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POSITIVENESS OF THE CAUCHY FUNCTION AND STABILITY OF A LINEAR DIFFERENTIAL EQUATION WITH DISTRIBUTED DELAY

Dedicated to the blessed memory of Professor N. V. Azbelev

Abstract. Effective (in terms of the parameters of the problem under consideration) conditions are presented for positiveness of the Cauchy function of a certain class of functional differential equations with distributed delay. From this result new conditions for exponential stability of solutions are obtained.

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1. Definitions and Notation

Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\Delta = \{(t,s) \in \mathbb{R}^2 \colon\ t \geq s\}$. Denote by L_∞ the space of measurable and essentially bounded on $\mathbb{R}$ functions with the natural norm.

Consider the functional differential equation

$$(Lx)(t) \equiv \dot{x}(t) + a \int\limits_{t-r(t)}^{t} x(s)\,ds = f(t), \ \ t \in \mathbb{R}_+ \tag{1}$$

$$\big(x(\xi) = 0 \ \text{ for } \ \xi < 0 \big).$$

We assume that the delay $r \colon \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and bounded on $\mathbb{R}_+$, the coefficient a is a real constant, and the function $f \colon \mathbb{R}_+ \to \mathbb{R}$ is locally integrable.

By a solution of the equation (1) we mean a function $x \colon \mathbb{R}_+ \to \mathbb{R}$ that is absolutely continuous on any segment and satisfies (1) almost everywhere.

As is known ([1], p. 35), under the assumptions described above the equation (1) with the initial condition $x(0) = c$ is uniquely solvable for any c and f. Moreover, there exists a function $C \colon \Delta \to \mathbb{R}$ such that any solution of (1) can be represented by the formula

$$x(t) = C(t,0)x(0) + \int\limits_0^t C(t,s)f(s)\,ds. \tag{2}$$

The function C is called *the Cauchy function* of the equation (1). It is the central object in studies of properties of solutions to (1), as the first and the second terms of the right-hand side of (2) describe the $x(0)$-dependence and the f-dependence respectively of solutions to (1). Clearly, any property of the Cauchy function determines a certain property of all solutions of (1).

2. Positiveness of the Cauchy Function

The problem on conditions for the Cauchy function to preserve a sign is worth taking into detailed consideration, in so far as it follows from the formula (2) that if the Cauchy function is positive, then the first term preserves its sign, and the second one is a nonnegative integral operator.

This problem for various classes of functional differential equations in terms of parameters of the equation was studied in the works [2], [3], [4]. In particular, the *lemma on differential inequality* was proved and usefully employed. We will also use that lemma.

Assume that $a \geq 0$ and r is bounded. Set $r = \sup\limits_{t \in \mathbb{R}_+} r(t)$.

The case $a = 0$ in (1) is trivial as here we have $C(t,s) = 1 > 0$.

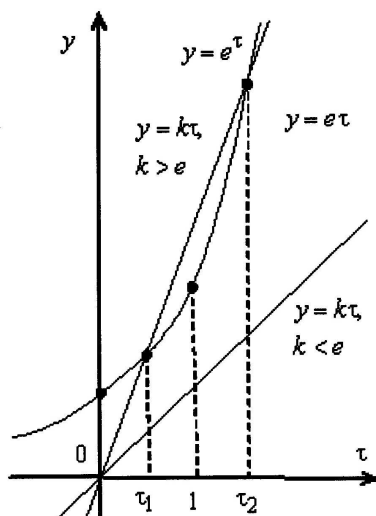Let us next assume $a \neq 0$. State the lemma on differential inequality in the convenient form.

FIGURE 1

Lemma 1 ([4], p. 65)**.** *Suppose $a > 0$ in the equation* (1)*, and there exists an absolutely continuous function ν such that $\nu(t) > 0$ for all $t \in \mathbb{R}_+$ and $(L\nu)(t) \leq 0$ for almost all $t \in \mathbb{R}_+$. Then $C(t,s) \geq \nu(t)/\nu(s)$ for all $t \geq s \geq 0$.*

Let $\nu(t) = e^{-\alpha t} > 0$ for $\alpha > 0$. Applying Lemma 1 to (1), we obtain

$$(L\nu)(t) = -\alpha e^{-\alpha t} - \frac{a}{\alpha}\left(e^{-\alpha t} - e^{-\alpha t + \alpha r(t)}\right) =$$
$$= -e^{-\alpha t}\left(\alpha + \frac{a}{\alpha}\left(1 - e^{\alpha r(t)}\right)\right) \leq -e^{-\alpha t}\left(\alpha + \frac{a}{\alpha}\left(1 - e^{\alpha r}\right)\right).$$

It is obvious that if there exists $\alpha > 0$ such that $\alpha + \frac{a}{\alpha}\left(1 - e^{\alpha r}\right) \geq 0$, then $(L\nu)(t) \leq 0$.

Let $\tau = \alpha r$ and let us define the function $\varphi(\tau) = \tau^2 + ar^2(1 - e^\tau)$ on the set $\tau > 0$.

As a result, our problem is reduced to the following one: *what conditions must the parameter ar^2 satisfy in order that there would exist at least one point τ such that $\varphi(\tau) \geq 0$?*

Let $k = \frac{2}{ar^2} > 0$. Find the derivatives of φ with respect to τ:

$$\varphi'(\tau) = ar^2(k\tau - e^\tau), \quad \varphi''(\tau) = ar^2(k - e^\tau).$$

At the point $\tau_0 = \ln k$ the function $\varphi''(\tau)$ changes its sign plus (for $\tau < \tau_0$) to minus (for $\tau > \tau_0$), that is, τ_0 is a point of maximum of φ'; we have $\varphi'(\tau_0) = ar^2 k(\ln k - 1)$. Now consider three cases, which are accompanied, for the sake of clearness, by graphic illustrations (see Figure 1).

1. Suppose $\ln k < 1$, that is, $k < e$. Then $\varphi'(\tau) < 0$ for all $\tau > 0$ (we see the graph of the function $y = k\tau$ lying lower than that of the function $y = e^\tau$), and the function φ monotonically decreases for all $\tau > 0$. Since $\lim_{\tau \to +0} \varphi(\tau) = 0$, we see that $\varphi(\tau) < 0$ for all $\tau > 0$. Hence, for $k < e$, there exists no point τ such that $\varphi(\tau) \geq 0$.

2. Suppose $\ln k = 1$, that is $k = e$ and $\tau_0 = 1$. Then $\varphi'(\tau) < 0$ for all $\tau \neq 1$, and $\varphi'(1) = 0$ (the graph of the function $y = e\tau$ is a tangent to that of the function $y = e^\tau$). So the function φ monotonically decreases for all $\tau > 0$, and the point $\tau_0 = 1$ is the point of inflection for the graph of φ. Since $\lim_{\tau \to +0} \varphi(\tau) = 0$, we have $\varphi(\tau) < 0$ for all $\tau > 0$. Hence, for $k = e$, there exists also no point τ such that $\varphi(\tau) \geq 0$.

3. Suppose $\ln k > 1$, that is $k > e$. Then the graphs of the functions $y = k\tau$ and $y = e^\tau$ have exactly two points of intersection. Denote by τ_1 and τ_2 the τ-coordinates of those points. It is obvious that $\tau_1 < 1 < \tau_2$. So we have $\varphi'(\tau_1) = \varphi'(\tau_2) = 0$, $\varphi'(\tau) < 0$ for $\tau < \tau_1$ or $\tau > \tau_2$, and $\varphi'(\tau) > 0$ for $\tau_1 < \tau < \tau_2$. Hence $\tau = \tau_1$ is a point of minimum for the function φ, and $\tau = \tau_2$ is its point of maximum. On the interval (τ_1, τ_2) the function φ monotonically increases, and its graph may go into the upper half-plane. It is clear that *this situation is possible if and only if $\varphi(\tau_2) \geq 0$.*

Thus, our problem is reduced to the following one: *for what values of the parameter $k > e$ there exists a point ξ such that both of the following relations hold:*

$$k\xi = e^\xi, \tag{3}$$

$$\xi^2 + \frac{2}{k}\left(1 - e^\xi\right) \geq 0. \tag{4}$$

Consider the function $\omega(\xi) = \xi e^{-\xi}$ for $\xi \geq 0$. For $0 \leq \xi < 1$ the function ω increases monotonically from 0 to $1/e$, reaching at $\xi = 1$ its maximum which equals $1/e$, for $\xi > 1$ it decreases monotonically and tends to zero asymptotically. Hence, for $\xi \geq 1$, there exists the inverse function ω^{-1} that is defined on half-interval $(0, 1/e]$ and decreases monotonically, with $\omega^{-1}(0, 1/e] = [1, \infty)$.

There is $\xi > 1$ and $k > e$ in the equality (3). Therefore $1/k \in (0, 1/e]$, and so (3) can be represented in the equivalent form $\xi = \omega^{-1}(1/k)$. Considering (3), rewrite the left-hand side of (4) in the following way:

$$\xi^2 + 2/k(1 - e^\xi) = \xi^2 + 2/k(1 - k\xi) = \xi^2 - 2\xi + 2/k =$$
$$= \left(\xi - \left(1 + \sqrt{1 - 2/k}\right)\right)\left(\xi - \left(1 - \sqrt{1 - 2/k}\right)\right). \tag{5}$$

Our concern is only with solutions of the inequality (4) such that $\xi > 1$. It follows from (5) that $\xi \geq 1 + \sqrt{1 - 2/k}$. This implies that ξ is a solution of the system (3)–(4) if and only if

$$\xi = \omega^{-1}(1/k) \text{ and } \xi \geq 1 + \sqrt{1 - 2/k}.$$

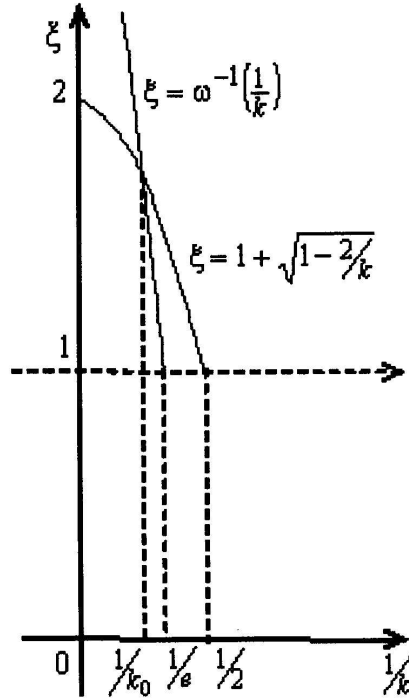$$\xi = \omega^{-1}\left(\frac{1}{k}\right)$$

$$\xi = 1 + \sqrt{1 - 2/k}$$

FIGURE 2

It is obvious that the graphs of the functions which are the right-hand sides of the latter two relations intersect each other at a single point (see Figure 2). Denote by $1/k_0$ the abscissa of this point. Then the solution of the system (3)–(4) is the set $0 < 1/k \leq 1/k_0$.

It remains to find k_0. By construction, k_0 is the unique root of the equation

$$\omega^{-1}(1/k) = 1 + \sqrt{1 - 2/k}. \tag{6}$$

Let $\mu = 1 + \sqrt{1 - 2/k_0}$ and use the designation for the function ω. Then we obtain from (6) that μ is the positive root of the equation

$$e^{-\mu} = 1 - \mu/2. \tag{7}$$

We have the approximation $1{,}59 < \mu < 1{,}6$.

Theorem 1. *Suppose*

$$\sqrt{a} \sup_{t \in \mathbb{R}_+} r(t) \leq \sqrt{\mu(2 - \mu)}, \tag{8}$$

where μ is the positive root of (7). *Then the Cauchy function of the equation* (1) *is positive.*

Remark 1. Approximate calculations give us the following estimate of the constant in the right-hand side of (8): $0{,}8 < \sqrt{\mu(2-\mu)} < 0{,}81$.

3. STABILITY

Let us demonstrate the way to apply the obtained conditions of the positiveness of the Cauchy function to investigation of stability of the equations (1). Note that it is suitable to formulate conditions for stability in terms of the Cauchy function.

The equation (1) is said to be *exponentially stable* if there exist positive constants N and α such that for all $(t, s) \in \Delta$ the following estimate holds:

$$|C(t, s)| \le N e^{-\alpha(t-s)}. \tag{9}$$

Consider (1) for $r(t) \equiv r \ge 0$ and $f(t) \equiv 0$:

$$\dot{x}(t) + a \int_{t-r}^{t} x(s)\, ds = 0, \quad t \in \mathbb{R}_+ \tag{10}$$

$$\big(\, x(\xi) = 0 \ \text{ for } \ \xi < 0 \,\big).$$

Denote by x_0 the solution of (10) satisfying the initial condition $x_0(0) = 1$. Since the equation (10) is autonomous, the function $x_0(t - s)$ is the Cauchy function. For the equation (10), a criterion of asymptotical (which is here exponential) stability is known.

Lemma 2 ([5]). *The equation* (10) *is exponentially stable if and only if* $0 < r\sqrt{a} < \pi/\sqrt{2}$.

Lemma 3. *Suppose the equation* (10) *is exponentially stable. Then* $\int_{0}^{\infty} x_0(s)\, ds = 1/ar$.

Proof. Substitute the function $x_0(t)$ into the equality (10) and integrate along the segment $[0, t]$:

$$x_0(t) - 1 = -a \int_{0}^{t} \int_{s-r}^{s} x(\tau)\, d\tau\, ds.$$

Change the integration order and pass to the limit as $t \to \infty$:

$$1 - \lim_{t \to \infty} x_0(t) = \lim_{t \to \infty} \left(a \int_{0}^{t-r} x_0(\tau) \int_{\tau}^{\tau+r} ds\, d\tau + a \int_{t-r}^{t} x_0(\tau) \int_{\tau}^{t} ds\, d\tau \right).$$

Since $x_0(t)$ has an exponential estimate, using the latter equality we obtain

$$1 = ar \int_0^\infty x_0(s)\, ds,$$

as was required.   □

In the proof of the next theorem we use a method suggested by S. A. Gusarenko in the paper [6].

Theorem 2. *Suppose in the equation* (1)

$$0 < \sqrt{a}\, \inf_{t\in\mathbb{R}_+} r(t) \le \sqrt{a}\, \sup_{t\in\mathbb{R}_+} r(t) < 2\sqrt{\mu(2-\mu)},$$

where μ is the positive root of the equation (7). *Then the equation* (1) *is exponentially stable.*

Proof. Let $r = \dfrac{\sqrt{\mu(2-\mu)}}{\sqrt{a}}$ and let us rewrite (1) in the form

$$\dot{x}(t) + a \int_{t-r}^{t} x(s)\, ds = a \int_{t-r}^{t-r(t)} x(s)\, ds + f(t), \quad t \in \mathbb{R}_+.$$

Applying the formula (2), we can represent the latter equality in the following integral form

$$x(t) = (Kx)(t) + g(t), \tag{11}$$

where

$$(Kx)(t) = a \int_0^t x_0(t-s) \int_{s-r}^{s-r(s)} x(\tau)\, d\tau\, ds,$$

$$g(t) = \int_0^t x_0(t-s) f(s)\, ds + x_0(t) x(0),$$

and $x_0(t-s)$ is the Cauchy function of the equation (10).

By virtue of the choise of r and according to the remark after Theorem 1, we have $r\sqrt{a} < 0{,}81 < \pi/\sqrt{2}$, that is, by Lemma 2 the function $x_0(t-s)$ admits the exponential estimate (9).

Suppose $f \in L_\infty$. Then $g \in L_\infty$, and the operator K maps L_∞ into L_∞. Let us estimate the norm of K:

$$\|Kx\| = \sup_{t \in \mathbb{R}_+} a \left| \int\limits_0^t x_0(t-s) \int\limits_{s-r}^{s-r(s)} x(\tau)\, d\tau\, ds \right| \leq$$

$$\leq \left(\sup_{t \in \mathbb{R}_+} a \int\limits_0^t |x_0(t-s)| \sup_{s \in \mathbb{R}_+} |r(s) - r|\, ds \right) \|x\| =$$

$$= \left(\sup_{t \in \mathbb{R}_+} a|r(t) - r| \int\limits_0^\infty |x_0(s)|\, ds \right) \|x\|.$$

From Theorem 1 and by virtue of the choice of r it follows that $x_0(t) > 0$, hence $|x_0(t)| = x_0(t)$. Now by Lemma 3, $\int\limits_0^\infty x_0(s)\, ds = 1/ar$.

Taking into account the assumptions of the theorem, we obtain $\|K\| < 1$. Applying the contraction mapping principle, we conclude that the equation (1) has a solution that is bounded in $\mathbb{R}_+$. According to the Bohl–Perron theorem ([4], p. 103, th. 3.3.1), it follows that the equation (1) is exponentially stable. $\qquad\square$

Corollary 1. *Suppose in the equation* (1)

$$0 < \sqrt{a}\, \varliminf_{t \to \infty} r(t) \leq \sqrt{a}\, \varlimsup_{t \to \infty} r(t) < 2\sqrt{\mu(1-\mu)},$$

where μ is the positive root of the equation (7). *Then the equation* (1) *is exponentially stable.*

Proof follows from the fact that the Cauchy function of the equation (1) is bounded on any strip of finite width $t - s \leq T$. $\qquad\square$

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