Mem. Differential Equations Math. Phys. 44 (2008), 155–160

IVAN KIGURADZE

ON PERIODIC TYPE BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER DIFFERENTIAL SYSTEMS

Abstract. For higher order systems of differential equations solvability and correctness conditions of periodic type boundary value problems are established.

რეზიუმე. მაღალი რიგის ჩვეულებრივ დიფერენციალურ განტოლებათა სისტემებისათვის დადგენილია პერიოდული ტიპის სასაზღვრო ამოცანების ამოხსნადობისა და კორექტულობის პირობები.

2000 Mathematics Subject Classification: 34B15.
Key words and phrases: Nonlinear differential system, periodic type problem, existence and uniqueness of a solution.

On the interval $I = [a, b]$, let us consider the nonlinear differential system

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{1}$$

with the nonlinear boundary conditions

$$\sum_{k=1}^{n} \Big(A_{ik}(u) u^{(k-1)}(a) + B_{ik}(u) u^{(k-1)}(b) \Big) = c_i(u) \quad (i = 1, \dots, n). \tag{2}$$

Here $f : I \times R^{nl} \to R^l$ is a vector function from the Carathéodory class, and

$$A_{ik} : C^{n-1}(I; R^l) \to R^{l \times l}, \quad B_{ik} : C^{n-1}(I; R^l) \to R^{l \times l},$$
$$c_i : C^{n-1}(I; R^l) \to R^l$$

are nonlinear continuous operators satisfying some additional conditions.

These conditions are rather general, and, in particular, they are satisfied for a periodic problem, the Dirichlet problem and the Neumann problem

$$u^{(i-1)}(a) = u^{(i-1)}(b) + c_i \quad (i = 1, \dots, n); \tag{3_1}$$

$$u^{(i-1)}(a) = c_i \;\; (i = 1, \dots, m), \;\; u^{(i-1)}(b) = c_{m+i} \;\; (i = 1, \dots, n-m); \tag{3_2}$$

$$u^{(n-i)}(a) = c_i \; (i = 1, \dots, n-m), \;\; u^{(i-1)}(b) = c_{n-m+i} \; (i = 1, \dots, m); \tag{3_3}$$

$$u^{(n-i)}(a) = c_i \; (i = 1, \dots, n-m), \;\; u^{(n-i)}(b) = c_{n-m+i} \; (i = 1, \dots, n-m), \tag{3_4}$$

where m is an integer part of the number $\frac{n}{2}$, $c_i \in R^l$ $(i = 1, \dots, n)$.

Reported on the Tbilisi Seminar on Qualitative Theory of Differential Equations on November 12, 2007.

In the case of even $n = 2m$, the above-mentioned conditions hold for the problem

$$u^{(2i-2)}(a) = c_i \quad u^{(2i-2)}(b) = c_{m+i} \quad (i = 1, \dots, m), \tag{3_5}$$

which is sometimes said to be the Lidston problem.

In the case where $l = 1$, these classical problems has been intensively investigated, and they are the subject of numerous investigations (see, e.g., [1]–[12] and the references therein).

We single out a class of boundary value problems of the type (1), (2) to which belong the above-mentioned two-point boundary value problems (1), (3_k) $(k = 1, 2, 3, 4, 5)$, and we propose a unique method for their investigation.

By the way, we are trying to find general properties which unite the problems from that class.

Before formulating the main results, we introduce the notation.

R^l is the l-dimensional real Euclidean space with the norm $\|\cdot\|_{R^l}$.

$R^{l\times l}$ is the space of real $l \times l$ matrices with the norm $\|\cdot\|_{R^{l\times l}}$.

$x \cdot y$ is the scalar product of the vectors $x, y \in R^l$.

$C^{n-1}(I; R^l)$ is the Banach space of $(n-1)$-times continuously differentiable vector functions $u : I \to R^l$ with the norm

$$\|u\|_{C^{n-1}} = \max\left\{ \sum_{k=1}^{n} \|u^{(k-1)}(t)\|_{R^l} : t \in I \right\}.$$

$\widetilde{C}^{n-1}(I; R^l)$ is the space of all vector functions $u \in C^{n-1}(I; R^l)$ for which $u^{(n-1)}$ is absolutely continuous.

Of course, we will seek for a solution of the problem (1), (2) in the space $\widetilde{C}^{n-1}(I; R^l)$.

For arbitrary x_k and $y_k \in R^l$ $(k = 1, \dots, n)$, we set

$$\nu(x_1, \dots, x_n; y_1, \dots, y_n) =$$
$$= \begin{cases} \sum\limits_{k=1}^{m} (-1)^k (x_k \cdot x_{n-k+1} - y_k \cdot y_{n-k+1}) & \text{for} \quad n = 2m, \\ \sum\limits_{k=1}^{m} (-1)^k (x_k \cdot x_{n-k+1} - y_k \cdot y_{n-k+1}) - \\ \quad -\frac{(-1)^m}{2} \left(\|x_{m+1}\|_{R^l}^2 - \|y_{m+1}\|_{R^l}^2 \right) & \text{for} \quad n = 2m+1. \end{cases}$$

We study the problem (1), (2) in the case where there exists a positive number μ such that for any $x_k, y_k \in R^l$ and $v \in C^{n-1}$ the operators A_{ik} and B_{ik} satisfy the inequality

$$(-1)^m \nu(x_1, \dots, x_n; y_1, \dots, y_n) \le$$
$$\le \mu \sum_{k=1}^{m} \left(\|x_k\|_{R^l} + \|y_k\|_{R^l} \right) \sum_{i=1}^{n} \left\| \sum_{k=1}^{n} \left(A_{ik}(v) x_k + B_{ik}(v) y_k \right) \right\|_{R^l}. \tag{4}$$

As for the vector function f, it satisfies the condition

$$p_*(t, \|x_1\|_{R^l}) \le (-1)^{m+1} f(t, x_1, \dots, x_n) \cdot x_1 \le p^*(t, \|x_1\|_{R^l}) \tag{5}$$

on the set $I \times R^n$, where p_* and $p^* : I \times [0, +\infty[\to R$ are non-decreasing with respect to a second argument Carathéodoty functions such that

$$\lim_{\rho \to +\infty} \int_a^b p_*(t, \rho)\, dt = +\infty. \tag{6}$$

We will also assume that

$$A_{ik}, \ \ B_{ik} \ \text{ and } \ c_i \ \ (i, k = 1, \dots, n) \ \text{ are bounded in } \ C^{n-1}(I; R^l). \tag{7}$$

Theorem 1. *If the conditions* (4)–(7) *hold, then the problem* (1), (2) *has at least one solution.*

As it is already said above, for the periodic problem, the Dirichlet problem, the Neumann and Lidston problems, and the mixed problem, the condition (4) is automatically satisfied. Therefore it is clear that if the conditions (5)–(7) hold, then each of the above-mentioned problems (1), (3_k), $k \in \{1, 2, 3, 4, 5\}$, has at least one solution.

We will give also examples of another well-known boundary value problems, for which the condition (4) is satisfied. More precisely, in the case of even $n = 2m$, we consider the problems

$$\begin{aligned} &\alpha_i u^{(i-1)}(a) + \alpha_{m+i} u^{(n-i)}(a) = c_i, \\ &\beta_i u^{(i-1)}(b) + \beta_{m+i} u^{(n-i)}(b) = c_{m+i}, \quad (i = 1, \dots, m) \end{aligned} \tag{8_{2m}}$$

and

$$\begin{aligned} &u^{(i-1)}(a) = \eta_i u^{(i-1)}(b) + c_i, \\ &u^{(n-i)}(b) = \eta_i u^{(n-i)}(a) + c_{m+i}, \quad (i = 1, \dots, m), \end{aligned} \tag{9_{2m}}$$

while in the case of $n = 2m + 1$, we consider the problems

$$\begin{aligned} \alpha_i u^{(i-1)}(a) + \alpha_{m+i} u^{(n-i)}(a) = c_i, \quad \beta_i u^{(i-1)}(b) + \beta_{m+i} u^{(n-i)}(b) = \\ = c_{m+i}, \quad (i = 1, \dots, m), \quad u^{(m)}(a) = \eta u^{(m)}(b) + c_n, \end{aligned} \tag{8_{2m+1}}$$

and

$$\begin{aligned} u^{(i-1)}(a) = \eta_i u^{(i-1)}(b) + c_i, \quad u^{(n-i)}(b) = \eta_i u^{(n-i)}(a) + \\ + c_{m+i}, \quad (i = 1, \dots, m), \quad u^{(m)}(a) = \eta u^{(m)}(b) + c_n, \end{aligned} \tag{9_{2m+1}}$$

where α_i, β_i, η_i, and η are real numbers, $c_i \in R^l$. Moreover,

$$\begin{gathered} (-1)^{m+i} \alpha_i \alpha_{m+i} \ge 0, \quad |\alpha_i| + |\alpha_{m+i}| > 0, \\ (-1)^{m+i} \beta_i \beta_{m+i} \le 0, \quad |\beta_i| + |\beta_{m+i}| > 0, \quad \eta_i \ne 0, \quad |\eta| \le 1. \end{gathered} \tag{10}$$

From Theorem 1 it follows the following

Corollary 1. *If the conditions* (5)–(7) *and* (10) *hold, then both problems* (1), (8_n) *and* (1), (9_n) *have at least one solution.*

Under the conditions of Theorem 1, the components of the vector function $(t, x_1, \dots, x_n) \to f(t, x_1, \dots, x_n)$ may have an arbitrary growth order with respect to x_1. As an example, we consider the nonlinear differential system

$$u_k^{(n)} = (-1)^{m+1} g_k(t, u_1, \dots, u_l)|u_k|^{\mu_k} \operatorname{sgn} u_k + \\ + h_k(t, u_1, \dots, u_l) \quad (k = 1, \dots, l) \tag{11}$$

with the periodic boundary conditions

$$u_k^{(i-1)}(a) = u_k^{(i-1)}(b) + c_{ik} \quad (i = 1, \dots, n; \quad k = 1, \dots, l), \tag{12}$$

or with the Neumann conditions

$$\begin{aligned} &u_k^{(i-1)}(a) = c_{ik} \ \ (i = 1, \dots, m; \quad k = 1, \dots, l), \\ &u_k^{(i-1)}(b) = c'_{ik} \ \ (i = 1, \dots, n-m; \quad k = 1, \dots, l), \end{aligned} \tag{13}$$

where $\mu_k > 0$, c_{ik}, c'_{ik} are constants, and $g_k : I \times R^2 \to R$ and $h_k : I \times R^l \to R^l$ are Carathéodory functions.

From Theorem 1 for these problem we have

Corollary 2. *Let there exist an integrable with respect to the first argument and non-decreasing with respect to the second argument function* $g_0 : I \times [0, +\infty[\to]0, +\infty[$ *, and a positive constant* r *such that on the set* $I \times R^l$ *the inequalities*

$$\begin{aligned} &g_k(t, u_1, \dots, u_l) \ge g_0\bigg(t, \sum_{i=1}^{l} |u_i|\bigg), \\ &|h_k(t, u_1, \dots, u_l)| \ge r g_0\bigg(t, \sum_{i=1}^{l} |u_i|\bigg) \quad (k = 1, \dots, l) \end{aligned} \tag{14}$$

hold. Then both problems (11), (12) *and* (11), (13) *have at least one solution.*

It is evident that under the conditions of Corollary 2, the numbers $\mu_k > 0$ $(k = 1, \dots, l)$ may be arbitrarily large, and $(t, u_1, \dots, u_l) \to g_k(t, u_1, \dots, u_l)$ $(k = 1, \dots, l)$ may be arbitrary functions, increasing rapidly with respect to $u_1, \dots, u_l$.

We can prove the unique solvability of the problem (1), (2) and its stability with respect to small perturbations of the right-hand side of the system (1) and boundary data only in the case where

$$\begin{gathered} f(t, x_1, \dots, x_n) \equiv f(t, x_1), \\ A_{ik}(v) \equiv A_{ik}, \quad B_{ik}(v) \equiv B_{ik}, \quad c_i(v) \equiv c_i \quad (i, k = 1, \dots, n), \end{gathered}$$

where A_{ik} and B_{ik} are constant $l \times l$ matrices, and c_i is a constant l-dimensional vector.

Thus let us consider the boundary value problem

$$u^{(n)} = f(t,u), \tag{15}$$

$$\sum_{k=1}^{n} \Big(A_{ik}u^{(k-1)}(a) + B_{ik}u^{(k-1)}(b)\Big) = c_i \quad (i=1,\dots,n) \tag{16}$$

and the corresponding perturbed problem

$$v^{(n)} = f(t,v) + h(t), \tag{15$'$}$$

$$\sum_{k=1}^{n} \Big(A_{ik}v^{(k-1)}(a) + B_{ik}v^{(k-1)}(b)\Big) = c_i' \quad (i=1,\dots,n). \tag{16$'$}$$

We will use the following definition of the well-posedness.

The problem (15), (16) is said to be well-posed if for any $\varepsilon > 0$ there exists $\delta > 0$, such that the problem (15′), (16′) is uniquely solvable for an arbitrary integrable vector function $h : I \to R^l$ and vectors $c_i' \in R^l$ $(i=1,\dots,n)$ satisfying the condition

$$\sum_{i=1}^{n} \|c_i - c_i'\|_{R^l} + \bigg\| \int_a^t h(s)\,ds \bigg\|_{R^l} < \delta \quad \text{for} \quad t \in I,$$

and the inequality

$$\|v-u\|_{C^{n-1}} < \varepsilon$$

holds, where u and v are solutions of the problems (15), (16) and (15′), (16′), respectively.

Theorem 2. *Let there exist $\mu > 0$ such that for arbitrary x_k and $y_k \in R^l$ $(k=1,\dots,n)$ the inequality*

$$(-1)^{m+1}\nu(x_1,\dots,x_n;y_1,\dots,y_n) \le \\ \le \mu \sum_{k=1}^{n} (\|x_k\|_l + \|y_k\|_l) \sum_{i=1}^{n} \bigg\| \sum_{k=1}^{n} (A_{ik}x_k + B_{ik}y_k) \bigg\|_{R^l} \tag{17}$$

holds. Moreover, the vector function f satisfies the condition

$$(-1)^{m+1}\,(f(t,x) - f(t,y))\cdot(x-y) \ge p(t)\,\|x-y\|^2_{R^l}, \tag{18}$$

where $p : I \to [0,+\infty[$ is an integrable function positive on the set of positive measure. Then the problem (15),(16) *is well-posed.*

From this theorem it follows, in particular, that if the condition (18) is satisfied, then the above-mentioned classical two-point boundary value problems for the system (16) are well-posed.

Acknowledgement

This work was supported by the Georgian National Science Foundation (Grant No GNSF/ST06/3-002).

References

1. R. P. Agarwal, Focal boundary value problems for differential and difference equations. *Kluwer Academic Publishers, Dordrecht–Boston–London,* 1998.
2. R. P. Agarwal and I. Kiguradze, Two-point boundary value problems for higher-order linear differential equations with strong singularities. *Bound. Value Probl.* **2006**, Art. ID 83910, 32 pp.
3. R. P. Agarwal and D. O'Regan, Singular differential and integral equations with applications. *Kluwer Academic Publishers, Dordrecht,* 2003.
4. P. W. Bates and J. R. Ward, Jr., Periodic solutions of higher order systems. *Pacific J. Math.* **84**(1979), No. 2, 275–282.
5. S. R. Bernfeld and V. Lakshmikantham, introduction to nonlinear boundary value problems. *Mathematics in Science and Engineering,* Vol. 109. *Academic Press, Inc., New York–London,* 1974.
6. R. E. Gaines and J. L. Mawhin, Coincidence degree, and nonlinear differential equations. *Lecture Notes in Mathematics,* Vol. 568. *Springer-Verlag, Berlin–New York,* 1977.
7. G. T. Gegelia, On boundary value problems of periodic type for ordinary odd order differential equations. *Arch. Math. (Brno)* **20**(1984), No. 4, 195–203.
8. G. T. Gegelia, Boundary value problems of periodic type for ordinary differential equations. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **17**(1986), 60–93.
9. I. Kiguradze and T. Kusano, On conditions for the existence and uniqueness of a periodic solution of nonautonomous differential equations. (Russian) *Differ. Uravn.* **36**(2000), No. 10, 1301–1306; English transl.: *Differ. Equ.* **36**(2000), No. 10, 1436–1442
10. I. Kiguradze and T. Kusano, On periodic solutions of even-order ordinary differential equations. *Ann. Mat. Pura Appl. (4)* **180**(2001), No. 3, 285–301.
11. I. Kiguradze, On nonlinear boundary value problems for higher-order ordinary differential equations. *Differential & difference equations and applications*, 529–540, *Hindawi Publ. Corp., New York,* 2006.
12. I. Kiguradze, On periodic type boundary value problems for high order ordinary differential equations. (Russian) *Differ. Uravn.* **42**(2006), No. 11, 1577–1579.

(Received 14.11.2007)

Author's address:

A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: kig@rmi.acnet.ge