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EXISTENCE OF EXTREMAL SOLUTIONS FOR NONLINEAR DISCONTINUOUS IMPULSIVE FUNCTIONAL ϕ-LAPLACIAN EQUATIONS WITH NONLINEAR DISCONTINUOUS FUNCTIONAL BOUNDARY CONDITIONS

Abstract. We derive sufficient conditions for the existence of extremal solutions for a second order nonlinear functional ϕ-Laplacian boundary value problems with impulses, subject to boundary value conditions of general type which cover Dirichlet and multipoint boundary data as particular cases. Our approach is that of upper and lower solutions together with growth restrictions of Nagumo's type. Discontinuous functional dependence of the nonlinear data and the boundary conditions are allowed.

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[illegible]

1. Introduction

In [2] it is considered the nonlinear impulsive boundary value problem

$$(\phi(u'(t)))' = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0, t] \setminus P, \tag{1.1}$$

$$\left.\begin{aligned} g_1(u(0), u) = 0, \\ g_2(u(T), u) = 0, \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} I_k(u(t_k), u) = 0, \\ M_k(u(t_k^+), u) = 0, \end{aligned}\right\} \text{ for } k = 1, \ldots, p. \tag{1.3}$$

Here $p \in \mathbb{N}$ is fixed, $P = \{t_1, \ldots, t_p\}$, and

$$0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T.$$

Moreover, the function $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function:

(i) for each $x \in \mathbb{R}^2$ the function $f(\cdot, x)$ is measurable on $[0, T]$;
(ii) for almost each $t \in [0, T]$ the function $f(t, \cdot)$ is continuous on $\mathbb{R}^2$;
(iii) for each compact set $K \subset \mathbb{R}^2$, there is a function $m_K(t) \in L^1[0, T]$ such that

$$|f(t, x)| \le m_K(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } x \in K.$$

The boundary conditions (1.2) cover, among others, the Dirichlet

$$u(0) = A, \quad u(T) = B,$$

and the multipoint boundary conditions

$$u(0) = \sum_{i=1}^{m} a_i \, u(\eta_i), \quad u(T) = \sum_{j=1}^{n} b_j \, u(\rho_j),$$

with $a_i \ge 0$ and $\eta_i \in (0, T]$ for $i = 1, \ldots, m$, $b_j \ge 0$ and $\rho_j \in [0, T)$ for $j = 1, \ldots, n$.

It is enough to define $g_1(x, v) = x - A$ and $g_2(x, v) = x - B$ for the first case and $g_1(x, v) = x - \sum_{i=1}^{m} a_i u(\eta_i)$, $g_2(x, v) = x - \sum_{j=1}^{n} b_j u(\rho_j)$ for the multipoint problem.

Moreover, various nonlinear functional boundary conditions as

$$u(0) = \int_J u^l(s) \, ds, \quad u(T) = \min_{t \in K} \{u(t)\},$$

with $l \in \mathbb{N}$ odd and J, $K \subset I$ two measurable sets, can be considered under this formulation.

To define the concept of solution of the problem (1.1)–(1.3), we introduce in this section some suitable notation and definitions.

For a real valued measurable function u defined on the interval $I \subset \mathbb{R}$, we put for all $q \ge 1$

$$\|u\|_q = \left(\int_I |u(s)|^q \mathrm{d}s \right)^{\frac{1}{q}}$$

and

$$\|u\|_\infty = \sup_{t\in I} \operatorname{ess} |u(t)|.$$

For given Banach spaces A and B, let $C^0(A;B)$ be the set of all functions $f : A \to B$ which are continuous on A. If $B = \mathbb{R}$, we write $C^0(A)$. Furthermore, let $C^m(I)$ be the set of the functions having continuous derivatives of order $i = 0, \ldots, m$ on I. For $1 \le q \le \infty$, we define $L^q(I)$ as the set of Lebesgue measurable on I functions u such that $\|u\|_q$ is finite. $W^{m,q}(I)$ will be the set of the functions $u \in C^{m-1}(I)$ with $u^{(m-1)}$ absolutely continuous in I and $u^{(m)} \in L^q(I)$.

It is well–known that $C^m(I)$ and $W^{m,q}(I)$ are Banach spaces with the norms

$$\|u\|_{C^m(I)} = \max_{k=0,\ldots,m} \|u^{(k)}\|_\infty$$

and

$$\|u\|_{W^{m,q}(I)} = \max_{k=0,\ldots,m} \|u^{(k)}\|_q.$$

We denote $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$ for all $k = 1, \ldots, p$. Moreover,

$$\begin{aligned} C^m_P &= \big\{u : [0,T] \to \mathbb{R} : \ u \in C^m(J_k), \ k = 0, \ldots, p, \text{ there exist } u^{(l)}(t_k^+), \\ &\qquad k = 1, \ldots, p, \ u^{(l)}(t_k^-) \equiv u^{(l)}(t_k), \ k = 1, \ldots, p+1; \ l = 0, \ldots, m\}, \\ W^{m,q}_P &= \big\{u : [0,T] \to \mathbb{R} : \ u_{|J_k} \in W^{m,q}(J_k), k = 0, \ldots, p\big\} \end{aligned}$$

for $m \in \mathbb{N} \cup \{0\}$ and $1 \le q \le \infty$.

It is not difficult to verify that the spaces C^m_P and $W^{m,q}_P$ are Banach spaces with the norms

$$\|u\|_{C^m_P} = \max_{k=0,\ldots,p} \|u_{|J_k}\|_{C^m(J_k)}$$

and

$$\|u\|_{W^{m,q}_P} = \max_{k=0,\ldots,p} \|u_{|J_k}\|_{W^{m,q}(J_k)}.$$

Remark 1.1. Let us note that the convergence of a sequence $\{u_n\} \subset C^m_P$ (resp. $W^{m,q}_P$) in this space is equivalent to the convergence of all sequences $\{u_{n|J_k}\}$ in $C^m(J_k)$ (resp. $W^{m,q}(J_k)$) for each $k = 0, \ldots, p$.

Given $v \le w$ in C^0_P, we denote by $[v, w]$ the following set:

$$[v, w] = \big\{u \in C^0_P, \ v(t) \le u(t) \le w(t) \text{ for all } t \in I\big\}.$$

Now we are in a position to define the concept of solution of the problem (1.1)–(1.3) as follows.

Definition 1.1. A function $u \in C^1_P$, such that $\phi \circ u' \in W^{1,1}_P$, which satisfies the equation (1.1) and fulfills the conditions (1.2) and (1.3) is called a solution of the problem (1.1)–(1.3).

We say that a solution $x \in V$ of the problem (1.1)–(1.3) is the maximal solution of this problem in the set V if given any other solution $w \in V$ it holds $w \leq x$. If the reversed inequalities hold, the solution x will be the minimal solution of the problem (1.1)–(1.3) in V. We refer to both functions as extremal solutions in V.

Now we define the concept of lower and upper solutions of the problem (1.1)–(1.3).

Definition 1.2. A function $\alpha \in W_P^{1,\infty}$ is called a lower solution of the problem (1.1)–(1.3) if for each $t_0 \in (0,T) \setminus P$ either

$$D_{-}\alpha(t_0) < D^{+}\alpha(t_0)$$

or there exists an open interval $I_0 \subset (0,T) \setminus P$ such that $t_0 \in I_0$, $\phi \circ \alpha' \in W^{1,1}(I_0)$ and

$$(\phi(\alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)) \text{ for a.e. } t \in I_0. \tag{1.4}$$

Moreover, for all $k = 1, \ldots, p$, the following boundary value conditions are satisfied:

$$I_k(\alpha(t_k), \alpha) \geq 0 \geq M_k(\alpha(t_k^{+}), \alpha)$$

and

$$g_1(\alpha(0), \alpha) \geq 0 \geq g_2(\alpha(T), \alpha).$$

Definition 1.3. A function $\beta \in W_P^{1,\infty}$ is called an upper solution of the problem (1.1)–(1.3) if for each $t_0 \in (0,T) \setminus P$ either

$$D_{-}\beta(t_0) > D^{+}\beta(t_0)$$

or there exists an open interval $I_0 \subset (0,T) \setminus P$ such that $t_0 \in I_0$, $\phi \circ \beta' \in W^{1,1}(I_0)$ and

$$(\phi(\beta'(t)))' \leq f(t, \beta(t), \beta'(t)) \text{ for a.e. } t \in I_0.$$

Moreover, for all $k = 1, \ldots, p$, the following boundary value conditions are satisfied

$$I_k(\beta(t_k), \beta) \leq 0 \leq M_k(\beta(t_k^{+}), \beta)$$

and

$$g_1(\beta(0), \beta) \leq 0 \leq g_2(\beta(T), \beta).$$

The existence of extremal solutions for (1.1)–(1.3) is proven in [2] under the following assumptions on the functions f, g_1, g_2, M_k and I_k:

(H_1) $f \in Car([0,T] \times \mathbb{R}^2)$.

(H_2) $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function.

(H_3) There exist a lower solution α and an upper solution β of the problem (1.1)–(1.3) such that

$$\alpha(t) \leq \beta(t) \text{ for each } t \in [0,T].$$

(H_4) $g_1 : \mathbb{R} \times C^0_P([0,T]) \to \mathbb{R}$ is continuous and $g_1(x,\cdot)$ is nondecreasing for all $x \in [\alpha(0), \beta(0)]$.
$g_2 : \mathbb{R} \times C^0_P([0,T]) \to \mathbb{R}$ is continuous and $g_2(x,\cdot)$ is nonincreasing for all $x \in [\alpha(T, \beta(T)]$.

(H_5) For all $k = 1, \ldots, p$, the functions $I_k : \mathbb{R} \times C^0_P([0,T]) \to \mathbb{R}$ are continuous and $I_k(x,\cdot)$ are nondecreasing for all $x \in [\alpha(t_k), \beta(t_k)]$.
For all $k = 1, \ldots, p$, the functions $M_k : \mathbb{R} \times C^0_P([0,T]) \to \mathbb{R}$ are continuous and $M_k(x,\cdot)$ are nonincreasing for all $x \in [\alpha(t_k^+), \beta(t_k^+)]$.

(H_6) There exists $\varphi \in C^0((0,\infty);(0,\infty))$ such that
$$|f(t,u,v)| \le \varphi(|v|) \tag{1.5}$$
for a.e. $t \in [0,T]$, $\alpha(t) \le u \le \beta(t)$ and each $v \in \mathbb{R}$. Moreover, there exists a constant $K \ge \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty, r\}$ such that
$$\min\left\{ \int\limits_{\phi(r)}^{\phi(K)} \frac{\phi^{-1}(s)\,\mathrm{d}s}{\varphi(\phi^{-1}(s))}, - \int\limits_{\phi(-K)}^{\phi(-r)} \frac{\phi^{-1}(s)\,\mathrm{d}s}{\varphi(-\phi^{-1}(s))} \right\} > \|\alpha\|_\infty + \|\beta\|_\infty \tag{1.6}$$
with
$$r = \max_{k=0,\ldots,p} \left\{ \frac{1}{t_{k+1} - t_k} \right\} (\|\alpha\|_\infty + \|\beta\|_\infty).$$

Under the assumptions listed above, we prove in [2, Theorem 14, Remark 15] the following existence result:

Theorem 1.1. *Assume the hypotheses* (H_1)–(H_6) *hold. Then the problem* (1.1)–(1.3) *has the minimal and the maximal solution lying between* α *and* β.

It is the aim of the present paper to extend Theorem 1.1 to cover a wider class of functional equations. To be concise, on the contrary to [2], in this paper it is allowed that the function f be discontinuously dependent on the solutions. Moreover, both the boundary value conditions and the impulsive functions can be discontinuous in one of its variables.

In Section 2 we introduce the problem that we are going to study and give the list of conditions that we will consider. Some preliminary results are also proven in that section.

In Section 3 we prove our main existence result.

Finally we present an example in Section 4, where approximation methods are developed.

2. Preliminaries

In this paper we study the functional equation
$$(\phi(u'(t)))' = f\big(t, u, u(t), u'(t)\big) \quad \text{for a.e. } t \in [0,T] \setminus P, \tag{2.1}$$
coupled with the functional boundary conditions (1.2)–(1.3). We refer this problem as the problem (P).

In this case we say that u is a solution of the problem (P) if $u \in C^1_P$, $\phi \circ u' \in W^{1,1}_P$, and u satisfies the equation (2.1) and fulfills the conditions (1.2) and (1.3).

The concept of lower solution of the problem (P) is parallel to that given for the problem (1.1)–(1.3), in this case the inequality (1.4) being replaced by an analogous one:

$$(\phi(\alpha'(t)))' \geq f(t, \alpha, \alpha(t), \alpha'(t)) \text{ for a.e. } t \in I_0.$$

The definition of upper solution is given similarly.

To deduce the existence of extremal solutions, we will assume the conditions (H_2) and (H_3) together with the following ones:

(H_1^*) $f : I \times C(I) \times \mathbb{R}^2 \to \mathbb{R}$ is such that for every $u \in C(I)$ the function $f_u : I \times \mathbb{R}^2 \to \mathbb{R}$ defined as $f_u(t, y, z) = f(t, u, y, z)$ satisfies the condition (H_1).

(H_4^*) $g_1(\cdot, u)$ is continuous for all $u \in C^0_P([0, T])$ and $g_1(x, \cdot)$ is nondecreasing for all $x \in [\alpha(0), \beta(0)]$.
$g_2(\cdot, u)$ is continuous for all $u \in C^0_P([0, T])$ and $g_2(x, \cdot)$ is nonincreasing for all $x \in [\alpha(t_k^+), \beta(t_k^+)]$.

(H_5^*) For all $k = 1, \ldots, p$, the functions $I_k(\cdot, u)$ are continuous for all $u \in C^0_P([0, T])$ and $I_k(x, \cdot)$ are nondecreasing for all $x \in [\alpha(t_k), \beta(t_k)]$.
For all $k = 1, \ldots, p$, the functions $M_k(\cdot, u)$ are continuous for all $u \in C^0_P([0, T])$ and $M_k(x, \cdot)$ are nonincreasing for all $x \in [\alpha(t_k^+), \beta(t_k^+)]$.

(H_6^*) $f : I \times C(I) \times \mathbb{R}^2 \to \mathbb{R}$ is such that for every $u \in [\alpha, \beta]$ the function $f_u : I \times \mathbb{R}^2 \to \mathbb{R}$ defined as $f_u(t, y, z) = f(t, u, y, z)$ satisfies the condition (H_6).

Note that we allow to the functions f, g_1, g_2, I_k and M_k to have discontinuities in the spacial variable u.

To deduce extremal solutions of the problem (P), we will use the following result which improves [1, Lemma 2.4]

Lemma 2.1. *Given an order interval $[a, b] \subset C^0_P$ and a mapping $G : [a, b] \to [a, b]$, assume that G is nondecreasing and that the sequence $\{Gv_n\}$ has a pointwise limit in C^0_P whenever $\{v_n\}$ is a monotone sequence in $[a, b]$. Then G has the least fixed point u_* and the greatest fixed point u^*. Moreover,*

$$u_* = \min\{u \in [a, b] : \ Gu \leq u\} \ \text{ and } \ u^* = \max\{u \in [a, b] : \ u \leq Gu\}. \quad (2.2)$$

Proof. For all $k = 0, \ldots, p$, we have that $a \leq v_n \leq b$ and $\{v_n\}$ is monotone in J_k. As a consequence, we deduce that the same property holds for the sequence $\{G v_n\}$. Moreover, the Dini theorem ensures that the sequence $\{Gv_n\}$ converges uniformly on J_k for all $k = 0, \ldots, p$, which is, from Remark 1.1, equivalent to the fact that $\{Gv_n\}$ converges in C^0_P. Thus the conclusions follow from [3, Theorem 1.2.2] when $X = Y = C^0_P$. $\square$

3. Main Results

In this section we prove the following existence result for the problem (P).

Theorem 3.1. *Assume that the conditions* (H_1^*), (H_2), (H_3), (H_4^*)–(H_6^*) *hold. Suppose that the function* $f(t, \cdot, x, y)$ *is nonincreasing in* C_P^0 *for a. e.* $t \in [0, T]$ *and all* x, $y \in \mathbb{R}$. *Then the problem* (P) *has the extremal solutions in* $[\alpha, \beta]$.

Proof. Let $v \in [\alpha, \beta]$ be an arbitrarily fixed function. Consider the nonlinear second-order impulsive problem

$$(P_v) \begin{cases} (\phi(u'(t)))' = f(t, v, u(t), u'(t)) & \text{for a.e. } t \in [0, T] \setminus P, \\ g_1(u(0), v) = 0, & \\ g_2(u(T), v) = 0, & \\ I_k(u(t_k), v) = 0 & \text{for } k = 1, \dots, p, \\ M_k(u(t_k^+), v) = 0 & \text{for } k = 1, \dots, p. \end{cases}$$

It is not difficult to verify that the conditions (H_1)–(H_6) hold for the problem (P_v). Therefore the problem (P_v) has the extremal solutions lying between α and β. As a consequence, we can define the mapping $G : [\alpha, \beta] \to [\alpha, \beta]$ as follows

$$Gv := \text{maximal solution in } [\alpha, \beta] \text{ of the problem } (P_v). \tag{3.1}$$

It is clear that the fixed points of the function G coincide with the solutions of the problem (P). So, to prove the result, we must ensure that such fixed points exist. To this end, we verify that the function G fulfills the conditions of Lemma 2.1.

To prove that G is nondecreasing, let v_1, $v_2 \in [\alpha, \beta]$ with $v_1 \le v_2$ on $[0, T]$, and put $u_1 := Gv_1$ and $u_2 := Gv_2$. Let's see that $u_1 \le u_2$ on [0,T].

From the definition of G and the monotonicity assumptions imposed on the functions f, g_1, g_2, I_k and M_k, we know that the following properties hold

$$(\phi(u_1'(t)))' = f\big(t, v_1, u_1(t), u_1'(t)\big) \ge f\big(t, v_2, u_1(t), u_1'(t)\big) \text{ for a.e. } t \in [0, T] \setminus P,$$
$$0 = g_1(u_1(0), v_1) \le g_1(u_1(0), v_2),$$
$$0 = g_2(u_1(T), v_1) \ge g_2(u_1(T), v_2),$$
$$0 = I_k(u_1(t_k), v_1) \le I_k(u_1(t_k), v_2) \text{ for } k = 1, \dots, p,$$
$$0 = M_k(u_1(t_k^+), v_1) \ge M_k(u_1(t_k^+), v_2) \text{ for } k = 1, \dots, p.$$

This implies that u_1 is a lower solution for the problem (P_{v_2}).

As a consequence, we know that the problem (P_{v_2}) has the extremal solutions in $[u_1, \beta]$. This property implies that the maximal solution in $[\alpha, \beta]$ is greater or equal to u_1, i.e., $u_1 \le u_2$ on $[0, T]$.

Now let $\{v_n\}$ be a monotone sequence in $[\alpha, \beta]$. From the monotonicity of the function G we have that, given $k \in \{0, \dots, p\}$ fixed, the sequence $\{Gv_n|_{J_k}\}$ is monotone and bounded on J_k. As a consequence, it has a pointwise limit in J_k. From Nagumo's condition we have that there is a positive constant K such that $|(Gv_n)'(t)| < K$ for all $t \in J_k$. This implies that the sequence $\{Gv_n|_{J_k}\}$ is equicontinuous and, in consequence, the limit is uniform in J_k, i.e., the sequence $\{Gv_n|_{J_k}\}$ converges in C^0_P.

Now from Lemma 2.1 we have that the function G has a maximal fixed point u^* that is a solution of the problem (P). Let's see that it is the maximal solution of this problem in $[\alpha, \beta]$.

From (2.2) we have that

$$u^* = \max\big\{u \in [\alpha, \beta] : \; u \leq Gu\big\}.$$

If there is a solution $u \in [u^*, \beta]$ of the problem (P), then it is a solution of the problem (P_u) and, from the monotonicity of the function f, the function u^* is a lower solution of such problem. The definition of G says us that $u \leq Gu$ which contradicts the definition of u^*.

The existence of the minimal solution of the problem (P) is proved similarly. □

4. An Example

In this section we present an example that explains the power of the techniques used here. We prove the existence of the extremal positive solutions, in some particular cases it is deduced the uniqueness of positive solutions of the considered problem. Moreover, we obtain the exact expression of such solutions. To deduce this expression, we use the following result, which is a direct consequence of [3, Corollary 1.2.2].

Lemma 4.1. *Assume that the hypotheses of Theorem* 3.1 *hold. Let G be defined as in the proof of Theorem* 3.1 *and define the sequence $\{\beta_n\}$ as*

$$\beta_0 = \beta \;\; \text{and} \;\; \beta_{n+1} = G\,\beta_n.$$

If the functions f, g_1, g_2, I_k and M_k are right continuous for all $k \in \{1, \ldots, p\}$, then the sequence $\{\beta_n\}$ converges in C^0_P to the maximal solution in $[\alpha, \beta]$ of the problem (P).

The obtained result is the following.

Example 4.1. Let $A, B > 0$ and $\eta \in (0,1)$ be fixed. Assume that one of the following conditions holds: $\xi \in (0,1]$ and $\rho \in [0,2)$ or $\xi \in (1,2)$ and $\rho \in [0,1]$. Denoting by $[x]$ the integer part of a real number x, we consider the following nonlinear impulsive boundary value problem

$$(E) \begin{cases} u''(t) = F([u(\xi)])|u'(t)| \quad \text{for all } \; t \in (0,2) \setminus \{1\}, \\ u(0) = A, \\ u(1) = u(\eta), \\ u^2(1^+) = u(\rho), \\ u(2) = B, \end{cases}$$

with $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = -\frac{x^3}{1+x^2}\,.$$

It is not difficult to verify that the problem (E) is of the form (P) for the particular case of $T=2$, $p=1$, $t_1=1$ and

$$\begin{aligned} \phi(x) &= x, \\ f(t,v,x,y) &= F([v(\xi)])|y|, \\ I_1(x,v) &= -x+v(\eta), \\ M_1(x,v) &= x^2-v(\rho), \\ g_1(x,v) &= -x+A, \end{aligned}$$

and

$$g_2(x,v) = x-B.$$

One can verify that $\alpha \equiv 0$ is a lower solution and any constant $\beta \geq \max\{A,\, B,\, 1\}$ is an upper solution of the problem (E). Moreover, the assumptions (H_1^*), (H_2), (H_3), (H_4^*)–(H_6^*) hold.

In this situation the problem (P_v) considered in the proof of Theorem 3.1 takes the form

$$(E_v)\ \begin{cases} u''(t) = F([v(\xi)])|u'(t)| & \text{for all } \ t \in (0,2) \setminus \{1\}, \\ u(0) = A, \\ u(1) = v(\eta), \\ u^2(1^+) = v(\rho), \\ u(2) = B. \end{cases}$$

It is not difficult to verify that the first derivative of every solution of the previous problem has constant sign in each subinterval $[0,1]$ and $(1,2]$. As a consequence, we deduce that for any constant $\beta \geq \max\{A,B,1\}$ and $v \in C_P^0$ such that $0 \leq v(t) \leq \beta$ for all $t \in [0,2]$, the problem (E_v) has a unique solution $u_v \in [0,\beta]$ given by the following expression:

$$u_v(t) = \begin{cases} A + (v(\eta)-A)H_v(t) & \text{for all } \ t \in [0,1], \\ \sqrt{v(\rho)} + \left(B - \sqrt{v(\rho)}\right) H_v(t) & \text{for all } \ t \in (1,2], \end{cases}$$

with $H_v : [0,2] \to \mathbb{R}$ defined as

$$H_v(t) = \begin{cases} \dfrac{e^{C_v F([v(\xi)])t}-1}{e^{C_v F([v(\xi)])}-1} & \text{for all } \ t \in [0,1] \ \text{ and } \ C_v\,[v(\xi)] \neq 0, \\ t & \text{for all } \ t \in [0,1] \ \text{ and } \ C_v[v(\xi)] = 0, \\ \dfrac{e^{\overline{C}_v F([v(\xi)])(t-1)}-1}{e^{\overline{C}_v F([v(\xi)])}-1} & \text{for all } \ t \in (1,2] \ \text{ and } \ \overline{C}_v[v(\xi)] \neq 0, \\ t-1 & \text{for all } \ t \in (1,2] \ \text{ and } \ \overline{C}_v[v(\xi)] = 0. \end{cases}$$

Here

$$C_v = \operatorname{sign}\{v(\eta)-A\} \ \text{ and } \ \overline{C}_v = \operatorname{sign}\left\{B - \sqrt{v(\rho)}\,\right\}.$$

From the fact that $H_v(t) > 0$ for all $v \geq 0$ and $t > 0$, it is clear that $u_v(t) - A$ has the same sign as $v(\eta) - A$ for all $t \in (0,1]$. The same property holds for $B - \sqrt{v(\rho)}$ and $u_v(t) - \sqrt{v(\rho)}$ in $(1,2]$.

So, starting at $\beta_0 \equiv \beta$, we have that the sequence $\{\beta_n\}$ satisfies the following recursive equation

$$\beta_{n+1}(t) = A + (\beta_n(\eta) - A)H_{\beta_n}(t) \ \text{ for all } \ t \in [0,1].$$

Moreover, the sequence $\{\beta_n\}$ is monotone nonincreasing and bounded, so there exists

$$u^*(t) = \lim_{n\to\infty} \beta_n(t), \ \ t \in [0,2].$$

Since the integer part of the real numbers is a right-continuous function, we deduce that

$$u^*(t) = A + (u^*(\eta) - A)\, H_{u^*}(t) \ \text{ for all } \ t \in [0,1].$$

Evaluating this expression at $t = \eta \in (0,1)$, we conclude that $u^*(\eta) = A$, i.e.,

$$u^*(t) = A \ \text{ for all } \ t \in [0,1].$$

On the other hand, we know that

$$\beta_{n+1}(t) = \sqrt{\beta_n(\rho)} + \big(B - \sqrt{\beta_n(\rho)}\big)H_{\beta_n}(t) \ \text{ for all } \ t \in (1,2].$$

We consider three different situations.

Case I: $\xi \in (0,1]$, $\rho \in [0,1]$.

We know that

$$u^*(t) = \sqrt{A} + (B - \sqrt{A})H(t) \ \text{ for all } \ t \in (1,2].$$

Here the function H is defined as

$$H(t) = \begin{cases} \dfrac{e^{\overline{C}F([A])(t-1)} - 1}{e^{\overline{C}F([A])} - 1} & \text{if } \overline{C}[A] \neq 0, \\ t-1 & \text{if } \overline{C}[A] = 0, \end{cases} \tag{4.1}$$

with $\overline{C} = \operatorname{sign}\{B - \sqrt{A}\}$.

Since β is arbitrarily large, we have that u^* is the greatest positive solution of the problem (E).

Setting $\alpha_0 = 0$ and $\alpha_{n+1} = G\alpha_n$, we have that the sequence $\{\alpha_n\}$ converges to a function $u_* \in C^0_P$. Arguing as for the sequence $\{\beta_n\}$, we prove that $u_* = u^* = A$ on $[0,1]$.

Note that from the recurrence formula for the sequence α_n we know that if there is $n_0 \geq 0$ such that $\alpha_{n_0}(\eta) = A$, then $\alpha_n \equiv A$ in $[0,1]$ for all $n \geq n_0$. On the other hand, since

$$A = \alpha_{n_0}(t) = A + (\alpha_{n_0-1}(\eta) - A)H_{\alpha_{n_0-1}}(t) \ \text{ for all } \ t \in [0,1],$$

we conclude that $\alpha_n \equiv A$ in $[0,1]$ for all $n \geq 0$, which contradicts the definition of α_0.

As a consequence, we have $\alpha_n(\eta) < A$ for all $n \in \mathbb{N}$. Moreover, it is clear that if $\alpha_n(\eta) = \alpha_{n+1}(\eta)$ for some $n \in \mathbb{N}$, we deduce that $\alpha_n(\eta) = A$, in contradiction with the previous proof. So we have that the sequence $\{\alpha_n(\eta)\}$ is strictly increasing and

$$\operatorname{sign}\{\alpha_n(\eta) - A\} = -1 \ \text{ for all } \ n \geq 0.$$

Now suppose that $\alpha_{n_1}(\xi) = \alpha_{n_1-1}(\xi)$ for some $n_1 \geq 1$.

In this situation, we have

$$H_{\alpha_{n_1}}(t) = H_{\alpha_{n_1-1}}(t) \text{ for all } t \in [0,1],$$

and

$$\alpha_{n_1+1}(\xi) - \alpha_{n_1}(\xi) = (\alpha_{n_1}(\eta) - \alpha_{n_1-1}(\eta))H_{\alpha_{n_1}}(\xi) > 0.$$

So we conclude that $\alpha_n(\xi) < A$ for all $n \in \mathbb{N}$.

Since $\alpha_n \leq A$ in $[0,1]$ for all $n \in \mathbb{N}$, we have that if $A \notin \mathbb{N}$, then $[\alpha_n(\xi)] = [A]$ for n large enough. In this situation we conclude that $u_* = u^*$ on $(1,2]$ and, as a consequence, the problem (E) has a unique positive solution.

When $A \in \mathbb{N}$, we have that

$$u_*(t) = \sqrt{A} + (B - \sqrt{A})\overline{H}(t) \text{ for all } t \in (1,2].$$

Here the function $\overline{H}$ is defined as

$$\overline{H}(t) = \begin{cases} \dfrac{e^{\overline{C}F([A-1])(t-1)} - 1}{e^{\overline{C}F([A-1])} - 1} & \text{if } \overline{C}[A-1] \neq 0, \\ t-1 & \text{if } \overline{C}[A-1] = 0, \end{cases}$$

with $\overline{C} = \operatorname{sign}\{B - \sqrt{A}\}$.

However, in this case the function u_* verifies

$$u_*''(t) = F[A-1]|u_*'(t)|, \quad t \in [0,2] \setminus \{1\},$$

which implies that u_* is not a solution of the problem (E).

In this situation, we have that $u^* = G\,u_*$ and, as a consequence, see [3, Corollary 1.2.2], the minimal positive solution u_* of the problem (E) is equal to the maximal one u^*, that is, the problem (E) has a unique positive solution given by the expression

$$u(t) = \begin{cases} A & \text{for all } t \in [0,1], \\ \sqrt{A} + (B - \sqrt{A})H(t) & \text{for all } t \in (1,2], \end{cases}$$

with H defined in (4.1).

Case II: $\xi \in (0,1]$, $\rho \in (1,2)$.

Now, reasoning as in the previous situation, we have that the problem (E) has a unique solution u given by the following expression

$$u(t) = \begin{cases} A & \text{for all } t \in [0,1], \\ C_\rho + (B - C_\rho)H(t) & \text{for all } t \in (1,2], \end{cases}$$

with

$$C_\rho = \frac{1 - H(\rho) + \sqrt{(1 - H(\rho))^2 + 4BH(\rho)}}{2}.$$

Case III: $\xi \in (1,2)$, $\rho \in [0,1]$.

As in the previous cases, we can construct the sequences $\{\alpha_n\}$, $\{\beta_n\}$, where $\alpha_0 \equiv 0$, $\beta_0 \equiv \beta$, $\alpha_{n+1} = G\alpha_n$, $\beta_{n+1} = G\beta_n$ for each $n \in \mathbb{N}$. First we will prove that β_n converges to the maximal solution u^* of the problem (E). Using the same arguments as previously, we conclude that

$$u^*(t) = A \text{ for all } t \in [0,1].$$

Let us assume that $B < \sqrt{A}$ (other cases are similar).

We know that $\{\beta_n\}$ is a nonincreasing sequence, so $[\beta_n(\xi)] \to [u^*(\xi)]$ and

$$u^*(t) = \sqrt{A} + (B - \sqrt{A})H(t),$$

where

$$H(t) = \begin{cases} \dfrac{e^{-F([u^*(\xi)])(t-1)} - 1}{e^{-F([u^*(\xi)])} - 1} & \text{if } u^*(\xi) \geq 1, \\ t-1 & \text{if } 0 \leq u^*(\xi) < 1. \end{cases}$$

Obviously, $u^*(\xi)$ satisfies the equality

$$x = \sqrt{A} + (B - \sqrt{A})g([x]) \equiv \overline{G}(x), \tag{4.2}$$

where

$$g(y) = \begin{cases} \dfrac{e^{-F(y)(\xi-1)} - 1}{e^{-F(y)} - 1} & \text{for all } y > 0, \\ \xi - 1 & \text{if } y = 0. \end{cases}$$

The function g is continuous and increasing. So $\overline{G}$ is right–continuous, nondecreasing and $G([B, \sqrt{A}]) \subset [B, \sqrt{A}]$. Then the equation $\overline{G}(x) = x$ is solvable in $[B, \sqrt{A}]$ (see [3, Theorem 1.2.2]). Let x^* be the greatest solution of this equation in $[B, \sqrt{A}]$. We have that u^* satisfying $u^*(\xi) = x^*$ is the maximal solution of the problem (E).

The sequence $\{\alpha_n\}$ is nondecreasing, so there exists α^1 such that $\alpha_n \to \alpha^1$ on $[0,2]$. As in *Case* I, one can verify that $\alpha_n(\rho) < A$ for all $n \in \mathbb{N}$, $\{\alpha_n(\rho)\}$ is strictly increasing,

$$\overline{C}_n = \operatorname{sign}\left\{B - \sqrt{\alpha_n(\rho)}\right\} = -1, \text{ for } n \text{ large enough},$$

and

$$\alpha^1(t) = A \text{ for all } t \in [0,1].$$

In the case where $\alpha_{n_1}(\xi) = \alpha_{n_1-1}(\xi)$ holds for some $n_1 \geq 1$, we arrive at the expression

$$\alpha_{n_1+1}(\xi) - \alpha_{n_1}(\xi) = \left(\sqrt{\alpha_{n_1}(\rho)} - \sqrt{\alpha_{n_1-1}(\rho)}\right)(1 - H_{\alpha_{n_1}}(\xi)) > 0.$$

So we conclude that $\alpha_n(\xi) < \alpha^1(\xi)$ for all $n \in \mathbb{N}$.

As a consequence, $H_{\alpha_n}(t) \to \overline{H}(t)$ for $t \in (1,2]$, where

$$\overline{H}(t) = \begin{cases} \dfrac{e^{-F([\alpha^1(\xi)])(t-1)} - 1}{e^{-F([\alpha^1(\xi)])} - 1} & \text{if } \alpha^1(\xi) > 1, \ \alpha^1(\xi) \notin \mathbb{N}, \\ \dfrac{e^{-F([\alpha^1(\xi)]-1)(t-1)} - 1}{e^{-F([\alpha^1(\xi)]-1)} - 1} & \text{if } \alpha^1(\xi) \in \mathbb{N}, \\ t-1 & \text{if } 0 \leq \alpha^1(\xi) < 1, \end{cases}$$

for $t \in (1,2]$.

Obviously

$$\alpha^1(t) = \sqrt{A} + (B - \sqrt{A})\overline{H}(t) \text{ for all } t \in (1, 2].$$

It is clear that if $\alpha^1(\xi) \notin \mathbb{N}$, then α^1 is a solution of (E). Moreover,

$$\alpha^1 = \min \{ y \in [0, \beta] : \; Gy \le y \}$$

and it is the minimal solution of (E) in $[0, \beta]$.

If $\alpha^1(\xi) \in \mathbb{N}$, then α^1 is not a solution of (E) and the function

$$\alpha_1^1(t) = G(\alpha^1)(t) = \sqrt{A} + (B - \sqrt{A}) \, \frac{e^{-F([\alpha^1(\xi)])(t-1)} - 1}{e^{-F([\alpha^1(\xi)])} - 1}$$

satisfies the inequality $\alpha_1^1(t) > \alpha^1(t)$ for $t \in (1, 2)$.

Let us denote $\alpha_n^1 \to \alpha^2$ on $[0, 2]$. If $\alpha^2(\xi) \in \mathbb{N}$, then we construct another sequence $\{\alpha_n^2\}$ converging to α^3, and so on. Since $[0, \beta] \cap \mathbb{N}$ is a finite set, it follows that this process is finite. So there exists $n_0 \in \mathbb{N}$ such that $\alpha^{n_0}(\xi)$ is not an integer and, as a consequence, $u_* \equiv \alpha^{n_0}$ is the minimal positive solution of (E).

In this situation, $u_*(\xi)$ is given as the minimal solution of the equation (4.2) on the interval $[B, \sqrt{A}]$. We illustrate this situation in the figures 1, 2 and 3, in which the particular case of $A = 13/2$, $B = 3/2$, $\eta = 3/5$, $\rho = 1/2$ and $\xi = 8/5$ is considered. The first six iterations of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ (with $\beta \equiv 10$) are represented in figures 1 and 2, respectively. The problem has two different positive solutions that take their values $u^*(\xi)$ and $u_*(\xi)$ at x^* and x_*, the two solutions of the equation (4.2), given by (see figure 3):

$$x_* \approx 1.9835 \qquad \text{and} \qquad x^* \approx 2.12161.$$

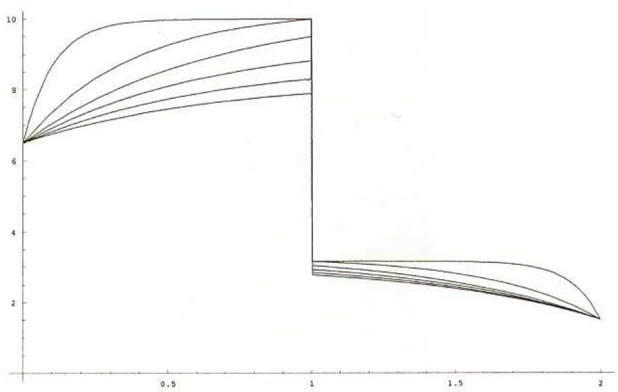

Figure 1: The sequence $\{\beta_n\}$

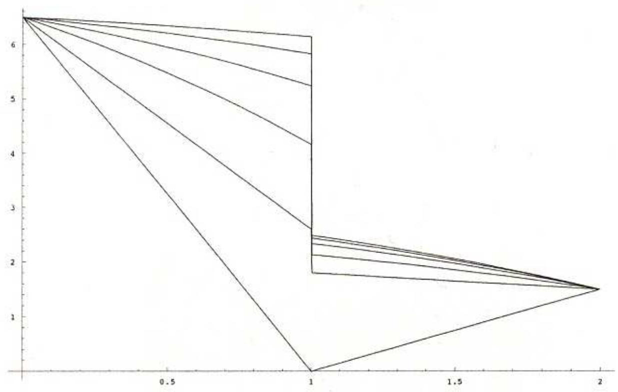

Figure 2: The sequence $\{\alpha_n\}$

It is clear that if the equation (4.2) has a unique solution in $[B, \sqrt{A}]$, then the problem (E) has a unique positive solution.

Remark 4.1. Note that in the previous example we obtain extremal and uniqueness results in an unbounded domain.

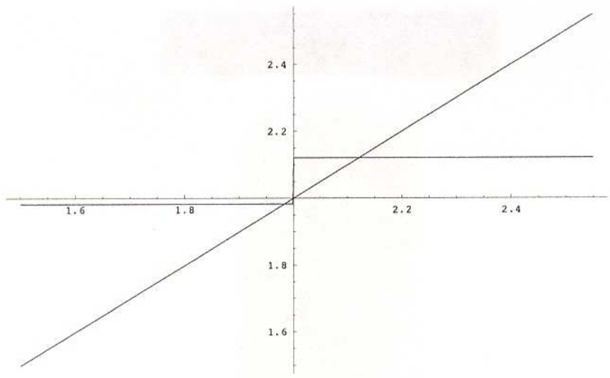

Figure 3: Fixed points of the function $\overline{G}$

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References

1. A. Cabada and S. Heikkilä, Extremality and comparison results for discontinuous third order functional initial-boundary value problems. *J. Math. Anal. Appl.* **255**(2001), No. 1, 195–212.
2. A. Cabada and J. Tomeček, Extremal solutions for nonlinear functional ϕ-Laplacian impulsive equations. *Nonlinear Anal.* (to appear).
3. S. Heikkilä and V. Lakshmikantham, Monotone iterative techniques for discontinuous nonlinear differential equations. *Marcel Dekker, Inc., New York*, 1994.

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