_k(y)| = \left|\frac{x}{3+x} - \frac{y}{3+y}\right| = \frac{3|x-y|}{(3+x)(3+y)} \leq \frac{1}{3}\,|x-y|, $$

and

$$ \left|\overline{I}_k(x) - \overline{I}_k(y)\right| \leq \frac{1}{5}\,|x-y|. $$

Hence the condition $(H2)$ holds with $l^* = \frac{1}{3}$ and $\overline{l}^{\,*} = \frac{1}{5}$. We will check that the condition (13) is satisfied with $T = 1$ and $m = 1$. Indeed,

$$ \left[\frac{mlT^\alpha}{\Gamma(\alpha+1)} + \frac{mlT^\alpha}{\Gamma(\alpha)} + \frac{lT^\alpha}{\Gamma(\alpha+1)} + m(l^* + T\overline{l}^{\,*})\right] < 1 \Longleftrightarrow $$
$$ \Longleftrightarrow \frac{1}{5\Gamma(\alpha+1)} + \frac{1}{10\Gamma(\alpha)} < \frac{7}{15}\,, \tag{28} $$

which is satisfied for appropriate values of $\alpha \in (1,2]$. Then by Theorem 3.5 the problem (24)–(27) has a unique solution on $[0,1]$ for such values of α.

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