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SOME MULTIDIMENSIONAL PROBLEMS FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS AND SYSTEMS

Abstract. For a class of first and second order hyperbolic systems with symmetric principal part, to which belong systems of Maxwell and Dirac equations, crystal optics equations, equations of the mathematical theory of elasticity and so on which are well known from the mathematical physics, we have developed a method allowing one to give correct formulations of boundary value problems in dihedral angles and conical domains in Sobolev spaces. For second order hyperbolic equations of various types of degeneration, we study the multidimensional versions of the Goursat and Darboux problems in dihedral angles and conical domains in the corresponding Sobolev spaces with weight. For the wave equation with one or two spatial variables, the correctness of some nonlocal problems is shown. The existence or nonexistence of global solutions of the characteristic Cauchy problem in a conic domain is studied for multidimensional wave equations with power nonlinearity.

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რეზიუმე. პირველი და მეორე რიგის ჰიპერბოლურ სისტემათა ერთი კლასისათვის სიმეტრიული მთავარი ნაწილით, რომელსაც მიეკუთვნება მათემატიკური ფიზიკიდან ცნობილი მაქსველის, დირაკის, კრისტალთა ოპტიკის, დრეკადობის მათემატიკური თეორიისა და სხვა განტოლებათა სისტემები, შემუშავებულია მიდგომა, რომელიც ორწახნაგა კუთხეში და კონუსურ არეებში იძლევა სასაზღვრო ამოცანათა კორექტულად დასმის საშუალებას სობოლევის სივრცეებში. სხვადასხვა სახის გადაგვარების მქონე მეორე რიგის ჰიპერბოლური განტოლებებისათვის ორწახნაგა კუთხეში და კონუსურ არეებში შესწავლილია გურსას და დარბუს ამოცანების მრავალგანზომილებიანი ვარიანტები. ტალღის განტოლებისათვის ერთი და ორი სივრცითი ცვლადის შემთხვევაში გამოკვლეულია ზოგიერთი არალოკალური ამოცანის კორექტულობის საკითხი. ხარისხოვანი არაწრფივობის შემცველი მრავალგანზომილებიანი ტალღის განტოლებისათვის შესწავლილია კოშის მახასიათებელი ამოცანის გლობალური ამონახსნის არსებობის და არარსებობის საკითხი კონუსურ არეში.

Introduction

In this work we investigate some multidimensional problems for hyperbolic partial differential equations and systems. It should be said that when passing from two to more than two independent variables difficulties may arise that are not only technical. They may arise even when formulating multidimensional versions of classical two-dimensional problems, for example, of the Goursat and Darboux problems.

As is known, the strict hyperbolicity of a system plays an important role in establishing the correctness of the posed initial, initial-boundary and other problems. At the same time, the investigation of some problems makes it possible to consider a class of systems wider than that of strictly hyperbolic ones. In the case of one equation this is the ultrahyperbolic equation. In the first section of Chapter I we consider second order systems with several independent variables hyperbolic with respect to some two-dimensional planes. For such systems, in dihedral domains of a certain orientation we consider boundary value problems in special weight function spaces with boundary conditions of Poincaré type imposed on the faces of the dihedral angle. The correctness of these problems is proved when the order of the weight function determining the function space is greater than a definite value [69]. A separate consideration is given to the case of ultrahyperbolic equation [70].

In the second section of Chapter I we develop methods of formulating correct boundary value problems for a class of second order hyperbolic systems with several independent variables with symmetric principal part in conic domains, taking into account the spatial orientation of the latter.

In the third section of the same chapter we investigate boundary value problems for a class of first order hyperbolic systems with symmetric principal part. To this class belong, in particular, the Maxwell and Dirac systems of differential equations and the equation of crystal optics which are well known from mathematical physics. We begin the subsection by considering boundary value problems in a conic domain whose boundary is one of the connected components of the characteristic conoid of the system [71], [72]. Certain difficulties arise even if the cone of normals of the system consists of infinitely smooth sheets and the connected components of the characteristic conoid of the system corresponding to these sheets may have strong singularities [24, p. 586]. Thus difficulties already arise when formulating a characteristic problem, when the carrier of boundary data must be indicated [72].

In the second part of Section 3 we consider boundary value problems in dihedral domains [73], [74]. To show that the formulation of a problem in terms of its correctness demands much care we give the following simple example of symmetric system [105]

$$E_0U_t + AU_x + BU_y = F,$$

where $E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $F = (F_1, F_2)$ is a given and $U = (u_1, u_2)$ is an unknown two-dimensional real vector. The characteristic polynomial of the system is $p(\xi_0, \xi_1, \xi_2) = \det(\xi_0 E_0 + \xi_1 A + \xi_2 B) = \xi_0^2 - \xi_1^2 - \xi_2^2$. We denote by $D: \ -t < x < t$, $0 < t < +\infty$ the dihedral angle bounded by the characteristic surfaces $S_1: \ t - x = 0$, $0 \leq t < +\infty$ and $S_2: \ t + x = 0$, $0 \leq t < +\infty$, of the system. As is shown in [71], the problem of finding a solution of the system under consideration in the domain D by the boundary conditions

$$u_2\big|_{S_1} = f_1, \quad u_1\big|_{S_2} = f_2$$

is posed correctly, whereas in the case of the boundary conditions

$$u_1\big|_{S_1} = f_1, \quad u_2\big|_{S_2} = f_2$$

for the problem to be solvable we need the fulfulment of a continual set of solvability conditions imposed on the right-hand sides F, f_1 and f_2 of the problem.

Note that in the second and third sections the approaches to stating correct boundary value problems make an essential use of the structure of quadratic forms which correspond to characteristic matrices of the systems and which, in particular, depend on the spatial orientation of the problem data carriers. We conclude the sections by presenting the correct statements of boundary value problems for Maxwell and Dirac systems of differential equations and those of crystal optics.

Problems in a certain sense close to the ones we consider in this chapter, were investigated by A. V. Bitsadze [8]–[10], K. O. Friedrichs [30], [31], K. O. Friedrichs and P. D. Lax [32], [33], A. A. Dezin [25]–[27], M. S. Agranovich [1]–[3], V. S. Vladimirov [122], [123], V. N. Vragov [125], [126], K. Kubota and T. Ohkubo [80], [81], T. Ohkubo [106], S. Kharibegashvili [61], [64], [65], O. Jokhadze [55], [56], P. Secchi [113], Y. Tanaka [117] and other authors.

In Chapter II we study some multidimensional versions of the Goursat and Darboux problems for degenerating hyperbolic equations of second order. Note that when passing from nondegenerating hyperbolic equations to degenerating ones there may arise essential differences in the correct statement of multidimensional versions of the Goursat and Darboux problems.

For example, if the characteristic conoid for a second order nondegenerating hyperbolic equation which is simultaneously the data carrier of the characteristic Cauchy problem (of the multidimensional version of the Goursat problem), consisting of bicharacteristic curves emanating from one point (conoid vertex), is homeomorphic to the conic surface of a circular cone, then in the case of degeneration this conoid may have a smaller dimension. For example, for the equation

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - x_3 u_{x_3x_3} = F$$

the characteristic conoid K_O with the vertex at the origin O degenerates into the two-dimensional conic manifold $\{(x_1, x_2, x_3, t) \in R^4 : \ t^2 - x_1^2 - x_2^2 = 0,\ x_3 = 0\}$, while for the equation

$$x_3^m u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = F$$

the characteristic conoid K_O consists only of one bicharacteristic curve $\{(x_1, x_2, x_3, t) \in R^4 : \ x_1 = x_2 = 0,\ t^2 = \frac{4}{9} x_3^3,\ x_3 > 0\}$ in the case $m = 1$ and it degenerates into one point $O(0, 0, 0, 0)$ in the case $m = 2$. It clearly follows that in such cases the statement of the characteristic Cauchy problem is out of question. Another peculiarity connected with degeneration of an equation is that the parts of the boundary where the equation undergoes characteristic degeneration must be completely free from any kind of boundary conditions.

In the first section of Chapter II, for the degenerating equation

$$u_{tt} - u_{x_1x_1} - x_3 u_{x_2x_2} - u_{x_3x_3} = F$$

we construct the characteristic conoids K_O and K_A, where $O = (0, 0, 0, 0)$, $A = (0, 0, 0, t_0)$, and study a multidimensional version of the first Darboux problem in a finite domain bounded by the hyperplane $x_3 = 0$ and by some parts of the conoids K_O and K_A lying in the half-space $x_3 \geq 0$ [76].

In the second section of that chapter we investigate the characteristic Cauchy problem for the equation

$$u_{tt} - t^m(u_{x_1x_1} + u_{x_2x_2}) + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \quad m = \text{const} > 0,$$

with noncharacteristic degeneration, and for the equation

$$(t^m u_t)_t - u_{x_1x_1} - u_{x_2x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_{x_t} + a_1 u = F, \quad 1 \leq m = \text{const} < 2,$$

with characteristic degeneration on the plane $t = 0$ [68].

Finally, in the last section of Chapter II we consider some multidimensional versions of the first Darboux problem in dihedral domains for the degenerating equations

$$u_{tt} - |x_2|^m u_{x_1x_1} - u_{x_2x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \quad m = \text{const} \geq 0,$$

and

$$u_{tt} - u_{x_1x_1} - (|x_2|^m u_{x_2})_{x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \quad 1 \leq m = \text{const} < 2,$$

respectively with noncharacteristic and characteristic degeneration on the plane $x_2 = 0$ [66], [67].

All the results of this chapter are obtained by the method of a priori estimates in negative Lax norms in special Sobolev weight spaces connected with the principal parts of degenerating equations.

The questions raised in this chapter were investigated by many authors (see A. V. Bitsadze [7], [10], A. M. Nakhushev [103], [104], A. V. Bitsadze and A. M. Nakhushev [11]–[13], R. W. Carrol and R. E. Showalter [22], M. M. Smirnov [116], D. Gvazava and S. Kharibegashvili [40], J. M. Rassias [109], N. I. Popivanov and M. F. Schneider [107] and other works).

Chapter III, consisting of two sections, deals with some nonlocal problems for wave equations. In the first section, we show for the wave equation with one spatial variable that the lowest term affects the correctness of the nonlocal problem: in some cases the problem has a unique solution, while in other cases the corresponding homogeneous problem has an infinite set of linearly independent solutions [75]. We give the correct formulation of a nonlocal problem with an integral condition. The multidimensional version of this problem is studied in the next section. In the second section we establish one property of solutions of the wave equation with two spatial variables. This property is of integral nature and defines solutions completely. Furthermore, we give the properties of wave potentials, by means of which the nonlocal problem is reduced to a Volterra type integral equation with a weakly singular kernel. The investigation of this integral equation made it possible to prove the correctness of the nonlocal problem both in the class of generalized solutions and in the class of regular classical solutions of arbitrary smoothness [75].

The active interest shown recently in nonlocal problems for partial differential equations is, to a certain extent, connected with the fact that nonlocal problems arise in the mathematical modelling of some physical, biological and other processes. For equations of parabolic and elliptic type, these problems were studied by J. R. Cannon [21], L. I. Kamynin [58], N. I. Ionkin [51], N. I. Yurchuk [129], A. Bouziani [18], S. Mesloub and A. Bouziani [94], A. M. Nakhushev [104], A. V. Bitsadze and A. A. Samarskii [14], A. V. Bitsadze [15], [16], V. A. Il'in and E. I. Moiseyev [49], E. Moiseyev [102], D. G. Gordeziani [36], A. L. Skubachevskii [115], A. K. Gushchin and V. P. Mikhailov [39], F. J. Correa and S. D. Menezes [23] and other authors. For equations of hyperbolic type, mention should be made of the works by Z. O. Mel'nik [90], Z. O. Mel'nik and V. M. Kirilich [91], T. I. Kiguradze [77], [78], V. A. Il'in and E. I. Moiseyev [50], S. Mesloub and A. Bouziani [93], A. Bouziani [19], S. Mesloub and N. Lekrine [95], G. Avalishvili and D. Gordeziani [4], D. G. Gordeziani and G. A. Avalishvili [37], [38], G. A. Avalishvili [5], L. S. Pul'kina [108], J. Gvazava [41], [42], B. Midodashvili [97], [98], G. G. Bogveradze and S. S. Kharibegashvili [17], M. Dohghan [28].

As is known, the characteristic Cauchy problem for linear hyperbolic equations of second order with the data carrier on a characteristic conoid (in particular, for the linear wave equation with the data carrier on the boundary of a light cone of the future) is globally solvable in the corresponding function spaces [20], [24], [43], [88]. This circumstance may significantly change if the equation involves nonlinear terms. In the last Chapter IV, consisting of two sections, we study the question of the existence and nonexistence of global solutions of the characteristic Cauchy problem in a light cone of the future for nonlinear wave Gordon equations

$$u_{tt} - \sum_{i=1}^{n} u_{x_i x_i} = f(u) + F(x,t), \quad n > 1,$$

with power nonlinearity of the type $f(u) = \lambda |u|^{\alpha}$, or $f(u) = -\lambda |u|^p u$ in the right-hand side, where λ, α and p are real constants, and $\lambda \neq 0$, $\alpha > 0$, $p > 0$. In the first section, in the case $f(u) = \lambda |u|^{\alpha}$, $1 < \alpha < \frac{n+1}{n-1}$, where n is the spatial dimension of the equation, the local solvability of that problem is proved; for $\lambda > 0$, the conditions on the right-hand sides of the problem are found when a global solution does not exist. The estimate of the time interval of solution's life is given. In the second section, in case $f(u) = -\lambda |u|^p u$, for $\lambda > 0$ the existence of the global solution of the characteristic Cauchy problem and for $\lambda < 0$ the nonexistence of such a solution is proved, when some additional conditions are imposed on the right-hand sides of the problem.

Note that the problems of existence or nonexistence of global solutions for nonlinear equations with the initial conditions $u|_{u=0} = u_0$, $\frac{\partial u}{\partial t}\big|_{t=0} = u_1$ have been considered and studied by K. Jörgens [57], H. A. Levin [85], F. John [52], [53], F. John and S. Klainerman [54], T. Kato [59], V. Georgiev, H. Lindblad and C. Sogge [35], L. Hörmander [47], E. Mitidieri and S. I. Pohozaev [101], M. Keel, H. F. Smith, and C. D. Sogge [60], C. Miao, B. Zhang, and D. Fang [96], Z. Yin [128], K. Hidano [45], G. Todorava and E. Vitillaro [119], F. Merle and H. Zaag [92], Y. Zhou [130], Z. Gan and J. Zhang [34], etc.

CHAPTER 1

Boundary Value Problems for Some Classes of Hyperbolic Systems in Conic and Dihedral Domains

1. Boundary Value Problems for a Class of Systems of Partial Differential Equations of Second Order, Hyperbolic with Respect to Some Two-Dimensional Planes

1.1. Statement of the problem and formulation of results. Consider in the real n-dimensional space R^n, $n > 2$, a system of linear partial differential equations of second order

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Cu = F, \tag{1.1}$$

where A_{ij}, B_i, C are given constant $(m \times m)$-matrices, F is a given and u is an unknown n-dimensional real vector.

Under strict hyperbolicity of the system (1.1) is meant the existence of the vector $\zeta \in R^n$, passing through the point $O(0,\dots,0)$, such that any two-dimensional plane π, passing through ζ, intersects the cone of normals of the system (1.1) $K:\ p(\xi) \equiv \det\big(\sum\limits_{i,j=1}^{n} A_{ij}\xi_i\xi_j\big) = 0$, $\xi = (\xi_1,\dots,\xi_n) \in R^n$, along $2m$ different real lines [24, p. 584].

Below, we will consider a wider class of systems of equations, when there exists a two-dimensional plane π_0 passing through the point $O(0,\dots,0)$ and intersecting the cone of normals $K:\ p(\xi) = 0$ of the system (1.1) along $2m$ different real lines. For the sake of simplicity, without restriction of generality, we can assume that $\pi_0:\ \xi_3 = \dots = \xi_n = 0$. For $m = 1$, an example of such equation is the ultrahyperbolic equation

$$u_{x_1x_1} - u_{x_2x_2} + u_{x_3x_3} - U_{x_4x_4} = 0 \tag{1.2}$$

for which $\pi_0 : \xi_3 = \xi_4 = 0$.

By $D:\ k_2x_2 < x_1 < k_1x_2$, $0 < x_2 < +\infty$, $k_i = \text{const}$, $i = 1,2$, $k_2 < k_1$, we denote the dihedral angle bounded by the plane surfaces $S_i:\ x_1 - k_ix_2 = 0$, $0 \le x_2 < +\infty$, $i = 1,2$. It will be assumed that the hyperplane $S:\ x_1 - k_0x_2 = 0$ with $k_2 \le k_0 \le k_1$ is not characteristic for the system (1.1).

Consider the boundary value problem formulated as follows: in the domain D, find a solution $u(x_1,\dots,x_n)$ of the system (1.1) satisfying the boundary conditions

$$\Big(\sum_{i=1}^{n} M_{ji}u_{x_i}+C_ju\Big)\Big|_{S_j}=f_j,\quad j=1,2, \tag{1.3}$$

where M_{ji}, C_j are given real $(s_j\times m)$-matrices, f_j are given s_j-dimensional real vectors with $s_j\geq 1$, $j=1,2$, and $s_1+s_2=2m$.

By our assumption, in the plane of variables x_1, x_2 the system of equations

$$\sum_{i,j=1}^{2} A_{ij}\widetilde{u}_{x_ix_j}=0 \tag{1.4}$$

is strictly hyperbolic. Without restriction of generality, we can assume that $p(0,1,0,\dots,0)=\det A_{22}\neq 0$. In this case under strict hyperbolicity of the system (1.4) is meant that the polynomial $p_0(\lambda)=\det(A_{11}+(A_{12}+A_{21})\lambda+A_{22}\lambda^2)$ has only simple real roots $\lambda_1,\dots,\lambda_{2m}$. The characteristics of the system (1.4) are the families of straight lines $x_1+\lambda_ix_2=\text{const}$, $i=1,\dots,2m$.

Denote by D_0 the section of the domain D by the two-dimensional plane $\pi_0:\ x_3=\dots=x_n=0$, i.e. D_0 is the angle in the half-plane $\{(x_1,x_2)\in R^2:\ x_2>0\}$ bounded by the rays $\gamma_i:\ x_1-k_ix_2=0,\ 0\leq x_2<+\infty$, $i=1,2$, coming out of the origin $(0,0)$. By the requirements on the domain D, the rays γ_1, γ_2 are not characteristics of the system (1.4). On γ_1 we fix arbitrarily a point P_1 different from the origin $(0,0)$, and enumerate the roots of the polynomial $p_0(\lambda)$ in such a way that the characteristic rays $\ell_1(P_1),\dots,\ell_{2m}(P_1)$ corresponding to the roots $\lambda_1,\dots,\lambda_m$ and coming out of the point P_1 to the inside of the angle D_0 were numbered counter-clockwise, starting from $\ell_1(P_1)$.

Let $P=P(x_1,x_2)\in D_0$. Denote by $D_{0P}\subset D_0$ the convex quadrangle with vertex at the origin $(0,0)$ bounded by the rays γ_1, γ_2 and the characteristics $L_{s_1}(P)$, $L_{s_1+1}(P)$ of the system (1.4) passing through the point P. Obviously, as $P\to P_0\in\partial D_0\setminus(0,0)$ the quadrangle D_{0P} degenerates into the corresponding triangle D_{0P}. If now $Q=Q(x_1,x_2,\dots,x_n)\in\overline{D}\setminus(S_1\cap S_2)$, then by $D_Q\subset D$ we denote the domain $D_Q=\big\{(x_1^0,x_2^0,\dots,x_n^0)\in D:\ (x_1^0,x_2^0)\in D_{0P},\ P=P(x_1,x_2)\big\}$.

Since all the roots $\lambda_1,\dots,\lambda_{2m}$ of the polynomial $p_0(\lambda)$ are simple, there take place the equalities $\dim\operatorname{Ker}(A_{11}+(A_{12}+A_{21})\lambda_i+A_{22}\lambda_i^2)=1$, $i=1,\dots,2m$. Denote by ν_i the vectors $\nu_i\in\operatorname{Ker}(A_{11}+(A_{12}+A_{21})\lambda_i+A_{22}\lambda_i^2)$, $\|\nu_i\|\neq 0$, $i=1,\dots,2m$, and form the matrices

$$V_1=\begin{pmatrix}\nu_1 & \dots & \nu_{s_1}\\ \lambda_1\nu_1 & \dots & \lambda_{s_1}\nu_{s_1}\end{pmatrix},\quad V_2=\begin{pmatrix}\nu_{s_1+1} & \dots & \nu_{2m}\\ \lambda_{s_1+1}\nu_{s_1+1} & \dots & \lambda_{2m}\nu_{2m}\end{pmatrix},$$
$$\Gamma_i=(M_{i1},M_{i2}),\quad i=1,2,$$

of dimensions $2m\times s_1$, $2m\times s_2$, $s_i\times 2m$, $i=1,2$, respectively.

Denote by $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$, $k \geq 2$, $\alpha \geq 0$, the space of the functions $u(x_1, \dots, x_n)$ of the class $C^k(\overline{D})$ for which $\partial^{i_1,i_2} u(0,0,x_3,\dots,x_n) = 0$, $-\infty < x_i < +\infty$, $i = 3,\dots,n$, $0 \leq i_1 + i_2 \leq k$, $\partial^{i_1,i_2} = \partial^{i_1+i_2}/\partial x_i^{i_1} \partial x_2^{i_2}$, and whose partial Fourier transforms $\widehat{u}(x_1, x_2, \xi_3, \dots, \xi_n)$ with respect to the variables $x_3 \dots, x_n$ are functions continuous in $\overline{G}_1 = \big\{(x_1, x_2, \xi_3, \dots, \xi_n) \in R^n : (x_1, x_2) \in \overline{D}_0,\ \xi^0 = (\xi_3, \dots, \xi_n) \in R^{n-2}\big\}$ together with partial derivatives with respect to the variables x_1 and x_2 up to the k-th order, inclusively, and satisfy the following estimates: for any natural N there exist positive numbers $\widetilde{C}_N = \widetilde{C}_N(x_1, x_2)$ and $\widetilde{K}_N = \widetilde{K}_N(x_1, x_2)$, independent of $\xi^0 = (\xi_3, \dots, \xi_n)$, such that for $(x_1, x_2) \in \overline{D}_0$ and $|\xi^0| = |\xi_3| + \dots + |\xi_n| > \widetilde{K}_N$ the inequalities

$$\big\|\partial^{i_1,i_2} \widehat{u}(x_1, x_2, \xi^0)\big\| \leq \widetilde{C}_N x_2^{k+\alpha-i_1-i_2} \exp(-N|\xi^0|), \quad 0 \leq i_1 + i_2 \leq k, \tag{1.5}$$

hold, where $\widetilde{C}^0_N(x_1, x_2) = \sup\limits_{(x_1^0, x_2^0) \in \overline{D}_{0P} \setminus (0,0)} \widetilde{C}_N(x_1^0, x_2^0) < +\infty$, $\widetilde{K}^0_N(x_1, x_2) = \sup\limits_{(x_1^0, x_2^0) \in \overline{D}_{0P} \setminus (0,0)} \widetilde{K}_N(x_1^0, x_2^0) < +\infty$, $P = P(x_1, x_2)$.

Analogously we introduce the spaces $\overset{\circ}{\Phi}{}^k_\alpha(S_i)$, $i = 1,2$. Note that the trace $u|_S$ of the function u from the space $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$ belongs to the space $\overset{\circ}{\Phi}{}^k_\alpha(S_i)$. It can be easily verified that the function $u(x_1, x_2, \dots, x_n) = x_2^{k+\alpha} \varphi(x_1, x_2) \exp\big(-\sum\limits_{i=3}^{n} \psi_i(x_1, x_2) x_i^2\big)$ belongs to the space $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$ for any φ, $\psi_i \in C^k(\overline{D}_0)$ if $\psi_i(x_1, x_2) \geq \text{const} > 0$, $i = 3, \dots, n$.

Remark 1.1. When considering the problem (1.1), (1.3) in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$, it is required of the functions F, f_j and the coefficients M_{ji}, C_j, $i = 1, \dots, n$, $j = 1,2$, that in the boundary conditions (1.3) $F \in \overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D})$, $f_j \in \overset{\circ}{\Phi}{}^{k-1}_\alpha(S_j)$, $j = 1,2$, M_{ji}, $C_j \in C^{k-1}(S_j)$, $j = 1,2$, $i = 1, \dots, n$. Below it will be assumed that the coefficients M_{ji} and C_j, $j = 1,2$, $i = 1, \dots, n$, depend only on the variables x_1, x_2.

In Subsection 3^0 we prove the following statements.

Theorem 1.1. *Let the conditions*

$$\det(\Gamma_i \times V_i)\big|_{S_i} \neq 0, \quad i = 1,2, \tag{1.6}$$

be fulfilled. Then if at least one of the equalities $(\Gamma_1 \times V_2)(O) = 0$ *or* $(\Gamma_2 \times V_1)(O) = 0$ *holds, where* $O = O(0, \dots, 0)$*, then for any* $F \in \overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D})$ *and* $f_j \in \overset{\circ}{\Phi}{}^{k-1}_\alpha(S_j)$*,* $j = 1,2$*, the problem* (1.1), (1.3) *is uniquely solvable in the class* $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$ *for* $k \geq 2$*,* $\alpha \geq 0$*, and the domain of dependence of the solution* u *of that problem for the point* $Q \in D$ *is contained in* $\overline{D}_Q$*.*

Theorem 1.2. *Let the conditions* (1.6) *and* $(\Gamma_1 \times V_2)(0) \neq 0$ *be fulfilled. Then there exists a positive number* ρ_0*, depending only of the coefficients* A_{ij} *and* M_{ij}*,* $1 \leq i,j \leq 2$*, such that for any* $F \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}(\overline{D})$ *and* $f_j \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}(S_j)$*,* $j = 1,2$*, the problem* (1.1)*,* (1.3) *is uniquely solvable in the class* $\overset{\circ}{\Phi}{}^{k}_{\alpha}(\overline{D})$ *for* $k + \alpha > \rho_0$*, and the domain of dependence of the solution* u *of that problem for the point* $Q \in D$ *is contained in* $\overline{D}_Q$*.*

In the case where the equation (1.2) is ultrahyperbolic, in the boundary conditions (1.3) we should assume that $s_1 = s_2 = 1$, i.e. the coefficients M_{ji}, C_j are scalar functions and $|k_i| < 1$, $i = 1,2$, $k_2 < 0$ and $k_1 > 0$. Suppose $\tau_0 = (1+k_2)(1-k_1)/((1+k_1)(1-k_2))$, $\sigma = \big[(M_{11} - M_{12})(M_{21} + M_{22})/((M_{11} + M_{12})(M_{21} - M_{22}))\big](0)$. Owing to our assumptions, it is obvious that $0 < \tau_0 < 1$.

Corollary 1.1. *Let the conditions* $(M_{11}+M_{12})\big|_{S_1} \neq 0$*,* $(M_{21}-M_{22})\big|_{S_2} \neq 0$ *be fulfilled. Then if at least one of the equalities* $(M_{11}-M_{12})(O) = 0$ *or* $(M_{21} + M_{22})(O) = 0$ *holds, then for any* $F \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}(\overline{D})$ *and* $f_j \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}(S_j)$*,* $j = 1,2$*, the problem* (1.2)*,* (1.3) *is uniquely solvable in the class* $\overset{\circ}{\Phi}{}^{k}_{\alpha}(\overline{D})$ *for* $k \geq 2$*,* $\alpha \geq 0$*, and the domain of dependence of the solution* u *of that problem for the point* $Q \in D$ *is contained in* $\overline{D}_Q$*.*

Corollary 1.2. *Let the conditions of Corollary* 1.1 *and* $(M_{11}-M_{12})\big|_{S_1} \neq 0$*,* $(M_{21} + M_{22})\big|_{S_2} \neq 0$ *be fulfilled. Then for any* $F \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}(\overline{D})$ *and* $f_j \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}(S_j)$*,* $j = 1,2$*, the problem* (1.2)*,* (1.3) *is uniquely solvable in the class* $\overset{\circ}{\Phi}{}^{k}_{\alpha}(\overline{D})$ *for* $k + \alpha > -\log|\sigma|/\log\tau_0 + 1$*, and the domain of dependence of the solution* u *of that problem for the point* $Q \in D$ *is contained in* $\overline{D}_Q$*.*

1.2. Reduction of the problem (1.1), (1.3) to a system of integro-functional equations with a parameter. Below, without restriction of generality it will be assumed that

$$k_1 > 0, \;\; k_2 < 0, \;\; \lambda_{s_1} > 0, \;\; \lambda_{s_1+1} < 0, \tag{1.7}$$

since otherwise, due to the above enumeration of the roots $\lambda_1, \dots, \lambda_{2m}$ of the polynomial $p_0(\lambda)$, one can achieve the fulfillment of the equalities (1.7) by a proper linear transformation of the variables x_1 and x_2. As far as $\det A_{22} \neq 0$, in the system (1.1) we assume $A_{22} = E$, where E is the unit $(m \times m)$-matrix, since otherwise one can achieve this by multiplying both parts of the system (1.1) by the inverse matrix A_{22}^{-1}.

In the notation $v_i = u_{x_i}$, $i = 1, \dots, n$, the system (1.1) is reduced to the following system of the first order:

$$u_{x_2} = v_2, \tag{1.8}$$

$$v_{1x_2} - v_{2x_1} = 0, \tag{1.9}$$

$$v_{2x_2}+A_{11}v_{1x_1}+(A_{12}+A_{21})v_{2x_1}+\sum_{i=1}^{2}\sum_{j=3}^{n}A_{ij}v_{ix_j}+$$
$$+\sum_{i=3}^{n}\sum_{j=1}^{2}A_{ij}v_{jx_i}+\sum_{i,j=3}^{n}A_{ij}v_{ix_j}+\sum_{i=1}^{n}B_iv_i+Cu=F, \qquad (1.10)$$
$$v_{ix_2}-v_{2x_i}=0,\ \ i=3,\dots,n, \qquad (1.11)$$

and the boundary conditions (1.3) can now be written as

$$\Big(\sum_{i=1}^{n}M_{ji}v_i+C_ju\Big)\Big|_{S_j}=f_j,\ \ j=1,2. \qquad (1.12)$$

Along with the conditions (1.12), let us consider the boundary conditions

$$(u_{x_i}-v_i)\big|_{S_1\cup S_2}=0,\ \ i=1,3\dots,n. \qquad (1.13_i)$$

It is evident that if u is a regular solution of the problem (1.1), (1.3) of the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$, then the system of functions u, v_i, $i=1,\dots,n$, will be a regular solution of the boundary value problem (1.8)–(1.13), where $v_i\in\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D})$, $i=1,\dots,n$. Conversely, let the system of functions u, v_i, $i=1,\dots,n$, of the class $\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D})$ be a solution of the problem (1.8)–(1.13). Let us prove that in this case $v_i=u_{x_i}$, $i=1,\dots,n$, and hence the function u is a solution of the problem (1.1), (1.3) in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$. Indeed, using the equality (1.9), we have $(u_{x_1}-v_1)_{x_2}=(u_{x_2})_{x_1}-v_{2x_1}=v_{2x_1}-v_{2x_1}=0$, whence by the boundary condition (1.13_1) we find that $v_1\equiv v_{x_1}$ in $\overline{D}$.

Further, applying the equality (1.11) we obtain $(u_{x_i}-v_i)_{x_2}=(u_{x_2})_{x_i}-v_{2x_i}=v_{2x_i}-v_{2x_i}=0$, whence by the boundary condition (1.13_i), $i\neq 1$, we obtain $v_i\equiv u_{x_i}$ in $\overline{D}$, $i=3,\dots,n$.

Thus the problem (1.1), (1.3) in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$ is equivalent to the problem of finding a system of functions u, v_i, $i=1,\dots,n$, in the class $\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D})$ satisfying the boundary value problem (1.8)–(1.13).

Introduce the following $(2m\times 2m)$-matrices:

$$A_0=\begin{pmatrix}0 & -E\\ A_{11} & (A_{12}+A_{21})\end{pmatrix},$$
$$quadK=(V_1,V_2)=\begin{pmatrix}\nu_1 & \nu_2 & \dots & \nu_{2m}\\ \lambda_1\nu_1 & \lambda_2\nu_2 & \dots & \lambda_{2m}\nu_{2m}\end{pmatrix},$$

where E is the unit $(m\times m)$-matrix.

Due to strict hyperbolicity of the system (1.4) it can be easily shown that

$$K^{-1}A_0K=D_1; \qquad (1.14)$$

here $D_1=\operatorname{diag}(-\lambda_1,\dots,-\lambda_{2m})$.

Suppose $v=(v_1,v_2)$. As a result of the substitution $v=Kw$, by virtue of (1.14) instead of the system (1.9), (1.10) we have

$$w_{x_2}+D_1 w_{x_1}+\sum_{j=3}^{n}A_j w_{x_j}+\sum_{p,j=3}^{n}A_{pj}^1 v_{px_j}+\sum_{j=3}^{n}B_j^1 v_j+B^0 w+C^1 u=F^1, \quad (1.15)$$

where A_j and B^0 are $(2m\times 2m)$-matrices, A_{pj}^1, B_j^1 and C^1 are $(2m\times 2m)$-matrices which are expressed in terms of the coefficients of the system (1.1), $F^1=K^{-1}F^0$, $F^0=(0,F)$.

Representing the matrix K in the form $K=\operatorname{colon}(K_1,K_2)$, where K_1, K_2 are matrices of order $m\times 2m$, from the equality $v=Kw$ we find that $v_j=K_j w$, $j=1,2$.

If u, v_j, $j=1,\dots,n$, is a solution of the problem (1.8)–(1.13), then after the Fourier transform with respect to the variables $x_3\dots,x_n$ the system of equations (1.8), (1.15), (1.11) and the boundary conditions (1.12), (1.13) take the form

$$\widehat{u}_{x_2}=K_2\widehat{w}, \quad (1.16)$$

$$\widehat{w}_{x_2}+D_1\widehat{w}_{x_1}+i\Big(\sum_{j=3}^{n}A_j\xi_j\Big)\widehat{w}+i\sum_{p=3}^{n}\Big(\sum_{j=3}^{n}A_{pj}^1\xi_j\Big)\widehat{v}_p+$$

$$+\sum_{j=3}^{n}B_j^1\widehat{v}_j+B^0\widehat{w}+C^1\widehat{u}=\widehat{F}^1, \quad (1.17)$$

$$\widehat{v}_{jx_2}-i\xi_j K_2\widehat{w}=0,\ \ j=3,\dots,n, \quad (1.18)$$

$$\Big[(M_{k1}K_1+M_{k2}K_2)\widehat{w}+\sum_{j=3}^{n}M_{kj}\widehat{v}_j+C_k\widehat{u}\Big]\Big|_{\gamma_k}=\widehat{f}_k,\ \ k=1,2, \quad (1.19)$$

$$(\widehat{u}_{x_1}-K_1\widehat{w})\big|_{\gamma_1\cup\gamma_2}=0, \quad (1.20)$$

$$(\widehat{v}_j-i\xi_j\widehat{u})\big|_{\gamma_1\cup\gamma_2}=0,\ \ j=3,\dots,n, \quad (1.21)$$

where $\widehat{u}$, $\widehat{w}$, $\widehat{v}_j$, $j=3,\dots,n$; $\widehat{F}^1$, $\widehat{f}_1$, $\widehat{f}_2$ are the Fourier transforms respectively of the functions u, v_j, $j=3,\dots,n$, F^1, f_1, f_2 with respect to the variables $x_3\dots,x_n$, and $\gamma_j:\ x_1-k_jx_2=0$, $0\le x_2<+\infty$, $j=1,2$, are the above-introduced rays bounding the angular domain D_0 in the plane of the variables x_1, x_2. Here in these equalities $i=-\sqrt{1}$.

Remark 1.2. Thus after the Fourier transform with respect to the variables $x_1,\dots,x_n$ the spatial problem (1.8)–(1.13) is reduced to the plane problem (1.16)–(1.21) with the parameters $\xi_3,\dots,\xi_n$ in the domain D_0 : $k_2x_2<x_1<k_1x_2$, $0<x_2<+\infty$ of the plane of the variables x_1, x_2. It is easy to see that in the class $\overset{\circ}{\Phi}{}_\alpha^k(\overline{D})$ of the functions defined by the inequalities (1.5), this reduction is equivalent.

Written parametrically, let $L_j(x_1^0,x_2^0):\ x_1=z_j(x_1^0,x_2^0,t)=x_1^0+\lambda_jx_2^0-\lambda_jt$, $x_2=t$ be the characteristic of the j-th family of the system (1.4) passing through the point $(x_1^0,x_2^0)\in\overline{D}_0$, $1\le j\le 2m$. Denote by $\omega_j(x_1,x_2)$ the ordinate of the point of intersection of the characteristic $L_j(x_1,x_2)$ with the curve γ_1 for $1\le j\le s_1$ and with γ_2 for $s_1<j\le 2m$, $(x_1,x_2)\in\overline{D}_0$. Here as the ordinate of the point (x_1,x_2) in the plane of the variables x_1 and x_2 we take x_2. Obviously, $\omega_j(x_1,x_2)\in C^\infty(\overline{D}_0)$, $1\le j\le 2m$.

By the inequalities (1.7), the domain D_{0P}, $P(x_1^0,x_2^0)\in\overline{D}_0\setminus(0,0)$ constructed above lies entirely in the half-plane $x_2\le x_2^0$. Therefore from the construction of the function $\omega_j(x_1,x_2)$ it follows that

$$0\le\omega_j(x_1,x_2)\le x_2,\ \ (x_1,x_2)\in D_0,\ \ j=1,\dots,2m, \tag{1.22}$$

since the segment of the characteristic $L_j(p)$ coming out of the point $P\in\overline{D}_0\setminus(0,0)$ up to the intersection with γ_1 for $1\le j\le s_1$ and with γ_2 for $s_1<j\le 2m$ lies entirely in $\overline{D}_{0P}$.

It can be easily verified that

$$\begin{gathered}\omega_j\big|_{\gamma_l}=\begin{cases}\tau_j^{l-1}x_2, & j=1,\dots,s_1,\\ \tau_j^{2-l}x_2, & j=s_1+1,\dots,2m,\end{cases}\quad l=1,2,\\ \tau_j=\begin{cases}(k_2+\lambda_j)(k_1+\lambda_j)^{-1}, & j=1,\dots,s_1,\\ (k_1+\lambda_j)(k_2+\lambda_j)^{-1}, & j=s_1+1,\dots,2m,\end{cases}\end{gathered} \tag{1.23}$$

and by virtue of (1.7) and the fact that γ_1 and γ_2 are not characteristic rays of the system (1.4), we have

$$0<\tau_j<1,\ \ j=1,\dots,2m. \tag{1.24}$$

Remark 1.3. The functions $\widehat{u}$, $\widehat{w}$, $\widehat{v}_j$, $j=3,\dots,n$, $\widehat{F}^1$, $\widehat{f}_1$, $\widehat{f}_2$, besides the independent variables x_1 and x_2, depend also on the parameters $\xi_3\dots,\xi_n$. For the sake of simplicity of writing, these parameters will be omitted below. For example, instead of $\widehat{u}(x_1,x_2,\xi_3,\dots,\xi_n)$ we will write $\widehat{u}(x_1,x_2)$.

By (1.16), (1.20) and the fact that $\widehat{u}(0,0)=0$, if $u\in\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$ we have

$$\begin{aligned}\widehat{u}(x_1,x_2)&=\int_0^{x_2}(k_j\widehat{u}_{x_1}+\widehat{u}_{x_2})(k_jt,t)\,dt=\\ &=\int_0^{x_2}(k_jK_1+K_2)\widehat{w}(k_jt,t)\,dt,\ \ (x_1,x_2)\in\gamma_j.\end{aligned} \tag{1.25}$$

Denote by $\widetilde{\sigma}(x_1^0,x_2^0)$ the ordinate of the point of intersection of the straight line $x_1=x_1^0$ passing through the point $P(x_1^0,x_2^0)\in\overline{D}_0$ with γ_1 for $x_1^0>0$ and with γ_2 for $x_1^0\le 0$. Obviously, $\widetilde{\sigma}(x_1,x_2)=\begin{cases}k_1^{-1}x_1 & \text{for } x_1>0,\\ k_2^{-1}x_2 & \text{for } x_1\le 0,\end{cases}$ and by (1.7) we have $0\le\widetilde{\sigma}(x_1,x_2)\le x_2$, $(x_1,x_2)\in\overline{D}_0$.

Suppose $\beta = \begin{cases} k_1 & \text{for } x_1 > 0, \\ k_2 & \text{for } x_1 \le 0, \end{cases}$. Integrating the equations (1.16) and (1.18) with respect to the variable x_2 and taking into account the boundary conditions (1.21) and (1.25), we obtain for $(x_1, x_2) \in \overline{D}_0$

$$\widehat{u}(x_1,x_2) = \int\limits_0^{\widetilde{\sigma}(x_1,x_2)} (\beta K_1 + K_2)\widehat{w}(\beta t, t)\,dt + \int\limits_{\widetilde{\sigma}(x_1,x_2)}^{x_2} K_2 \widehat{w}(x_1,t)\,dt, \quad (1.26)$$

$$\widehat{v}_j(x_1,x_2) = i\xi_j \int\limits_0^{\widetilde{\sigma}(x_1,x_2)} (\beta K_1 + K_2)\widehat{w}(\beta t, t)\,dt + i\xi_j \int\limits_{\widetilde{\sigma}(x_1,x_2)}^{x_2} K_2 \widehat{w}(x_1,t)\,dt, \quad (1.27)$$

$$j = 3, \dots, n.$$

Suppose

$$\varphi_j(x_2) = \begin{cases} w_j\big|_{\gamma_1} = w_j(k_1 x_2, x_2), & j = 1, \dots, s_1, \\ w_j\big|_{\gamma_2} = w_j(k_2 x_2, x_2), & j = s_1 + 1, \dots, 2m. \end{cases}$$

Integrating now the j-th equation of the system (1.17) along the j-th characteristic $L_j(x_1,x_2)$ from the point $P(x_1,x_2) \in \overline{D}_0$ to the point of intersection $L_j(x_1,x_2)$ with γ_1 for $j \le s_1$ and with γ_2 for $j > s_1$, we obtain

$$\widehat{w}_j(x_1,x_2) = \varphi_j\big(\omega_j(x_1,x_2)\big) + \int\limits_{\omega_j(x_1,x_2)}^{x_2} \Big[\sum_{p=1}^{2m} E_{1jp}\widehat{w}_p + \sum_{p=3}^{n}\sum_{q=1}^{m} E_{2jpq}\widehat{v}_{pq} +$$

$$+ \sum_{q=1}^{m} E_{3jq}\widehat{u}_q\Big]\big(z_j(x_1,x_2;t),t\big)\,dt + F_{2j}(x_1,x_2), \quad j = 1, \dots, 2m, \qquad (1.28)$$

where E_{1jp}, E_{2jpq}, E_{3jq} are quite definite linear scalar functions with respect to the parameters $\xi_3, \dots, \xi_n$, $\widehat{v}_p = (\widehat{v}_{p1}, \dots, \widehat{v}_{pm})$,

$$F_{2j}(x_1,x_2) = \int\limits_{\omega_j(x_1,x_2)}^{x_2} \widehat{F}_j^1\big(z_j(x_1,x_2;t),t\big)\,dt, \quad j = 1, \dots, 2m.$$

Rewrite the system of equations (1.28) in the form of one equation

$$\widehat{w}(x_1,x_2) = \varphi(x_1,x_2) +$$

$$+ \sum_{j=1}^{2m} \int\limits_{\omega_j(x_1,x_2)}^{x_2} \Big[E_{4j}\widehat{w} + \sum_{q=3}^{n} E_{5jq}\widehat{v}_q + E_{6j}\widehat{u}\Big]\big(z_j(x_1,x_2;t),t\big)\,dt + \widehat{F}(x_1,x_2), \quad (1.29)$$

where E_{4j}, E_{5jp} and E_{6j} are matrices of orders $2m \times 2m$, $2m \times m$ and $2m \times m$, respectively, whose elements are linear functions with respect to the parameters $\xi_3, \dots, \xi_n$; $\varphi(x_1,x_2) = (\varphi_1(w_1(x_1,x_2)), \dots, \varphi_{2m}(w_{2m}(x_1,x_2)))$.

Substituting the expressions for the values $\widehat{u}$, $\widehat{w}$, $\widehat{v}_j$, $j=3,\dots,n$, from (1.26), (1.27), (1.29) into the boundary conditions (1.19) and using the equalities (1.23), we obtain for $0\le x_2<+\infty$

$$G_0^1(x_2)\varphi(x_2)+\sum_{j=s_1+1}^{2m}G_j^1(x_2)\psi(\tau_j x_2)+ \\ +\big[T_1(\widehat{u},\widehat{w},\widehat{v}_3,\dots,\widehat{v}_n)\big](x_2)=f_3(x_2), \tag{1.30}$$

$$G_0^2(x_2)\psi(x_2)+\sum_{j=1}^{s_1}G_j^2(x_2)\varphi(\tau_j x_2)+ \\ +\big[T_2(\widehat{u},\widehat{w},\widehat{v}_3,\dots,\widehat{v}_n)\big](x_2)=f_4(x_2), \tag{1.31}$$

where $\varphi(x_2)=(\varphi_1(x_2),\dots,\varphi_{s_1}(x_2))$, $\psi(x_2)=(\varphi_{s_1+1}(x_2),\dots,\varphi_{2m}(x_2))$, G_j^1, G_j^2 are quite definite matrices of the class $C^{k-1}([0,+\infty))$, and T_1 and T_2 are linear integral operators.

It is obvious that G_0^j, $j=1,2$, from (1.30) and (1.31) are matrices of order $s_j\times s_j$ representable in the form of a product $G_0^j=\Gamma_j\times V_j$, $j=1,2$. Therefore if the conditions (1.6) are fulfilled, the matrices G_0^1 and G_0^2 are invertible, and resolving the equations (1.30) and (1.31) with respect to φ and ψ, we obtain

$$\varphi(x_2)-\sum_{j=1}^{s_1}\sum_{p=s_1+1}^{2m}G_{1jp}\varphi(\tau_j\tau_p x_2)= \\ =\big[T_3(\widehat{u},\widehat{w},\widehat{v}_3,\dots,\widehat{v}_n)\big](x_2)+f_5(x_2),\ \ 0\le x_2<+\infty, \tag{1.32}$$

$$\psi(x_2)-\sum_{j=1}^{s_1}\sum_{p=s_1+1}^{2m}G_{2jp}\psi(\tau_j\tau_p x_2)= \\ =\big[T_4(\widehat{u},\widehat{w},\widehat{v}_3,\dots,\widehat{v}_n)\big](x_2)+f_6(x_2),\ \ 0\le x_2<+\infty, \tag{1.33}$$

where G_{1jp} and G_{2jp} are matrices of the class $C^{k-1}([0,+\infty))$ which are defined through the matrices G_j^1, G_j^2, and T_3, T_4 are linear integral operators with kernels linearly depending on the parameters $\xi_3,\dots,\xi_n$.

Let $P\in D_0$. Denote by P_1 and P_2 the vertices of the above-constructed quadrangle D_{0P} which lie, respectively, on γ_1 and γ_2 and are different from the origin $(0,0)$.

Remark 1.4. As is seen from our reasoning above, if the conditions (1.6) are fulfilled, the problem (1.1), (1.3) in the class $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$ is equivalent to the problem of finding a system of functions $\widehat{u}$, $\widehat{w}$, $\widehat{v}_j$, $j=3,\dots,n$, φ and ψ from the system of integro-functional equations (1.26), (1.27), (1.29), (1.32), (1.33), where $\widehat{u}$, $\widehat{w}$, $\widehat{v}_j\in\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D}_0)$, φ, $\psi\in\overset{\circ}{\Phi}{}^{k-1}_\alpha([0,+\infty))$, $\widetilde{F}\in\overset{\circ}{\Phi}{}^{k-1}_\alpha(\overline{D}_0)$, f_5, $f_6\in\overset{\circ}{\Phi}{}^{k-1}_\alpha([0,+\infty))$. Note also that in considering the problem (1.16)–(1.21) in the domain D_{0P}, it is sufficient to investigate the equations (1.32) and (1.33) respectively on the segments $[0,d_1]$ and $[0,d_2]$, where d_1 and d_2

are the ordinates of the points P_1 and P_2 which are the points of intersection of the characteristics $L_{s_1}(P)$ and $L_{s_1+1}(P)$ respectively with the curves γ_1 and γ_2.

1.3. Investigation of the system of integro-functional equations (1.26), (1.27), (1.29), (1.32), (1.33) and proof of the theorems. Introduce into consideration the functions

$$h_q(\rho) = \sum_{j=1}^{s_1} \sum_{p=s_1+1}^{2m} (\tau_j \tau_p)^{\rho-1} \|G_{qjp}(0)\|, \quad q = 1, 2,$$

where G_{qjp}, $\tau_j\tau_p$ are defined in (1.32), (1.33), and $\|\cdot\|$ is the norm of the matrix operator in the space R^{s_q}. If all the values $\|G_{qjp}(O)\| = 0$ for $j = 1, \dots, s_1$, $p = s_1 + 1, \dots, 2m$, then we put $\rho_q = -\infty$. Let now for some values of the indices q, j, p the number $\|G_{qjp}(O)\|$ be different from zero. In this case, by virtue of (1.24), the function $h_q(\rho)$ is continuous and strictly monotonically decreases on $(-\infty, +\infty)$ with $\lim\limits_{\rho\to-\infty} h_q(\rho) = +\infty$ and $\lim\limits_{\rho\to+\infty} h_q(\rho) = 0$. Therefore there exists a unique real number ρ_q such that $h_q(\rho_q) = 1$. Assume that $\rho_0 = \max(\rho_1, \rho_2)$. It can be easily verified that if at least one of the equalities $(\Gamma_1 \times V_2)(O) = 0$ or $(\Gamma_2 \times V_1)(O) = 0$ given in the conditions of Theorem 1.1 holds, then $\rho_0 = -\infty$. Note also that in the case of the problem (1.2), (1.3) if at least one of the equalities $(M_{11} - M_{12})(O) = 0$ or $(M_{21} + M_{22})(O) = 0$ holds, then $\rho_0 = -\infty$, while otherwise $\rho_0 = -(\log|\sigma|)/\log\tau_0 + 1$, where σ and τ_0 are introduced in Subsection 1.1.

Consider the functional equations

$$(\Lambda_{1i}(\varphi))(x_2) = \varphi(x_2) - \sum_{j=1}^{s_1} \sum_{p=s_1+1}^{2m} (\tau_j\tau_p)^i G_{1jp}\varphi(\tau_j\tau_p x_2) = \\ = \chi_1(x_2), \ \ 0 \le x_2 \le d_1, \ \ i = 0, 1, \dots, k-1, \tag{1.34}$$

$$(\Lambda_{2i}(\psi))(x_2) = \psi(x_2) - \sum_{j=1}^{s_1} \sum_{p=s_1+1}^{2m} (\tau_j\tau_p)^i G_{2jp}\psi(\tau_j\tau_p x_2) = \\ = \chi_2(x_2), \ \ 0 \le x_2 \le d_2, \ \ i = 0, 1, \dots, k-1. \tag{1.35}$$

Note that if one differentiates i times the expression $(\Lambda_{10}(\varphi))(x_2)$ in the left-hand side of the equation (1.32) with respect to x_2, then in the obtained expression the sum of the summands in which the function $\varphi(x_2)$ appears in the form of the derivative $\varphi^{(i)}(x_2)$ yields $(\Lambda_{1i}(\varphi^{(i)}))(x_2)$. A similar remark is valid for the operators Λ_{2i}.

Let in the equations (1.34), (1.35) the left-hand sides χ_q be in $\overset{\circ}{\Phi}_{k-1+\alpha-i}([0, d_q])$, $q = 1, 2$. Here we agree to write $\overset{\circ}{\Phi}{}^k_\alpha([0, d_q]) = \overset{\circ}{\Phi}_\alpha([0, d_q])$ for $k = 0$. Then by the definition of the space $\overset{\circ}{\Phi}_{k-1+\alpha-i}([0, d_q])$ for any natural N there exist positive numbers $\widetilde{C}_q = \widetilde{C}_q(x_2, N, \chi_q)$, $\widetilde{K}_q = \widetilde{K}_q(x_2, N, \chi_q)$

independent of $\xi^0 = (\xi_3, \dots, \xi_n)$ and such that for $0 \le x_2 \le d_q$ and $|\xi^0| > \widetilde{K}_q$ the inequality $\|\chi_q(x_2)\| \le \widetilde{C}_q x_2^{k-1+\alpha-i} \exp(-N|\xi^0|)$ holds, where $\widetilde{C}_q^0 = \sup\limits_{0\le x_2^0\le x_2} \widetilde{C}_q(x_2^0) < +\infty$, $\widetilde{K}_q^0 = \sup\limits_{0\le x_2^0\le x_2} \widetilde{K}_q(x_2^0) < +\infty$.

Lemma 1.1. *For $k+\alpha > \rho_0$, the equations* (1.34), (1.35) *are uniquely solvable in the spaces $\overset{\circ}{\Phi}_{k-1+\alpha-i}([0,d_1])$ and $\overset{\circ}{\Phi}_{k-1+\alpha-i}([0,d_2])$, and for $|\xi^0| > \widetilde{K}_1$ and $|\xi^0| > \widetilde{K}_2$ respectively the estimates*

$$\big\|(\Lambda_{1i}^{-1}(\chi_1))(x_2)\big\| = \|\varphi(x_2)\| \le \delta_1 \widetilde{C}_1 x_2^{k-1+\alpha-i} \exp(-N|\xi^0|), \tag{1.36}$$

$$\big\|(\Lambda_{2i}^{-1}(\chi_2))(x_2)\big\| = \|\varphi(x_2)\| \le \delta_2 \widetilde{C}_2 x_2^{k-1+\alpha-i} \exp(-N|\xi^0|) \tag{1.37}$$

are valid, where the positive constants δ_1 and δ_2 do not depend on N, ξ^0 and on the functions χ_1, χ_2.

The proof of Lemma 1.1 word for word repeats the reasoning of [62], [63].

We solve the system of equations (1.26), (1.27), (1.29), (1.32), (1.33) with respect to the unknowns $\widehat{u}$, $\widehat{w}$, $\widehat{v}_j \in \overset{\circ}{\Phi}{}_\alpha^{k-1}(\overline{D}_{0P})$, $j = 3, \dots, n$, $\varphi \in \overset{\circ}{\Phi}{}_\alpha^{k-1}([0,d_1])$ and $\psi \in \overset{\circ}{\Phi}{}_\alpha^{k-1}([0,d_2])$ by the method of successive approximations.

Assume $\widehat{u}_0(x_1,x_2) \equiv 0$, $\widehat{w}_0(x_1,x_2) \equiv 0$, $\widehat{v}_{j,0}(x_1,x_2) \equiv 0$, $j = 3, \dots, n$, $\varphi(x_2) \equiv 0$, $\psi_0(x_2) \equiv 0$,

$$\widehat{u}_p(x_1,x_2) = \int\limits_0^{\widetilde{\sigma}(x_1,x_2)} (\beta K_1 + K_2)\widehat{w}_{p-1}(\beta t, t)\,dt + \int\limits_{\widetilde{\sigma}(x_1,x_2)}^{x_2} K_2 \widehat{w}_{p-1}(x_1,t)\,dt, \tag{1.38}$$

$$\begin{aligned} \widehat{v}_{q,p}(x_1,x_2) = i\xi_q \int\limits_0^{\widetilde{\sigma}(x_1,x_2)} (\beta K_1 + K_2)\widehat{w}_{p-1}(\beta t, t)\,dt + \\ + i\xi_q \int\limits_{\widetilde{\sigma}(x_1,x_2)}^{x_2} K_2 \widehat{w}_{p-1}(x_1,t)\,dt, \quad q = 3, \dots, n, \end{aligned} \tag{1.39}$$

$$\begin{gathered} \widehat{w}_p(x_1,x_2) = \varphi_p(x_1,x_2) + \\ + \sum_{j=1}^{2m} \int\limits_{\omega_j(x_1,x_2)}^{x_2} \Big[E_{4j}\widehat{w}_{p-1} + \sum_{q=3}^{n} E_{5jq}\widehat{v}_{q,p-1} + E_{6j}\widehat{u}_{p-1}\Big]\big(z_j(x_1,x_2;t),t\big)\,dt + \\ + \widetilde{F}(x_1,x_2), \end{gathered} \tag{1.40}$$

and define the functions $\varphi_p(x_2)$ and $\psi_p(x_2)$ from the equations

$$\begin{aligned} (\Lambda_{10}(\varphi_p))(x_2) &= \big[T_3\big(\widehat{u}_{p-1}, \widehat{w}_{p-1}, \widehat{v}_{3,p-1}, \dots, \widehat{v}_{n,p-1}\big)\big](x_2) + f_5(x_2), \\ (\Lambda_{20}(\psi_p))(x_2) &= \big[T_4\big(\widehat{u}_{p-1}, \widehat{w}_{p-1}, \widehat{v}_{3,p-1}, \dots, \widehat{v}_{n,p-1}\big)\big](x_2) + f_6(x_2). \end{aligned} \tag{1.41}$$

By (1.22), (1.24) and the inequality $0 \le \widetilde{\sigma}(x_1,x_2) \le x_2$, the integral operators in the equalities (1.26), (1.27), (1.29), (1.32), (1.33) are of Volterra structure. Therefore in the domain D_{0P}, $P(x_1^0,x_2^0) \in D_0$, using the estimates (1.36) and (1.37), for $|\xi^0| > \widetilde{K}_3$ by the method of mathematical induction we obtain

$$\big\|\big[\partial^{i_1,j_1}(\widehat{u}_{p+1}-\widehat{u}_p\big](x_1,x_2)\big\| \le$$
$$\le M^*(M_*^p/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-i_1-j_1-1}\exp(-N|\xi^0|), \tag{1.42}$$
$$\big\|\big[\partial^{i_1,j_1}(\widehat{w}_{p+1}-\widehat{w}_p\big](x_1,x_2)\big\| \le$$
$$\le M^*(M_*^p/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-i_1-j_1-1}\exp(-N|\xi^0|), \tag{1.43}$$
$$\big\|\big[\partial^{i_1,j_1}(\widehat{v}_{q,p+1}-\widehat{v}_{q,p}\big](x_1,x_2)\big\| \le$$
$$\le M^*(M_*^p/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-i_1-j_1-1}\exp(-N|\xi^0|),\ \ q=3,\dots,n, \tag{1.44}$$
$$\Big\|\big[d^{i_1+j_1}(\varphi_{p+1}-\varphi_p)/dx_2^{i_1+j_1}\big](x_2)\Big\| \le$$
$$\le M^*(M_*^p/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-i_1-j_1-1}\exp(-N|\xi^0|), \tag{1.45}$$
$$\Big\|\big[d^{i_1+j_1}(\psi_{p+1}-\psi_p)/dx_2^{i_1+j_1}\big](x_2)\Big\| \le$$
$$\le M^*(M_*^p/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-i_1-j_1-1}\exp(-N|\xi^0|), \tag{1.46}$$

where $\partial^{i_1,j_1} = \frac{\partial^{i_1+j_1}}{\partial x_1^{i_1}\partial x_2^{j_1}}$, $0 \le i_1+j_1 \le k-1$, $\widetilde{K}_3 = \widetilde{K}_3(x_1^0,x_2^0,N,f_1,f_2,F)$, $M^* = M^*(x_1^0,x_2^0,N,f_1,f_2,F,\delta_1,\delta_2)$ and $M_* = M_*(x_1^0,x_2^0,N,f_1,f_2,F,\delta_1,\delta_2)$ do not depend on ξ^0, and δ_1 and δ_2 are the constants from (1.36), (1.37).

Remark 1.5. By the definition of $\widetilde{\sigma}(x_1,x_2)$ and β, the inequalities (1.38) and (1.39) define the functions $\widehat{u}_p$, $\widehat{v}_{q,p}$, $q=3,\dots,n$, using different formulas for $x_1>0$ and $x_1 \le 0$. But this does not imply the existence of discontinuities along the axis $Ox_2:\ x_1=0$ of the functions $\widehat{u}_p$, $\widehat{v}_{q,p}$, $q=3,\dots,n$, and their derivatives with respect to x_1 and x_2 up to the order $(k-1)$ inclusively since the functions $\widetilde{F}$, f_5, f_6 from (1.40), (1.41) and their derivatives up to the order $(k-1)$ inclusively are equal to zero at the point $(0,0)$, by the condition.

It follows from (1.42) that for $0 \le i_1+j_1 \le k-1$ the series

$$\widehat{u}_{i_1,j_1}(x_1,x_2) = \lim_{p\to\infty}\big[\partial^{i_1,j_1}\widehat{u}_p\big](x_1,x_2) = \sum_{p=1}^{\infty}\big[\partial^{i_1,j_1}(\widehat{u}_p-\widehat{u}_{p-1})\big](x_1,x_2)$$

converges uniformly in $\overline{D}_{0P}$, and for its sum the estimate

$$\big\|\widehat{u}_{i_1,j_1}(x_1,x_2)\big\| \le M^* x_2^{k+\alpha-i_1-j_1-1}\exp\big[M_*(1+|\xi^0|)x_2\big]\exp(-N|\xi^0|) \tag{1.47}$$

is valid, from which it follows that $\widehat{u}_{i_1,j_1} \in \overset{\circ}{\Phi}_{k-1+\alpha-i_1-j_1}(\overline{D}_{0P})$ since, as it can be easily verified, the operator of multiplication by the function $\exp[M_*(1+|\xi^0|)x_2]$ maps the space $\overset{\circ}{\Phi}_{k-1+\alpha-i_1-j_1}(\overline{D}_{0P})$ into itself. In its turn, this implies that the function $\widehat{u}^1 \equiv \widehat{u}_{0,0}(x_1,x_2)$ belongs to $\overset{\circ}{\Phi}{}^{k-1}_{\alpha}(\overline{D}_{0P})$, where $\widehat{u}_{i_1,j_1}(x_1,x_2) \equiv \partial^{i_1,j_1}\widehat{u}^1(x_1,x_2)$.

Analogously, from (1.43)–(1.46) we obtain that the series

$$\widehat{w}^1(x_1,x_2) = \lim_{p\to\infty} \widehat{w}_p(x_1,x_2) = \sum_{p=1}^{\infty} \big(\widehat{w}_p(x_1,x_2) - \widehat{w}_{p-1}(x_1,x_2)\big),$$

$$\widehat{v}^1_q(x_1,x_2) = \lim_{p\to\infty} \widehat{v}_{q,p}(x_1,x_2) = \sum_{p=1}^{\infty} \big(\widehat{v}_{q,p}(x_1,x_2) - \widehat{v}_{q,p-1}(x_1,x_2)\big),$$
$$q = 3,\dots,n,$$

converge in the space $\overset{\circ}{\Phi}{}^{k-1}_{\alpha}(\overline{D}_{0P})$, and the series

$$\varphi^1(x_2) = \lim_{p\to\infty} \varphi_p(x_2) = \sum_{p=1}^{\infty} \big(\varphi_p(x_2) - \varphi_{p-1}(x_2)\big),$$

$$\psi^1(x_2) = \lim_{p\to\infty} \psi_p(x_2) = \sum_{p=1}^{\infty} \big(\psi_p(x_2) - \psi_{p-1}(x_2)\big)$$

converge respectively in the spaces $\overset{\circ}{\Phi}{}^{k-1}_{\alpha}([0,d_1])$ and $\overset{\circ}{\Phi}{}^{k-1}_{\alpha}([0,d_2])$. By virtue of (1.38)–(1.41) it follows that the limiting functions $\widehat{u}^1$, $\widehat{w}^1$, $\widehat{v}^1_q \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}(\overline{D}_{0P})$, $q = 3,\dots,n$, $\varphi^1 \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}([0,d_1])$ and $\psi^1 \in \overset{\circ}{\Phi}{}^{k-1}_{\alpha}([0,d_2])$ satisfy the system of equations (1.26), (1.27), (1.29), (1.32), (1.33).

Thus to prove Theorems 1.1 and 1.2, it remains to show that this system of equations has no other solutions in the classes under consideration. Indeed, assume that the functions $\widehat{u}^*$, $\widehat{w}^*$, $\widehat{v}^*_q$, $q = 3,\dots,n$, φ^* and ψ^* from the above-mentioned classes satisfy the homogeneous system of equations corresponding to (1.26), (1.27), (1.29), (1.32), (1.33), i.e. for $\widetilde{F} = 0$, $f_5 = 0$, $f_6 = 0$. To this system we apply the method of successive approximations, taking the functions $\widehat{u}^*$, $\widehat{w}^*$, $\widehat{v}^*_q$, $q = 3,\dots,n$, φ^* and ψ^* themselves as zero approximations. Since these functions satisfy the homogeneous system of equations, then every next approximation will coincide with them, i.e. $\widehat{u}^*_p(x_1,x_2) \equiv \widehat{u}^*(x_1,x_2)$, $\widehat{w}^*_p(x_1,x_2) \equiv \widehat{w}^*(x_1,x_2)$, $\widehat{v}^*_{q,p}(x_1,x_2) \equiv \widehat{v}^*(x_1,x_2)$, $q = 3,\dots,n$, $\varphi^*_p(x_2) \equiv \varphi^*(x_2)$, $\psi^*_p(x_2) \equiv \psi^*(x_2)$. The same reasoning as in deducing the estimates (1.42)–(1.46) allows one to obtain

$$\|\widehat{u}^*(x_1,x_2)\| = \|\widehat{u}^*_p(x_1,x_2)\| \le \widetilde{M}^*(\widetilde{M}^p_*/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-1}\exp(-N|\xi^0|),$$
$$\|\widehat{w}^*(x_1,x_2)\| = \|\widehat{w}^*_p(x_1,x_2)\| \le \widetilde{M}^*(\widetilde{M}^p_*/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-1}\exp(-N|\xi^0|),$$
$$\|\widehat{v}^*_q(x_1,x_2)\| = \|\widehat{v}^*_{q,p}(x_1,x_2)\| \le$$
$$\le \widetilde{M}^*(\widetilde{M}^p_*/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-1}\exp(-N|\xi^0|),\quad q=3,\dots,n,$$
$$\|\varphi^*(x_2)\| = \|\varphi^*_p(x_2)\| \le \widetilde{M}^*(\widetilde{M}^p_*/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-1}\exp(-N|\xi^0|),$$
$$\|\psi^*(x_2)\| = \|\psi^*_p(x_2)\| \le \widetilde{M}^*(\widetilde{M}^p_*/p!)(1+|\xi^0|)^p x_2^{p+k+\alpha-1}\exp(-N|\xi^0|),$$

whence in the limit as $p \to \infty$ we find that $\widehat{u}^* \equiv 0$, $\widehat{w}^* \equiv 0$, $\widehat{v}_q^* \equiv 0$, $q = 3, \dots, n$, $\varphi^* \equiv 0$, $\psi^* \equiv 0$.

2. Boundary Value Problems for a Class of Hyperbolic Systems of Second Order with Symmetric Principal Part

2.1. Statement of the problem. In the Euclidean space R^{n+1} of the variables $x = (x_1, \dots, x_n)$ and t let us consider a system of linear differential equations of the type

$$Lu \equiv u_{tt} - \sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Cu = F, \tag{2.1}$$

where A_{ij} $(A_{ij} = A_{ji})$, B_i and C are given real $(m \times m)$-matrices, F is a given and u is an unknown m-dimensional real vector, $n \geq 2$, $m > 1$.

The matrices A_{ij} below will be assumed to be symmetric and constant, and for any m-dimensional real vectors η_i, $i = 1, \dots, n$, the inequality

$$\sum_{i,j=1}^{n} A_{ij} \eta_i \eta_j \geq c_0 \sum_{i=1}^{n} |\eta_i|^2, \ \ c_0 = \text{const} > 0, \tag{2.2}$$

is assumed to be valid.

It can be easily verified that by the condition (2.2) the system (2.1) is hyperbolic.

Let D be the conic domain $\big\{(x,t) \in R^{n+1} : \ |x| g(x/|x|) < t < +\infty\big\}$ lying in the half-space $t > 0$ and bounded by the conic manifold $S = \big\{(x,t) \in R^{n+1} : \ t = |x| g(x/|x|)\big\}$, where g is a positive continuous piecewise-smooth function given on the unit sphere of the space R^n. For $\tau > 0$ we denote by $D_\tau := \big\{(x,t) \in R^{n+1} : \ |x| g(x/|x|) < t < \tau\big\}$ the domain in the half-space $t > 0$ bounded by the cone S and the hyperplane $t = \tau$.

Let $S_0 = \partial D_{\tau_0} \cap S$ be the conic portion of the boundary of D_{τ_0} for an arbitrary $\tau_0 > 0$. Assume that $S_1, \dots, S_{k_1}, S_{k_1+1}, \dots, S_{k_1+k_2}$ are nonintersecting smooth conic open hypersurfaces, where $S_1, \dots, S_{k_1}$ are characteristic manifolds of the system (2.1), and $S_0 = \bigcup\limits_{i=1}^{k_1+k_2} \overline{S}_i$, where $\overline{S}_i$ is the closure of S_i.

Consider the boundary value problem which is stated as follows: find in the domain D_{τ_0} a solution $u(x,t)$ of the system (2.1) satisfying the conditions

$$u\big|_{S_0} = f_0, \tag{2.3}$$

$$\Gamma^i u_t\big|_{S_i} = f_i, \ \ i = 1, \dots, k_1 + k_2, \tag{2.4}$$

where f_i, $i = 0, 1, \dots, k_1 + k_2$, are given real $\varkappa_i$-dimensional vectors, Γ^i, $i = 1, \dots, k_1 + k_2$, are given real constant $(\varkappa_i \times m)$-matrices, and $\varkappa_0 = m$, $0 \leq \varkappa_i \leq m$, $i = 1, \dots, k_1 + k_2$. Here the number $\varkappa_i$, $1 \leq i \leq m$, shows to what extent the part S_i of the boundary ∂D_{τ_0} is occupied; in particular, $\varkappa_0 = 0$ shows that the corresponding part S_i (2.4) is completely free of boundary conditions. Below we will see that in order to ensure the correctness of the problem (2.1), (2.3), (2.4) the number $\varkappa_i$ must be chosen in a quite definite way, depending on geometric properties of the hypersurface S_i.

Below the elements of the matrices B_i, C in the system (2.1) will be assumed to be bounded measurable functions in the domain D_{τ_0}, and the right-hand side of that system F to belong to $L_2(D_{\tau_0})$.

2.2. The method of the choice of the numbers $\varkappa_i$ and matrices Γ^i in the boundary conditions (2.4) depending on geometric properties of S_i. By the condition (2.2), the symmetrical matrix $Q(\xi') = \sum\limits_{i,j=1}^{n} A_{ij}\xi_i\xi_j$, $\xi' = (\xi, \dots, \xi_n) \in R^n \setminus \{(0,\dots,0)\}$, is positive definite. Therefore there exists an orthogonal matrix $T = T(\xi')$ such that the matrix $T^{-1}(\xi')Q(\xi')T(\xi')$ is diagonal, and its elements $\mu_1, \dots, \mu_m$ on the diagonal are positive, i.e. $\mu_i = \widetilde{\lambda}_i^2(\xi') > 0$, $\widetilde{\lambda}_i > 0$, $i = 1, \dots, m$. In addition, without restriction of generality, we can assume that $\widetilde{\lambda}_m(\xi') \geq \cdots \geq \widetilde{\lambda}_1(\xi') > 0$, $\forall\, \xi' \in R^n \setminus \{(0,\dots,0)\}$. Below it will be assumed that the multiplicities of the values $\ell_1, \dots, \ell_s$ do not depend on ξ', and we assume that

$$\lambda(\xi') = \widetilde{\lambda}_1(\xi') = \cdots = \widetilde{\lambda}_{\ell_1}(\xi') < \lambda_2(\xi') = \widetilde{\lambda}_{\ell_1+1}(\xi') = \cdots \widetilde{\lambda}_{\ell_1+\ell_2}(\xi') <$$
$$< \lambda_s(\xi') = \widetilde{\lambda}_{m-\ell_s+1}(\xi') = \cdots = \widetilde{\lambda}_m(\xi'), \ \ \xi' \in R^n \setminus \{(0,\dots,0)\}. \qquad (2.5)$$

Note that by virtue of (2.5) and the continuous dependence of the roots of the characteristic polynomial of a symmetric matrix on its elements, $\lambda_1(\xi'), \dots, \lambda_2(\xi')$ are continuous first degree homogeneous functions [46, p. 634].

It can be easily seen that the roots of the characteristic polynomial $\det(E\xi_{n+1}^2 - Q(\xi'))$ of the system (2.1) with respect to ξ_{n+1} are the numbers $\xi_{n+1} = \pm\lambda_i(\xi_1, \dots, \xi_n)$, $i = 1, \dots, s$, with multiplicities $\ell_1, \dots, \ell_s$, respectively, where E is the unit $(m \times m)$-matrix. Therefore the cone of the normals $K = \big\{\xi = (\xi_1, \dots, \xi_n, \xi_{n+1}) \in R^{n+1} : \ \det(E\xi_{n+1}^2 - Q(\xi')) = 0\big\}$ of the system (2.1) consists of its separate connected components $K_i^{\pm} = \big\{\xi = (\xi', \xi_{n+1}) \in R^{n+1} : \ \xi_{n+1} \mp \lambda_i(\xi') = 0\big\}$, $i = 1, \dots, s$.

Denote by $D_i^- = \big\{\xi = (\xi', \xi_{n+1}) \in R^{n+1} : \ \xi_{n+1} + \lambda_i(\xi') < 0\big\}$ the conic domain whose boundary is the hypersurface K_i^-, $i = 1, \dots, s$. By (2.5), we have $D_1^- \supset D_2^- \supset \cdots \supset D_s^-$. Let $G_i = D_{i-1}^- \setminus \overline{D_i^-}$ for $1 < i \leq s$, $G_1 = R_-^{n+1} \setminus \overline{D}_1^-$ with $R_-^{n+1} = \{\xi \in R^{n+1} : \ \xi_{n+1} < 0\}$, and $G_{s+1} = D_s^-$.

Since for the unit vector of the outer normal $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ at the points of the cone S different from its vertex $O(0, \dots, 0)$ we have

$$\alpha_i = \frac{\frac{\partial g_0}{\partial x_i}}{\sqrt{1 + |\nabla_x g_0|^2}}\,, \ \ i = 1, \dots, n, \quad \alpha_{n+1} = \frac{-1}{\sqrt{1 + |\nabla_x g_0|^2}}\,,$$

where $\nabla_x = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$ and $g_0(x) = |x| g(x/|x|)$, therefore

$$\alpha_{n+1}\big|_{S\setminus O} < 0. \tag{2.6}$$

According to our assumption, the smooth conic hypersurface S_i for $1 \le i \le k_1$ is a characteristic one. Therefore by virtue of the fact that $S_i \subset S_0 \subset S$ and the condition (2.6) is fulfilled, for some index m_i, $1 \le m_i \le s$, we have

$$\alpha\big|_{S_i} \in K^-_{m_i}, \ \ i = 1, \dots, k_1. \tag{2.7}$$

Since S_i for $k_1 + 1 \le i \le k_1 + k_2$ at none of its point is characteristic, by virtue of $S_i \subset S_0 \subset S$ and (2.6), and by the definition of the domains G_j there is an index n_i, $1 \le n_i \le s + 1$, such that

$$\alpha\big|_{S_i} \in G_{n_i}, \ \ i = k_1 + 1, \dots, k_1 + k_2. \tag{2.8}$$

Below, without restriction of generality, it will be assumed that $m_1 \le \dots \le m_{k_1}$ and $n_{k_1+1} \le \dots \le m_{k_1+k_2}$.

By $Q_0(\xi) = E\xi^2_{n+1} - Q(\xi')$ we denote the characteristic matrix of the system (2.1) and consider the question on reduction of the quadratic form $(Q_0(\xi)\eta, \eta)$ to the canonic form when $\xi = \alpha$ is the unit vector of the normal to the hypersurface S_i, $1 \le i \le k_1 + k_2$, exterior with respect to the domain D_{τ_0}. Here $\eta \in R^m$ and $(\cdot\,,\cdot)$ is the scalar product in the Euclidean space R^m.

Since

$$\begin{gathered} T^{-1}(\alpha')Q(\alpha)T(\alpha') = \\ = \operatorname{diag}\Big(\underbrace{\lambda_1^2(\alpha'), \dots, \lambda_1^2(\alpha')}_{\ell_1}, \dots, \underbrace{\lambda_s^2(\alpha'), \dots, \lambda_s^2(\alpha')}_{\ell_s}\Big), \ \alpha' = (\alpha_1, \dots, \alpha_n), \end{gathered} \tag{2.9}$$

for $\eta = T\zeta$ we have

$$\begin{gathered} (Q_0(\alpha)\eta, \eta) = \big((T^{-1}Q_0T)(\alpha)\zeta, \zeta\big) = \big(\big(E\alpha^2_{n+1} - (T^{-1}QT)(\alpha')\big)\zeta, \zeta\big) = \\ = (\alpha^2_{n+1} - \lambda_1^2(\alpha'))\zeta_1^2 + \dots + (\alpha^2_{n+1} - \lambda_1^2(\alpha'))\zeta^2_{\ell_1} + \\ + (\alpha^2_{n+1} - \lambda_2^2(\alpha'))\zeta^2_{\ell_1+1} + \dots + (\alpha^2_{n+1} - \lambda_2^2(\alpha'))\zeta^2_{\ell_1+\ell_2} + \dots + \\ + (\alpha^2_{n+1} - \lambda_s^2(\alpha'))\zeta^2_{m-\ell_s+1} + \dots + (\alpha^2_{n+1} - \lambda_s^2(\alpha'))\zeta^2_m. \end{gathered} \tag{2.10}$$

For $1 \le i \le k_1$, i.e. in the case (2.7), since $\alpha^2_{n+1} - \lambda^2_{m_i}(\alpha') = 0$, by virtue of (2.5) we have

$$\begin{gathered} [\alpha^2_{n+1} - \lambda_j^2(\alpha')]\big|_{K^-_{m_i}} > 0, \ j = 1, \dots, m_i - 1; \ \ [\alpha^2_{n+1} - \lambda^2_{m_i}(\alpha')]\big|_{K^-_{m_i}} = 0, \\ [\alpha^2_{n+1} - \lambda_j^2(\alpha')]\big|_{K^-_{m_i}} < 0, \ \ j = m_i + 1, \dots, s. \end{gathered} \tag{2.11}$$

If $k_1+1\le i\le k_1+k_2$, i.e. in the case (2.8), by the definition of the domain G_{n_i} from (2.5) it follows that for $n_i\le s$

$$
\begin{gathered}
[\alpha_{n+1}^2-\lambda_j^2(\alpha')]\big|_{G_{n_i}}>0,\ \ j=1,\dots,n_i-1,\\
[\alpha_{n+1}^2-\lambda_j^2(\alpha')]\big|_{G_{n_i}}<0,\ \ j=n_i,\dots,s,
\end{gathered}
$$

and for $n_i=s+1$

$$
[\alpha_{n+1}^2-\lambda_j^2(\alpha')]\big|_{G_{n_i}}>0,\ \ j=1,\dots,s. \tag{2.12}
$$

Denote by $\varkappa_i^+$ and $\varkappa_i^-$ the positive and the negative indices of inertia of the quadratic form $(Q_0(\alpha)\eta,\eta)$ for $\alpha\in K_{m_i}^-$ when $1\le i\le k_1$ and for $\alpha\in G_{n_i}$ when $k_1+1\le i\le k_1+k_2$. For $1\le i\le k_1$, by (2.10) and (2.11) we have

$$
\varkappa_i^+=\ell_1+\dots+\ell_{m_i-1},\quad \varkappa_i^-=\ell_{m_i+1}+\dots+\ell_s,\quad (def)_{m_i}=\ell_{m_i}, \tag{2.13}
$$

where $(def)_{m_i}$ is the defect of that form, and in addition $\varkappa_i^+=0$ for $m_i=1$. In the case $k_1+1\le i\le k_1+k_2$, by (2.10) and (2.12) we have

$$
\varkappa_i^+=\ell_1+\dots+\ell_{n_i-1},\quad \varkappa_i^-=\ell_{n_i}+\dots+\ell_s, \tag{2.14}
$$

where $\varkappa_i^+=0$ for $n_i=1$.

If now $\zeta=C^i\eta$ is an arbitrary nondegenerate linear transformation reducing the quadratic form $(Q_0(\alpha)\eta,\eta)$ in the case (2.13) or (2.14) to the canonic form, then by the invariance of indices of inertia of the quadratic form with respect to nondegenerate linear transformations we have

$$
(Q_0(\alpha)\eta,\eta)=\sum_{j=1}^{\varkappa_i^+}[\Lambda_{ij}^+(\alpha,\eta)]^2-\sum_{j=1}^{\varkappa_i^-}[\Lambda_{ij}^-(\alpha,\eta)]^2,\ \ 1\le i\le k_1+k_2. \tag{2.15}
$$

Here

$$
\begin{gathered}
\Lambda_{ij}^+(\alpha,\eta)=\sum_{p=1}^m c_{jp}^i(\alpha)\eta_p,\quad \Lambda_{ij}^-(\alpha,\eta)=\sum_{p=1}^m c_{\varkappa_i+j,p}^i(\alpha)\eta_p,\\
C^i=C^i(\alpha)=(c_{jp}^i(\alpha))_{j,p=1}^m,\ \ 1\le i\le k_1+k_2.
\end{gathered} \tag{2.16}
$$

In accordance with (2.16), in the boundary conditions (2.4) as the matrix Γ^i we take the matrix of order $(\varkappa_i\times m)$ whose elements $\varkappa_i=\varkappa_i^+$, $1\le i\le k_1+k_2$, are given by the equalities

$$
\Gamma_{jp}^i=c_{jp}^i(\alpha),\ \ j=1,\dots,\varkappa_i^+;\ \ p=1,\dots,m, \tag{2.17}
$$

where $\alpha\in K_{m_i}^-$ for $1\le i\le k_1$ and $\alpha\in G_{n_i}$ for $k_1+1\le i\le k_1+k_2$.

Below it will be assumed that in the boundary conditions (2.4) the elements Γ_{jp}^i of the matrices Γ^i on S_i are bounded measurable functions. It will also be assumed that the domain D_{τ_0} is a Lipschitz domain [89, p. 68].

2.3. Deduction of an a priori estimate for the solution of the problem (2.1), (2.3), (2.4). Below, if it will not cause ambiguity, instead of $u=(u_1,\dots,u_m)\in[W_2^k(D_{\tau_0})]^m$ we will write $u\in W_2^k(D_{\tau_0})$. The condition $F=(F_1,\dots,F_m)\in L_2(D_{\tau_0})$ should be understood analogously. Let $u\in W_2^2(D_{\tau_0})$ be a solution of the problem (2.1), (2.3), (2.4). Multiplying both parts of the system of equations (2.1) scalarly by the vector $2u_t$ and integrating the obtained expression over D_τ, $0<\tau\le\tau_0$, we obtain

$$
\begin{aligned}
&2\int\limits_{D_\tau}\Big(F-\sum_{i=1}^n B_iu_{x_i}-Cu\Big)u_t\,dx\,dt=\int\limits_{D_\tau}\Big[\frac{\partial(u_t,u_t)}{\partial t}+2\sum_{i,j=1}^n A_{ij}u_{x_i}u_{tx_j}\Big]\,dx\,dt-\\
&\quad-2\int\limits_{S_0\cap\{t\le\tau\}}\sum_{i,j=1}^n A_{ij}u_tu_{x_j}\alpha_i ds=\int\limits_{\partial D_\tau\setminus S_0}\Big(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\Big)\,dx+\\
&\quad+\int\limits_{S_0\cap\{t\le\tau\}}\Big[\Big(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\Big)\alpha_{n+1}-2\sum_{i,j=1}^n A_{ij}u_tu_{x_j}\alpha_i\Big]\,ds=\\
&\qquad=\int\limits_{\partial D_\tau\setminus S_0}\Big(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\Big)\,dx+\\
&\qquad+\int\limits_{S_0\cap\{t\le\tau\}}\alpha_{n+1}^{-1}\Big[\sum_{i,j=1}^n A_{ij}(\alpha_{n+1}u_{x_i}-\alpha_iu_t)(\alpha_{n+1}u_{x_j}-\alpha_ju_t)+\\
&+\Big(E\alpha_{n+1}^2-\sum_{i,j=1}^n A_{ij}\alpha_i\alpha_j\Big)u_tu_t\Big]\,ds=\int\limits_{\partial D_\tau\setminus S_0}\Big(u_tu_t+\sum_{i,j=1}^n A_{ij}u_{x_i}u_{x_j}\Big)\,dx+\\
&\quad+\int\limits_{S_0\cap\{t\le\tau\}}\alpha_{n+1}^{-1}\Big[\sum_{i,j=1}^n A_{ij}(\alpha_{n+1}u_{x_i}-\alpha_iu_t)(\alpha_{n+1}u_{x_j}-\alpha_ju_t)\Big]\,ds+\\
&\qquad\qquad+\int\limits_{S_0\cap\{t\le\tau\}}\alpha_{n+1}^{-1}(Q_0(\alpha)u_t,u_t)\,ds. \qquad (2.18)
\end{aligned}
$$

Since $\big(\alpha_{n+1}\frac{\partial}{\partial x_i}-\alpha_i\frac{\partial}{\partial t}\big)$ is inner differential operator on the conic hypersurface S_0, by virtue of (2.3) and the boundedness of $|\alpha_{n+1}^{-1}|$ on S_0 we have

$$
\begin{aligned}
&\Big|\int\limits_{S_0\cap\{t\le\tau\}}\alpha_{n+1}^{-1}\Big[\sum_{i,j=1}^n A_{ij}(\alpha_{n+1}u_{x_i}-\alpha_iu_t)(\alpha_{n+1}u_{x_j}-\alpha_ju_t)\Big]\,ds\Big|\le\\
&\qquad\le c_1\|f_0\|_{W_2^1(S_0\cap\{t\le\tau\})},\ \ c_1=\mathrm{const}>0. \qquad (2.19)
\end{aligned}
$$

On the other hand, by (2.15), (2.16), (2.17) and (2.4), (2.6) we have

$$\int\limits_{S_0\cap\{t\le\tau\}} \alpha_{n+1}^{-1}(Q_0(\alpha)u_t,u_t)\,ds = -\sum_{i=1}^{k_1+k_2}\int\limits_{S_i\cap\{t\le\tau\}}\Big\{|\alpha_{n+1}^{-1}|\sum_{j=1}^{\varkappa_i^+}\big[\Lambda_{ij}^{+}(\alpha,u_t)\big]^2\Big\}\,ds+$$

$$+\sum_{i=1}^{k_1+k_2}\int\limits_{S_i\cap\{t\le\tau\}}\Big\{|\alpha_{n+1}^{-1}|\sum_{j=1}^{\varkappa_i^-}\big[\Lambda_{ij}^{-}(\alpha,u_t)\big]^2\Big\}\,ds\ge$$

$$\ge-\sum_{i=1}^{k_1+k_2}\int\limits_{S_i\cap\{t\le\tau\}}\Big\{|\alpha_{n+1}^{-1}|\sum_{j=1}^{\varkappa_i^+}\big[\Lambda_{ij}^{+}(\alpha,u_t)\big]^2\Big\}\,ds\ge$$

$$\ge-c_2\sum_{i=1}^{k_1+k_2}\int\limits_{S_i\cap\{t\le\tau\}}\Big\{\sum_{j=1}^{\varkappa_i^+}\big[\Lambda_{ij}^{+}(\alpha,u_t)\big]^2\Big\}\,ds=-c_2\sum_{i=1}^{k_1+k_2}\|f_i\|^2_{L_2(S_i\cap\{t\le\tau\})}, \quad (2.20)$$

where $0<c_2=\sup\limits_{S_0}|\alpha_{n+1}^{-1}|<+\infty$.

Assume

$$w(\tau)=\int\limits_{\partial D_\tau\setminus S_0}\Big(u_tu_t+\sum_{i,j=1}^{n}A_{ij}u_{x_i}u_{x_j}\Big)\,dx,\quad \widetilde{u}_i=\alpha_{n+1}u_{x_i}-\alpha_iu_t.$$

Then since the elements of the matrices B_i and C in the system (2.1) are bounded and measurable, as well as by (2.18), (2.19) and (2.20), we have

$$w(\tau)\le c_3\int\limits_0^\tau w(t)\,dt+c_4\int\limits_{D_\tau}u\,u\;dx\,dt+c_5\|f_0\|_{W_2^1(S_0\cap\{t\le\tau\})}+$$

$$+c_6\sum_{i=1}^{k_1+k_2}\|f_i\|^2_{L_2(S_0\cap\{t\le\tau\})}+c_7\|F\|_{L_2(D_\tau)}. \quad (2.21)$$

Here and in what follows, all the values c_i, $i\ge1$, are positive constants independent of u.

Let (x,τ_x) be the point of intersection of the conic hypersurface S with the straight line parallel to the axis t and passing through the point $(x,0)$. We have

$$u(x,t)=u(x,\tau_x)+\int\limits_{\tau_x}^{\tau}u_t(x,t)\,dt,\quad \tau\ge\tau_x,$$

whence with regard for (2.3) we find that

$$\int\limits_{\partial D_\tau\setminus S_0}u(x,\tau)u(x,\tau)\,dx\le$$

$$\leq 2 \int\limits_{\partial D_\tau \setminus S_0} u(x,\tau_x)u(x,\tau_x)\,dx + 2|\tau-\tau_x| \int\limits_{\partial D_\tau \setminus S_0} dx \int\limits_{\tau_x}^{\tau} u_t(x,t)u_t(x,t)\,dt \leq$$

$$\leq c_8 \int\limits_{S_0\cap\{t\leq\tau\}} u\,u\,ds + c_9\int\limits_0^\tau w(t)\,dt = c_8\|f_0\|^2_{L_2(S_0\cap\{t\leq\tau\})} + c_9\int\limits_0^\tau w(t)\,dt. \quad (2.22)$$

Introduce the notation

$$w_0(\tau) = \int\limits_{\partial D_\tau \setminus S_0} \Big(u\,u + u_t u_t + \sum_{i,j=1}^n A_{ij} u_{x_i} u_{x_j}\Big)\,dx.$$

Summing up the inequalities (2.21) and (2.22), we obtain

$$w_0(\tau) \leq$$

$$\leq c_{10}\bigg[\int\limits_0^\tau w_0(t)\,dt + \|f_0\|^2_{W_2^1(S_0\cap\{t\leq\tau\})} + \sum_{i=1}^{k_1+k_2} \|f_i\|^2_{L_2(S_i\cap\{t\leq\tau\})} + \|F\|^2_{L_2(D_\tau)}\bigg],$$

whence by the Gronwall lemma we find that

$$w_0(\tau) \leq c_{11}\Big(\|f_0\|_{W_2^1(S_0\cap\{t\leq\tau\})} + \sum_{i=1}^{k_1+k_2} \|f_i\|^2_{L_2(S_i\cap\{t\leq\tau\})} + \|F\|^2_{L_2(D_\tau)}\Big). \quad (2.23)$$

Integrating both parts of the inequality (2.23) with respect to τ, we arrive at the following a priori estimate for the solution $u \in W_2^2(D_{\tau_0})$ of the problem (2.1), (2.3), (2.4):

$$\|u\|_{W_2^1(D_{\tau_0})} \leq c\Big(\|f_0\|_{W_2^1(S_0)} + \sum_{i=1}^{k_1+k_2} \|f_i\|^2_{L_2(S_i)} + \|F\|^2_{L_2(D_{\tau_0})}\Big) \quad (2.24)$$

with a positive constant c independent of u.

Introduce the notion of the strong generalized solution of the problem (2.1), (2.3), (2.4) of the class W_2^1.

Definition 2.1. Let $f_0 \in W_2^1(S_0)$, $f_i \in L_2(S_i)$, $i = 1,\dots,k_1+k_2$, and $F \in L_2(D_{\tau_0})$. A vector function $u = (u_1,\dots,u_m)$ is said to be a strong generalized solution of the problem (2.1), (2.3), (2.4) of the class W_2^1 if $u \in W_2^1(D_{\tau_0})$ and there exists a sequence of vector functions $\{u_k\}_{k=1}^\infty$ from the space $W_2^2(D_{\tau_0})$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{W_2^1(D_{\tau_0})} = 0, \quad \lim_{k\to\infty} \big\|u_k\big|_{S_0} - f_0\big\|_{W_2^1(S_0)} = 0,$$

$$\lim_{k\to\infty} \Big\|\Gamma^i \frac{\partial u_k}{\partial t}\Big|_{S_i} - f_i\Big\|_{L_2(S_i)} = 0, \quad i = 1,\dots,k_1+k_2,$$

$$\lim_{k\to\infty} \|Lu_k - F|_{L_2(D_{\tau_0})} = 0.$$

Below we will prove the existence of a strong generalized solution of the problem (2.1), (2.3), (2.4) of the class W_2^1 for the case when the conic hypersurface S_0 is of temporal type, i.e. when the characteristic matrix of the system (2.1) is negative definite on $S_0 \setminus O$. The latter can be written as follows:

$$\Big(\Big[E\alpha_{n+1}^2 - \sum_{i,j=1}^{n} A_{ij}\alpha_i\alpha_j\Big]\eta,\eta\Big) < 0 \quad \forall\eta \in R^n \setminus \{0,\dots,0)\}, \tag{2.25}$$

where the vector $\alpha = (\alpha_1,\dots,\alpha_n,\alpha_{n+1})$ is the outer unit normal to the cone S_0 at the points different from its vertex O.

In the case (2.25), by (2.13)–(2.17) and according to our choice of Γ^i, $i = 1,\dots,k_1+k_2$, in (2.4) we have $\varkappa_i = 0$, $i = 1,\dots,k_1+k_2$, i.e. the problem (2.1), (2.3), (2.4) is free from the boundary conditions (2.4), and the a priori estimate (2.24) for the solution $u \in W_2^2(D_{\tau_0})$ of the problem (2.1), (2.3) takes the form

$$\|u\|_{W_2^1(D_{\tau_0})} \le c\big(\|f_0\|_{W_2^1(S_0)} + \|F\|_{L_2(D_{\tau_0})}\big). \tag{2.26}$$

Note that the geometric meaning of the condition (2.25) has been elucidated in [61], and also therein for the solution $u \in W_2^2(D_{\tau_0})$ of the problem (2.1), (2.3) the a priori estimate (2.26) is obtained, although there is not proved the existence of a strong generalized solution of the problem (2.1), (2.3) of the class W_2^1 whose uniqueness directly follows from the estimate (2.26).

2.4. Proof of the existence of a strong generalized solution of the problem (2.1), (2.3) of the class W_2^1. Consider the question on the solvability of the above-mentioned problem, when the conic hypersurface is of temporal type. For the sake of simplicity of our discussion, we restrict ourselves to the case where the boundary condition (2.3) is homogeneous, i.e.

$$u\big|_{S_0} = 0. \tag{2.27}$$

The system (2.1) after the change of variables

$$y = \frac{x}{t}\,, \quad z = t \ \text{ or } \ x = zy, \quad t = z \tag{2.28}$$

with respect to the unknown vector function $v(y,z) = u(zy,y)$ takes the form

$$L_1 v = v_{zz} - \frac{1}{z^2}\sum_{i,j=1}^{n}\widetilde{A}_{ij}v_{y_iy_j} - \frac{2}{z}\sum_{i=1}^{n} y_i v_{zy_i} + \frac{1}{z}\sum_{i=1}^{n}\widetilde{B}_i v_{y_i} + \widetilde{C}v = \widetilde{F}. \tag{2.29}$$

Here

$$\begin{gathered}\widetilde{A}_{ij} = Ey_iy_j + A_{ij},\\ \widetilde{B}_i = B_i(zy,z), \quad \widetilde{C} = C(zy,z), \ \ ;\widetilde{F} = F(zy,z).\end{gathered} \tag{2.30}$$

Denote by G the n-dimensional domain being the intersection of the conic domain $D:\ t > |x|g(x/|x|)$ and the hyperplane $t = 1$ in which the variable x is replaced by y. Obviously $\partial G = \{y \in R^n:\ 1 = |y|g(y/|y|)\}$. Upon the transformation $(x,t) \to (y,z)$, by the equalities (2.28) the domain D_τ transforms into the cylindrical domain $\Omega_\tau = G \times (0,\tau) = \{(y,z) \in$

$R^{n+1}: \ y \in G, \ z \in (0,\tau)\}$ lying in the space of the variables y, z. Denote by $\Gamma_\tau = \partial G \times [0,\tau]$ the lateral surface of the cylinder Ω_τ. The boundary condition (2.27) with respect to the vector function v takes the form

$$v\big|_{\Gamma_{\tau_0}} = 0. \tag{2.31}$$

The proof of existence of a strong generalized solution of the problem (2.1), (2.3) of the class W_2^1 will be presented in several steps.

1^0. First of all we deduce an a priori estimate for the solution $v = (v_1,\dots,v_m)$ of the problem (2.29), (2.31) from the space $W_2^2(\Omega_{\tau_0})$, equal to zero in the domain Ω_δ, $0<\delta<\tau_0$.

Let v be a solution of the problem (2.29), (2.31) from the space $W_2^2(\Omega_{\tau_0})$ such that for some positive δ

$$v\big|_{\Omega_\delta} = 0, \ \ 0<\delta<\tau_0. \tag{2.32}$$

Under the assumption that $(0,\dots,0)\in G$ and $\operatorname{diam} G$ is sufficiently small, by (2.2) and (2.30) for any m-dimensional vectors η_i, $i=1,\dots,n$, the inequality

$$\sum_{i,j=1}^n \widetilde{A}_{ij}(y)\eta_i\eta_j \geq \widetilde{c}_0 \sum_{i=1}^n |\eta_i|^2, \ \ \widetilde{c}_0 = \text{const} > 0, \quad \forall\, y\in G, \tag{2.33}$$

is valid.

If $\nu = (\nu_1,\dots,\nu_n,\nu_{n+1})$ is the unit vector of the outer normal to the boundary $\partial\Omega_0$ of the cylinder Ω_{τ_0} at the points (y,z) where it exists, then with regard for (2.32) we can easily see that

$$\nu_{n+1}\big|_{\Gamma_{\tau_0}} = 0, \ \ \nu_i\big|_{\partial\Omega_{\tau_0}\cap\{z=\tau_0\}} = 0, \ \ i=1,\dots,n, \quad v_z\big|_{\Gamma_{\tau_0}} = 0. \tag{2.34}$$

Assume $G_\tau = \Omega_{\tau_0}\cap\{z=\tau\}$.

Multiplying both parts of the system (2.29) by the vector $2v_z$ and integrating the obtained expression over Ω_τ, $\delta<\tau\le\tau_0$, and also taking into account (2.30), (2.31), (2.32) and (2.34), we obtain

$$2\int\limits_{\Omega_\tau} \Big(\widetilde{F} - \frac{1}{z}\sum_{i=1}^n \widetilde{B}_i v_{y_i} - \widetilde{C}v\Big) v_z \, dy\, dz =$$

$$= \int\limits_{\Omega_\tau} \Big[2v_{zz}v_z - \frac{2}{z^2}\sum_{i,j=1}^n \widetilde{A}_{ij}(y) v_{y_iy_j} v_z - \frac{4}{z}\sum_{i=1}^n y_i v_{zy_i} v_z\Big]\, dy\, dz =$$

$$= \int\limits_{\Omega_\tau} \Big[\frac{\partial(v_z,v_z)}{\partial z} + \frac{2}{z^2}\sum_{i,j=1}^n \widetilde{A}_{ij}(y) v_{y_i} v_{zy_j} + \frac{2}{z^2}\sum_{i,j=1}^n \frac{\partial \widetilde{A}_{ij}(y)}{\partial y_i}\, v_{y_i} v_z -$$

$$-\frac{2}{z}\sum_{i=1}^n y_i\,\frac{\partial(v_z v_z)}{\partial y_i}\Big]\, dy\, dz = \int\limits_{\Omega_\tau} \Big[\frac{\partial(v_z v_z)}{\partial z} + \frac{1}{z^2}\,\frac{\partial}{\partial z}\Big(\sum_{i,j=1}^n \widetilde{A}_{ij}(y) v_{y_i} v_{y_j}\Big) +$$

$$+\frac{2}{z^2}\sum_{i,j=1}^{n} Ey_i v_{y_i} v_z + \frac{2}{z}\sum_{i=1}^{n}\frac{\partial y_i}{\partial y_i}\, v_z v_z\Big]\, dy\, dz =$$

$$=\int\limits_{G_\tau}\Big[v_z v_z + \frac{1}{z^2}\sum_{i,j=1}^{n}\widetilde{A}_{ij}(y) v_{y_i} v_{y_j}\Big]\, dy +$$

$$+\int\limits_{\Omega_\tau\setminus\Omega_\delta}\Big[\frac{2}{z^3}\sum_{i,j=1}^{n}\widetilde{A}_{ij} v_{y_i} v_{y_j} + \frac{2}{z^2}\sum_{i,j=1}^{n} Ey_i v_{y_i} v_z + \frac{2n}{z}\, v_z v_z\Big]\, dy\, dz. \qquad (2.35)$$

Since the ranges of the variables y_i in G are bounded, i.e. $\sup\limits_{G}|y_i| \le d$, $i = 1,\dots,n$, by (2.30) and (2.33) for some $\widetilde{c}_1 = \mathrm{const} > 0$ the inequality

$$\sum_{i,j=1}^{n}\widetilde{A}_{ij}(y)\eta_i\eta_j \le \widetilde{c}_1\sum_{i=1}^{n}|\eta_i|^2 \quad \forall\, \eta_i \in R^n, \quad \forall\, y \in G, \qquad (2.36)$$

holds.

Under the notation

$$\widetilde{w}(\tau) = \int\limits_{G_\tau}\Big[v_z v_z + \sum_{i=1}^{n} v_{y_i} v_{y_i}\Big]\, dy, \quad \widetilde{w}_0(\tau) = \int\limits_{G_\tau}\Big[v\, v + v_z v_z + \sum_{i=1}^{n} v_{y_i} v_{y_i}\Big]\, dy,$$

by (2.33), (2.35) and (2.36) we have

$$\min\Big(1, \frac{\widetilde{c}_0}{\tau^2}\Big)\widetilde{w}(\tau) \le$$

$$\le \int\limits_{\Omega_\tau\setminus\Omega_\delta}\Big[\frac{2\widetilde{c}_1}{z^3}\sum_{i,j=1}^{n} v_{y_i} v_{y_i} + \frac{dn}{z^2}\sum_{i=1}^{n}(v_{y_i} v_{y_i} + v_z v_z) + \frac{2n}{z}\, v_z v_z\Big]\, dy\, dz +$$

$$+\int\limits_{\Omega_\tau\setminus\Omega_\delta}\Big[\widetilde{F}\,\widetilde{F} + v_z v_z + \frac{1}{z}\sum_{i=1}^{n}\|\widetilde{B}_i\|_{L_\infty}(v_{y_i} v_{y_i} + v_z v_z) +$$

$$+\|\widetilde{C}\|_{L_\infty}(v\, v + v_z v_z)\Big]\, dy\, dz \le$$

$$\le \Big(\frac{2\widetilde{c}_1}{\delta^3} + \frac{dn}{z^2} + \frac{1}{\delta}\max_{1\le i\le n}\|\widetilde{B}_i\|_{L_\infty}\Big)\int\limits_{\Omega_\tau\setminus\Omega_\delta}\Big(\sum_{i=1}^{n} v_{y_i} v_{y_i}\Big)\, dy\, dz +$$

$$+\Big(\frac{dn^2}{\delta^2} + \frac{2n}{\delta} + 1 + \|\widetilde{C}\|_{L_\infty}\Big)\int\limits_{\Omega_\tau\setminus\Omega_\delta} v_z v_z\, dy\, dz +$$

$$+\|\widetilde{C}\|_{L_\infty}\int\limits_{\Omega_\tau\setminus\Omega_\delta} v\, v\, dy\, dz + \int\limits_{\Omega_\tau\setminus\Omega_\delta}\widetilde{F}\,\widetilde{F}\, dy\, dz \le$$

$$\le c_2(\delta)\int\limits_{\Omega_\tau}\Big[v\, v + v_z v_z + \sum_{i=1}^{n} v_{y_i} v_{y_i}\Big]\, dy\, dz + \int\limits_{\Omega_\tau}\widetilde{F}\,\widetilde{F}\, dy\, dz =$$

$$= c_2(\delta)\int\limits_0^\tau \widetilde{w}_0(\sigma)\,d\sigma + \int\limits_{\Omega_\tau} \widetilde{F}\,\widetilde{F}\,dy\,dz, \tag{2.37}$$

where $c_2(\delta) = \text{const} > 0$, $\delta < \tau \le \tau_0$, and $\|\widetilde{B}_i\|_{L_\infty}$ and $\|\widetilde{C}\|_{L_\infty}$ are the upper bounds of the norms of the matrices $\widetilde{B}_i$ and $\widetilde{C}$ in Ω_{τ_0}.

By (2.32) we have

$$v(y,z) = \int\limits_0^\tau v_z(y,\sigma)\,d\sigma,$$

whence

$$\int\limits_{G_\tau} v(y,\tau)v(y,\tau)\,dy \le \int\limits_G \Big[\int\limits_0^\tau |v_z(y,\sigma)|\,d\sigma\Big]^2 dy \le$$

$$\le \int\limits_G \Big[\Big(\int\limits_0^\tau 1^2\,d\sigma\Big)^{1/2}\Big(\int\limits_0^\tau |v_z(y,\sigma)|^2\,d\sigma\Big)^{1/2}\Big]^2 dy \le$$

$$\le \tau\int\limits_G\int\limits_0^\tau v_z^2(y,\sigma)\;d\sigma\,dy = \tau\int\limits_{\Omega_\tau} v_z^2\;dy\,dz. \tag{2.38}$$

Taking into account (2.38), from (2.37) we have

$$\widetilde{w}_0(\tau) \le c_3(\delta)\int\limits_0^\tau \widetilde{w}_0(\sigma) + c_4(\delta)\int\limits_{\Omega_\tau} \widetilde{F}\,\widetilde{F}\;dy\,dz,$$

where $c_i(\delta) = \text{const} > 0$, $i = 3, 4$. From the above reasoning, on the basis of the Gronwall lemma we can conclude that

$$\widetilde{w}_0(z) \le c(\delta)\int\limits_\Omega \widetilde{F}\,\widetilde{F}\;dy\,dz,\quad 0 < \tau \le \tau_0, \tag{2.39}$$

with $c(\delta) = \text{const} > 0$.

In turn, from (2.39) it follows that

$$\|v\|_{W_2^1(\Omega_{\tau_0})} \le \widetilde{c}(\delta)\|\widetilde{F}\|_{L_2(\Omega_{\tau_0})},\quad \widetilde{c}(\delta) = \text{const} > 0. \tag{2.40}$$

Remark 2.1. To construct for

$$\widetilde{F}\big|_{\Omega_\delta} = 0,\quad 0 < \delta < \tau_0, \tag{2.41}$$

a solution v of the problem (2.29), (2.31) from the space $W_2^2(\Omega_{\tau_0})$ satisfying the condition (2.32), we apply Galerkin's method [84, pp. 213–220]. Note that unlike equations and systems of hyperbolic type considered in [84], the system (2.29) contains terms with mixed derivatives v_{zy_i}.

2^0. Here we prove the existence of a weak solution of the problem (2.29), (2.31), (2.32) of the class W_2^1. Let $\{\varphi_k(y)\}_{k=1}^{\infty}$ be an orthogonal basis in a separable Hilbert space $[\overset{\circ}{W}{}_2^1(G)]^m$. As elements of the basis $\{\varphi_k(y)\}_{k=1}^{\infty}$ in the space $[\overset{\circ}{W}{}_2^1(G)]^m$ we take proper vector functions of the Laplace operator: $\Delta\varphi_k = \lambda_k\varphi_k$, $\varphi_k|_{\partial G} = 0$ [84, pp. 110, 248]. In addition, in the space $[\overset{\circ}{W}{}_2^1(G)]^m$ as an equivalent norm we can take

$$\|v\|^2_{\overset{\circ}{W}{}_2^1(G)} = \int_G \Big(\sum_{i=1}^n v_{y_i} v_{y_i}\Big)\,dy,$$

$$v = (v_1,\dots,v_m),\ \ v_i \in \overset{\circ}{W}{}_2^1(G),\ \ i = 1,\dots,m.$$

An approximate solution $v^N(y,z)$ of the problem (2.29), (2.31) will be sought in the form of the sum

$$v^N(y,z) = \sum_{k=1}^N C_k^N(z)\varphi_k(y), \tag{2.42}$$

whose coefficients $C_k^N(z)$ are defined from the relations

$$\Big(\frac{\partial^2 v^N}{\partial z^2},\varphi_l\Big)_{L_2(G)} + \frac{1}{z^2}\int_G \Big\{\sum_{i,j=1}^n \Big[\widetilde{A}_{ij}(y)v^N_{y_i}\varphi_{ly_j} + \frac{\partial \widetilde{A}_{ij}}{\partial y_i}\,v^N_{y_i}\varphi_l\Big]\Big\}\,dy +$$
$$+\frac{2}{z}\int_G\Big\{\sum_{i=1}^n\big[y_i v_z^N\varphi_{ly_i} + v_z^N\varphi_l\big]\Big\}\,dy + \int_G\Big[\frac{1}{z}\sum_{i=1}^n \widetilde{B}_i v^N_{y_i}\varphi_l + \widetilde{C}v^N\varphi_l\Big]\,dy =$$
$$= (\widetilde{F},\varphi_l)_{L_2(G)},\ \ \delta \le z \le \tau_0,\ \ l = 1,\dots,N, \tag{2.43}$$

$$\frac{d}{dz}\,C_k^N(z)\Big|_{z=\delta} = 0,\ \ \ C_k^N(z)\big|_{z=\delta} = 0,\ \ k = 1,\dots,N, \tag{2.44}$$

$$C_k^N(z) = 0,\ \ 0 \le z < \delta,\ \ k = 1,\dots,N. \tag{2.45}$$

The equalities (2.43) represent a system of linear ordinary differential equations of second order in z with respect to unknown functions C_k^N, $k = 1,\dots,N$, with the constant matrix containing the elements which are in fact the coefficients of the second order derivatives $\frac{d^2 C_k^N(z)}{dz^2}$, and with the determinant different from zero, since it is the Gramm determinant with respect to the scalar product in $L_2(G)$ of a linearly independent system of vector functions $\varphi_1(y),\dots,\varphi_N(y)$. The coefficients of each of the equations of that system are measurable bounded functions, and the right-hand sides $g_l(z) = (\widetilde{F},\varphi_l)_{L_2(G)}$ belong to $L_1((0,\tau_0))$.

As is known [84, p. 214], the system (2.43) has a unique solution satisfying both the initial conditions (2.44) and the condition (2.45) by virtue of (2.41), where $\frac{d^2 C_k^N}{dz^2} \in L_1((0,\tau_0))$.

Let us now show that for $v = v^N$ the estimates (2.39) and (2.40) are valid. Indeed, multiplying each of the equalities (2.43) by $\frac{d}{dz}\,C_l^N(z)$ and summing up with respect to l from 1 to N, we obtain the equality

$$\Big(\frac{\partial^2 v^N}{\partial z^2},\frac{\partial v^N}{\partial z}\Big)_{L_2(G)}+\frac{1}{z^2}\int\limits_G\Big\{\sum_{i,j=1}^n\Big[\widetilde{A}_{ij}(y)v^N_{y_i}v^N_{zy_j}+\frac{\partial\widetilde{A}_{ij}}{\partial y_j}\,v^N_{y_i}v^N_z\Big]\Big\}\,dy+$$
$$+\frac{1}{z}\int\limits_G\Big\{\sum_{i=1}^n\big[y_iv^N_zv^N_{zy_i}+v^N_zv^N_z\big]\Big\}\,dy+\int\limits_G\Big[\frac{1}{z^2}\sum_{i=1}^n\widetilde{B}_iv^N_{y_i}v^N_z+\widetilde{C}v^Nv^N_z\Big]\,dy=$$
$$=(\widetilde{F},v^N_z)_{L_2(G)}, \tag{2.46}$$

from which after integration with respect to z from 0 to τ_0, with regard for (2.45) and subsequent transformations we have actually deduced the inequalities (2.39) and (2.40). In addition, by (2.45) it is obvious that

$$v^N\big|_{\Omega_\delta}=0,\ \ N=1,2,\dots. \tag{2.47}$$

Thus the estimates

$$\begin{gathered}\int\limits_{G_{\tau_0}}\Big[v^Nv^N+v^N_zv^N_z\sum_{i=1}^nv^N_{y_i}v^N_{y_i}\Big]\,dy\le c_5(\delta)\|\widetilde{F}\|^2_{L_2(\Omega_\tau)},\ \ 0<\tau\le\tau_0,\ \ N\ge1,\\ \|v^N\|_{W^1_2(\Omega_{\tau_0})}\le c_6(\delta)\|\widetilde{F}\|_{L_2(\Omega_{\tau_0})},\ \ N\ge1,\end{gathered} \tag{2.48}$$

are valid, where the positive constants $c_5(\delta)$ and $c_6(\delta)$ do not depend on N.

Owing to (2.48) and weak compactness of the closed ball in the Hilbert space $W^1_2(\Omega_{\tau_0})$, from the sequence $\{v^N\}$ we can choose a subsequence, without changing the notation, converging weakly in $W^1_2(\Omega_{\tau_0})$ to some element $v\in W^1_2(\Omega_{\tau_0})$ for which the equality (2.32) is valid by virtue of (2.47). Note also that since $v^N|_{\Gamma_{\tau_0}}=0$, $N\ge1$, by the compactness of the operation of taking the trace $v\to v|_{\Gamma_{\tau_0}}$ from the space $W^1_2(\Omega_{\tau_0})$ into $L_2(\Gamma_{\tau_0})$, the element v satisfies the homogeneous boundary condition (2.31) [84, p. 71].

Let us now show that v is a weak generalized solution of the system (2.29), i.e. the identity

$$\int\limits_{\Omega_{\tau_0}}\Big[-v_zw_z+\frac{1}{z^2}\sum_{i,j=1}^nv_{y_i}(\widetilde{A}_{ij}w)_{y_j}+\frac{2}{z}\sum_{i=1}^nv_z(y_iw)_{y_i}+$$
$$+\frac{1}{z}\sum_{i=1}^n\widetilde{B}_iv_{y_i}w+\widetilde{C}vw\Big]\,dy\,dz=\int\limits_{\Omega_{\tau_0}}\widetilde{F}w\,dy\,dz \tag{2.49}$$

holds for any $w\in V$ satisfying the following homogeneous boundary conditions:

$$w\big|_{\Gamma_{\tau_0}}=0,\quad w\big|_{z=\tau_0}=0, \tag{2.50}$$

where V is the closure of the space $W^1_2(\Omega_{\tau_0})$ of the vector functions $\omega=(\omega_1,\dots,\omega_m)$ of the class $C^2(\overline{\Omega}_{\tau_0})$.

Towards this end, we multiply each of the equalities (2.43) by its function $d_l(z)\in C^2([0,\tau_0])$, $d_l(\tau_0)=0$, and then we sum the obtained equalities with respect to all l from 1 to N and integrate with respect to z from 0 to τ_0. Further, integration by parts in the first term results in the identity

$$\int\limits_{\Omega_{\tau_0}} \Big[- v_z^N w_z + \frac{1}{z^2} \sum_{i,j=1}^{n} v_{y_i}^N (\widetilde{A}_{ij} w)_{y_j} + \frac{2}{z} \sum_{i=1}^{n} v_z^N (y_i w)_{y_i} + \\ + \frac{1}{z} \sum_{i=1}^{n} \widetilde{B}_i v_{y_i}^N w + \widetilde{C} v^N w \Big] \, dy \, dz = \int\limits_{\Omega_{\tau_0}} \widetilde{F} w \, dy \, dz, \tag{2.51}$$

which is valid for every w of the type $\sum\limits_{l=1}^{N} d_l(z)\varphi_l(y)$. We denote the family of such v by V_N. If we pass in (2.51) to the limit by means of the above chosen subsequence for a fixed w from some V_N, then we will arrive at the identity (2.49) for the limiting function $v \in W_2^1(\Omega_{\tau_0})$, valid for every $w \in \bigcup\limits_{N=1}^{\infty} V_N$. Now we show that $\bigcup\limits_{N=1}^{\infty} V_N$ is dense in V.

Indeed, let $w \in C^2(\overline{\Omega}_{\tau_0})$, and let the equalities (2.50) be fulfilled. Then there exists an extension w_0 of the vector function w to a larger cylinder $\Omega_* = \big\{(y,z) \in R^{n+1} : \ y \in G, \ z \in (-\tau_0, \tau_0)\big\}$ of the class $C^2(\overline{\Omega}_*)$ such that $w_0|_{\partial\Omega_*} = 0$, $w_0|_{\Omega_{\tau_0}} = w$ [29, p. 591]. Consequently, $w_0 \in \overset{\circ}{W}{}_2^1(\Omega_*)$, and since the system of functions

$$\Big\{\varphi_l(y) \sin \frac{\pi k(z+\tau_0)}{2\tau_0}\Big\}_{k,l=1}^{\infty} \tag{2.52}$$

is fundamental in the space $\overset{\circ}{W}{}_2^1(\Omega_*)$ [100, pp. 112, 165], for every $\varepsilon > 0$ there exists a linear combination $\sum\limits_{i=1}^{n} \alpha_i \widetilde{w}_i$ of vector functions from the system (2.52) such that

$$\Big\| w_0 - \sum_{i=1}^{k} \alpha_i \widetilde{w}_i \Big\|_{W_2^1(\Omega_*)} < \varepsilon, \tag{2.53}$$

because $\|\widetilde{w}\|_{\overset{\circ}{W}{}_2^1(\Omega_*)} = \|\widetilde{w}\|_{W_2^1(\Omega_*)}$. By (2.50) and the fact that $w_0|_{\Omega_{\tau_0}} = w$, we have

$$\Big\| w - \sum_{i=1}^{k} \alpha_i \widetilde{w}_i \Big\|_V = \Big\| w - \sum_{i=1}^{k} \alpha_i \widetilde{w}_i \Big\|_{W_2^1(\Omega_{\tau_0})} \le \Big\| w_0 - \sum_{i=1}^{k} \alpha_i \widetilde{w}_i \Big\|_{W_2^1(\Omega_*)} < \varepsilon. \tag{2.54}$$

But $\sum\limits_{i=1}^{n} \alpha_i \widetilde{w}_i \in \bigcup\limits_{N=1}^{\infty} V_N$. Therefore from (2.54) and the fact that the set $\big\{w \in C^2(\overline{\Omega}_{\tau_0}) : \ w|_{\Gamma_{\tau_0}} = 0, \ w|_{z=\tau_0} = 0\big\}$ is dense in the space V, we find that $\bigcup\limits_{N=1}^{\infty} V_N$ is dense in V. Since $v \in W_2^1(\Omega_{\tau_0})$, this in its turn implies that the identity (2.49), valid for every $w \in \bigcup\limits_{N=1}^{\infty} V_N$, is likewise valid for every

$w \in V$. Thus the limiting vector function $v = v(y, z)$ is a weak generalized solution of the equation (2.29) satisfying the equalities (2.31) and (2.32).

3^0. Let us show that if the supplementary conditions

$$\partial G \in C^2; \quad B_{ix_j}\, B_{it},\, C_{x_j},\, C_t \in L_\infty(D_{\tau_0}), \;\; i, j = 1, \dots, n, \tag{2.55}$$

$$F \in W_2^1(D_{\tau_0}), \quad F\big|_{D_\delta} = 0 \tag{2.56}$$

are fulfilled, then the above-obtained vector function v is a solution of the problem (2.29), (2.31), (2.39) from the space $W_2^1(\Omega_{\tau_0})$, where $L_\infty(D_{\tau_0})$ is the space of measurable functions bounded in D_{τ_0}.

First we multiply the expression obtained after differentiation of the equality (2.43) with respect to z by $\frac{d^2}{dz^2}\, C_l^N(z)$ and then sum with respect to l from 1 to N. Reasoning as in deducing the inequality (2.39) and using the already proven estimate (2.39), we obtain

$$\overline{w}_0(\tau) \le c_{10}(\delta) \int\limits_{\Omega_\tau} (\widetilde{F}\, \widetilde{F} + \widetilde{F}_z \widetilde{F}_z)\, dy\, dz, \quad c_{10}(\delta) = \text{const} > 0,$$

where $\overline{w}_0(\tau) = \int\limits_{G_\tau} \big[v_{zz}^N v_{zz}^N + \sum\limits_{i=1}^{n} v_{zy_i}^N v_{zy_i}^N \big]\, dy$, whence in its turn we have

$$\|v_{zz}^N\|_{L_2(\Omega_{\tau_0})} + \sum_{i=1}^{n} \|v_{zy_i}^N\|_{L_2(\Omega_{\tau_0})} \le c_{11}(\delta) \big[\|\widetilde{F}\|_{L_2(\Omega_{\tau_0})} + \|\widetilde{F}_z\|_{L_2(\Omega_{\tau_0})} \big], \tag{2.57}$$

where $c_{11}(\delta) = \text{const} > 0$.

By the estimates (2.48) and (2.57), some subsequence $\{v^{N_k}\}$ converges weakly in L_2 together with the first order derivatives v_z^N, $v_{y_i}^N$, $i = 1, \dots, n$, and the derivatives $v_{zz}^{N_k}$, $v_{zy_i}^{N_k}$, $i = 1, \dots, n$, to the above-constructed solution v and respectively to v_z, v_{y_i}, v_{zz}, v_{zy_i}, $i = 1, \dots, n$. In addition, for v the inequality

$$\|v_{zz}\|_{L_2(\Omega_{\tau_0})} + \sum_{i=1}^{n} \|v_{zy_i}\|_{L_2(\Omega_{\tau_0})} \le c_{12}(\delta) \big[\|\widetilde{F}\|_{L_2(\Omega_{\tau_0})} + \|\widetilde{F}_z\|_{L_2(\Omega_{\tau_0})} \big] \tag{2.58}$$

is valid, where $c_{12}(\delta) = \text{const} > 0$.

By (2.40) and (2.58), the vector function v will belong to the space $W_2^2(\Omega_{\tau_0})$ if we show that v has generalized derivatives $v_{y_i y_j}$ from $L_2(\Omega_{\tau_0})$, $i, j = 1, \dots, n$.

By $\widetilde{V}$ we denote the space of all vector functions $w = (w_1, \dots, w_m) \in L_2(\Omega_{\tau_0})$ which have generalized derivatives $w_{y_i y_j}$, $i, j = 1, \dots, n$, from $L_2(\Omega_{\tau_0})$ and satisfy the homogeneous boundary condition (2.31), i.e. $w|_{\Gamma_{\tau_0}} = 0$.

Analogously to our reasoning when we obtained (2.49) from (2.43), it follows from (2.43) that the vector function v satisfies the following integral identity

$$\int\limits_{\Omega_{\tau_0}}\Big[v_{zz}w+\frac{1}{z^2}\sum_{i,j=1}^{n}v_{y_i}(\widetilde{A}_{ij}w)_{y_j}-\frac{2}{z}\sum_{i=1}^{n}v_{zy_i}y_iw+\frac{1}{z}\sum_{i=1}^{n}\widetilde{B}_iv_{y_i}w+\widetilde{C}vw\Big]\,dy\,dz=$$

$$=\int\limits_{\Omega_{\tau_0}}\widetilde{F}w\,dy\,dz\quad\forall\,w\in\widetilde{V}.\tag{2.59}$$

If in (2.59) we take as $w\in\widetilde{V}$ the vector function $w(y,z)=\psi(z)\Psi(y)$, where the scalar function $\psi(t)$ and the vector function $\Psi(y)$ are arbitrary elements respectively from $L_2((0,\tau_0))$ and $\overset{\circ}{W}{}^1_2(G)$, then by Fubini's theorem the equality (2.59) can be rewritten in the form

$$\int\limits_{0}^{\tau_0}\psi(z)\Big\{\int\limits_{G_z}\Big[v_{zz}\Psi+\frac{1}{z^2}\sum_{i,j=1}^{n}v_{y_i}(\widetilde{A}_{ij}\Psi)_{y_j}-\frac{2}{z}\sum_{i=1}^{n}v_{zy_i}y_i\Psi+$$

$$+\frac{1}{z}\sum_{i=1}^{n}\widetilde{B}_iv_{y_i}\Psi+\widetilde{C}v\Psi\Big]\,dy\Big\}\,dz=\int\limits_{\Omega_{\tau_0}}\psi(z)\Big[\int\limits_{G_z}\widetilde{F}\Psi\,dy\Big]\,dz,\tag{2.60}$$

whence because of the arbitrary choice of $\psi(z)\in L_2((0,\tau_0))$, for all $z\in(0,\tau_0)$ we get

$$\int\limits_{G_z}\Big[\sum_{i,j=1}^{n}v_{y_i}(\widetilde{A}_{ij}\Psi)_{y_j}+z\sum_{i=1}^{n}\widetilde{B}_iv_{y_i}\Psi+z^2\widetilde{C}v\Psi\Big]\,dy=$$

$$=\int\limits_{G_z}\Big(-z^2v_{zz}+2z\sum_{i=1}^{n}v_{zy_i}y_i+z^2\widetilde{F}\Big)\Psi\,dy\quad\forall\,\Psi\in\overset{\circ}{W}{}^1_2(G).\tag{2.61}$$

Since for such $z\in(0,\tau_0)$ the vector function

$$\widehat{F}=\Big[-z^2v_{zz}+2z\sum_{i=1}^{n}v_{zy_i}y_i+z^2\widetilde{F}\Big]$$

belongs to $L_2(G)$, the identity (2.61) means that the vector function $v=(v_1,\dots,v_m)$ is a generalized solution from the space $\overset{\circ}{W}{}^1_2(G)$ for the following elliptic system of equations:

$$-\sum_{i,j=1}^{n}\widetilde{A}_{ij}v_{y_iy_j}+z\sum_{i=1}^{n}\widetilde{B}_iv_{y_i}+z^2\widetilde{C}v=\widehat{F}.\tag{2.62}$$

By the inequality (2.33), the system (2.62) is strongly elliptic. Therefore under the assumption that $\partial G\in C^2$, i.e. the appearing in the definition of the conic domain D function g belongs to C^2, we have that in the system (2.1) B_i, $C\in C^1(\overline{D}_{\tau_0})$ and hence in the system (2.62) $\widetilde{B}_i$, $\widetilde{C}\in C^1(\overline{\Omega}_{\tau_0})$, the generalized solution v of the system (2.62) from the space $\overset{\circ}{W}{}_2^1(G)$ belongs likewise to the space $W_2^2(G)$ for these $z \in (0, \tau_0)$ [89, p. 109], and

$$\|v\|_{W_2^2(G)} \leq c_{12}\|\widehat{F}\|_{L_2(G_z)} \leq c_{14}\Big[\|v_{zz}\|_{L_2(G_z)}+ \\ +\sum_{i=1}^{n}\|v_{zy_i}\|_{L_2(G)} + \|\widehat{F}\|_{L_2(G_z)}\Big], \ \ c_{13},\, c_{14} = \text{const} > 0. \tag{2.63}$$

Thus for such $z \in (0, \tau_0)$ the vector function v has the generalized derivatives $v_{y_iy_j}$, $i, j = 1, \dots, n$, and by virtue of (2.58) and (2.63) we have $\widetilde{g}_{ij}(z) = \|v_{y_iy_j}\|_{L_2(G_z)} \in L_2((0, \tau_0))$. Therefore it remains only to note that the function $\widehat{g}(y, z) \in L_2(\Omega_{\tau_0})$ has the generalized derivative $\widehat{g}_{y_i}(y, z) \in L_2(\Omega_{\tau_0})$, $1 \leq i \leq n$, if and only if for almost all $z \in (0, \tau_0)$ the function $\widehat{g}$ has the generalized derivative $\widehat{g}_{y_i} \in L_2(G_z)$, and $\widehat{\varphi}_i(z) = \|\widehat{g}_{y_i}\|_{L_2(G_2)} \in L_2((0, \tau_0))$.

Getting back from y, z to the initial variables x, t, the vector function $u(x, t) = v(\frac{x}{t}, t)$ due to the equalities (2.28) will be a solution of the system (2.1) from the space $W_2^2(D_{\tau_0})$ satisfying the homogeneous boundary condition (2.27).

Thus we have proved the following

Lemma 2.1. *Let $g \in C^2$, $B_i, C \in C^1(\overline{D}_{\tau_0})$, $i = 1, \dots, n$, $F \in W_2^1(D_{\tau_0})$, $F|_{D_\delta} = 0$, $0 < \delta < \tau_0$, and let the condition* (2.33) *be fulfilled. Then the problem* (2.1), (2.27) *has a unique solution u from the space $W_2^2(D_{\tau_0})$, where $u|_{D_\delta} = 0$.*

In the case where $F \in L_2(D_{\tau_0})$, since the space of infinitely differentiable finite functions $C_0^\infty(D)$ is dense in $L_2(D_{\tau_0})$, there exists a sequence of vector functions $F_k \in C_0^\infty(D_{\tau_0})$ such that $F_k \to F$ in $L_2(D_{\tau_0})$. Since $F_k \in C_0^\infty(D_{\tau_0})$, we have $F_k \in W_2^1(D_{\tau_0})$, and for a sufficiently small positive δ_k, $\delta_k < \tau_0$, we have $F_k|_{D_{\delta_k}} = 0$. Therefore, according to Lemma 2.1, there exists a unique solution $u_k \in W_2^2(D_{\tau_0})$ of the problem (2.1), (2.27). By (2.27), from the inequality (2.26) we have

$$\|u_k - u_p\|_{W_2^1(D_{\tau_0})} \leq c\|F_k - F_p\|_{L_2(D_{\tau_0})},$$

whence it follows that the sequence $\{u_k\}_{k=1}^\infty$ is fundamental in $W_2^1(D_{\tau_0})$, because $F_k \to F$ in $L_2(D_{\tau_0})$.

Since the space $W_2^1(D_{\tau_0})$ is complete, there exists a vector function $u \in W_2^1(D_{\tau_0})$ such that $u_k \to u$ in $W_2^1(D_{\tau_0})$ and $Lu_k = F_k \to F$ in $L_2(D_{\tau_0})$. Consequently, u is a strong generalized solution of the problem (2.1), (2.27) of the class W_2^1, for which the estimate

$$\|u\|_{W_2^1(D_{\tau_0})} \leq c\|F\|_{L_2(D_{\tau_0})} \tag{2.64}$$

holds by virtue of (2.26).

Thus the following theorem is valid.

Theorem 2.1. *Let $g \in C^2$; $B_i, C \in C^1(\overline{D}_{\tau_0})$, $i = 1, \dots, n$, and let the condition* (2.33) *be fulfilled. Then for every $F \in L_2(D_{\tau_0})$ there exists a unique strong generalized solution of the problem* (2.1), (2.27) *of the class W_2^1 for which the estimate* (2.64) *is valid.*

3. Boundary Value Problems for a Class of First Order Hyperbolic Systems with Symmetric Principal Part

3.1. Statement of the problem in conic domains. In the space of variables $x_1, \dots, x_n$ and t we consider a system of differential equations of the first order of the type

$$Lu \equiv Eu_t + \sum_{i=1}^{n} A_i u_{x_i} + Bu = F, \tag{3.1}$$

where A_i and B are given real $(m \times m)$-matrices, E is the unit $(m \times m)$-matrix, F is a given and u is an unknown m-dimensional real vector, $n > 1$, $m > 1$.

Below the matrices A_i will be assumed to be symmetric and constant. In this case the system (3.1) is hyperbolic [24, p. 587].

Since the matrix $Q(\xi') = -\sum\limits_{i=1}^{n} A_i \xi_i$, $\xi' = (\xi_1, \dots, \xi_n) \in R^N$ is symmetric, its characteristic roots are real. We enumerate them in decreasing order: $\widetilde{\lambda}_1(\xi') \geq \widetilde{\lambda}_2(\xi') \geq \dots \geq \widetilde{\lambda}_m(\xi')$. The multiplicities $k_1, \dots, k_s$ of these roots are assumed to be constant, i.e. do not depend on ξ', and we put

$$\lambda_1(\xi') = \widetilde{\lambda}_1(\xi') = \dots = \widetilde{\lambda}_{k_1}(\xi') > \lambda_2(\xi') = \widetilde{\lambda}_{k_1+1}(\xi') = \dots = \widetilde{\lambda}_{k_1+k_2}(\xi') >$$
$$> \lambda_s(\xi') = \widetilde{\lambda}_{m-k_s+1}(\xi') = \dots = \widetilde{\lambda}_m(\xi'), \;\; \xi' \in R^n \setminus \{(0, \dots, 0)\}. \tag{3.2}$$

Note that due to (3.2) and continuous dependence of the roots of a polynomial on its coefficients, $\lambda_1(\xi'), \dots, \lambda_s(\xi')$ are continuous homogeneous functions of degree 1 [46].

As far as the matrix $Q(\xi')$ is symmetric, there exists an orthogonal matrix $T = T(\xi')$ such that

$$(T^{-1}QT)(\xi') = \operatorname{diag}\big(\underbrace{\lambda_1(\xi'), \dots, \lambda_1(\xi')}_{k_1}, \dots, \underbrace{\lambda_s(\xi'), \dots, \lambda_s(\xi')}_{k_s}\big). \tag{3.3}$$

By (3.2) and (3.3), the cone of the normals $K = \big\{\xi = (\xi_1, \dots, \xi_n, \xi_0) \in R^{n+1} : \det(E\xi_0 - Q(\xi')) = 0\big\}$ of the system (3.1) consists of separate sheets $K_i = \Big\{\xi = (\xi', \xi_0) \in R^{n+1} : \xi_0 - \lambda_i(\xi') = 0\Big\}$, $i = 1, \dots, s$.

Since

$$\lambda_j(\xi') = -\lambda_{s+1-j}(-\xi'), \;\; 0 \leq j \leq \Big[\frac{s+1}{2}\Big], \tag{3.4}$$

the cones K_j and K_{s+1-j} are centrally symmetric with respect to the point $(0, \dots, 0)$, where $[a]$ denotes the integer part of the number a.

Remark 3.1. In the case where s is an odd number, we have $j = s+1-j$ for $j = \left[\frac{s+1}{2}\right]$. Therefore the cone K_j for $j = \left[\frac{s+1}{2}\right]$ is centrally symmetric with respect to the point $(0,\dots,0)$. In this case, for the sake of simplicity of our presentation, for $s = 2s_0 + 1$ we assume that

$$\lambda_{s_0+1}(\xi') \equiv 0, \quad \left[\frac{s+1}{2}\right] = s_0 + 1, \tag{3.5}$$

where K_{s_0+1} is the hyperplane $\pi_0 :\ \xi_0 = 0$.

Remark 3.2. Below it will be assumed that $\pi_0 \cap K_{s_0} = \{(0,\dots,0)\}$ for even $s = 2s_0$. By (3.2) and (3.4) this implies that the cones $K_1,\dots,K_{s_0}$ lie on one side from $\pi_0 :\ \xi_0 = 0$, while $K_{s_0+1},\dots,K_{2s_0}$ on the other side, i.e.

$$\lambda_1(\xi') > \cdots > \lambda_{s_0}(\xi') > 0 > \lambda_{s_0+1}(\xi') > \cdots > \lambda_{2s_0}(\xi'), \qquad \xi' \in R^n \setminus \{(0,\dots,0)\}. \tag{3.6}$$

If $s = 2s_0 + 1$ is odd, by virtue of (3.2), (3.4) and (3.5) we have $\pi_0 \cap K_{s_0} = \{(0,\dots,0)\}$, and hence

$$\lambda_1(\xi') > \cdots > \lambda_{s_0}(\xi') > \lambda_{s_0+1}(\xi') \equiv 0 > \\ > \lambda_{s_0+2}(\xi') > \cdots > \lambda_{2s_0+1}(\xi'), \quad \xi' \in R^n \setminus \{(0,\dots,0)\}. \tag{3.7}$$

In this case $K_1,\dots,K_{s_0}$ lie on one side from $\pi_0 = K_{s_0+1}$, while $K_{s_0+2},\dots,K_{2s_0+1}$ on the other side. From (3.4)–(3.7) it follows that for the multiplicities k_j of the roots λ_j the equalities

$$k_j = k_{s+1-j}, \quad j = 1,\dots,\left[\frac{s+1}{2}\right] \tag{3.8}$$

are valid.

Let $K_i^* = \bigcap\limits_{\eta \in K_i} \big\{\zeta \in R^{n+1} :\ \zeta \cdot \eta < 0\big\}$, where $\zeta \cdot \eta$ is the scalar product of the vectors ζ and η. By (3.4) or (3.5) we have $\pi_0 \cap K_i = \{(0,\dots,0)\}$, $1 \le i \le s$, and if s is odd we assume that $i \ne \left[\frac{s+1}{2}\right]$. Therefore K_i^* is a conic domain, and if $s = 2s_0$ is even we have $K_{s_0}^* \subset K_{s_0-1}^* \subset \cdots \subset K_1^*$, $K_{s_0+1}^* \subset K_{s_0+2}^* \subset \cdots \subset K_{2s_0}^*$, while for $s = 2s_0 + 1$ there takes place $K_{s_0}^* \subset K_{s_0-1}^* \subset \cdots \subset K_1^*$, $K_{s_0+2}^* \subset K_{s_0+3}^* \subset \cdots \subset K_{2s_0+1}^*$.

Remark 3.3. Note that $\partial(K_i^*)$ is a convex cone, where $i \ne \left[\frac{s+1}{2}\right]$ for odd s, and at those points P of the cone $\partial(K_i^*)$ which contain the unit vector α_P of the outer normal to $\partial(K_i^*)$ we have [61]

$$\alpha_P \in K_i, \quad P \in \partial(K_i^*). \tag{3.9}$$

It follows from (3.9) that at the points at which there exists the tangent plane, the conic surface $\partial(K_i^*)$ is a characteristic one. Below, as examples, we will consider some symmetric first order hyperbolic systems of the mathematical physics for which $\partial(K_i^*)$ is a smooth or piecewise smooth characteristic cone.

By $Q_0(\xi) \equiv E\xi_0 + \sum\limits_{i=1}^{n} A_i\xi_i = E\xi_0 - Q(\xi')$ we denote the characteristic matrix of the system (3.1) and consider the question on the reduction of the quadratic form $(Q_0(\xi)\eta, \eta)$ to the canonical form when $\xi \in K_i' = K_i \setminus \{(0, \dots, 0)\}$, where $\eta \in R^m$ and $(\cdot, \cdot)$ denotes the scalar product in the Euclidean space R^m.

By (3.3), for $\eta = T\zeta$ we have

$$\begin{gathered}(Q_0(\xi)\eta, \eta) = \big((T^{-1}Q_0T)(\xi)\zeta, \zeta\big) = \Big(\big[E\xi_0 - (T^{-1}QT)(\xi')\big]\zeta, \zeta\Big) = \\ = (\xi_0 - \lambda_1(\xi'))\zeta_1^2 + \dots + (\xi_0 - \lambda_1(\xi'))\zeta_{k_1}^2 + \\ +(\xi_0 - \lambda_2(\xi'))\zeta_{k_1+1}^2 + \dots + (\xi_0 - \lambda_2(\xi'))\zeta_{k_1+k_2}^2 + \dots + \\ +(\xi_0 - \lambda_s(\xi'))\zeta_{m-k_s+1}^2 + \dots + (\xi_0 - \lambda_s(\xi'))\zeta_m^2. \end{gathered} \tag{3.10}$$

Since for $\xi = (\xi', \xi_0) \in K_i'$ there takes place the equality $\xi_0 = \lambda_i(\xi')$, with regard for (3.2) we have

$$\begin{gathered}[\xi_0 - \lambda_j(\xi')]\big|_{K_i'} < 0, \ \ j = 1, \dots, i-1; \quad [\xi_0 - \lambda_i(\xi')]\big|_{K_i'} = 0, \\ [\xi_0 - \lambda_j(\xi')]\big|_{K_i'} > 0, \ \ j = i+1, \dots, s. \end{gathered} \tag{3.11}$$

Denoting by $\varkappa_i^+$ and $\varkappa_i^-$ the positive and the negative index of inertia of the quadratic form $(Q_0(\xi)\eta, \eta)\big|_{\xi \in K_i'}$, by virtue of (3.10) and (3.11) we find that

$$\varkappa_i^- = k_1 + \dots + k_{i-1}, \quad \varkappa_i^+ = k_{i+1} + \dots + k_s, \quad (def)_i = k_i, \tag{3.12}$$

where $(def)_i$ is the defect of that form, and $\varkappa_i^- = 0$ for $i = 1$.

If now $\zeta = C^i\eta$ is some nondegenerate linear transformation reducing the quadratic form $(Q_0(\xi)\eta, \eta)\big|_{\xi \in K_i'}$ to the canonical form, then due to (3.12) and the invariance of indices of inertia of the quadratic form with respect to nondegenerate linear transformations we have

$$(Q_0(\xi)\eta, \eta)\big|_{\xi \in K_i'} = \sum_{j=1}^{\varkappa_i^+} \big[\Lambda_j^+(\xi, \eta)\big]^2 - \sum_{j=1}^{\varkappa_i^-} \big[\Lambda_j^-(\xi, \eta)\big]^2. \tag{3.13}$$

Here

$$\begin{gathered}\Lambda_j^-(\xi, \eta) = \sum_{p=1}^{m} c_{jp}^i(\xi)\eta_p, \quad \Lambda_j^+(\xi, \eta) = \sum_{p=1}^{m} c_{\varkappa_i^- + j, p}^i(\xi)\eta_p, \\ C^i = C^i(\xi) = (c_{jp}^i(\xi)), \ \ \xi \in K_i'. \end{gathered} \tag{3.14}$$

Remark 3.4. Below it will be assumed that the elements $c_{jp}^i(\xi)$ of the matrix $C^i = C^i(\xi)$ are bounded piecewise continuous functions with respect to ξ on every compact subset of the conic surface K_i, and according to Remark 3.3 we assume that $S_i = \partial(K_i^*)$, where $i \neq \left[\frac{s+1}{2}\right]$ if s is odd, is a smooth or piecewise smooth characteristic cone, i.e. the surface $S_i \setminus \{(0, \dots, 0)\}$ is smooth or piecewise smooth. Note also that by (3.4) the conic surfaces S_i and S_{s+1-i} are centrally symmetric with respect to the point $(0,\dots,0)$, where $i \neq \left[\frac{s+1}{2}\right]$ if s is odd.

Let $P_0(x^0,t^0) \in K_i^*$, where $i > \left[\frac{s+1}{2}\right]$. Denote by $S_1(P_0)$ the cone with vertex at the point P_0 obtained by parallel transfer of the cone S_1, i.e. $S_1(P_0) = \big\{(x,t) \in R^{n+1} :\ (x-x^0, t-t^0) \in S_1 = \partial(K_1^*)\big\}$. Note that by virtue of the inequalities (3.6) or (3.7), the cone S_i for $i > \left[\frac{s+1}{2}\right]$ is directed towards the increasing values of t, while $S_1(P_0)$ towards the decreasing ones. Denote by $D_i \subset K_i^*$ the finite domain bounded by the cones S_i and $S_1(P_0)$, and let $S_i^0 = \partial D_i \cap S_i$, $S_1^0 = \partial D_i \cap S_1(P_0)$.

In accordance with (3.14), by Γ_{ij}^- we denote the boundary operator acting by the formula

$$\Gamma_{ij}^-(u)\big|_{S_i^0} \equiv \Lambda_{ij}^-(\alpha,u)\big|_{S_i^0} \equiv \Big[\sum_{p=1}^{m} c_{jp}^i(\alpha)u_p\Big]\Big|_{S_i^0},$$

where α is the unit vector of the outer normal to S_i^0, $u=(u_1,\dots,u_m)$.

Let us consider the characteristic problem which is formulated as follows: in the domain D_i, find a solution u of the system (3.1) by the boundary conditions

$$\Gamma_{ij}^-(u)\big|_{S_i^0} = f_j, \ \ j=1,\dots,k_i^-, \tag{3.15}$$

where f_j are given real scalar functions, and the number $\varkappa_i^-$ is defined in (3.12).

Below we assume that the elements of the matrix B in the system (3.1) are bounded measurable functions in D_i.

3.2. A priori estimate.

Lemma 3.1. *For any solution $u \in W_2^1(D_i)$ of the problem* (3.1), (3.15) *the following a priori estimate*

$$\|u\|_{L_2(D_i)} \le C\Big(\sum_{j=1}^{\varkappa_i^-} \|f_j\|_{L_2(S_i^0)} + \|F\|_{L_2(D_i)}\Big) \tag{3.16}$$

is valid, where $W_2^1(D_i)$ is the Sobolev space, and the positive constant C does not depend on u, f_j and F.

Proof. For any $u \in W_2^1(D_i)$ and $\lambda = \text{const} > 0$, integrating by parts we obtain

$$2\int\limits_{D_i} \big(Lu, u\exp(-\lambda t)\big)\,dD_i =$$

$$= \int\limits_{\partial D_i} (Q_0(\alpha)u,u)\exp(-\lambda t)\,ds + \int\limits_{D_i} \big((\lambda E + 2B)u, u\exp(-\lambda t)\big)\,dD_i, \tag{3.17}$$

where $Q_0(\alpha)=E\alpha_0+\sum\limits_{j=1}^{n}A_j\alpha_j$, $\alpha=(\alpha_1,\dots,\alpha_n,\alpha_0)$ is the unit vector of the outer normal to ∂D_i, $i>\left[\frac{s+1}{2}\right]$.

From (3.9) it follows that

$$\alpha\big|_{S_i^0}\in K_i',\quad \alpha\big|_{S_1^0}\in K_1'. \tag{3.18}$$

Taking now into account that $\partial D_i=S_i^0\cup S_1^0$, by virtue of (3.12), (3.13), (3.15) and (3.18) we have

$$(Q_0(\alpha)u,u)\big|_{S_1^0}=\Big(\sum_{j=1}^{\varkappa_1^+}\big[\Lambda_{1j}^+(\alpha,u)\big]^2\Big)\Big|_{S_1^0}, \tag{3.19}$$

$$\begin{aligned}(Q_0(\alpha)u,u)\big|_{S_i^0}&=\Big(\sum_{j=1}^{\varkappa_i^+}\big[\Lambda_{ij}^+(\alpha,u)\big]^2\Big)\Big|_{S_i^0}-\Big(\sum_{j=1}^{\varkappa_i^-}\big[\Lambda_{ij}^-(\alpha,u)\big]^2\Big)\Big|_{S_i^0}=\\&=\Big(\sum_{j=1}^{\varkappa_i^+}\big[\Lambda_{ij}^+(\alpha,u)\big]^2\Big)\Big|_{S_i^0}-\Big(\sum_{j=1}^{\varkappa_i^-}\big[\Gamma_{ij}^-(u)\big]^2\Big)\Big|_{S_i^0}\geq-\sum_{j=1}^{\varkappa_i^-}f_j^2.\end{aligned} \tag{3.20}$$

From (3.17)–(3.20) we find that

$$\begin{aligned}2\int\limits_{D_i}\big(F,u\exp(-\lambda t)\big)\,dD_i&=2\int\limits_{D_i}\big(Lu,u\exp(-\lambda t)\big)\,dD_i=\\=\int\limits_{S_1^0}(Q_0(\alpha)u,u)\exp(-\lambda t)\,ds&+\int\limits_{S_i^0}(Q_0(\alpha)u,u)\exp(-\lambda t)\,ds+\\&+\int\limits_{D_i}\big((\lambda E+2B)u,u\exp(-\lambda t)\big)\,dD_i\geq\\\geq-\sum_{j=1}^{\varkappa_i^-}\int\limits_{S_i^0}f_j^2\exp(-\lambda t)\,ds&+\int\limits_{D_i}\big((\lambda E+2B)u,u\exp(-\lambda t)\big)\,dD_i.\end{aligned} \tag{3.21}$$

Taking now into account that D_i is a bounded domain and the elements of the matrix B are bounded measurable functions, from (3.21) for sufficiently large λ it follows (3.16). Thus the proof of the lemma is complete. □

3.3. The existence and uniqueness theorems. Here we introduce into consideration a new unknown vector function $v(x,t)=u(x,t)\exp(-\lambda t)$, $\lambda=\text{const}>0$. Then for $v(x,t)$ we obtain the following system of equations:

$$L_\lambda v\equiv Ev_t+\sum_{i=1}^{n}A_iv_{x_i}+B_\lambda v=F_\lambda, \tag{3.22}$$

where $B_\lambda=B+\lambda E$, $F_\lambda=F\exp(-\lambda t)$.

Denote by $G = K_s^*$ the unbounded conic domain whose boundary is the characteristic cone S_s. If $v \in W_2^2(G) \cap \overset{\circ}{W}{}_2^1(G)$, then on the boundary ∂G we have $v = 0$, and hence

$$\partial G : v_t = \alpha_0 v_\alpha, \quad v_{x_i} = \alpha_i v_\alpha, \;\; i = 1, \dots, n, \tag{3.23}$$

where $v_\alpha = \alpha_0 v_t + \sum\limits_{i=1}^{n} \alpha_i v_{x_i}$, $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_0)$ is the unit vector of the outer normal to ∂G.

Further, for the sake of simplicity we introduce the notation $t = x_{n+1}$, $\alpha_0 = \alpha_{n+1}$, $A_{n+1} = E$. Then the principal part of the system (3.22) can be written as $L_\lambda^0 v \equiv \sum\limits_{i=1}^{n+1} A_i v_{x_i}$. For $v \in W_2^2(G) \cap \overset{\circ}{W}{}_2^1(G)$, simple integration by parts yields

$$\int\limits_G (A_i v_{x_i}, v_{x_j x_j})\, dG = \frac{1}{2} \int\limits_{\partial G} (A_j \alpha_j v_{x_j}, v_{x_j})\, ds, \;\; i = j,$$

$$\int\limits_G (A_i v_{x_i}, v_{x_j x_j})\, dG = \int\limits_{\partial G} (A_i \alpha_j v_{x_i}, v_{x_j})\, ds - \frac{1}{2} \int\limits_{\partial G} (A_i \alpha_i v_{x_j}, v_{x_j})\, ds, \;\; i \neq j,$$

whence it directly follows that

$$\int\limits_G (L_\lambda^0 v, v_{x_j x_j})\, dG = \int\limits_{\partial G} \Big(\sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_j v_{x_i}, v_{x_j} \Big)\, ds -$$

$$-\frac{1}{2} \int\limits_{\partial G} \Big(\sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_i v_{x_j}, v_{x_j} \Big)\, ds + \frac{1}{2} \int\limits_{\partial G} (A_j \alpha_j v_{x_j}, v_{x_j})\, ds. \tag{3.24}$$

By (3.23), (3.24) and also by the equality $\alpha^2 = \sum\limits_{i=1}^{n+1} \alpha_i^2 = 1$, we have

$$\int\limits_G (L_\lambda^0 v, \Delta v)\, dG = \int\limits_G \Big(L_\lambda^0 v, \sum_{j=1}^{n+1} v_{x_j x_j} \Big)\, dG =$$

$$= \sum_{j=1}^{n+1} \int\limits_{\partial G} \Big(\sum_{i=1}^{n+1} A_i \alpha_j v_{x_i}, v_{x_j} \Big)\, ds + \frac{1}{2} \sum_{j=1}^{n+1} \int\limits_{\partial G} \Big(\Big[A_j \alpha_j - \sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_i \Big] \alpha_j^2 v_\alpha, v_\alpha \Big)\, ds =$$

$$= \frac{1}{2} \sum_{j=1}^{n+1} \int\limits_{\partial G} \Big(\Big[A_j \alpha_j + \sum_{\substack{i=1 \\ i \neq j}}^{n+1} A_i \alpha_i \Big] \alpha_j^2 v_\alpha, v_\alpha \Big)\, ds =$$

$$= \frac{1}{2} \sum_{j=1}^{n+1} \int\limits_{\partial G} \Big(\Big(\sum_{i=1}^{n+1} A_i \alpha_i \Big) \alpha_j^2 v_\alpha, v_\alpha \Big)\, ds =$$

$$= \Big[\frac{1}{2}\int\limits_{\partial G}\Big(\sum_{i=1}^{n+1}A_i\alpha_i v_\alpha, v_\alpha\Big)\,ds\Big]\Big(\sum_{j=1}^{n+1}\alpha_j^2\Big) = \frac{1}{2}\int\limits_{\partial G}\big(Q_0(\alpha)v_\alpha, v_\alpha\big)\,ds, \quad (3.25)$$

where $Q_0(\alpha) = \sum\limits_{i=1}^{n+1} A_i\alpha_i = E\alpha_0 + \sum\limits_{i=1}^{n} A_i\alpha_i$ is the characteristic matrix of the system (3.22).

Below it will be assumed that the elements of the matrix B are bounded in the closed domain $\overline{G}$ together with their partial first order derivatives.

Reasoning analogously, owing to (3.22) and (3.25) we obtain

$$-\int\limits_G (L_\lambda v, \Delta v - v)\,dG = -\int\limits_G \Big(L_\lambda v, \sum_{j=1}^{n+1} v_{x_j x_j} - v\Big)\,dG =$$

$$= \sum_{j=1}^{n+1}\int\limits_G (B_\lambda v_{x_j}, v_{x_j})\,dG + \sum_{j=1}^{n+1}\int\limits_G (B_{\lambda x_j} v, v_{x_j})\,dG +$$

$$+ \int\limits_G (B_\lambda v, v)\,dG + \frac{1}{2}\int\limits_{\partial G}(Q_0(\alpha)v, v)\,ds - \frac{1}{2}\int\limits_{\partial G}\big(Q_0(\alpha)v_\alpha, v_\alpha\big)\,ds. \quad (3.26)$$

Since $\partial G = \partial K_s^*$, by (3.9), (3.10) and (3.11) the matrix $Q_0(\alpha)$, where α is the unit vector of the outer normal to ∂K_s^*, is nonpositive. Therefore

$$-\frac{1}{2}\int\limits_{\partial G}\big(Q_0(\alpha)v_\alpha, v_\alpha\big)\,ds \geq 0. \quad (3.27)$$

As far as $v \in W_2^2(G) \cap \overset{\circ}{W}{}_2^1(G)$, we have $v|_{\partial G} = 0$ and

$$\frac{1}{2}\int\limits_{\partial G}(Q_0(\alpha)v, v)\,ds = 0. \quad (3.28)$$

Under the assumption, the matrices B and B_{x_i}, $i = 1, \dots, n+1$, are bounded in $\overline{G}$, i.e. there exists $M = \mathrm{const} > 0$ such that

$$\|B\| + \sum_{i=1}^{n+1}\|B_{x_i}\| \leq M. \quad (3.29)$$

From (3.29) for $\lambda \geq \frac{1}{2}\,M(n+3)+1$, using the Cauchy inequality we can easily get

$$\sum_{j=1}^{n+1}\int\limits_G (B_\lambda v_{x_j}, v_{x_j})\,dG + \sum_{j=1}^{n+1}\int\limits_G (B_{\lambda x_j}, v_{x_j})\,dG + \int\limits_G (B_\lambda v, v)\,dG \geq$$

$$\geq (\lambda - M)\int\limits_G \Big(\sum_{j=1}^{n+1} v_{x_j}^2\Big)\,dG -$$

$$-\frac{M}{2}\int\limits_G\Big[\sum_{j=1}^{n+1}(v^2+v_{x_j}^2)\Big]\,dG+(\lambda-M)\int\limits_G v^2\,dG=$$

$$=\Big(\lambda-\frac{3}{2}\,M\Big)\int\limits_G\Big(\sum_{j=1}^{n+1}v_{x_j}^2\Big)\,dG+\Big(\lambda-\frac{M}{2}\,(n+3)\Big)\int\limits_G v^2\,dG\geq$$

$$\geq\int\limits_G\Big(v^2+\sum_{j=1}^{n+1}v_{x_j}^2\Big)\,dG=\|v\|^2_{W_2^1(G)}. \tag{3.30}$$

Next, by (3.27)–(3.30) from (3.26) it follows that for $\lambda\geq\frac{1}{2}\,M(n+3)+1$ and $v\in W_2^2(G)\cap\overset{\circ}{W}{}_2^1(G)$, the inequality

$$\Big|\int\limits_G\big(L_\lambda v,\Delta v-v\big)\,dG\Big|\geq\|v\|^2_{W_2^1(G)} \tag{3.31}$$

holds.

From (3.31) in the well-known manner we obtain the inequality [126, p. 51]

$$\|L_\lambda^* w\|_{-1}\geq c\|w\|_{-1}\quad\forall\, w\in W_2^1(G), \tag{3.32}$$

where $L_\lambda^* w\equiv -Ew_t-\sum\limits_{i=1}^{n+1}A_i w_{x_i}+B_\lambda^T w$, the positive constant c does not depend on w and

$$\|w\|_{-1}=\sup_{v\in\overset{\circ}{W}{}_2^1(G)}\frac{(w,v)_{L_2(G)}}{\|v\|_{\overset{\circ}{W}{}_2^1(G)}}$$

is a norm in the negative Lax space $\overset{\circ}{W}{}_2^{-1}(G)$, $(\,\cdot\,)^T$ is the operation of transposition.

Consider now the corresponding to (3.1), (3.15) characteristic problem for $i=s$ with homogeneous boundary conditions, which after the change $v(x,t)=u(x,t)\exp(-\lambda t)$ can be written as follows:

$$L_\lambda v=F_\lambda, \tag{3.33}$$

$$\Gamma_{sj}^-(v)\big|_{S_s^0}=0,\ \ j=1,\dots,\varkappa_s^-. \tag{3.34}$$

Definition 3.1. The vector function $v\in L_2(G)$ is said to be a weak generalized solution of the problem (3.33), (3.34), where $F_\lambda\in L_2(G)$, if for any $w\in W_2^1(G)$ the identity

$$(v,L_\lambda^* w)_{L_2(G)}=(F_\lambda,w)_{L_2(G)} \tag{3.35}$$

is valid.

Let us show that for any vector function $F_\lambda\in\overset{\circ}{W}{}_2^1(G)$ there exists a unique weak generalized solution of the problem (3.33), (3.34) from the space $\overset{\circ}{W}{}_2^1(G)$. Obviously, this solution $v\in\overset{\circ}{W}{}_2^1(G)$ will satisfy the system (3.33) a.e., and the boundary conditions (3.34) in the sense of the trace theory [84]. Indeed, considering the linear functional $(v, L^*_\lambda w)_{L_2(G)}$ over the space $\overset{\circ}{W}{}_2^{-1}(G)$, by virtue of (3.32) for any $w \in W_2^1(G)$ we have

$$\big|(F_\lambda, w)_{L_2(G)}\big| \leq \|F_\lambda\|_1 \|w\|_{-1} \leq c^{-1} \|L^*_\lambda w\|_{-1} \|F_\lambda\|_1, \tag{3.36}$$

where $\|F_\lambda\|_1 = \|F\|_{\overset{\circ}{W}{}_2^1(G)}$. On the basis of (3.36), we can extend the above functional to the whole space $\overset{\circ}{W}{}_2^{-1}(G)$. Further, using the Riesz theorem on the representation of the functional over the space $\overset{\circ}{W}{}_2^{-1}(G)$, we find that there exists a vector function $v \in \overset{\circ}{W}{}_2^1(G)$ satisfying the identity (3.35). To prove the uniqueness of solution, it should be noted that for a weak generalized solution v of the problem (3.33), (3.34) from the space $\overset{\circ}{W}{}_2^1(G)$ we have $v|_{\partial G} = 0$. Therefore integrating the identity (3.35) by parts, we obtain $L_\lambda v = F_\lambda$, $(x,t) \in G$. It remains only to note that analogously to the above proven a priori estimate (3.16), for any $v \in \overset{\circ}{W}{}_2^1(G)$ the inequality

$$\|v\|_{L_2(G)} \leq C_1 \|L_\lambda v\|_{L_2(G)}, \ \ C_1 = \text{const} > 0,$$

holds.

Now we get back to the problem (3.1), (3.15) when $i = s$, i.e. in the domain D_s under the homogeneous boundary conditions

$$\Gamma^-_{sj}(u)\big|_{S^0_s} = 0, \ \ j = 1, \dots, \varkappa^-_s. \tag{3.37}$$

Let $F \in W_2^1(D_s)$ and $F|_{S^0_s} = 0$. We extend F from the domain D_s to G in such a way that $F \in \overset{\circ}{W}{}_2^1(G)$, and hence $F_\lambda = F \exp(-\lambda t) \in \overset{\circ}{W}{}_2^1(G)$, where $\lambda \geq \frac{1}{2} M(n+3) + 1$. As is shown above, the problem (3.33), (3.34) has a solution v from the space $\overset{\circ}{W}{}_2^1(G)$. But in this case in the domain D_s the vector function $u = v \exp(\lambda t)$ is a solution of the problem (3.1), (3.15) from the space $W_2^1(D_s)$. The uniqueness of that solution in the space $W_2^1(D_s)$ follows from the a priori estimate (3.16). Thus the following theorem is valid.

Theorem 3.1. *For any $F \in W_2^1(D_s)$ such that $F|_{S^0_s} = 0$, there exists a unique solution of the problem* (3.1), (3.17) *from the space $W_2^1(D_s)$.*

Definition 3.2. Let $F \in L_2(D_s)$. We call the function $u \in L_2(D_s)$ a strong generalized solution of the problem (3.1), (3.37) of the class L_2 if there exists a sequence of functions $u_k \in W_2^1(D_s)$ satisfying the homogeneous boundary conditions (1.37) such that

$$\lim_{k\to\infty} \|u - u_k\|_{L_2(D_s)} = 0, \quad \lim_{k\to\infty} \|F - Lu_k\|_{L_2(D_s)} = 0.$$

If $F \in L_2(D_s)$, then since the space $C_0^1(D_s) = \{v \in C^1(D_s) : \operatorname{supp} v \subset D_s\}$ is dense in $L_2(D_s)$, there exists a sequence $F_k \in C_0^1(D_s)$ such that $F_k \to F$ in $L_2(D_s)$. Since $F_k \in C_0^1(D_s)$, we have $F_k \in W_2^1(D_s)$ and

$F_k|_{S_s^0} = 0$. Thus by Theorem 3.1 for $F = F_k$ there exists a unique solution $u_k \in W_2^1(D_s)$ of the problem (3.1), (3.37). From the inequality (3.16) we have

$$\|u_k - u_l\|_{L_2(D_s)} \le C\|F_k - F_l\|_{L_2(D_s)},$$

whence it follows that the sequence $\{u_k\}$ is fundamental in $L_2(D_s)$, because $F_k \to F$ in $L_2(D_s)$. Since the space $L_2(D_s)$ is complete, there exists a function $u \in L_2(D_s)$ such that $u_k \to u$ and $Lu_k = F_k \to F$ in $L_2(D_s)$. Consequently, u is a strong generalized solution of the problem (3.1), (3.37) of the class L_2. The uniqueness of the strong generalized solution of the problem (3.1), (3.37) of the class L_2 follows from the inequality (3.16).

Thus the following theorem is proved.

Theorem 3.2. *For any $F \in L_2(D_s)$, there exists a unique strong generalized solution u of the problem* (3.1), (3.37) *of the class L_2 for which the estimate*

$$\|u\|_{L_2(D_s)} \le C\|F\|_{L_2(D_s)}$$

is valid with a positive constant C independent of F.

3.4. Some examples of systems of differential equations of mathematical physics.

1^0. In the space of the variables x, y, z and t we consider the nonhomogeneous system of Maxwell differential equations for electromagnetic field in vacuum [24, p. 640]

$$\widetilde{E}_t - \operatorname{rot} H = F_1, \quad H_1 + \operatorname{rot} \widetilde{E} = F_2, \tag{3.38}$$

where $\widetilde{E} = (E_1, E_2, E_3)$ is the electromagnetic field vector and $H = (H_1, H_2, H_3)$ is the magnetic field vector. Light velocity is assumed to be equal to unity.

Assuming $U = (\widetilde{E}, H)$, $F = (F_1, F_2)$, we can rewrite the system (3.38) in the form

$$\widetilde{L}_1 U \equiv U_t + A_1 U_x + A_2 U_y + A_3 U_z = F, \tag{3.39}$$

where A_1, A_2, A_3 are quite definite real symmetric (6×6)-matrices. The characteristic determinant of the system (3.39) is equal to $\xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in R^4$, $Q_0(\xi) = \xi_0 E + \xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3$ is the characteristic matrix of that system. Here E is the unit (6×6)-matrix.

In accordance with (3.2), (3.5) and (3.7), for the system (3.39) we have

$$s_0 = 1, \ \ s = 2s_0 + 1, \ \ k_1 = k_2 = k_3 = 2, \ \ \lambda_1(\xi') = (\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2},$$
$$\lambda_2(\xi') \equiv 0, \ \ \lambda_2(\xi') = -(\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2},$$
$$K_i: \ \xi_0 - \lambda_i(\xi') = 0, \ \ i = 1, 2, 3,$$
$$S_1: \ t = -(x^2 + y^2 + z^2)^{1/2}, \quad S_3: \ t = (x^2 + y^2 + z^2)^{1/2}.$$

For the unit vector of the outer normal $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_0) = (\widetilde{\alpha}, \alpha_0)$ on S_3 we have [24, p. 642]

$$(Q_0(\alpha), U, U) = \frac{1}{\alpha_0}\Big[\big(\widetilde{E}\alpha_0 + [H \times \widetilde{\alpha}]\big)^2 + (H, \widetilde{\alpha})^2\Big]. \tag{3.40}$$

Since $\alpha_0 < 0$ on S_3, according to (3.13) and (3.40) $\varkappa_3^- = 4$ and the boundary conditions (3.15) for the system (3.38) take the form

$$\big(\widetilde{E}\alpha_0 + [H \times \widetilde{\alpha}]\big)\Big|_{S_3^0} = f_1, \tag{3.41}$$

$$(H, \widetilde{\alpha})\big|_{S_3^0} = f_2. \tag{3.42}$$

Owing to (3.16), for every solution $U = (\widetilde{E}, H) \in W_2^1(D_3)$ of the problem (3.38), (3.41), (3.42) the a priori estimate

$$\|U\|_{L_2(D_3)} \le C\Big(\|f_1\|_{L_2(S_3^0)} + \|f_2\|_{L_2(S_3^0)} + \|F\|_{L_2(D_3)}\Big)$$

is valid with a positive constant C, independent of U. Next, by the above proven theorems, for every $F \in L_2(D_3)$ there exists a unique strong generalized solution of the problem (3.38), (3.41), (3.42) of the class L_2 with the homogeneous boundary conditions $f_1 = f_2 = 0$. Moreover, if $F \in W_2^1(D_3)$ and $F|_{S_3^0} = 0$, then this solution belongs to the class $W_2^1(D_3)$.

$\mathbf{2^0}$. Consider the nonhomogeneous system of Dirac differential equations in the complex form [24, p. 183]

$$\sum_{k=1}^{4} \mu_k\Big(\frac{\partial}{\partial x_k} - a_k\Big)u - \beta b u = F, \tag{3.43}$$

where the vector (a_1, a_2, a_3) is proportional to the magnetic potential, a_4 to the electric potential, and b to the rest-mass; $F = (F_1, F_2, F_3, F_4)$ is a given and $u = (u_1, u_2, u_3, u_4)$ is an unknown 4-dimensional complex vector function of the variables x_1, x_2, x_3, $x_4 = t$. The coefficients in the system (3.43) are the following matrices:

$$\mu_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

If $u = w + iv$, then the system (3.43) with respect to the unknown 8-dimensional real vector $U = (w, v)$ can be written as

$$\widetilde{L}_2 U \equiv \sum_{k=1}^{4} \sigma_4 U_{x_k} + \sigma U = F_1, \tag{3.44}$$

where $F_1 = (\operatorname{Re} F, \operatorname{Im} F)$, σ is some real (8×8)-matrix, and

$$\sigma_1 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i\mu_2 \\ -i\mu_2 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \mu_3 & 0 \\ 0 & \mu_3 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} \mu_4 & 0 \\ 0 & \mu_4 \end{pmatrix}.$$

The system (3.44) is the first order symmetric hyperbolic system whose characteristic polynomial is equal to

$$p(\xi) = \det Q_0(\xi) = (\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_0^2)^4,$$

where $\xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in R^4$, $Q_0(\xi) = \xi_0\sigma_4 + \xi_1\sigma_1 + \xi_2\sigma_2 + \xi_3\sigma_3$ is the characteristic matrix of the system.

Taking into account (3.2) and (3.6), for the system (3.44) we have

$$s_0 = 1,\ \ s = 2s_0 = 2,\ \ k_1 = k_2 = 4,\ \ \lambda_1(\xi') = (\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2},$$
$$\lambda_2(\xi') = -(\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2},\ \ K_j:\ \xi_0 - \lambda_j(\xi') = 0,\ \ j = 1,2,$$
$$S_1:\ t = -(x_1^2 + x_2^2 + x_3^2)^{1/2},\quad S_2:\ t = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

We rewrite the system (3.44) in the form of scalar equations and multiply the equations under numbers 1, 2, 5 and 8 by -1. Then we transpose these equations according to the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 8 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$.

Then under the new notation $\Lambda = (W, V, \omega_1, \omega_2)$, where $W = (v_1, w_1, -v_2)$, $V = (w_3, -v_3, -w_4)$, $\omega_1 = v_4$, $\omega_2 = -w_2$, we obtain the system of equations which in the matrix form is written as follows:

$$\widetilde{L}_3\Lambda \equiv E\Lambda_t + \sum_{j=1}^{3} \widetilde{A}_j \Lambda_{x_j} + \widetilde{B}\Lambda = F_2, \tag{3.45}$$

where

$$\widetilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{A}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

F_2 is the known vector function, and $\widetilde{B}$ is a quite definite real (8×8)-matrix. Obviously, the systems (3.44) and (3.45) are equivalent.

It can be easily verified that if $\widetilde{L}_3^0 \equiv E\,\frac{\partial}{\partial t} + \sum\limits_{j=1}^{3} \widetilde{A}_j\,\frac{\partial}{\partial x_j}$ is the principal part of the operator from (3.45), then

$$2(\widetilde{L}_3^0\Lambda)\Lambda = 2\,\frac{\partial W}{\partial t}\,V - 2W\operatorname{rot}V - 2W\operatorname{grad}\omega_1 + 2V\,\frac{\partial V}{\partial t} + 2V\operatorname{rot}W +$$
$$+ 2V\operatorname{grad}\omega_2 + 2\omega_1\,\frac{\partial \omega_1}{\partial t} - 2\omega_1\operatorname{div}W + 2\omega_2\,\frac{\partial \omega_2}{\partial t} + 2\omega_2\operatorname{div}V,$$

whence

$$2(\widetilde{L}^0\Lambda)\Lambda = (W^2+V^2)_t + 2\operatorname{div}[W\times V] +$$
$$+ (\omega_1^2+\omega_2^2)_t + 2\operatorname{div}[\omega_2 V - \omega_1 W], \tag{3.46}$$

where $[W\times V]$ is the vector product of the vectors W and V.

On the other hand, for every $\Lambda \in W_2^1(D_2)$ analogously to (3.17) we have

$$2\int\limits_{D_2} (\widetilde{L}_3^0\Lambda)\Lambda\,dx\,dt = \int\limits_{\partial D_2} (Q_0(\alpha)\Lambda,\Lambda)\,ds,$$

where $Q_0(\alpha) = E\alpha_0 + \sum\limits_{j=1}^{3} \widetilde{A}_j\alpha_j$, $\alpha = (\widetilde{\alpha},\alpha_0) = (\alpha_1,\alpha_2,\alpha_3,\alpha_0)$ is the unit vector of the outer normal to ∂D_2. Therefore taking into account that

$$\alpha_0\big|_{S_2^0} < 0, \quad \big[(\alpha_0)^2 - |\widetilde{\alpha}|^2\big]\Big|_{S_2^0} = 0$$

and using the well-known vector relations [24, p. 642]

$$W^2\widetilde{\alpha}^2 = [W\times\widetilde{\alpha}]^2 + [W\cdot\widetilde{\alpha}]^2, \quad V^2\widetilde{\alpha}^2 = [V\times\widetilde{\alpha}]^2 + [V\cdot\widetilde{\alpha}]^2,$$

from (3.46) we have

$$(Q_0(\alpha)\Lambda,\Lambda) =$$
$$= \frac{1}{\alpha_0}\Big[(W^2+V^2)\alpha_0^2 + 2[W\times V]\widetilde{\alpha}\alpha_0 + (\omega_1^2+\omega_2^2)\alpha_0^2 + 2[\omega_2 V - \omega_1 W]\widetilde{\alpha}\alpha_0\Big] =$$
$$= \frac{1}{\alpha_0}\Big[(\omega_1\alpha_0 - W\widetilde{\alpha})^2 + (\omega_2\alpha_0 + V\widetilde{\alpha})^2 + (W^2+V^2)\alpha_0^2 +$$
$$+ 2[W\times V]\widetilde{\alpha}\alpha_0 - (W\widetilde{\alpha})^2 - (V\widetilde{\alpha})^2\Big] =$$

$$= \frac{1}{\alpha_0}\Big[(\omega_1\alpha_0 - W\widetilde{\alpha})^2 + (\omega_2\alpha_0 + V\widetilde{\alpha})^2 + [W\times\widetilde{\alpha}]^2 + $$
$$+2[W\times V]\widetilde{\alpha}\alpha_0 + [V\times\widetilde{\alpha}]^2\Big]. \tag{3.47}$$

Assume

$$I = [W\times\widetilde{\alpha}]^2 + 2[W\times V]\widetilde{\alpha}\alpha_0 + [V\times\widetilde{\alpha}]^2. \tag{3.48}$$

Let first

$$\widetilde{\alpha} = \widetilde{\alpha}_0 = (0,0,|\alpha_0|) = |\alpha_0|(0,0,1). \tag{3.49}$$

Then it can be easily verified that

$$\begin{gathered}[W\times\widetilde{\alpha}]^2 = |\alpha_0|^2(W_1^2+W_2^2),\\ 2[W\times V]\widetilde{\alpha}\alpha_0 = 2\alpha_0|\alpha_0|(W_1V_2 - W_2V_1),\\ [V\times\widetilde{\alpha}]^2 = |\alpha_0|^2(V_1^2+V_2^2).\end{gathered} \tag{3.50}$$

Therefore in the case (3.49) due to the fact that $|\alpha_0| = |\widetilde{\alpha}|$, $\alpha_0 < 0$, $\alpha_0^2 + |\widetilde{\alpha}|^2 = 1$ and hence $|\alpha_0|^2 = \frac{1}{2}$, from (3.48) and (3.50) we have

$$I = \frac{1}{2}(W_1 - V_2)^2 + \frac{1}{2}(W_2 + V_1)^2. \tag{3.51}$$

Let T be the matrix of the orthogonal transformation which transforms the vector $\widetilde{\alpha}$ into $\widetilde{\alpha}_0 = (0,0,|\alpha_0|)$ not changing the space orientation. As is known, the action of that transformation on the vector $x = (x_1,x_2,x_3)$ for $\widetilde{\alpha} \neq -\widetilde{\alpha}_0$ is given by the following equality [99, p. 68]:

$$Tx = x - \frac{(\widetilde{\alpha}+\widetilde{\alpha}_0)\cdot x}{\alpha_0^2 + \widetilde{\alpha}\cdot\widetilde{\alpha}_0}(\widetilde{\alpha}+\widetilde{\alpha}_0) + \frac{2}{\alpha_0^2}(\widetilde{\alpha}\cdot x)\widetilde{\alpha}_0, \;\; \widetilde{\alpha} \neq -\widetilde{\alpha}_0.$$

Using the properties of the vector and mixed product of vectors, we can see that

$$\begin{aligned} I &= [W\times\widetilde{\alpha}]^2 + 2[W\times V]\widetilde{\alpha}\alpha_0 + [V\times\widetilde{\alpha}]^2 = \\ &= [TW\times T\widetilde{\alpha}]^2 + 2[TW\times TV]T\widetilde{\alpha}\alpha_0 + [TV\times T\widetilde{\alpha}]^2 = \\ &= [TW\times\widetilde{\alpha}_0]^2 + 2[TW\times TV]\widetilde{\alpha}_0\alpha_0 + [TV\times\widetilde{\alpha}_0]. \end{aligned} \tag{3.52}$$

Let ν_1, ν_2 and ν_3 be the rows of the matrix T, i.e.

$$T = \begin{pmatrix} \nu_{11} & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} & \nu_{23} \\ \nu_{31} & \nu_{32} & \nu_{33} \end{pmatrix} = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}.$$

By (3.49)–(3.51), from (3.52) we obtain

$$I = \frac{1}{2}(\nu_1 W - \nu_2 V)^2 + \frac{1}{2}(\nu_2 W + \nu_2 V)^2. \tag{3.53}$$

Now from (3.47),(3.48) and (3.53) it follows that

$$(Q_0(\alpha)\Lambda,\Lambda) = \frac{1}{\alpha_0}\Big[(\omega_1\alpha_0 - W\widetilde{\alpha})^2 + (\omega_2\alpha_0 + V\widetilde{\alpha})^2 + $$
$$+\frac{1}{2}(\nu_1 W - \nu_2 V)^2 + \frac{1}{2}(\nu_2 W + \nu_1 V)^2\Big]. \tag{3.54}$$

According to (3.13)–(3.15), from (3.54) for $\varkappa_2^- = 4$ we obtain the following boundary conditions for the system (3.45),

$$\begin{aligned} &(\omega_1\alpha_0 - W\widetilde{\alpha})\big|_{S_2^0} = f_1, \quad (\omega_2\alpha_0 + V\widetilde{\alpha})\big|_{S_2^0} = f_2,\\ &(\nu_1 W - \nu_2 V)\big|_{S_2^0} = f_3, \quad (\nu_2 W + \nu_1 V)\big|_{S_2^0} = f_4. \end{aligned} \tag{3.55}$$

By (3.16), for every solution $u \in W_2^1(D_2)$ of the problem (3.43), (3.55) the a priori estimate

$$\|u\|_{L_2(D_2)} \leq C\Big(\sum_{j=1}^{4} \|f_j\|_{L_2(S_2^0)} + \|F\|_{L_2(D_2)}\Big), \quad C = \text{const} > 0,$$

is valid.

Further, just as when considering the problem (3.38), (3.41), (3.42) we find that for every $F \in L_2(D_2)$ there exists a unique strong generalized solution of the problem (3.43), (3.55) of the class L_2 with the homogeneous boundary conditions $f_i = 0$, $i = 1, \dots, 4$. Moreover, if $F \in W_2^1(D_2)$ and $F|_{S_2^0} = 0$, then this solution belongs to the class $W_2^1(D_2)$.

3^0. The system of equations of the crystal optics has the form

$$\frac{1}{c}\, E_\varepsilon \widetilde{E}_t - \operatorname{rot} H = F_1, \quad \frac{1}{c}\, \mu H_t + \operatorname{rot} \widetilde{E} = F_2, \tag{3.56}$$

where $\widetilde{E}$ and H are the same as for the Maxwell equations (3.38), c is the light velocity, μ is the constant of magnetic permeability, E_3 is the diagonal (3×3)-matrix with elements ε_1, ε_2 and ε_3 on the diagonal, and ε_i are the dielectric constants is valid in the direction of three coordinate axes.

In the notation $U = (\widetilde{E}, H)$, $F = (F_1, F_2)$ the system (3.56) takes the form

$$\widetilde{E}_\varepsilon U_t + A_1 U_x + A_2 U_y + A_3 U_z = F, \tag{3.57}$$

where A_i, $i = 1, 2, 3$, are the same matrices as in (3.39), and

$$\widetilde{E}_\varepsilon = \frac{1}{c}\, \operatorname{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \mu, \mu, \mu).$$

Since all the coefficients in (3.57) are real symmetric matrices and $\widetilde{E}_\varepsilon$ is positive definite, the system (3.57) is hyperbolic [24, p. 587].

Assume $\sigma_i = (\mu/c^2)\varepsilon_i$, $i = 1, 2, 3$, and $\sigma_1 > \sigma_2 > \sigma_3$. If K and S are respectively the cone of normals and that of rays for the system (3.56), then as is known, they are algebraic surfaces of sixth order given by the equations [24, p. 599]

$$\begin{aligned} &K: \prod_{i=1}^{3} (\rho^2 - \sigma_i \xi_0)\Big[1 - \sum_{i=1}^{3} \frac{\xi_i^2}{\rho^2 - \sigma_i \xi_0^2}\Big] = 0,\\ &S: 1 - \sum_{i=1}^{3} \frac{x_i^2}{r^2 - \sigma_i^{-1} t^2} = 0, \end{aligned} \tag{3.58}$$

where $\rho^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$, $r^2 = x_1^2 + x_2^2 + x_3^3$, and instead of x, y and z are written x_1, x_2 and x_3, respectively.

In accordance with (3.2), (3.5) and (3.7), in this case we have

$$s_0 = 2, \quad s = 2s_0 + 1 = 5, \quad k_1 = k_2 = k_4 = k_5 = 1,$$
$$k_3 = 2, \quad \varkappa_5^- = 5, \quad \lambda_3(\xi') \equiv 0,$$

the remaining $\lambda_i(\xi')$ are the roots of the first of the equations (3.58) with respect to ξ_0 which determine the sheets $K_i : \xi_0 - \lambda_i(\xi') = 0$ of the cone of normals K. If $S^+ = S \cap \{t \geq 0\}$, then $S_5 = \partial(K_5^*)$ is the convex shell of S^+, and S_5 is a piecewise smooth conic manifold [24, p. 602].

Remark 3.5. Note that in the given case the constancy of multiplicities k_i of the roots $\lambda_i(\xi')$, $\xi' \in R^n \setminus \{(0,\dots,0)\}$, or what is the same, the fulfilment of the inequalities (3.2) is violated only on the set of measure zero in R^n, but this fact, as it can be easily noticed, does not affect the validity of the a priori estimate (3.16) and the theorems proven above. Analogous remark is true for the coefficients $c_{jp}^i(\alpha)$ in the boundary conditions (3.15) which can be assumed to be bounded and measurable.

As for the unknown vector function $V = (\widetilde{E}_\varepsilon)^{1/2}U$, the system (3.57) can be rewritten equivalently as

$$V_t + \sum_{i=1}^{3} \widetilde{A}_i V_{x_i} = \widetilde{F}, \tag{3.59}$$

where $\widetilde{A}_i = (\widetilde{E}_\varepsilon)^{-1/2} A_i (\widetilde{E}_\varepsilon)^{-1/2}$, $i = 1,2,3$, are likewise real symmetric matrices, $\widetilde{F} = (\widetilde{E}_\varepsilon)^{-1/2} F$. Let $\widetilde{Q}_0(\xi)$ be the characteristic matrix of the system (3.59), and let $\widetilde{T}$ be the orthogonal matrix from the corresponding equality (3.3). Then by (3.10) and (3.11) applied to the system (3.59) we have

$$\begin{aligned}
(\widetilde{Q}_0(\alpha)\eta, \eta) &= -\sum_{j=1}^{\varkappa_5^-} \Big[\big(\widetilde{\lambda}_j(\alpha) - \lambda_5(\alpha)\big)^{1/2} \zeta_j \Big]^2 = \\
&= -\sum_{j=1}^{5} \Big[\big(\widetilde{\lambda}_j(\alpha) - \lambda_5(\alpha)\big)^{1/2} (\widetilde{T}'\eta)_j \Big]^2 = \\
&= -\sum_{j=1}^{5} \Big[\big(\widetilde{\lambda}_j(\alpha) - \lambda_5(\alpha)\big)^{1/2} \sum_{i=1}^{6} \widetilde{T}_{ij} \eta_i \Big]^2 = \\
&= -\sum_{j=1}^{5} \Big[\sum_{i=1}^{6} \widetilde{c}_{ji}(\alpha) \eta_i \Big]^2, \quad \alpha \in K_5',
\end{aligned} \tag{3.60}$$

where according to (3.2) we obtain $\widetilde{\lambda}_i = \lambda_1$, $\widetilde{\lambda}_2 = \lambda_2$, $\widetilde{\lambda}_3 = \widetilde{\lambda}_4 = \lambda_3$, $\widetilde{\lambda}_5 = \lambda_4$, $\eta \in R^6$, $(\widetilde{T}'\eta)_j$ is the j-th component of the vector $\widetilde{T}'\eta$, $\widetilde{c}_{ji}(\alpha) = (\widetilde{\lambda}_j(\alpha) - \lambda_5(\alpha))^{1/2} \widetilde{T}_{ij}$ are bounded measurable functions.

By virtue of (3.13) and (3.60), the boundary conditions (3.37) for the system (3.59) take the form

$$\Big(\sum_{i=1}^{6}\widetilde{c}_{ji}(\alpha)V_i\Big)\Big|_{S_5^0}=0,\ \ j=1,\dots,5,$$

which with respect to the unknown $U=(\widetilde{E}_\varepsilon)^{-1/2}V$ of the initial system (3.57) can be written in the form

$$\Big(\sum_{i=1}^{6}\widetilde{c}_{ji}(\alpha)\,d_iU_i\Big)\Big|_{S_5^0}=0,\ \ j=1,\dots,5, \tag{3.61}$$

where d_i, $i=1,\dots,6$, are the diagonal elements of the matrix $(\widetilde{E}_\varepsilon)^{1/2}$, and α is the unit vector of the outer normal to S_5^0. Therefore by the above-proven Theorem 3.1, for every $F\in W_2^1(D_5)$ such that $F|_{S_5^0}=0$ there exists a unique solution of the problem (3.57), (3.61) from the space $W_2^1(D_5)$. If, however, $F\in L_2(D_5)$, then there exists a strong generalized solution of the same problem of the class L_2.

3.5. Boundary value problems in dihedral domains. In this subsection we will give a brief scheme of investigation of boundary value problems for the system (3.1) with the symmetric principal part in the dihedral domain $D=\big\{(x_1,\dots,x_n,t)\in R^{n+1}:\ \alpha_0^it+\sum\limits_{j=1}^{n}\alpha_j^ix_j<0,\ i=1,2\big\}$ bounded by the hypersurfaces $\widetilde{\Pi}_1:\ \alpha_0^1t+\sum\limits_{j=1}^{n}\alpha_j^1x_j=0$ and $\widetilde{\Pi}_2:\ \alpha_0^2t+\sum\limits_{j=1}^{n}\alpha_j^2x_j=0$, where $\alpha^j=(\alpha_1^j,\dots,\alpha_n^j,\alpha_0^j)$ is the unit vector of the outer normal to ∂D at the point of the face $\Pi_j=\widetilde{\Pi}_j\cap\partial D$, $j\neq 1,2$, $\alpha^1\neq\alpha^2$. For the sake of simplicity it will be assumed that $\alpha_0^j<0$, $j=1,2$.

Consider the boundary value problem which is formulated as follows: find in the domain D a solution u of the system (3.1) by the boundary conditions

$$\Gamma^ju\big|_{\Pi_j}=f^j,\ \ j=1,2, \tag{3.62}$$

where Γ^j are given real constant $(\varkappa_j\times m)$-matrices, and $f^j=(f_1^j,\dots,f_{\varkappa_j}^j)$ are given $\varkappa_j$-dimensional real vectors, $j=1,2$.

Below, the elements of the matrix B in the system (3.1) will be assumed to be bounded measurable functions in D, i.e. $B\in C_\infty(\overline{D})$. Introduce into consideration the following weight spaces:

$$\begin{aligned}
W_{2,\lambda}^1(D)&=\big\{u\in L_{2,\mathrm{loc}}(D):\ u\exp(-\lambda t)\in W_2^1(D)\big\},\\
\|u\|_{W_{2,\lambda}^1(D)}&=\big\|u\exp(-\lambda t)\big\|_{W_2^1(D)},\\
L_{2,\lambda}(D)&=\big\{F\in L_{2,\mathrm{loc}}(D):\ F\exp(-\lambda t)\in L_2(D)\big\},\\
\|F\|_{L_{2,\lambda}(D)}&=\big\|F\exp(-\lambda t)\big\|_{L_2(D)},
\end{aligned}$$

$$L_{2,\lambda}(\Pi_j) = \big\{ f \in L_{2,\mathrm{loc}}(\Pi_j) : \ f\exp(-\lambda t) \in L_2(\Pi_j) \big\}, \quad j=1,2,$$
$$\|f\|_{L_{2,\lambda}(\Pi_j)} = \big\| f\exp(-\lambda t)\big\|_{L_2(\Pi_j)},$$

where λ is a real parameter, and $L_{2,\mathrm{loc}}(D)$, $W_2^1(D)$, $L_{2,\mathrm{loc}}(\Pi_j)$, $j=1,2$, are well-known function spaces [121, p. 384].

Let $\lambda_{\max}(P)$ be the largest characteristic number of the nonpositively defined symmetric matrix $B^T B$ at the point $P \in \overline{D}$, where $(\,\cdot\,)^T$ denotes transposition. Then due to the fact that $B \in L_\infty(\overline{D})$, we have

$$\lambda_0^2 = \sup_{P\in\overline{D}} \lambda_{\max}(P) < +\infty. \tag{3.63}$$

Let the faces Π_1 and Π_2 of the dihedral angle D be characteristic ones, namely $\alpha^j \in K_{s_j}$, where $s_j > \left[\frac{s+1}{2}\right]$, $j=1,2$, and K_i is the i-th connected component of the cone of normals K of the system (3.1) considered in the Subsection 3.1. In accordance with (3.13) and (3.14), in the boundary conditions (3.62) we put $\varkappa_j = \varkappa_{s_j}^-$, and as Γ^j we take the matrix of order $(\varkappa_{s_j}^- \times m)$ which is composed of the first $\varkappa_{s_j}^-$ rows of the matrix $C^{s_j}(\alpha^j)$, $j=1,2$, and the number $\varkappa_{s_j}^-$ is defined in (3.12). Under these assumptions the following proposition is valid.

Lemma 3.2. *For every solution $u \in W_{2,\lambda}^1(D)$ of the problem* (3.1),(3.62) *for $\lambda > \lambda_0$ the a priori estimate*

$$\|u\|_{L_{2,\lambda}(D)} \le \frac{1}{\sqrt{\lambda-\lambda_0}} \sum_{j=1}^{2}\sum_{i=1}^{\varkappa_j} \|f_i^j\|_{L_{2,\lambda}(\Pi_j)} + \frac{1}{\lambda-\lambda_0}\|F\|_{L_{2,\lambda}(D)} \tag{3.64}$$

is valid, where $\varkappa_j = \varkappa_{s_j}^-$, $j=1,2$, and the number $\lambda_0 \ge 0$ is defined from (3.63).

Proof of Lemma 3.2 is similar to that of Lemma 3.1 [73].

Let $f^j = 0$, $j=1,2$, and $F \in L_{2,\lambda}(D)$. Analogously to Definition 3.2 from Subsection 3.3 we call a function $u \in L_{2,\lambda}(D)$ strong generalized solution of the problem (3.1), (3.62) of the class $L_{2,\lambda}$ if there exists a sequence of functions $u_k \in W_2^1(D)$ satisfying the homogeneous condition corresponding to (3.62) such that

$$\lim_{k\to\infty} \|u-u_k\|_{L_{2,\lambda}(D)} = \lim_{k\to\infty} \|F-Lu_k\|_{L_{2,\lambda}(D)} = 0.$$

Below the elements of the matrix B will be assumed to be bounded in the closed domain $\overline{D}$ together with their partial first order derivatives, and the faces Π_1 and Π_2 of the dihedral angle D be tangent to the outer cone of rays of the characteristic conoid of the system (3.1), i.e.

$$s_1 = s_2 = s, \quad \alpha^j \in K_s, \quad j=1,2. \tag{3.65}$$

Let $\lambda_{\max}^i(P)$ be the largest characteristic number of the nonnegatively defined symmetric matrix $B_{x_i}^T B_{x_i}$ at the point $P \in \overline{D}$, $i=1,\dots,n+1$, where we have introduced the notation $x_{n+1} = t$. Then taking into account that by our assumption the elements of the matrices B_{x_i}, $i = 1, \dots, n+1$, are bounded in $\overline{D}$, we have $\lambda_*^2 = \max\limits_{1 \le i \le n+1} \sup\limits_{P \in D} \lambda^i_{\max}(P) < +\infty$.

Consider the condition

$$\lambda > \lambda_0 + \frac{1}{2}\,(n+1)\lambda_*. \tag{3.66}$$

Analogously to Theorems 3.1 and 3.2 from Subsection 3.3 we prove the following

Theorem 3.3. *Let $f^j = 0$, $j = 1, 2$, and let the conditions* (3.65) *and* (3.66) *be fulfilled. Then for every $F \in L_{2,\lambda}(D)$ there exists a unique strong generalized solution u of the problem* (3.1), (3.62) *of the class $L_{2,\lambda}$. For $F \in \overset{\circ}{W}{}^1_{2,\lambda} = \{u \in W^1_{2,\lambda}(D) : \ u|_{\partial D} = 0\}$, this solution belongs to the space $W^1_{2,\lambda}(D)$.*

Here we give examples of systems of differential equations from mathematical physics and boundary value problems corresponding to these systems for which Theorem 3.3 is valid [73].

As $D \subset R^4$ we take the dihedral angle $D : \ t > |x_3|$, $n = 3$, whose faces are the characteristic surfaces $\Pi_1 : \ t - x_3 = 0$, $t \ge 0$, and $\Pi_2 : \ t + x_3 = 0$, $t \ge 0$.

1^0. For the system of Maxwell differential equations (3.38) such kind of boundary conditions are

$$\big(H_2 + (-1)^j E_1\big)\big|_{\Pi_j} = f_1^j, \quad \big(H_1 + (-1)^{j+1} E_2\big)\big|_{\Pi_j} = f_2^j,$$
$$E_3\big|_{\Pi_j} = f_3^j, \quad H_3\big|_{\Pi_j} = f_4^j, \ \ j = 1, 2.$$

2^0. For the Dirac differential equations, as the boundary conditions (when Theorem 3.3 holds) we can take

$$\big(v_4 + (-1)^j v_2\big)\big|_{\Pi_j} = f_1^j, \quad \big(w_4 + (-1)^j w_2\big)\big|_{\Pi_j} = f_2^j,$$
$$\big(v_1 + (-1)^{j+1} v_3\big)\big|_{\Pi_j} = f_3^j, \quad \big(w_1 + (-1)^{j+1} w_3\big)\big|_{\Pi_j} = f_4^j, \ \ j = 1, 2.$$

Remark 3.6. The method of proving the existence of solutions of the problems presented in Theorems 3.1–3.3 in fact uses the requirement that the data carriers of the problem be tangent to the outer cone of rays of the characteristic conoid of the system (3.1). For the dihedral domain this requirement is reflected in the condition (3.65). When the condition (3.65) is violated, the proof of the existence of a solution of the problem (3.1), (3.62) is carried out according to a different scheme in several steps [74]:

(i) using the Fourier transform with respect to the variables varying in the subspace $\Pi_1 \cap \Pi_2$, similarly to Section 1 of Chapter I we prove the existence of a solution of the problem (3.1), (3.62) in the weight functional space $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$;

(ii) it is proved that the space $\overset{\circ}{\Phi}{}^k_\alpha(\overline{D})$ $(\overset{\circ}{\Phi}{}^k_\alpha(\overline{\Pi}_j))$ is densely embedded into the space $L_2(D)$ $(L_2(\Pi_j))$;

(iii) on the basis of the facts cited in (i) and (ii), we prove the existence of a strong generalized solution of the problem (3.1), (3.62) by using the a priori estimate (3.64).

The same approach can be used for the investigation of the above problem when one or both faces Π_1 and Π_2 are noncharacteristic [74].

CHAPTER 2

Multidimensional Versions of the Goursat and Darboux Problems for Degenerating Second Order Hyperbolic Equations

1. A Multidimensional Version of the First Darboux Problem for a Model Degenerating Second Order Hyperbolic Equation in a Cone-Shaped Domain

1.1. Statement of the problem. In the space of the variables x_1, x_2, x_3, t we consider a degenerating second order hyperbolic equation of the type

$$Lu \equiv u_{tt} - u_{x_1x_1} - x_3^m u_{x_2x_2} - u_{x_3x_3} = F, \tag{1.1}$$

where F is a given, and u is an unknown real function, m is a natural number. The equation (1.1) degenerates parabolically for $x_3 = 0$ and is strictly hyperbolic for $x_3 > 0$. For $x_3 < 0$, the equation (1.1) is ultrahyperbolic when m is odd and strictly hyperbolic when m is even.

Below for the equation (1.1) in the case $m = 1$ we will construct characteristic conoids K_O and K_A with vertices at the point $O(0,0,0,0)$ and $A(0,0,0,t_0)$, $t_0 > 0$, respectively, and consider the boundary value problem whose data carriers are a part of the hypersurface $x_3 = 0$ and some parts of the conoids K_O and K_A contained in the half-layer $\big\{(x_1,x_2,x_3,t) \in R^4 :$ $0 \le t \le t_0,\ x_3 \ge 0\big\}$. Note that already for $m = 2$ the characteristic conoid K_O of the equation (1.1) has sufficiently complicated geometric structure which makes it difficult to formulate the boundary value problem. Of interest is the fact that for the equation

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} - x_3 u_{x_3x_3} = F$$

the characteristic conoid K_O degenerates into the two-dimensional conic manifold $\big\{(x_1,x_2,x_3,t) \in R^4 :\ t^2 - x_1^2 - x_2^2 = 0,\ x_3 = 0\big\}$. For the equation

$$x_3^m u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = F$$

the characteristic conoid K_O for $m = 1$ consists of only one bicharacteristic curve $\big\{(x_1,x_2,x_3,t) \in R^4 :\ x_1 = x_2 = 0,\ t^2 = \frac{4}{9}x_3^3,\ x_3 > 0\big\}$, and for $m = 2$ the conoid K_O degenerates into one point $O(0,0,0,0)$.

By K_O^+ and K_A^+ we denote the parts of the characteristic conoids K_O and K_A with vertices at the points $O(0,0,0,0)$ and $A(0,0,0,t_0)$ which are contained respectively in the dihedral angles $\big\{(x_1,x_2,x_3,t) \in R^4 :\ x_3 \ge$

$0,\ t \geq 0\big\}$ and $\big\{(x_1, x_2, x_3, t) \in R^4 :\ x_3 \geq 0,\ t \leq t_0\big\}$. Let the domain D, lying in the half-layer $\big\{(x_1, x_2, x_3, t) \in R^4 : \leq t \leq t_0,\ x_3 \geq 0\big\}$, be bounded by the hyperplane $x_3 = 0$ and by the characteristic hypersurfaces K_O^+ and K_A^+ of the equation (1.1). Assume $S_0 = \partial D \cap \{x_3 = 0\}$, $S_1 = \partial D \cap K_0^+$, $S_2 = \partial D \cap K_A^-$.

For the equation (1.1) we consider a multidimensional version of the Darboux problem formulated as follows: in the domain D, find a solution u of the equation (1.1) satisfying the boundary condition

$$u\big|_{S_0 \cup S_1} = 0. \tag{1.2}$$

Note that the operator L appearing in (1.1) is formally self-conjugate, i.e. $L^* = L$.

The problem for the equation (1.1) in the domain D is formulated analogously by means of the boundary condition

$$u\big|_{S_0 \cup S_2} = 0. \tag{1.3}$$

By E and E^* we denote the classes of functions from the space $C^2(\overline{D})$ satisfying respectively the boundary condition (1.2) or (1.3). Let W_+ (W_+^*) be the Hilbert space with weight which is obtained by closing the space E (E^*) by the norm

$$\|u\|_1^2 = \int\limits_D \big[u^2 + u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2 + u_t^2\big]\, dD.$$

Let W_- (W_-^*) be the space with the negative norm constructed on the basis of $L_2(D)$ and W_+ (W_+^*) [6, p. 46].

Definition. If $F \in L_2(D)(W_-^*)$, then the function u is said to be a strong generalized solution of the problem (1.1), (1.2) of the class $W_+(L_2)$ if $u \in W_+(L_2(D))$ and there exists a sequence of functions $u_n \in E$ such that $u_n \to u$ in the space $W_+(L_2(D))$ and $Lu_n \to F$ in the space W_-^* (W_-^*).

1.2. Construction of the characteristic conoids K_O and K_A of the equation (1.1). The system of ordinary differential equations of a bicharacteristic strip [24, p. 577]:

$$\dot{x}_i = p_{\xi_i}, \quad \dot{\xi}_i = -p_{x_i}, \quad i = 1, \dots, n,$$

for the equation (1.1), i.e. for $p = \xi_4^2 - \xi_1^2 - x_3\xi_2^2 - \xi_3^2$, $x_4 = t$, $n = 4$, has the form

$$\dot{x}_1 = -2\xi_1, \quad \dot{x}_2 = -2x_3\xi_2, \quad \dot{x}_3 = -2\xi_3, \quad \dot{x}_4 = 2\xi_4, \tag{1.4}$$

$$\dot{\xi}_1 = 0, \quad \dot{\xi}_2 = 0, \quad \dot{\xi}_3 = \xi_2^2, \quad \dot{\xi}_4 = 0, \tag{1.5}$$

where $\dot{x}_i$ $(\dot{\xi}_i)$ is the ordinary derivative of the function $x = x_i(\tau)$ $(\xi_i = \xi_i(\tau))$ with respect to the parameter τ. To construct the characteristic conoid K_O, the initial conditions for $x_i = x_i(\tau)$ and $\xi_i = \xi_i(\tau)$, $1 \le i \le 4$, when $\tau = 0$ must be the following:

$$x_i(0)=0,\ \ 1\le i\le 4;\quad \xi_1(0)=\alpha,\quad \xi_2(0)=\beta,\quad \xi_3(0)=\gamma,\quad \xi_4(0)=\delta. \tag{1.6}$$

In addition, since we are interested in zero bicharacteristic strips, i.e. in strips along which the first integral $p = p(x,\xi)$ of the system (1.4), (1.5) vanishes, the real parameters α, β, γ and δ from (1.6) with regard for $x_3(0) = 0$ should obey the equality

$$\delta^2 - \alpha^2 - \gamma^2 = 0$$

or

$$\delta = \pm\sqrt{\alpha^2 + \gamma^2}. \tag{1.7}$$

Integrating the differential equations (1.4), (1.5) and taking into account the initial conditions (1.6), we obtain

$$\xi_1 = \alpha, \quad \xi_2 = \beta, \quad \xi_3 = \gamma + \beta^2\tau, \quad \xi_4 = \delta, \ \ \tau \ge 0, \tag{1.8}$$

$$\begin{gathered} x_1 = -2\alpha\tau, \quad x_2 = 2\gamma\beta\tau^2 + \frac{2}{3}\,\beta^3\tau^3, \\ x_3 = -2\gamma\tau - \beta^2\tau^2, \quad x_4 = 2\delta\tau, \ \ \tau \ge 0. \end{gathered} \tag{1.9}$$

The equalities (1.8) and (1.9) provide us with a parametric representation of the bicharacteristic strip of the equation (1.1) satisfying the initial conditions (1.6).

By (1.7) and (1.9), we have

$$4(\gamma\tau)^2 = x_4^2 - x_1^2, \quad (\beta\tau)^2 = -x_3 - 2\gamma\tau. \tag{1.10}$$

For $\gamma \ge 0$, since $\tau \ge 0$, from (1.10) we find that

$$2\gamma\tau = (x_4^2 - x_1^2)^{1/2}, \quad \beta\tau = \big(-x_3 - (x_4^2 - x_1^2)^{1/2}\big)^{1/2}\operatorname{sgn}\beta. \tag{1.11}$$

It follows from (1.10) and (1.11) that for $\gamma \ge 0$ the condition

$$x_3 + \sqrt{x_4^2 - x_1^2} \le 0 \tag{1.12}$$

must be fulfilled along the bicharacteristic strip.

By (1.9) and (1.11), we have

$$\begin{gathered} x_2 = \beta\tau\Big[2\gamma\tau + \frac{2}{3}\,(\beta\tau)^2\Big] = \big(-x_3^2 - (x_4^2 - x_1^2)^{1/2}\big)^{1/2}\times \\ \times\Big[(x_4^2 - x_1^2)^{1/2} + \frac{2}{3}\,\big(-x_3 - (x_4^2 - x_1^2)^{1/2}\big)^{1/2}\operatorname{sgn}\beta = \\ = \frac{1}{3}\,\big(-x_3^2 - (x_4^2 - x_1^2)^{1/2}\big)^{1/2}\big[(x_4^2 - x_1^2)^{1/2} - 2x_3\big]\operatorname{sgn}\beta. \end{gathered} \tag{1.13}$$

Squaring both parts of the equality (1.13), we obtain

$$x_2^2 = \frac{1}{9}\,\big[-x_3 - \sqrt{x_4^2 - x_1^2}\big]\,\big[\sqrt{x_4^2 - x_1^2} - 2x_3\big]^2. \tag{1.14}$$

Thus all bicharacteristics of the equation (1.1) coming out of the origin $O(0,0,0,0)$ for $\gamma \geq 0$ lie on the characteristic hypersurface which is described by the equations (1.14).

For $\gamma \leq 0$ we analogously obtain

$$2\gamma\tau = -(x_4^2 - x_1^2)^{1/2}, \quad \beta\tau = \big(- x_3 + (x_4^2 - x_1^2)^{1/2}\big)^{1/2} \operatorname{sgn} \beta, \tag{1.15}$$

and hence

$$\begin{aligned} x_2 = \beta\tau \Big[2\gamma\tau + \frac{2}{3}\,(\beta\tau)^2 \Big] = \big(- x_3^2 + (x_4^2 - x_1^2)^{1/2}\big)^{1/2} \times \\ \times \Big[- (x_4^2 - x_1^2)^{1/2} + \frac{2}{3}\,\big(- x_3 + (x_4^2 - x_1^2)^{1/2}\big)\Big] \operatorname{sgn} \beta = \\ = -\frac{1}{3}\,\big(- x_3^2 + (x_4^2 - x_1^2)^{1/2}\big)^{1/2} \big[(x_4^2 - x_1^2)^{1/2} + 2x_3\big] \operatorname{sgn} \beta. \end{aligned} \tag{1.16}$$

It follows from (1.15) that all bicharacteristics of the equation (1.1) coming out of the point O for $\gamma \leq 0$ lie on the characteristic hypersurface which is described by the equation

$$x_2^2 = \frac{1}{9}\,\big[\sqrt{x_4^2 - x_1^2} - x_3\big]\,\big[\sqrt{x_4^2 - x_1^2} + 2x_3\big]^2. \tag{1.17}$$

Obviously the equalities (1.16) and (1.17) make sense under the condition $x_4^2 \geq x_2^2 + x_3^2$.

Remark 1.1. In accordance with the above-considered cases, the characteristic conoid K_O with the vertex at the point O consists of two parts K_O^1 and K_O^2. K_O^1 is given by the equality (1.14) and lies, by (1.12), entirely in the half-space $x_3 \leq 0$, i.e. in the closed domain where the equation (1.1) is ultrahyperbolic, while K_O^2 is given by the equation (1.17) and lies in the closed domain $\big\{(x_1, x_2, x_3, t) \in R^4 :\ x_4^2 \geq x_1^2 + x_3^2\big\}$. Analogously we can show that the characteristic conoid K_A with the vertex at the point $A(0,0,0,t_0)$ consists of two parts K_A^1 and K_O^2 which are described, respectively, by the equations

$$\begin{aligned} K_A^1 : x_2^2 = \frac{1}{9}\,\big[- x_3 - \sqrt{(x_4 - t_0)^2 - x_1^2}\big]\,\big[\sqrt{(x_4 - t_0)^2 - x_1^2} - 2x_3\big]^2, \\ K_A^2 : x_2^2 = \frac{1}{9}\,\big[\sqrt{(x_4 - t_0)^2 - x_1^2} - x_3\big]\,\big[\sqrt{(x_4 - t_0)^2 - x_1^2} + 2x_3\big]^2, \end{aligned} \tag{1.18}$$

where, just as above, $x_4 = t$.

Let us now show that $K_O^+ = K_O \cap \big\{(x_1, x_2, x_3, t) \in R^4 :\ x_3 \geq 0,\ t \geq 0\big\}$ can be represented in the form

$$K_O^+ :\ t = g^+(x_1, x_2, x_3) \in C^\infty(\Omega) \cap C(\overline{\Omega}), \tag{1.19}$$

where $\Omega = \big\{(x_1, x_2, x_3) \in R^3 :\ x_3 > 0\big\}$.

Indeed, it is obvious that $K_O^+ \subset K_O^2$, and since this case corresponds to $\gamma \leq 0$, in the notation $z = \beta\tau$ from (1.9) and (1.15) we have

$$x_2 = -\sqrt{x_4^2 - x_1^2}\, z + \frac{2}{3}\, z^2, \quad x_3 = \sqrt{x_4^2 - x_1^2} - z^2,$$

whence

$$\sqrt{x_4^2 - x_1^2} = x_3 + z^2 \tag{1.20}$$

and $x_2 = -(x_2 + z^2)z + \frac{2}{3}\,z^3$ or

$$z^3 + 3x_3 z + 3x_2 = 0. \tag{1.21}$$

As far as for $p = 3x_3$, $q = 3x_2$ the discriminant

$$\Delta = -4p^3 - 27q^2 = -4 \cdot 27x_3^3 - 27 \cdot 9x_2^2 = -27(4x_3^3 + 9x_2^2)$$

of the cubic equation (1.21) is negative for $x_3 \geq 0$, this equation, as is known, has only one real root $z = z_0(x_2, x_3)$ for $x_3 \geq 0$, $|x_2| + |x_3| \neq 0$, which is given by the Cardano formula [83, p. 237]

$$z_0(x_2, x_3) = \sqrt[3]{-\frac{3}{2}\,x_2 + \sqrt{\frac{9}{4}\,x_2^2 + x_3^3}} + \sqrt[3]{-\frac{3}{2}\,x_2 - \sqrt{\frac{9}{4}\,x_2^2 + x_3^3}}. \tag{1.22}$$

By (1.20), (1.22) and the fact that the coordinate $t = x_4$ of the points of the manifold K_O^+ is nonnegative, we obtain

$$K_O^+ :\ t = \sqrt{x_1^2 + \big(x_3 + z_0^2(x_2, x_3)\big)^2}, \ \ x_3 \geq 0. \tag{1.23}$$

The relation (1.23) implies (1.19), since by (1.22)

$$z_0(x_2, x_3) \in C^\infty(\Omega_1) \cap C(\overline{\Omega}_1),$$

where $\Omega_1 = \big\{(x_2, x_3) \in R^2 :\ x_3 > 0\big\}$.

Similarly to (1.19) and (1.23), taking into account (1.18), the manifold $K_A^- \subset K_A^2$ can be represented as

$$K_A^- :\ t = g^-(x_1, x_2, x_3) \in C^\infty(\Omega) \cap C(\overline{\Omega}), \tag{1.24}$$

where

$$g^-(x_1, x_2, x_3) = t_0 - \sqrt{x_1^2 + \big(x_3 + z_0^2(x_2, x_3)\big)^2},$$

and the function $z_0(x_2, x_3)$ is given by the equality (1.22).

By the definition of the domain D we can see that

$$S_1 \subset K_O^+, \quad S_2 \subset K_A^-, \quad \partial D = S_0 \cup S_1 \cup S_2. \tag{1.25}$$

1.3. Self-conjugacy of the problems (1.1), (1.2) and (1.1), (1.3). Since the class E (E^*) of functions vanishing in some (its own for every function) neighborhood of S_0 is likewise dense in the space W_+ (W_+^*) [89, p. 81], the class E (E^*) will be assumed below to possess this property.

Let us show that for every $u \in E$ the inequality

$$\|Lu\|_{W_-^*} \leq c_1 \|u\|_{W_+} \tag{1.26}$$

holds, where the positive constant c_1 does not depend on u, $\|\cdot\|_{W_+} = \|\cdot\|_{W_+^*} = \|\cdot\|_1$.

Indeed, let $\nu = (\nu_1, \nu_2, \nu_3, \nu_0)$ be the unit vector of the outer normal to ∂D, i.e. $\nu_i = \cos(\widehat{n, x_i})$, $i = 1, 2, 3$, $\nu_0 = \cos(\widehat{n, t})$. Since the derivative with respect to the conormal $\frac{\partial}{\partial N} = \nu_0\,\frac{\partial}{\partial t} - \nu_1\,\frac{\partial}{\partial x_1} - x_3\nu_2\,\frac{\partial}{\partial x_2} - \nu_3\,\frac{\partial}{\partial x_3}$

corresponding to the operator L is an interior differential operator on the characteristic hypersurfaces of the equation (1.1), by (1.12) for the function $u \in E$ we have

$$\frac{\partial u}{\partial N}\Big|_{S_1} = 0. \tag{1.27}$$

By definition of the negative norm, the equalities (1.2), (1.27) for $u \in E$ and the equality (1.3) for $v \in E^* \subset W_+^*$, we have

$$\|Lu\|_{W^*} = \sup_{\widetilde{v}\in W_+^*} \|\widetilde{v}\|^{-1}_{W_+^*} (Lu, \widetilde{v})_{L_2(D)} = \sup_{v\in E^*} \|v\|^{-1}_{W_+^*} (Lu, v)_{L_2(D)} =$$

$$= \sup_{v\in E^*} \|v\|^{-1}_{W_+^*} \int\limits_{S_0\cup S_1\cup S_2} \frac{\partial u}{\partial N}\, v\, ds +$$

$$+ \sup_{v\in E^*} \|v\|^{-1}_{W_+^*} \int\limits_{D} \big[- u_t v_t + u_{x_1} v_{x_1} + x_3 u_{x_2} v_{x_2} + u_{x_3} v_{x_3} \big]\, dD =$$

$$= \sup_{v\in E^*} \|v\|^{-1}_{W_+^*} \int\limits_{D} \big[- u_t v_t + u_{x_1} v_{x_1} + x_3 u_{x_2} v_{x_2} + u_{x_3} v_{x_3} \big]\, dD. \tag{1.28}$$

Using the Cauchy and Schwartz inequalities, from (1.28) we have

$$\|Lu\|_{W^*} \le$$

$$\le \sup_{v\in E^*} \|v\|^{-1}_{W_+^*} \int\limits_{D} \Big[\big(u_t^2 + u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big)^{1/2} \big(v_t^2 + v_{x_1}^2 + x_3 v_{x_2}^2 + v_{x_3}^2\big)^{1/2} \Big]\, dD \le$$

$$\le \sup_{v\in E^*} \|v\|^{-1}_{W_+^*} \Big[\int\limits_{D} \big(u_t^2 + u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big)\, dD \Big]^{1/2} \times$$

$$\times \Big[\int\limits_{D} \big(v_t^2 + v_{x_1}^2 + x_3 v_{x_2}^2 + v_{x_3}^2\big)\, dD \Big]^{1/2} \le \sup_{v\in E^*} \|v\|^{-1}_{W_+^*} \|u\|_{W_+} \|v\|_{W_+^*} = \|u\|_{W_+},$$

which proves the inequality (1.26).

Analogously it can be proved that for every $v \in E^*$ the inequality

$$\|L^* v\|_{W_-} \le c_2 \|v\|_{W_-^*} \tag{1.29}$$

is valid with a positive constant c_2 independent of v.

Remark 1.2. By (1.26) ((1.29)), the operator $L : W_+ \to W_-^*$ ($L^* = L : W_+^* \to W_-$) with the dense domain of definition E (E^*) admits the closure which in fact is a continuous operator from the space W_+ (W_+^*) to the space W_-^* (W_-). Leaving for that closure the same notation L ($L^* = L$), we can say that it is defined on the whole Hilbert space W_+ (W_+^*).

Let us now show that the problems (1.1), (1.2) and (1.1), (1.3) are self-conjugate, i.e. the inequality

$$(Lu, v) = (u, Lv) \quad \forall\, u \in W_+, \ \ \forall\, v \in W_+^* \tag{1.30}$$

holds.

Indeed, by Remark 1.2 it suffices to prove the equality (1.30) when $u \in E$ and $v \in E^*$. We have

$$(Lu, v) = (Lu, v)_{L_2(D)} = \int\limits_{\partial D} \Big[v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \Big]\, ds + (u, Lv)_{L_2(D)}. \tag{1.31}$$

Since the equality (1.3) is valid for $v \in E^*$, analogously to (1.27) we have

$$\frac{\partial v}{\partial N}\Big|_{S_2} = 0. \tag{1.32}$$

From (1.31), owing to (1.2), (1.37) as well as (1.3), (1.32) for v, immediately follows (1.30).

1.4. A priori estimates. The existence and uniqueness theorem. We have the following

Lemma 1.1. *For any function $u \in W_+$ the estimate*

$$c\|u\|_{L_2(D)} \le \|Lu\|_{W_-^*} \tag{1.33}$$

is valid with a positive constant c independent of u.

Proof. By Remark 1.2, it suffices to prove the estimate (1.33) for $u \in E$. In this case, since the function u vanishes in some neighborhood $S_0 \subset \partial D$, as it can be easily verified the function

$$v(x,t) = \int\limits_t^{g^-(x)} e^{-\lambda\tau} u(x,\tau)\, d\tau, \quad \lambda = \text{const} > 0, \quad x = (x_1, x_2, x_3),$$

where $t = g^-(x)$ represents by virtue of (1.24), (1.25) the equation of the characteristic hypersurface S_2, belongs to the space E^*, and the equalities

$$v_t(x,t) = -e^{-\lambda t} u(x,t), \quad u_t(x,t) = -e^{\lambda t} v_t(x,t) \tag{1.34}$$

are valid.

By (1.2), (1.3), (1.27), (1.32) and (1.34) we have

$$(Lu, v)_{L_2(D)} =$$

$$= \int\limits_{\partial D} v \frac{\partial u}{\partial N}\, ds + \int\limits_D \big[-u_t v_t + u_{x_1} v_{x_1} + x_3 u_{x_2} v_{x_2} + u_{x_3} v_{x_3} \big]\, dD =$$

$$= \int\limits_D \big[-u_t v_t + u_{x_1} v_{x_1} + x_3 u_{x_2} v_{x_2} + u_{x_3} v_{x_3} \big]\, dD =$$

$$= \int\limits_D e^{-\lambda t} u_t u\, dD + \int\limits_D e^{-\lambda t} \big[-u_{x_1 t} v_{x_1} - x_3 v_{x_2 t} v_{x_2} - v_{x_3 t} v_{x_3} \big]\, dD, \tag{1.35}$$

$$\int\limits_D e^{-\lambda t} u_t u\, dD = \frac{1}{2} \int\limits_{\partial D} e^{-\lambda t} u^2 \nu_0\, ds + \frac{1}{2} \int\limits_D e^{-\lambda t} \lambda u^2\, dD =$$

$$=\frac{1}{2}\int\limits_{S_2} e^{-\lambda t}u^2\nu_0\,ds+\frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda v_t^2\,dD=\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds+\frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda v_t^2\,dD,\quad(1.36)$$

$$\int\limits_{D} e^{\lambda t}\big[-v_{x_1t}v_{x_1}-x_3v_{x_2t}v_{x_2}-v_{x_3t}v_{x_3}\big]\,dD=$$

$$=-\frac{1}{2}\int\limits_{\partial D} e^{\lambda t}\big[v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\nu_0\,ds+\frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda\big[v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\,dD.\quad(1.37)$$

As far as $v|_{S_2}=0$, the gradient $\nabla v=(v_{x_1},v_{x_2},v_{x_3},v_t)$ is proportional to the vector of the outer normal ν on S_2, i.e. for some α we have $v_{x_1}=\alpha\nu_1$, $v_{x_2}=\alpha\nu_2$, $v_{x_3}=\alpha\nu_3$, $v_t=\alpha\nu_0$ on S_2. Thus taking into account that S_2 is a characteristic surface, we get

$$\big(v_t^2-v_{x_1}^2-x_3v_{x_2}^2-v_{x_3}^2\big)\Big|_{S_2}=\alpha^2\big(\nu_0^2-\nu_1^2-x_3\nu_2^2-\nu_3^2\big)\Big|_{S_2}=0.\quad(1.38)$$

It is easy to see that

$$\nu_0\big|_{S_0}=0,\quad \nu_0\big|_{S_1\setminus 0}<0,\quad \nu_0\big|_{S_2\setminus 0}>0.\quad(1.39)$$

By virtue of (1.3),(1.38), (1.39) we arrive at

$$\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds-\frac{1}{2}\int\limits_{D} e^{\lambda t}\big[v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\nu_0\,ds=$$

$$=\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds-\frac{1}{2}\int\limits_{S_1} e^{\lambda t}\big[v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\nu_0\,ds-$$

$$-\frac{1}{2}\int\limits_{S_2} e^{\lambda t}\big[v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\nu_0\,ds\geq$$

$$\geq\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds-\frac{1}{2}\int\limits_{S_2} e^{\lambda t}\big[v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\nu_0\,ds=$$

$$=\frac{1}{2}\int\limits_{S_2} e^{\lambda t}\big[v_t^2-v_{x_1}^2-x_3v_{x_2}^2-v_{x_3}^2\big]\nu_0\,ds=0.\quad(1.40)$$

Taking now into account (1.34), (1.36), (1.37) and (1.40), from (1.35) for the fixed $\lambda>0$ we obtain

$$(Lu,v)_{L_2(D)}=\frac{1}{2}\int\limits_{S_2} e^{\lambda t}v_t^2\nu_0\,ds+\frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda v_t^2\,dD-$$

$$-\frac{1}{2}\int\limits_{\partial D} e^{\lambda t}\big[v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\nu_0\,ds+\frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda\big[v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\,dD\geq$$

$$\geq\frac{\lambda}{2}\int\limits_{D} e^{\lambda t}\big[v_t^2+v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big]\,dD\geq$$

$$\geq \frac{\lambda}{2}\Big[\int\limits_D e^{\lambda t} v_t^2\,dD\Big]^{1/2}\Big[\int\limits_D e^{\lambda t}\big(v_t^2+v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big)\,dD\Big]^{1/2} =$$

$$= \frac{\lambda}{2}\Big[\int\limits_D e^{-\lambda t} u^2\,dD\Big]^{1/2}\Big[\int\limits_D e^{\lambda t}\big(v_t^2+v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big)\,dD\Big]^{1/2} \geq$$

$$= \frac{\lambda}{2}\,e^{-\frac{1}{2}\lambda t_0}\Big[\int\limits_D u^2\,dD\Big]^{1/2}\Big[\int\limits_D \big(v_t^2+v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big)\,dD\Big]^{1/2} \tag{1.41}$$

because $t|_D \geq 0$ and $e^{-\frac{1}{2}\lambda t_0} = \big(\inf\limits_D e^{-\lambda t}\big)^{1/2} > 0$.

Since $v|_{S_2} = 0$ $(u|_{S_1} = 0)$, standard reasoning allows one to prove that the inequalities

$$\int\limits_D v^2\,dD \leq c_0 \int\limits_D v_t^2\,dD \quad \Big(\int\limits_D u^2\,dD \leq c_0 \int\limits_D u_t^2\,dD\Big)$$

are valid for some $c_0 = \text{const} > 0$ independent of $v \in E^*$ $(u \in E)$. This implies that in the space W_+ (W_+^*) the norm

$$\|v\|^2_{W_+\ (W_+^*)} = \int\limits_D \big(v^2+v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2+v_t^2\big)\,dD$$

is equivalent to the norm

$$\|v\|^2 = \int\limits_D \big(v_t^2+v_{x_1}^2+x_3v_{x_2}^2+v_{x_3}^2\big)\,dD. \tag{1.42}$$

Therefore leaving for the norm (1.42) the same notation $\|v\|_{W_+\ (W_+^*)}$, from (1.41) we obtain

$$(Lu,v)_{L_2(D)} \geq \mu \|u\|_{L_2(D)} \|v\|_{W_+^*}, \tag{1.43}$$

where $\mu = \frac{\lambda}{2}\,e^{-\frac{1}{2}\lambda t_0} > 0$. Thus we can easily verify that $\sup\limits_{\lambda>0}\mu(\lambda) = \mu\big(\frac{2}{t_0}\big) = (e\,t_0)^{-1}$.

Applying the generalized Schwartz inequality

$$(Lu,v) \leq \|Lu\|_{W_-^*}\|v\|_{W_+^*}$$

to the left-hand side of (1.43), after reduction by $\|v\|_{W_+^*}$ we obtain the inequality (1.33) with $c = \mu > 0$. Thus the lemma is proved. □

Lemma 1.2. *For every function $v \in W_+^*$ the estimate*

$$c\|v\|_{L_2(D)} \leq \|Lv\|_{W_-} \tag{1.44}$$

is valid with a positive constant c independent of v.

Proof. Just as in the case of Lemma 1.1, by Remark 1.2 it is sufficient to prove that the inequality (1.44) is valid for $v \in E^*$. Let $v \in E^*$. We introduce into consideration the function

$$u(x,t) = \int\limits_{g^+(x)}^{t} e^{\lambda\tau} v(x,\tau)\,d\tau, \quad \lambda = \text{const} > 0,$$

where $t = g^+(x)$ is by virtue of (1.19), (1.25) the equation of the characteristic hypersurface S_1. The function $u(x,t)$ belongs to the space E, and the equality

$$u_t(x,t) = e^{\lambda t} v(x,t), \quad v(x,t) = e^{-\lambda t} u_t(x,t) \tag{1.45}$$

holds.

By virtue of (1.2), (1.3), (1.27), (1.32) and (1.45), analogously to (1.35)–(1.40) we have

$$(Lv,u)_{L_2(D)} = -\int\limits_D e^{\lambda t} v_t v\,dD + \\ + \int\limits_D e^{-\lambda t}\big[u_{x_1 t} u_{x_1} + x_3 u_{x_2 t} u_{x_2} + u_{x_3 t} u_{x_3}\big]\,dD, \tag{1.46}$$

$$-\int\limits_D e^{\lambda t} v_t v\,dD = -\frac{1}{2}\int\limits_D e^{\lambda t} v^2 \nu_0\,ds + \frac{1}{2}\int\limits_D e^{\lambda t}\lambda v^2\,dD = \\ = -\frac{1}{2}\int\limits_{S_1} e^{\lambda t} v^2 \nu_0\,ds + \frac{1}{2}\int\limits_D e^{-\lambda t}\lambda u_t^2\,dD = \\ = -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0\,ds + \frac{1}{2}\int\limits_D e^{-\lambda t}\lambda u_t^2\,dD, \tag{1.47}$$

$$\int\limits_D e^{-\lambda t}\big[u_{x_1 t} u_{x_1} + x_3 u_{x_2 t} u_{x_2} + u_{x_3 t} u_{x_3}\big]\,dD = \\ = \frac{1}{2}\int\limits_{\partial D} e^{-\lambda t}\big[u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big]\nu_0\,ds + \frac{1}{2}\int\limits_D e^{-\lambda t}\lambda\big[u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big]\,dD, \tag{1.48}$$

$$\big(u_t^2 - u_{x_1}^2 - x_3 u_{x_2}^2 - u_{x_3}^2\big)\Big|_{S_1} = 0, \tag{1.49}$$

$$-\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0\,ds + \frac{1}{2}\int\limits_{\partial D} e^{-\lambda t}\big[u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big]\nu_0\,ds = \\ = -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0\,ds + \frac{1}{2}\int\limits_{S_1} e^{-\lambda t}\big[u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big]\nu_0\,ds + \\ + \frac{1}{2}\int\limits_{S_2} e^{-\lambda t}\big[u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big]\nu_0\,ds \geq$$

$$\geq -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t}\big[u_t^2 - u_{x_1}^2 - x_3 u_{x_2}^2 - u_{x_3}^2\big]\nu_0\, ds = 0. \tag{1.50}$$

By (1.45) and (1.47)–(1.50), from (1.46) it follows that

$$(Lv,u)_{L_2(D)} \geq \frac{\lambda}{2}\int\limits_D e^{-\lambda t}\big[u_t^2 + u_{x_1}^2 + x_3 u_{x_2 t}^2 + u_{x_3}^2\big]\, dD \geq$$

$$\geq \frac{\lambda}{2}\Big[\int\limits_D e^{-\lambda t} u_t^2\, dD\Big]^{1/2}\Big[\int\limits_D e^{-\lambda t}\big(u_t^2 + u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big)\, dD\Big]^{1/2} =$$

$$= \frac{\lambda}{2}\Big[\int\limits_D e^{\lambda t} v^2\, dD\Big]^{1/2}\Big[\int\limits_D e^{-\lambda t}\big(u_t^2 + u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big)\, dD\Big]^{1/2} \geq$$

$$\geq \frac{\lambda}{2}\, e^{-\frac{1}{2}\lambda t_0}\Big[\int\limits_D v^2\, dD\Big]^{1/2}\Big[\int\limits_D \big(u_t^2 + u_{x_1}^2 + x_3 u_{x_2}^2 + u_{x_3}^2\big)\, dD\Big]^{1/2},$$

whence just in the same way as in obtaining the inequality (1.33) from (1.44) in Lemma 1.1, we arrive at the inequality (1.44). Thus the lemma is proved. □

Due to the results of [86, pp. 184–186], the consequence of the inequalities (1.26) and (1.29), the equality (1.30) and Lemmas 1.1 and 1.2 is the following

Theorem 1.1. *For every $F \in L_2(D)$ (W_-^*) there exists a unique strong generalized solution u of the problem* (1.1), (1.2) *of the class W_+ (L_2) for which the estimate*

$$\|u\|_{L_2(D)} \leq c_0 \|F\|_{W_-^*}$$

is valid with a positive constant c_0 independent of F.

2. The Characteristic Cauchy problem for Some Degenerating Second Order Hyperbolic Equations in Cone-Shaped Domains

2.1. The case of equation with noncharacteristic degeneration. In the space of variables x_1, x_2, t we consider a degenerating second order hyperbolic equation of the type

$$Lu \equiv u_{tt} - t^m(u_{x_1x_1} + u_{x_2x_2}) + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \tag{2.1}$$

where a_i, $i = 1,\dots,4$, F are given and u is an unknown real function, $m = \text{const} > 0$.

By

$$D:\ 0 < t < \Big[1 - \frac{2+m}{2}\, r\Big]^{\frac{2}{2+m}},\ \ r = (x_1^2 + x_2^2)^{1/2} < \frac{2}{2+m}$$

we denote the bounded domain lying in a half-space $t > 0$, bounded above by the characteristic conoid

$$S: \ t = \Big[1 - \frac{2+m}{2}\, r\Big]^{\frac{2}{2+m}}, \ \ r \leq \frac{2}{2+m}$$

of the equation (2.1) with vertex at the point $(0, 0, 1)$ and below by the basis

$$S_0: \ t = 0, \ \ r \leq \frac{2}{2+m}$$

of the conoid on which the equation (2.1) has a noncharacteristic degeneration. Below in the domain D the coefficients a_i, $i = 1, \dots, 4$, of the equation (2.1) will be assumed to be functions of the class $C^2(\overline{D})$.

For the equation (2.1) we consider the characteristic problem which is formulated as follows: in the domain D, find a solution $u(x_1, x_2, t)$ of the equation (2.1) satisfying the boundary condition

$$u\big|_S = 0. \tag{2.2}$$

As it will be shown, the Cauchy problem on finding a solution of the equation

$$L^* v \equiv v_{tt} - t^m (v_{x_1x_1} + v_{x_2x_2}) - (a_1 v)_{x_1} - (a_2 v)_{x_2} - (a_3 v)_t + a_4 v = F \tag{2.3}$$

in the domain D by the boundary conditions

$$v\big|_{S_0} = 0, \quad v_t\big|_{S_0} = 0 \tag{2.4}$$

is the problem conjugate to the problem (2.1), (2.2), where L^* is the operator formally conjugate to L.

Here by E and E^* we denote the classes of functions from the Sobolev space $W_2^2(D)$ which satisfy respectively the boundary condition (2.2) or (2.4) and vanish in some (its own for every function) three-dimensional neighborhood of the circumference $\Gamma = S \cap S_0: \ r = \frac{2}{2+m}$, $t = 0$ and of the segment $I: x_1 = x_2 = 0$, $0 \leq t \leq 1$.

Let W_+ (W_+^*) be the Hilbert space with weight obtained by closing the space E (E^*) by the norm

$$\|u\|_1^2 = \int\limits_D \big[u_t^2 + t^m (u_{x_1}^2 + u_{x_2}^2) + u^2\big]\, dD.$$

Denote by W_- (W_-^*) the space with the negative norm constructed with respect to $L_2(D)$ and W_+ (W_+^*) [6].

Let $\nu = (\nu_1, \nu_2, \nu_3)$ be the unit vector of the outer normal to ∂D, i.e. $\nu_1 = \cos(\widehat{\nu, x_1})$, $\nu_2 = \cos(\widehat{\nu, x_2})$, $\nu_0 = \cos(\widehat{\nu, t})$. By definition, the derivative with respect to the conormal on the boundary ∂D of D for the operator L is calculated by the formula

$$\frac{\partial}{\partial N} = \nu_0 \, \frac{\partial}{\partial t} - t^m \nu_1 \frac{\partial}{\partial x_1} - t^m \nu_2 \, \frac{\partial}{\partial x_2} \, .$$

Remark 2.1. Since the derivative with respect to the conormal $\frac{\partial}{\partial N}$ for the operator L is an inner differential operator on the characteristic surfaces of the equation (2.1), by virtue of (2.2) and (2.4) for the functions $u \in E$ and $v \in E^*$ we have

$$\frac{\partial u}{\partial N}\Big|_S = 0, \quad \frac{\partial v}{\partial N}\Big|_{S_0} = 0. \tag{2.5}$$

In the equation (2.1) we impose on the coefficients a_1 and a_2 the following restrictions:

$$M_i = \sup_{\overline{D}} \left| t^{-\frac{m}{2}} a_i(x_1, x_2, t) \right| < +\infty, \quad i = 1, 2. \tag{2.6}$$

Lemma 2.1. *For all functions $u \in E$, $v \in E^*$ the inequalities*

$$\|Lu\|_{W_-^*} \leq c_1 \|u\|_{W_+}, \tag{2.7}$$

$$\|L^* v\|_{W_-} \leq c_2 \|v\|_{W_+^*} \tag{2.8}$$

hold, where the positive constants c_1 and c_2 do not depend respectively on u and v, $\|\cdot\|_{W_+} = \|\cdot\|_{W_+^*} = \|\cdot\|_1$.

Proof. By definition of negative norm, for $u \in E$, by virtue of the equalities (2.2), (2.4) and (2.5), using the integration by parts we obtain

$$\|Lu\|_{W_-^*} = \sup_{v \in W_+^*} \|v\|_{W_+^*}^{-1} (Lu, v)_{L_2(D)} = \sup_{v \in E^*} \|v\|_{W_+^*}^{-1} (Lu, v)_{L_2(D)} =$$

$$= \sup_{v \in E^*} \|v\|_{W_+^*}^{-1} \int\limits_D \Big[-u_t v_t + t^m (u_{x_1} v_{x_1} + u_{x_2} v_{x_2}) +$$

$$+ a_1 u_{x_1} v + a_2 u_{x_2} v + a_3 u_t v + a_4 uv \Big] \, dD. \tag{2.9}$$

From (2.6) and the Cauchy and Schwartz inequalities it follows that

$$\left| \int\limits_D \big[-u_t v_t + t^m (u_{x_1} v_{x_1} + u_{x_2} v_{x_2}) \big] \, dD \right| \leq \|u\|_{W_+} \|v\|_{W_+^*}, \tag{2.10}$$

$$\left| \int\limits_D \big[a_1 u_{x_1} v + a_2 u_{x_2} v + a_3 u_t v + a_4 uv \big] \, dD \right| \leq \widetilde{c} \|u\|_{W_+} \|v\|_{W_+^*}, \tag{2.11}$$

where $\widetilde{c} = \sum\limits_{=1}^{2} \big(M_i + \sup\limits_{\overline{D}} |a_{2+i}| \big)$, and from (2.9)–(2.11) it follows

$$\|Lu\|_{W_-^*} \leq (1 + \widetilde{c}) \sup_{v \in E^*} \|v\|_{W_+^*}^{-1} \|u\|_{W_+} \|v\|_{W_+^*} = (1 + \widetilde{c}) \|u\|_{W_+},$$

i.e. we have obtained the inequality (2.7) for $c_1 = 1 + \widetilde{c}$. Since the proof of the inequality (2.8) repeats word for word that of the inequality (2.7), we can conclude that Lemma 2.1 is proved completely. □

Remark 2.2. By the inequality (2.7) ((2.8)), the operator $L : W_+ \to W_-^*$ ($L^* : W_+^* \to W_-$) with the dense domain of definition E (E^*) admits a closure which itself is a continuous operator from the space W_+ (W_+^*) to the space W_-^* (W_-). Leaving for that closure the same notation L (L^*), we can say that it is defined on the entire Hilbert space W_+ (W_+^*).

Lemma 2.2. *The problems* (2.1), (2.2) *and* (2.3), (2.4) *are self-conjugate, i.e. for every* $u \in W_+$ *and* $v \in W_+^*$ *the equality*

$$(Lu, v) = (u, L^*v) \tag{2.12}$$

holds.

Proof. By Remark 2.2 it is sufficient to prove the equality (2.12) in the case where $u \in E$ and $v \in E^*$. In this case it is evident that $(Lu, v) = (Lu, v)_{L_2(D)}$. Therefore the integration by parts yields

$$(Lu, v) = \\ = \int\limits_{\partial D} \Big[\Big(v\,\frac{\partial u}{\partial N} - u\,\frac{\partial v}{\partial N}\Big) + (a_1\nu_1 + a_2\nu_2 + a_3\nu_0)uv\Big]\,ds + (u, L^*v)_{L_2(D)}. \tag{2.13}$$

From the equalities (2.2), (2.4), (2.5) and (2.13) immediately follows (2.12). □

Consider the following conditions:

$$\Omega\big|_S \le 0, \quad \big[t\Omega_t - (\lambda t + m)\Omega\big]\Big|_D \ge 0, \tag{2.14}$$

where the second inequality holds if λ is sufficiently large, and $\Omega = a_{1x_1} + a_{2x_2} + a_{3t} - a_4$.

Remark 2.3. It is easy to see that the inequalities (2.14) are a consequence of the condition

$$\Omega|_{\overline{D}} \le \text{const} < 0.$$

Lemma 2.3. *Let the conditions* (2.6) *and* (2.14) *be fulfilled. Then for every* $u \in W_+$ *the inequality*

$$c\big\|t^{\frac{1}{2}(m-1)}u\big\|_{L_2(D)} \le \|Lu\|_{W_-^*} \tag{2.15}$$

is valid with a positive constant c *independent of* u.

Proof. By Remark 2.2, it suffices to prove the inequality (2.15) for $u \in E$. If $u \in E$, then for $\alpha = \text{const} > 0$ and $\lambda = \text{const} > 0$ the function

$$v(x_1, x_2, t) = \int\limits_0^t e^{\lambda\tau}\tau^{\alpha}u(x_1, x_2, \tau)\,d\tau \tag{2.16}$$

belongs to the space E^*. We can easily verify that for $\alpha \ge 1$ the function v belongs to E^*, and for $0 < \alpha < 1$ this statement follows from the well-known

Hardy's inequality [118, p. 405]

$$\int_0^1 t^{-2} g^2(t)\,dt \le 4\int_0^1 f^2(t)\,dt,$$

where $f \in L_2(0,1)$ and $g(t) = \int\limits_0^t f(\tau)\,d\tau$.

By virtue of (2.16), the equalities

$$v_t(x_1,x_2,t) = e^{\lambda t} t^{\alpha} u(x_1,x_2,t), \quad u(x_1,x_2,t) = e^{-\lambda t} t^{-\alpha} v_t(x_1,x_2,t) \tag{2.17}$$

are valid.

Taking into account (2.2), (2.4), (2.5) and (2.17), we have

$$\begin{gathered}
(Lu,v)_{L_2(D)} = \int\limits_{\partial D} \Big[v\,\frac{\partial u}{\partial N} + (a_1\nu_1 + a_2\nu_2 + a_3\nu_0)uv \Big]\,ds + \\
+ \int\limits_D \Big[-u_t v_t + t^m u_{x_1} v_{x_1} + t^m u_{x_2} v_{x_2} - u(a_1 v)_{x_1} - u(a_2 v)_{x_2} - u(a_3 v)_t + a_4 uv \Big]\,dD = \\
= -\int\limits_D e^{\lambda t} t^{\alpha} u u_t\,dD + \int\limits_D e^{-\lambda t} t^{-\alpha} \Big[t^m (v_{x_1 t} v_{x_1} + v_{x_2 t} v_{x_2}) - \\
-(a_1 v_{x_1} + a_2 v_{x_2}) v_t - (a_{1x_1 t} + a_{2x_2} + a_{3t} - a_4) v_t v - a_3 v_t^2 \Big]\,dD,
\end{gathered} \tag{2.18}$$

and by (2.2) we find that

$$\begin{gathered}
-\int\limits_D e^{\lambda t} t^{\alpha} u u_t\,dD = -\frac{1}{2}\int\limits_D e^{\lambda t} t^{\alpha} (u^2)_t\,dD = -\frac{1}{2}\int\limits_{\partial D} e^{\lambda t} t^{\alpha} u^2 \nu_0\,ds + \\
+ \frac{1}{2}\int\limits_D e^{\lambda t} (\alpha t^{\alpha-1} + \lambda t^{\alpha}) u^2\,dD = \frac{1}{2}\int\limits_D e^{\lambda t} (\alpha t^{\alpha-1} + \lambda t^{\alpha}) u^2\,dD = \\
= \frac{\alpha}{2}\int\limits_D e^{\lambda t} t^{\alpha-1} u^2\,dD + \frac{1}{2}\int\limits_D \lambda e^{-\lambda t} t^{-\alpha} v_t^2 dD,
\end{gathered} \tag{2.19}$$

$$\begin{gathered}
\int\limits_D e^{-\lambda t} t^{m-\alpha} (v_{x_1 t} v_{x_1} + v_{x_2 t} v_{x_2})\,dD = \frac{1}{2}\int\limits_{\partial D} e^{-\lambda t} t^{m-\alpha} (v_{x_1}^2 + v_{x_2}^2)\nu_0\,ds + \\
+ \frac{1}{2}\int\limits_D e^{-\lambda t} \big[\lambda t^{m-\alpha} + (\alpha - m) t^{m-\alpha-1} \big] (v_{x_1}^2 + v_{x_2}^2)\,dD \ge \\
\ge \frac{1}{2}\int\limits_D e^{-\lambda t} \big[\lambda t^{m-\alpha} + (\alpha - m) t^{m-\alpha-1} \big] (v_{x_1}^2 + v_{x_2}^2)\,dD.
\end{gathered} \tag{2.20}$$

In deducing the inequality (2.20) we took into account that

$$\nu_0\big|_S \ge 0, \quad (v_{x_1}^2 + v_{x_2}^2)\big|_{S_0} = 0.$$

From (2.19) we have

$$-\int\limits_D e^{\lambda t}t^{\alpha}uu_t\,dD \geq \frac{\alpha}{2}\,\big\|t^{\frac{1}{2}\,(\alpha-1)}u\big\|^2_{L_2(D)} + \frac{1}{2}\int\limits_D \lambda e^{-\lambda t}t^{-\alpha}v_t^2\,dD. \tag{2.21}$$

Below it will be assumed that $\alpha = m$.

By (2.6) we have

$$\begin{gathered}\Big|\int\limits_D e^{-\lambda t}t^{-m}(a_1v_{x_1}+a_2v_{x_2})v_t\,dD\Big| \leq \\ \leq M\int\limits_D e^{-\lambda t}t^{-m}\Big[v_t^2+\frac{1}{2}\,t^m(v_{x_1}^2+v_{x_2}^2)\Big]\,dD \leq \\ \leq M\int\limits_D e^{-\lambda t}t^{-m}v_t^2\,dD + \frac{M}{2}\int\limits_D e^{-\lambda t}(v_{x_1}^2+v_{x_2}^2)\,dD,\end{gathered} \tag{2.22}$$

where $M = \max(M_1, M_2)$.

Since $\nu_0|_S \geq 0$, taking into account the conditions (2.4) and (2.14) and integrating by parts, we obtain

$$\begin{gathered}-\int\limits_D e^{-\lambda t}t^{-m}(a_{1x_1}+a_{2x_2}+a_{3t}-a_4)v_tv\,dD = \\ = -\frac{1}{2}\int\limits_D e^{-\lambda t}t^{-m}\Omega(v^2)_t\,dD = -\frac{1}{2}\int\limits_{\partial D} e^{-\lambda t}t^{-m}\Omega v^2\nu_0\,ds + \\ +\frac{1}{2}\int\limits_D e^{-\lambda t}t^{-m-1}\big[t\Omega_t-(\lambda t+m)\Omega\big]v^2\,dD \geq 0.\end{gathered} \tag{2.23}$$

In deducing the inequality (2.23) we used the fact that the function $t^{-m}v^2$ has the zero trace on S_0, i.e. $t^{-m}v^2|_{S_0} = 0$.

From (2.18) and (2.20)–(2.23) we have

$$\begin{gathered}(Lu,v)_{L_2(D)} \geq \frac{m}{2}\,\|t^{\frac{1}{2}\,(m-1)}u\|^2_{L_2(D)} + \frac{1}{2}\int\limits_D \lambda e^{-\lambda t}t^{-m}v_t^2\,dD + \\ +\frac{1}{2}\int\limits_D \lambda e^{-\lambda t}(v_{x_1}^2+v_{x_2}^2)\,dD - M\int\limits_D e^{-\lambda t}t^{-m}v_t^2\,dD - \\ -\frac{M}{2}\int\limits_D e^{-\lambda t}(v_{x_1}^2+v_{x_2}^2)\,dD - \sup_{\overline{D}}|a_3|\int\limits_D e^{-\lambda t}t^{-m}v_t^2\,dD = \\ = \frac{m}{2}\,\|t^{\frac{1}{2}\,(m-1)}u\|^2_{L_2(D)} + \Big(\frac{\lambda}{2}-M-\sup_{\overline{D}}|a_3|\Big)\int\limits_D e^{-\lambda t}t^{-m}v_t^2\,dD + \\ +\frac{1}{2}\,(\lambda-M)\int\limits_D e^{-\lambda t}(v_{x_1}^2+v_{x_2}^2)\,dD \geq\end{gathered}$$

$$\geq \frac{m}{2}\,\|t^{\frac{1}{2}\,(m-1)}u\|^2_{L_2(D)}+\sigma\int\limits_D e^{-\lambda t}(v_t^2+v_{x_1}^2+v_{x_2}^2)\,dD\geq$$

$$\geq\sqrt{2m\sigma\inf_{\overline{D}} e^{-\lambda t}}\,\|t^{\frac{1}{2}\,(m-1)}u\|_{L_2(D)}\bigg(\int\limits_D\big[v_t^2+t^m(v_{x_1}^2+v_{x_2}^2)\big]\,dD\bigg)^{1/2}, \quad (2.24)$$

where $\sigma=\big[\frac{\lambda}{2}-M-\sup\limits_{\overline{D}}|a_3|\big]>0$ for sufficiently large λ, and $\inf\limits_{\overline{D}} e^{-\lambda t}=e^{-\lambda}>0$. In deducing the inequality (2.24) we took into account that $t^{-m}|_D\geq 1$.

If $u\in W_+$ (W_+^*), then due to the fact that $u|_S=0$ $(u|_{S_0}=0)$ it is not difficult to prove the inequality

$$\int\limits_D u^2\,dD\leq c_0\int\limits_D u_t^2\,dD$$

for some $c_0=\text{const}>0$ independent of u. This implies that in the space W_+ (W_+^*) the norm

$$\|u\|^2_{W_+\,(W_+^*)}=\int\limits_D\big[u_t^2+t^m(u_{x_1}^2+u_{x_2}^2)+u^2\big]\,dD$$

is equivalent to the norm

$$\|u\|^2=\int\limits_D\big[u_t^2+t^m(u_{x_1}^2+u_{x_2}^2)\big]\,dD. \quad (2.25)$$

Therefore leaving for the latter the same notation $\|u\|_{W_+\,(W_+^*)}$, from (2.24) we obtain

$$(Lu,v)_{L_2(D)}\geq\sqrt{2m\sigma e^{-\lambda}}\,\|t^{\frac{1}{2}\,(m-1)}u\|_{L_2(D)}\|v\|_{W_+^*}. \quad (2.26)$$

Applying now the generalized Schwartz inequality

$$(Lu,v)\leq\|Lu\|_{W_-^*}\|v\|_{W_+^*}$$

to the left-hand side of (2.26), after reduction by $\|v\|_{W_+^*}$ we obtain the inequality (2.15) with $c=\sqrt{2m\sigma e^{-\lambda}}$. Thus the proof of Lemma 2.3 is complete. □

Consider the conditions

$$a_4\big|_{S_0}\geq 0,\quad (\lambda a_4+a_{4t})\big|_D\geq 0, \quad (2.27)$$

where the second inequality holds for sufficiently large λ.

Lemma 2.4. *Let the conditions* (2.6) *and* (2.27) *be fulfilled. Then for every $v\in W_+^*$ the inequality*

$$c\|v\|_{L_2(D)}\leq\|L^*v\|_{W_-} \quad (2.28)$$

is valid for some $c=\text{const}>0$ independent of $v\in W_+^*$.

Proof. Just as in the case of Lemma 2.3, by Remark 2.2 it is sufficient to prove that the inequality (2.28) is valid for $v \in E^*$. Let $v \in E^*$, and introduce into consideration the function

$$u(x_1, x_2, t) = \int\limits_t^{\varphi(x_1,x_2)} e^{-\lambda\tau} v(x_1, x_2, \tau)\, d\tau, \quad \lambda = \text{const} > 0, \tag{2.29}$$

where $t = \varphi(x_1, x_2)$ is the equation of the characteristic conoid S. Although the function

$$\varphi(x_1, x_2) = \Big[1 - \frac{2+m}{2}\, r\Big]^{\frac{2}{2+m}}$$

has singularities on the circumference $r = \frac{2}{2+m}$ and at the origin $x_1 = x_2 = 0$, since by the definition of the space E^* the function v vanishes in some neighborhood of the circumference $\Gamma = S \cap S_0$ and the segment $I:\ x_1 = x_2 = 0,\ 0 \le t \le 1$, the function u defined by the equality (2.29) belongs to the space E. In addition, it is obvious that the equalities

$$u_t(x_1, x_2, t) = -e^{-\lambda t} v(x_1, x_2, t), \quad v(x_1, x_2, t) = -e^{-\lambda t} u_t(x_1, x_2, t) \tag{2.30}$$

hold.

By (2.6), (2.27), (2.29) and (2.30), the same reasoning as in proving the estimate (2.28) allows us to prove the estimate (2.15) in Lemma 2.3. □

Denote by $L_{2,\alpha}(D)$ the space of functions u such that $t^\alpha u \in L_2(D)$. Assume

$$\|u\|_{L_{2,\alpha}(D)} = \|t^\alpha u\|_{L_2(D)}, \quad \alpha_m = \frac{1}{2}\,(m-1).$$

Definition 2.1. If $F \in L_2(D)$ (W^*_-), then the function u is said to be a strong generalized solution of the problem (2.1), (2.2) of the class W_+ (L_{2,α_m}) if $u \in W_+$ $(L_{2,\alpha_m}(D))$ and there exists a sequence of functions $u_n \in E$ such that $u_n \to u$ in the space W_+ $(L_{2,\alpha_m}(D))$ and $Lu_n \to F$ in the space W^*_- (W^*_-).

By the results of [86, p. 184–186], a consequence of Lemmas 2.1–2.4 is the following

Theorem 2.1. *Let the conditions* (2.6), (2.14) *and* (2.27) *be fulfilled. Then for every* $F \in L_2(D)$ (W^*_-) *there exists a unique strong generalized solution* u *of the problem* (2.1), (2.2) *of the class* W_+ (L_{2,α_m}) *for which the estimate*

$$\|u\|_{L_{2,\alpha_m}(D)} \le c\|F\|_{W^*_-}$$

is valid with a positive constant c *independent of* F.

2.2. The case of equation with characteristic degeneration. Consider a hyperbolic second order equation with characteristic degeneration of the type

$$L_1 u \equiv (t^m u_t)_t - u_{x_1x_1} - u_{x_2x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \tag{2.31}$$

where $1 \le m = \text{const} < 2$.

Denote by

$$D_1:\ 0<t<\Big[1-\frac{2-m}{2}\,r\Big]^{\frac{2}{2-m}},\quad r=(x_1^2+x_2^2)^{1/2}<\frac{2}{2-m}$$

a finite domain lying in the half-space $t>0$ which is bounded above by the characteristic conoid

$$S_2:\ t=\Big[1-\frac{2-m}{2}\,r\Big]^{\frac{2}{2-m}},\quad r\le\frac{2}{2-m}$$

of the equation (2.31) with the vertex at the point $(0,0,1)$ and below by the basis

$$S_1:\ t=0,\ \ r\le\frac{2}{2-m}$$

of that conoid on which the equation (2.31) has a characteristic degeneration. Just as in the case of the equation (2.1), we assume that in the domain D_1 the coefficients a_i, $i=1,\dots,4$, of the equation (2.31) are functions of the class $C^2(\overline{D})$.

For the equation (2.31) we consider a characteristic problem which is formulated as follows: in the domain D_1, find a solution $u(x_1,x_2,t)$ of the equation (2.31) satisfying the boundary condition

$$u\big|_{S_1}=0 \tag{2.32}$$

on the plane characteristic surface S_1.

Analogously we formulate a characteristic problem for the equation

$$L_1^*v\equiv(t^mv_t)_t-v_{x_1x_1}-v_{x_2x_2}-(a_1v)_{x_1}-(a_2v)_{x_2}-(a_3v)_t+a_4v=F \tag{2.33}$$

in the domain D_1 by means of the boundary condition

$$v\big|_{S_2}=0, \tag{2.34}$$

where L_1^* is the operator formally conjugate to the operator L_1.

By E_1 and E_1^* we denote the classes of functions from the Sobolev space $W_2^2(D_1)$ which satisfy respectively the boundary condition (2.32) or (2.34) and vanish in some (its own for every function) three-dimensional neighborhood of the segment $I:\ x_1=0,\ x_2=0,\ 0\le t\le 1$. Let W_{1+} (W_{1+}^*) be the Hilbert space obtained by closing the space E_1 (E_1^*) by the norm

$$\|u\|^2=\int\limits_{D_1}\big[u_t^2+u_{x_1}^2+u_{x_2}^2+u^2\big]\,dD_1.$$

Denote by W_{1-} (W_{1-}^*) the space with the negative norm which is constructed with respect to $L_2(D_1)$ and W_{1+} (W_{1+}^*).

Analogously to Lemmas 2.1 and 2.2 we prove

Lemma 2.5. *For all functions $u\in E_1$, $v\in E_1^*$ the inequalities*

$$\|L_1\|_{W_{1-}^*}\le c_1\|u\|_{W_{1+}},\quad \|L_1^*v\|_{W_{1-}}\le c_2\|u\|_{W_{1+}^*}$$

hold, and the problems (2.31), (2.32), *and* (2.33), (2.34) *are self-conjugate, i.e. for every* $u \in W_{1+}$ *and* $v \in W^*_{1+}$ *the equality*

$$(L_1 u, v) = (u, L^*_1 v)$$

holds.

Consider the conditions

$$\inf_{\overline{D}_1}(a_4 - a_{1x_1} - a_{2x_2} - a_{3t}) > 0, \tag{2.35}$$

$$\inf_{S_1} a_3 > \frac{1}{2} \ \text{ for } \ m = 1, \quad \inf_{S_1} a_3 > 0 \ \text{ for } \ m > 1. \tag{2.36}$$

Lemma 2.6. *Let the conditions* (2.35) *and* (2.36) *be fulfilled. Then for every* $u \in W_{1+}$ *the inequality*

$$c\|u\|_{L_2(D_1)} \le \|L_1 u\|_{W^*_{1-}}$$

is valid with a positive constant c *independent of* u.

Consider now the conditions

$$\inf_{\overline{D}_1} a_4 > 0, \tag{2.37}$$

$$\inf_{S_1} a_3 > -\frac{1}{2} \ \text{ for } \ m = 1, \quad \inf_{S_1} a_3 > 0 \ \text{ for } \ m > 1. \tag{2.38}$$

Note the the condition (2.38) follows from (2.36).

Lemma 2.7. *Let the conditions* (2.37) *and* (2.38) *be fulfilled. Then for every* $v \in W^*_{1+}$ *the inequality*

$$c\|u\|_{L_2(D_1)} \le \|L^*_1 v\|_{W_{1-}}$$

is valid with a positive constant c *independent of* v.

From Lemmas 2.5–2.7, the proof of which to a certain extent repeats that of Lemmas 2.1–2.4, we have the following [68]

Theorem 2.2. *Let the conditions* (2.35), (2.36), (2.37) *be fulfilled. Then for every* $F \in L_2(D_1)$ (W^*_{1-}) *there exists a unique strong generalized solution* u *of the problem* (2.31), (2.32) *of the class* W_{1+} (L_2) *for which the estimate*

$$\|u\|_{L_2(D_1)} \le c\|F\|_{W^*_{1-}}$$

is valid with a positive constant c *independent of* F.

Theorem 2.3. *Let the conditions* (2.35), (2.36), (2.37) *be fulfilled. Then for every* $F \in L_2(D_1)$ (W_{1-}) *there exists a unique strong generalized solution* v *of the problem* (2.33), (2.34) *of the class* W^*_{1+} (L_2) *for which the estimate*

$$\|v\|_{L_2(D_1)} \le c\|F\|_{W_{1-}}$$

is valid with a positive constant c *independent of* F.

Here, just as in Definition 2.1, for $F \in L_2(D_1)$ (W^*_{1-}) the function u is said to be a strong generalized solution of the problem (2.31), (2.32) of the class W_{1+} (L_2) if $u \in W_{1+}$ $(L_2(D_1))$ and there exists a sequence of functions $u_n \in E_1$ such that $u_n \to u$ in the space W_{1+} $(L_2(D_1))$ and $L_1 u_n \to F$ in the space W^*_{1-} (W^*_{1-}). For $F \in L_2(D)$ (W_{1-}), the function v is called a strong generalized solution of the problem (2.33), (2.34) of the class W^*_{1+} (L_2) if $v \in W^*_{1+}$ $(L_2(D_1))$ and there exists a sequence of functions $v_n \in E^*_1$ such that $v_n \to v$ in the space W^*_{1+} $(L_2(D_1))$ and $L^*_1 v_n \to F$ in the space W_{1-} (W_{1-}).

3. Multi-Dimensional Versions of the First Darboux Problem for Some Degenerating Second Order Hyperbolic Equations in Dihedral Domains

3.1. The case of equation with noncharacteristic degeneration. Consider a degenerating second order hyperbolic equation of the type

$$L_1 u \equiv u_{tt} - |x_2|^m u_{x_1 x_1} - u_{x_2 x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \quad (3.1)$$

where $m = \mathrm{const} > 0$.

By $D: \ x_2 < t < 1 - x_2,\ 0 < x_2 < 1/2$ we denote the unbounded domain lying in the half-plane $x_2 > 0$ and bounded by the characteristic surfaces $S_1: \ t - x_2 = 0,\ 0 \le x_2 \le 1/2$, $S_2: \ t + x_2 - 1 = 0,\ 0 \le x_2 \le 1/2$ of the equation (3.1) and by the plane surface $S_0: \ x_2 = 0,\ 0 \le t \le 1$ of temporal type on which the above equation degenerates. We assume that in the domain D the coefficients a_i, $i = 1, \dots, 4$, of the equation (3.1) are bounded functions of the class $C^1(\overline{D})$.

For the equation (3.1), we consider a multidimensional version of the first Darboux problem which is formulated as follows: in the domain D, find a solution $u(x_1, x_2, t)$ of the equation (3.1) satisfying the boundary condition

$$u\big|_{S_0 \cap S_1} = 0. \quad (3.2)$$

Analogously we formulate the problem for the equation

$$L^* v \equiv v_{tt} - |x_2|^m v_{x_1 x_1} - v_{x_2 x_2} - (a_1 v)_{x_1} - (a_2 v)_{x_2} - (a_3 v)_t + a_4 v = F \quad (3.3)$$

in the domain D by the boundary condition

$$v\big|_{S_0 \cap S_2} = 0, \quad (3.4)$$

where L^* is the operator formally conjugate to the operator L.

Denote by E and E^* the classes of functions from the Sobolev space $W^2_2(D)$ which satisfy respectively the boundary condition (3.2) or (3.4). Let W_+ (W^*_+) be the Hilbert space obtained by closing the space E (E^*) by the norm

$$\|u\|^2 = \int_D \left(u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2 + u^2\right) dD.$$

By W_- (W^*_-) we denote the space with negative norm which is constructed with respect to $L_2(D)$ and W_+ (W^*_+).

Below we impose on the coefficient a_1 in the equation (3.1) the following restriction:

$$\sup_{\overline{D}} \left| x_2^{-\frac{m}{2}} a_1(x_1, x_2, t) \right| < +\infty. \tag{3.5}$$

For $F \in L_2(D)$ (W_-^*), the function u is said to be a strong generalized solution of the problem (3.1), (3.2) of the class W_+ (L_2) if $u \in W_+$ $(L_2(D))$ and there exists a sequence $u_n \in E$, such that $u_n \to u$ in the space W_+ $(L_2(D))$ and $Lu_n \to F$ in the space W_-^* (W_-^*).

Similarly to the results obtained in the above subsections of this chapter we prove that the estimates

$$c\|u\|_{L_2(D)} \le \|Lu\|_{W_-^*} \le c^{-1}\|u\|_{W_+}, \quad c\|v\|_{L_2(D)} \le \|L^*v\|_{W_-} \le c\|v\|_{W_+^*}$$

$\forall\, u\ (v) \in E\ (E^*)$ are valid and the problems (3.1), (3.2) and (3.3), (3.4) are self-conjugate. In its turn, this results in the following [66]

Theorem 3.1. *Let the condition* (3.5) *be fulfilled. Then for every* $F \in L_2(D)$ (W_-^*) *there exists a unique strong generalized solution* u *of the problem* (3.1), (3.2) *of the class* W_+ (L_2) *for which the estimate*

$$c\|u\|_{L_2(D)} \le \|F\|_{W_-^*}$$

is valid with a positive constant c *independent of* F.

3.2. The case of equation with characteristic degeneration. Consider the second order equation of the type

$$L_1 u \equiv u_{tt} - u_{x_1x_1} - (|x_2|^m u_{x_2})_{x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \tag{3.6}$$

where $1 \le m = \text{const} < 2$. We denote by

$$D_1:\ \frac{2}{2-m}\, x_2^{\frac{2-m}{2}} < t < 1 - \frac{2}{2-m}\, x_2^{\frac{2-m}{2}}, \quad 0 < x_2 < \Big(\frac{2-m}{4}\Big)^{\frac{2}{2-m}}$$

the unbounded domain lying in the half-space $x_2 > 0$ and bounded by the characteristic surfaces

$$S_1: t - \frac{2}{2-m}\, x_2^{\frac{2-m}{2}} = 0, \ \ 0 \le x_2 \le \Big(\frac{2-m}{4}\Big)^{\frac{2}{2-m}},$$
$$\widetilde{S}_2: t + \frac{2}{2-m}\, x_2^{\frac{2-m}{2}} = 1, \ \ 0 \le x_2 \le \Big(\frac{2-m}{4}\Big)^{\frac{2}{2-m}}$$

of the equation (3.6) and by the plane surface $\widetilde{S}_0:\ x_2 = 0,\ 0 \le t \le 1$, on which this equation has a characteristic degeneration. It is assumed that in the domain D_1 the coefficients a_i, $i = 1, \dots, 4$, of the equation (3.6) are bounded functions of the class $C^2(\overline{D}_1)$.

For the equation (3.6) we consider a multidimensional version of the Darboux problem which is formulated as follows: in the domain D, find a solution $u(x_1, x_2, t)$ of the equation (3.6) satisfying the boundary condition

$$u\big|_{S_1} = 0. \tag{3.7}$$

Analogously is formulated the problem for the equation

$$L_1^* u \equiv v_{tt} - v_{x_1x_1} - (|x_2|^m v_{x_2})_{x_2} - (a_1 v)_{x_1} - (a_2 v)_{x_2} - (a_3 v)_t + a_4 v = F \quad (3.8)$$

in the domain D by means of the boundary condition

$$v\big|_{\widetilde{S}_2} = 0, \quad (3.9)$$

where L_1^* is the operator formally conjugate to the operator L_1.

Here E_1 and E_1^* are the classes of functions from the Sobolev space $W_2^2(D_1)$ which satisfy respectively the boundary condition (3.7) or (3.9). Let W_{1+} (W_{1+}^*) be the Hilbert Sobolev space with weight obtained by closing the space E_1 (E_1^*) by the norm

$$\|u\|^2 = \int\limits_{D_1} \left(u_t^2 + u_{x_1}^2 + x_2^m u_{x_2}^2 + u^2\right) dD.$$

Denote by W_{1-} (W_{1-}^*) the space with negative norm constructed with respect to $L_2(D_1)$ and W_{1+} (W_{1+}^*). We impose on the coefficient of the equation (3.6) the following conditions:

$$\sup_{\overline{D}} \left|x_2^{-\frac{m}{2}} a_2(x_1, x_2, t)\right| < +\infty, \quad (3.10)$$

$$\Omega\big|_{S_1} \leq 0, \quad (\lambda\Omega + \Omega_t)\big|_D \leq 0, \quad a_4\big|_{\widetilde{S}_2} \geq 0, \quad (\lambda a_4 - a_{4t})\big|_D \geq 0, \quad (3.11)$$

where $\Omega = a_{1x_1} + a_{2x_2} + a_{3t} - a_4$, and λ is a sufficient large number.

For $F \in L_2(D)$ (W_{1-}^*), just as in the previous section, we introduce the notion of a strong generalized solution of the problem (3.6), (3.7) of the class W_{1+} (L_2), prove the two-sided estimates for the values $\|L_1 u\|_{W_{1-}^*}$ and $\|L_1^* v\|_{W_{1-}}$, and show that the problems (3.6), (3.7) and (3.8), (3.9) are self-conjugate. As a consequence we have the following [67]

Theorem 3.2. *Let the conditions* (3.10), (3.11) *be fulfilled. Then for every $F \in L_2(D)$ (W_{1-}^*) the problem* (3.6), (3.7) *has a unique strong generalized solution u of the class W_{1+} (L_2) for which the estimate*

$$c\|u\|_{L_2(D_1)} \leq \|F\|_{W_{1-}^*}$$

is valid with a positive constant c independent of F.

CHAPTER 3

Some Nonlocal Problems for Wave Equations

1. A Nonlocal Problem for the Wave Equation with One Spatial Variable

1.1. Statement of the problem. Consider the wave equation with one spatial variable

$$\square_1 u := u_{tt} - u_{xx} = 0. \tag{1.1}$$

By D we denote the characteristic quadrangle of the equation (1.1) with vertices at the points $O(0,0)$, $A(1,1)$, $B(-1,1)$ and $C(0,2)$. Let $J : OA \to OC$ be the mapping transforming the point $P \in OA$ into the point $J(P) \in OC$ lying on the characteristic of the family $x + t = \text{const}$ passing through the point P, i.e. if $P = (x,x) \in OA$, then $J(P) = (0,2x) \in OC$.

For the equation (1.1), in the domain D we consider a nonlocal problem which is formulated as follows: find a regular in the domain D solution $u(x,t)$ of the equation (1.1), continuous in $\overline{D}$ and satisfying the conditions

$$u(P) = \varphi(P), \quad P \in OB, \tag{1.2}$$

$$u(J(P)) = u(P), \quad P \in OA, \tag{1.3}$$

where φ is a given function continuous on the segment OB of the characteristic $x + t = 0$.

It is easy to verify that the problem (1.1)–(1.3) is not correctly posed because the corresponding homogeneous problem has an infinite set of linearly independent solutions of the type $u(x,t) = \psi(x+t)$, where $\psi(x)$ is an arbitrary function of the class $C([0,2]) \cap C^2([0,2])$ satisfying $\psi(0) = 0$.

Consider now the same problem for the equation

$$(\square_1 + \lambda)u : u_{tt} - u_{xx} + \lambda u = 0, \quad \lambda = \text{const} \neq 0. \tag{1.4}$$

In new variables $\xi = 2^{-1}(t+x)$, $\eta = 2^{-1}(t-x)$, the problem (1.4), (1.2), (1.3) in the domain $\Omega :\ 0 < \xi < 1,\ 0 < \eta < 1$ of the plane of the variables ξ, η can be rewritten as

$$v_{\xi\eta} + \lambda v = 0, \tag{1.5}$$

$$v(0,\eta) = \varphi(\eta), \quad 0 \le \eta \le 1, \tag{1.6}$$

$$v(\xi,\xi) = v(\xi,0), \quad 0 \le \xi \le 1, \tag{1.7}$$

where $v(\xi,\eta) := u(\xi - \eta, \xi + \eta)$.

1.2. The existence and uniqueness theorem. As is known [10, p. 66], every solution $v(\xi,\eta)$ of the equation (1.5) of the class $C(\overline{\Omega})\cap C^2(\Omega)$ is representable in the form

$$v(\xi,\eta)=R(\xi,0;\xi,\eta)v(\xi,0)+R(0,\eta;\xi,\eta)v(0,\eta)-R(0,0;\xi,\eta)v(0,0)-$$
$$-\int_0^\xi \frac{\partial R(\sigma,0;\xi,\eta)}{\partial\sigma}\,v(\sigma,0)\,d\sigma-\int_0^\eta \frac{\partial R(0,\tau;\xi,\eta)}{\partial\tau}\,v(0,\tau)\,d\tau, \tag{1.8}$$

where $R(\xi_1,\eta_1;\xi,\eta)$ is the Riemann function of the equation (1.5).

This function for the equation (1.5) can be represented in terms of the Bessel function of zero order in the form [24, p. 455]

$$R(\xi_1,\eta_1;\xi,\eta)=J_0\big(2\sqrt{\lambda(\xi-\xi_1)(\eta-\eta_1)}\big)=$$
$$=\sum_{k=0}^{\infty}(-1)^k\frac{\lambda^k}{(k!)^2}\,(\xi-\xi_1)^k(\eta-\eta_1)^k. \tag{1.9}$$

Substituting (1.8) in (1.6) and (1.7) and taking into account (1.9), with respect to the unknown function $\psi(\xi)=v(\xi,0)$ we obtain the first order Volterra equation

$$\int_0^\xi K(\xi,\sigma;\lambda)\psi(\sigma)\,d\sigma=f(\xi),\;\;0\le\xi\le 1; \tag{1.10}$$

here

$$K(\xi,\sigma;\lambda):=\lambda+\sum_{k=2}^{\infty}(-1)^{k+1}\frac{k\lambda^k}{(k!)^2}\,(\xi-\sigma)^{k-1}\xi^{k-1}, \tag{1.11}$$

$$f(\xi):=-\int_0^\xi K(\xi,\tau;\lambda)\varphi(\tau)\,d\tau+\frac{1}{\xi}\,\big(\varphi(\xi)-R(0,0;\xi,\xi)\varphi(0)\big). \tag{1.12}$$

If the equation (1.10) is solvable in the class of continuous functions $C([0,1])$, then $f(\xi)\in C^1([0,1])$ and, differentiating both parts of the equation (1.10), by means of (1.11) we obtain

$$\lambda\psi(\xi)+\int_0^\xi \frac{\partial K(\xi,\sigma;\lambda)}{\partial\xi}\,\psi(\sigma)\,d\sigma=f'(\xi),\;\;0\le\xi\le 1. \tag{1.13}$$

With regard for (1.11) and (1.12), we have

$$f'(\xi)=-\lambda\varphi(\xi)-\int_0^\xi \frac{\partial K(\xi,\tau;\lambda)}{\partial\xi}\,\varphi(\tau)\,d\tau+\Big(\frac{\varphi(\xi)-\varphi(0)}{\xi}\Big)'-$$
$$-\varphi(0)\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)\lambda^k}{(k!)^2}\,\xi^{2(k-1)}. \tag{1.14}$$

From (1.13) and (1.14) we immediately obtain

Theorem 1.1. *The problem* (1.4), (1.2), (1.3) *in the class* $C(\overline{D}) \cap C^2(D)$ *cannot have more than one solution. For every function* φ *such that* $(\varphi(\xi) - \varphi(0))/\xi \in C^3([0,1])$, *the problem* (1.4), (1.2), (1.3) *has a unique solution* $u(x,t)$ *of the class* $C^2(\overline{D})$.

Remark 1.1. Since $(\varphi(\xi) - \varphi(0))/\xi = \int\limits_0^1 \varphi'(\xi\tau)\, d\tau$, it is enough to require in the Theorem 1.1 that $\varphi \in C^4([0,1])$.

Remark 1.2. Note that the theorem on the uniqueness for the problem (1.4), (1.2), (1.3) is likewise valid in the class of generalized solutions $u(x,t)$ of the class $C(\overline{D})$, i.e. when $u(x,t) \in C(\overline{D})$ and $(u, \square_1\omega + \lambda\omega)_{L_2(D)} = 0$ for any $\omega \in C_0^\infty(D)$. In addition, the representation (1.8) holds for generalized solutions of the equation (1.4) of the class $C(\overline{D})$ as well. For a generalized solution $u(x,t)$ of the problem (1.4), (1.2), (1.3) of the class $C(\overline{D})$ to exist, it is sufficient that $\varphi \in C^1([0,1])$ and $\varphi \in C^2([0,\varepsilon])$ for arbitrarily small $\varepsilon > 0$. Here $(\cdot\,,\cdot)_{L_2(D)}$ is the scalar product in the space $L_2(D)$.

Remark 1.3. As it can be easily verified, for the equation (1.1) the nonlocal problem in which instead of (1.3) the condition

$$\int\limits_{OJ(P)} u\, ds := \int\limits_0^{2x} u(0,t)\, dt = u(P), \quad P(x,t) \in OA, \tag{1.15}$$

is prescribed and the condition (1.2) remains as before, i.e. the problem (1.1), (1.2), (1.15), is well-posed, where $OJ(P)$ is the rectilinear segment connecting the points O and $J(P)$.

2. A Nonlocal Problem with the Integral Condition for the Wave Equation with Two Spatial Variables

2.1. Statement of the problem. One integral property of solutions of the wave equation. Consider the wave equation

$$\square_2 u := u_{tt} - u_{x_1x_1} - u_{x_2x_2} = 0 \tag{2.1}$$

in the dihedral angle $D:\ -t < x_2 < t,\ 0 < t < +\infty$ whose faces are the characteristic surfaces $S_1:\ t - x_2 = 0,\ 0 \le t < +\infty$ and $S_2:\ t + x_2 = 0,\ 0 \le t < +\infty$ of the equation (2.1). Let $J_\pm(P)$ be the point of intersection with the plane $x_2 = 0$ of the bicharacteristic ray $\ell_\pm(P):\ x_1 = x_1^0,\ x_2 = x_2^0 - \tau,\ t = t^0 \pm \tau,\ 0 \le \tau < +\infty$ of the equation (2.1) passing through the point $P(x_1^0, x_2^0, t^0) \in S_1$, i.e. $t^0 = x_2^0 \ge 0$. Obviously, $J_\pm(P) = (x_1^0, 0, x_2^0 \pm x_2^0)$. Denote by $t_+(P)$ the coordinate of the point $J_+(P)$ with respect to the axis t, i.e. $t_+(P) = 2x_2^0$.

Consider a nonlocal problem which is formulated as follows: find in the domain D a solution $u(x_1,x_2,t)$ of the equation (2.1) satisfying the conditions

$$u(P)=\varphi(P),\ \ P\in S_2, \tag{2.2}$$

$$\int\limits_{J_-(P)J_+(P)} u\,ds := \int\limits_0^{t_+(P)} u(x_1,0,t)\,dt = u(P)+\mu(P),\ \ P(x_1^0,x_2^0,t^0)\in S_1, \tag{2.3}$$

where μ and φ are given functions on S_1 and S_2 satisfying on $S_1\cap S_2$ the consistency condition $(\mu+\varphi)|_{S_1\cap S_2}=0$, and $J_-(P)J_+(P)$ is the rectilinear segment connecting the points $J_-(P)$ and $J_+(P)$.

Definition 2.1. The function $u(x_1,x_2,t)$ is said to be a weak generalized solution of the equation (2.1) of the class $C(\overline{D})$ if $u\in C(\overline{D})$ and this function satisfies the equation (2.1) in terms of generalized functions [48, p. 8], i.e. $(u,\square_2\omega)_{L_2(D)} := \int\limits_D u\square_2\omega\,dD=0\ \forall\,\omega\in C_0^\infty(D)$.

Definition 2.2. The function $u(x_1,x_2,t)$ is said to be a strong generalized solution of the equation (2.1) of the class $C(\overline{D})$ if $u\in C(\overline{D})$ and for every subdomain D_1 compactly imbedded in D (i.e. $\overline{D}_1$ is a compact and $\overline{D}_1\subset D$) there exists a sequence u_n of regular solutions of the equation (2.1) of the class $C^2(\overline{D}_1)$ tending to u in the space $C(\overline{D}_1)$: $\|u_n-u\|_{C(\overline{D}_1)}\to 0$ for $n\to\infty$.

Denote by $E(r,t,\tau)$ the Volterra function [10, p. 83]

$$E(r,t,\tau)=\frac{1}{2\pi}\log\frac{t-\tau-\sqrt{(t-\tau)^2-r^2}}{r}\,,\ \ r^2=\sum_{i=1}^2(x_i-y_i)^2, \tag{2.4}$$

which is a solution of the equation (2.1) inside the cone

$$K_{x,t}:\ t-\tau-r=0,\ \ x=(x_1,x_2), \tag{2.5}$$

characteristic for the equation (2.1). This solution vanishes on the cone (2.5), but has singularities along its axis $r=0$. Let $S_{x,t}^i$, $(x,t)\in D$, be a part of the surface S_i lying inside the cone $K_{x,t}$, $i=1,2$, and $S_{x,t}=S_{x,t}^1\cup S_{x,t}^2$. As is known [64, p. 96], for every regular solution $u(x,t)$ of the equation (2.1) of the class $C^2(\overline{D})$ the integral equality

$$\int\limits_{x_2}^t u(x_1,x_2,\tau)\,d\tau=\int\limits_{S_{x,t}}\Big[u\,\frac{\partial E(r,t,\tau)}{\partial N}-E(r,t,\tau)\,\frac{\partial u}{\partial N}\Big]\,ds,\ \ \forall\,(x,t)\in D, \tag{2.6}$$

holds, where N is the unit vector of the conormal to $S_{x,t}$ at the point $(y,\tau)\in S_{x,t}$, i.e. $N=(\cos\widehat{nx_1},\cos\widehat{nx_2},-\cos\widehat{nt})$, $n=(\cos\widehat{nx_1},\cos\widehat{nx_2},\cos\widehat{nt})$ is the unit vector of the outer normal to $S_{x,t}$ at the point $(y,\tau)\in S_{x,t}$, and $y=(y_1,y_2)$ and τ are the variables of integration in the right-hand side of the equality (2.6). It is evident that $N|_{S^1_{x,t}} = (0, 1/\sqrt{2}, 1/\sqrt{2})$, $N|_{S^2_{x,t}} = (0, -1/\sqrt{2}, 1/\sqrt{2})$.

Taking into account that $E|_{K_{x,t}} = 0$ and on the characteristic surface $S_{x,t}$ the differentiation with respect to the conormal $\partial/\partial N$ is an inner differential operator and integrating by parts the equality (2.6), we obtain

$$\int_{x_2}^{t} u(x_1, x_2, \tau)\, d\tau =$$

$$= \frac{1}{\pi} \int_{x_1-\sqrt{t^2-x_2^2}}^{x_1+\sqrt{t^2-x_2^2}} u(y_1, 0, 0) \log \frac{(x_1-y_1)^2 + x_2^2}{\sqrt{(x_1-y_1)^2+x_2^2}(t+\sqrt{t^2-(x_1-y_1)^2-x_2^2})}\, dy_1 +$$

$$+2 \int_{S_{x,t}} u\, \frac{\partial E(r,t,\tau)}{\partial N}\, ds. \tag{2.7}$$

2.2. Some properties of wave potentials. First of all we make the following

Remark 2.1. In the characteristic half-plane $S_1:\ t - x_2 = 0,\ 0 \le t < +\infty$ introduce a Cartesian coordinate system of the points y_1, y_2': one of its axes Oy_1 coincides with Ox_1 and the other axis Oy_2' is directed along the bicharacteristic ray with the directional vector $(0, 1/\sqrt{2}, 1/\sqrt{2})$. Below, for a function g defined on the surface S_1 or on $S^1_{x,t} \subset S_1$, we will assume that it is a function of the variables y_1, y_2', i.e. $g = g(y_1, y_2')$. Obviously, $\partial g/\partial N|_{S_1} = \partial g(y_1, y_2')/\partial y_2'$, and on S_1 $ds = dy_1 dy_2'$.

Lemma 2.1. *The operator T_1 acting by the formula*

$$(T_1 g)(x,t) := \int_{S^1_{x,t}} g \frac{\partial E(r,t,\tau)}{\partial N}\, ds, \quad (x,t) \in \overline{D}, \tag{2.8}$$

is a linear continuous operator from the space $C(S_1)$ into the space $C(\overline{D})$, and

$$(T_1 g)(x,t) = 0, \quad (x,t) \in S_1 \cup S_2. \tag{2.9}$$

The operator T_0 acting by the formula $(T_0 g)(x,t) := \int\limits_{S^1_{x,t}} g \frac{\partial E(r,t,\tau)}{\partial t}\, ds$, $(x,t) \in \overline{D}$, is likewise a linear continuous operator from the space $C(S_1)$ into the space $C(\overline{D})$, and

$$\begin{aligned} &(T_0 g)(x,t) = \frac{1}{2} \int_0^{\sqrt{2}\,t} g(x_1, y_2')\, dy_2', \quad (x,t) \in S_1; \\ &(T_0 g)(x,t) = 0, \quad (x,t) \in S_2, \end{aligned} \tag{2.10}$$

Proof. Taking into account the fact that on the space S_1 the variables y_1, y_2, τ, where $y_2=\tau$, are connected with the variables y_1, y_2' by the equality $\tau=y_2'/\sqrt{2}$, we have

$$2\sqrt{2\pi}\,\frac{\partial E(r,t,\tau)}{\partial N}\Big|_{S_1}=\big((t-\tau)^2-(x_1-y_1)^2-(x_2-y_2)^2\big)^{-\frac12}-$$
$$-(t-\tau)(x_2-y_2)\big[(x_1-y_1)^2+(x_2-y_2)^2\big]^{-1}\times$$
$$\times\big((t-\tau)^2-(x_1-y_1)^2-(x_2-y_2)^2\big)^{-\frac12}=$$
$$=\Big(\Big(t-\frac{y_2'}{\sqrt2}\Big)^2-(x_1-y_1)^2-\Big(x_2-\frac{y_2'}{\sqrt2}\Big)^2\Big)^{-\frac12}-$$
$$-\Big(t-\frac{y_2'}{\sqrt2}\Big)\Big(x_2-\frac{y_2'}{\sqrt2}\Big)\Big[(x_1-y_1)^2+\Big(x_2-\frac{y_2'}{\sqrt2}\Big)\Big]^{-1}\times$$
$$\times\Big(\Big(t-\frac{y_2'}{\sqrt2}\Big)^2-(x_1-y_1)^2-\Big(x_2-\frac{y_2'}{\sqrt2}\Big)^2\Big)^{-\frac12}. \tag{2.11}$$

As is easily seen, when the point (x,t) belongs to D, the boundary $\partial S^1_{x,t}$ of the plane domain $S^1_{x,t}$ consists of the upper part of the parabola $\gamma^1_{x,t}:\ y_2'=-(\sqrt2\,(t-x_2))^{-1}(y_1-x_1)^2+(t+x_2)/\sqrt2,\ y_2'\ge0$ and the segment $\delta^1_{x,t}:\ x_1-\sqrt{t_2^2-x_2^2}\le y_1\le x_1+\sqrt{t^2-x_2^2},\ y_2'=0$, in the plane of the variables y_1, y_2'.

Under the new variables z_1, z_2

$$y_2'-\sqrt2\,x_2=(t-x_2)z_2,\quad y_1-x_1=(t-x_2)z_1 \tag{2.12}$$

we have

$$\Big(t-\frac{y_2'}{\sqrt2}\Big)^2-(x_1-y_1)^2-\Big(x_2-\frac{y_2'}{\sqrt2}\Big)^2=(t-x_2)^2\big[1-\sqrt2\,z_2-z_1^2\big],$$
$$(x_1-y_1)^2+\Big(x_2-\frac{y_2'}{\sqrt2}\Big)^2=(t-x_2)^2\Big[z_1^2+\frac{z_2^2}{2}\Big], \tag{2.13}$$
$$\Big(t-\frac{y_2'}{\sqrt2}\Big)^2\Big(x_2-\frac{y_2'}{\sqrt2}\Big)^2=2^{-1}(t-x_2)[z_2-\sqrt2\,]z_2.$$

After the transformation of (2.25), the domain $S^1_{x,t}$ transforms into the plane domain

$$\Omega_{x_2,t}:\ -\Big(\frac{\sqrt2}{t-x_2}\Big)x_2\le z_2\le\Big(\frac{1}{\sqrt2}\Big)(1-z_1^2),$$
$$-\Big(\frac{t+x_2}{t-x_2}\Big)^{1/2}\le z_1\le\Big(\frac{t+x_2}{t-x_2}\Big)^{1/2},$$

which is bounded by the parabola $z_2=(1/\sqrt2)(1-z_1^2)$ and the straight line $z_2=-(\sqrt2/(t-x_2))x_2$ in the plane of the variables z_1, z_2.

By virtue of (2.11) -(2.13), we rewrite the equality (2.8) in the form

$$(T_1g)(x,t)=$$

$$= (t-x_2) \int\limits_{\Omega_{x_2,t}} G_1(z_1,z_2) g\big(x_1+(t-x_2)z_1, \sqrt{2}\,x_2+(t-x_2)z_2\big)\, dz_1\, dz_2, \quad (2.14)$$

where

$$G_1(z_1,z_2) := (2\sqrt{2}\,\pi)^{-1}(1-\sqrt{2}\,z_2 - z_1^2)^{-\frac{1}{2}} -$$
$$- (4\sqrt{2}\,\pi)^{-1} z_2(z_2-\sqrt{2}) \Big[z_1^2 + \frac{z_2^2}{2}\Big]^{-1} (1-\sqrt{2}\,z_2 - z_1^2)^{-\frac{1}{2}}.$$

For $(x,t) \in D$, i.e. for $t > |x_2|$, it can be easily shown that $\int\limits_{\Omega_{x_2,t}} |G_1(z_1,z_2)|\, dz_1\, dz_2 < +\infty$, whence by the representation (2.14) it directly follows that the function $T_1 g$ is continuous at the point (x,t) if $g \in C(S_1)$.

Let now $(x^0,t^0) \in S_1$, $t^0 > 0$. Denote by $\Pi_{x,t}$, $(x,t) \in D$, the rectangle $|y-x_1| \le \sqrt{t^2-x_2^2}$, $0 \le y_2' \le ((t+x_2)/\sqrt{2})$ in the plane of the variables y_1, y_2'. Obviously, $S^1_{x,t} \subset \Pi_{x,t}$. Therefore as $D \ni (x,t) \to (x^0,t^0)$, the plane domain $S^1_{x,t}$ shrinks into the segment $I = \big\{(x_1,x_2,t) \in S_1 :\ x_1 = x_1^0,\ x_2 = \tau,\ t = \tau,\ 0 \le \tau \le t^0\big\}$ which in the plane of the variables y_1, y_2' (see Remark 2.1) represents the segment $\widetilde{I}:\ y_1 = x_1^0$, $0 \le y_2' \le \sqrt{2}\,t^0$. Since $g \in C(S)$, for every $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$, $\delta < \varepsilon$, such that for $|x-x^0| < \delta$, $|t-t^0| < \delta$ we have

$$\big|g(y_1,y_2') - g(x_1^0,y_2')\big| < \varepsilon, \ \ (y_1,y_2') \in S^1_{x,t}. \quad (2.15)$$

Assuming $a^2 = (t-y_2'/\sqrt{2})^2 - (x_2-y_2'/\sqrt{2})^2 = (t-x_2)(t+x_2-\sqrt{2}\,y_2')$, $a > 0$, we can see that

$$\int\limits_{S^1_{x,t}} \frac{1}{2\sqrt{2}\,\pi} \frac{g(x_1^0,y_2')}{\big[(t-y_2'/\sqrt{2})^2 - (x_1-y_1)^2 - (x_2-y_2'/\sqrt{2})^2\big]^{1/2}}\, dy_1\, dy_2' =$$

$$= \frac{1}{2\sqrt{2}\,\pi} \int\limits_0^{(t+x_2)/\sqrt{2}} dy_2' \int\limits_{x_1-a}^{x_1+a} \frac{g(x_1^0,y_2')}{[a^2-(y_1-x_1)^2]^{1/2}}\, dy_1 =$$

$$= \frac{1}{2\sqrt{2}\,\pi} \int\limits_0^{(t+x_2)/\sqrt{2}} g(x_1^0,y_2') \arcsin \frac{y_1-x_1}{a}\Big|_{y_1=x_1-a}^{y_1=x_1+a} dy_2' =$$

$$= \frac{1}{2\sqrt{2}} \int\limits_0^{(t+x_2)/\sqrt{2}} g(x_1^0,y_2')\, dy_2'. \quad (2.16)$$

Obviously, the latter equality can be rewritten as follows:

$$\frac{1}{\sqrt{2}}\,(T_0\widetilde{g})(x,t) =$$

$$= -\frac{1}{\sqrt{2}} \int\limits_{S^1_{x,t}} g(x^0_1, y'_2) \frac{\partial E(r,t,\tau)}{\partial t}\, ds = \frac{1}{2\sqrt{2}} \int\limits_0^{(t+x_2)/\sqrt{2}} g(x^0_1, y'_2)\, dy'_2, \quad (2.17)$$

where $\widetilde{g}(y_1, y'_2) := g(x^0_1, y'_2)$, T_0 is the operator from (2.10), and $-\partial E(r,t,\tau)/\partial t = \partial E(r,t,\tau)/\partial\tau = (2\pi)^{-1}((t-\tau)^2 - r^2)^{-\frac{1}{2}}$, $r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$.

From (2.17) it follows that

$$2\sqrt{2}\,(T_0 1)(x,t) = t + x_2. \quad (2.18)$$

Assuming that $M = \max\limits_{0 \le y'_2 \le \sqrt{2}\,(t^0+\varepsilon)} |g(x^0_1, y'_2)|$ and taking into account that $t^0 = x^0_2$ on S_1, for $|x - x^0| < \delta$, $|t - t^0| < \delta$, by virtue of (2.15)–(2.18) we have

$$\Big|(T_0 g)(x,t) - \frac{1}{2} \int\limits_0^{\sqrt{2}\,t^0} g(x^0_1, y'_2)\, dy'_2\Big| = \Big|\big(T_0\big[g(y_1, y'_2) - g(x^0_1, y'_2)\big]\big)(x,t) +$$

$$+ \frac{1}{2} \int\limits_0^{(t+x_2)/\sqrt{2}} g(x^0_1, y'_2)\, dy'_2 - \frac{1}{2} \int\limits_0^{\sqrt{2}\,t^0} g(x^0_1, y'_2)\, dy'_2\Big| \le \varepsilon\big|(T_0 1)(x,t)\big| +$$

$$+ \frac{1}{2}\Big| \int\limits_{\sqrt{2}\,t}^{(t+x_2)/\sqrt{2}} g(x^0_1, y'_2)\, dy'_2\Big| \le \varepsilon\,\frac{1}{2\sqrt{2}}\,|t + x_2| + \frac{1}{2}\,M\Big|\frac{1}{\sqrt{2}}\,(t + x_2) - \sqrt{2}\,t^0\Big| =$$

$$= \frac{\big[\varepsilon|t + x_2| + M\big|(t - t^0) + (x_2 - x^0_2)\big|\big]}{2\sqrt{2}} \le$$

$$\le \frac{\varepsilon\big(|t - t^0| + |x_2 - x^0_2| + |t + x^0_2|\big) + M\big(|t - t^0| + |x_2 - x^0_2|\big)}{2\sqrt{2}} \le$$

$$\le \frac{\varepsilon(2\delta + 2t^0) + 2\delta M}{2\sqrt{2}} \le \frac{(\varepsilon + t^0 + M)\varepsilon}{\sqrt{2}}\,. \quad (2.19)$$

From (2.19) immediately follows the continuity of the function $T_0 g$ at the point $(x^0, t^0) \in S_1$, as well as the first of the equalities (2.10). The continuity of $T_0 g$ at an arbitrary point $(x^0, t^0) \in S_2$ and the second of the equalities (2.10) are proved analogously.

We introduce now the operator

$$(L_0 g)(x,t) := \int\limits_{S^1_{x,t}} G_0(x,t; y_1, y'_2) g(y_1, y'_2)\, dy_1\, dy_2,$$

where

$$G_0(x,t; y_1, y'_2) := \frac{1}{2\sqrt{2}\,\pi} \times$$

$$\times \frac{(t-y_2'/\sqrt{2})(x_2-y_2'/\sqrt{2})}{\big[(x_1-y_1)^2+(x_2-y_2'/\sqrt{2})^2\big]\big((t-y_2'/\sqrt{2})^2-(x_1-y_1)^2-(x_2-y_2'/\sqrt{2})^2\big)^{1/2}}\,.$$

Note that on the segment $0 \le y_2' \le (t+x_2)/\sqrt{2}$, where $(x,t)\in D$, i.e. $t>|x_2|$, the function $(x_2-y_2'/\sqrt{2})$ is positive for $0<y_2'<\sqrt{2}\,x_2$ and negative for $\sqrt{2}\,x_2<y_2'<(t+x_2)/\sqrt{2}$. Taking this fact and the equality

$$\int \frac{b}{(b^2+x^2)\sqrt{a^2-x^2}}\,dx =$$

$$= \frac{1}{\sqrt{a^2+b^2}}\,\operatorname{arctg}\frac{x\sqrt{a^2+b^2}}{b\sqrt{a^2-x^2}} + \text{const}, \quad b\neq 0, \quad a\neq 0,$$

into account, by analogy with (2.16) we have

$$(L_0\widetilde{g})(x,t) = \int\limits_{S^1_{x,t}} G_0(x,t;y_1,y_2')g(x_1^0,y_2')\,dy_1\,dy_2' =$$

$$= \frac{1}{2\sqrt{2}\,\pi}\Big(\int\limits_0^{\sqrt{2}\,x_2} + \int\limits_{\sqrt{2}\,x_2}^{(t+x_2)/\sqrt{2}}\Big)\,dy_2'\Big(t-\frac{y_2'}{\sqrt{2}}\Big)\times$$

$$\times g(x_1^0,y_2')\int\limits_{x_1-a}^{x_1+a}\frac{(x_2-y_2'/\sqrt{2})}{\big[(x_2-y_2'/\sqrt{2})^2+(y_1-x_1)^2\big]\sqrt{a^2-(y_1-x_1)^2}}\,dy_1 =$$

$$= \frac{1}{2\sqrt{2}\,\pi}\Big(\int\limits_0^{\sqrt{2}\,x_2} + \int\limits_{\sqrt{2}\,x_2}^{(t+x_2)/\sqrt{2}}\Big)\times$$

$$\times g(x_1^0,y_2')\,\operatorname{arctg}\frac{(y_1-x_1)(t-y_2'/\sqrt{2})}{(x_2-y_2'/\sqrt{2})\sqrt{a^2-(y_1-x_1)^2}}\Bigg|_{y_1-x_1=-a}^{y_1-x_1=a}\,dy_2' =$$

$$= \frac{1}{2\sqrt{2}}\int\limits_0^{\sqrt{2}\,x_2} g(x_1^0,y_2')\,dy_2' - \frac{1}{2\sqrt{2}}\int\limits_{\sqrt{2}\,x_2}^{(t+x_2)/\sqrt{2}} g(x_1^0,y_2')\,dy_2', \tag{2.20}$$

where $\widetilde{g}(y_1,y_2') := g(x_1^0,y_2')$.

It should be noted that the second summand in (2.20) tends to zero as $D\ni(x,t)\to(x^0,t^0)\in S_1$ since $\sqrt{2}\,x_2\to\sqrt{2}\,t^0$ and $(t+x_2)/\sqrt{2}\to\sqrt{2}\,t^0$. From (2.20), just as in proving the continuity of the function T_0g at the point $(x^0,t^0)\in S_1$, it follows that the function L_0g is continuous at the point $(x^0,t^0)\in S_1$, where

$$(L_og)(x^0,t^0) = \frac{1}{2\sqrt{2}}\int\limits_0^{\sqrt{2}t^0} g(x_1^0,y_2')\,dy_2', \quad (x^0,t^0)\in S_1. \tag{2.21}$$

Since now $T_1 = T_0\sqrt{2} - L_0$, we find that the function $T_1 g$ is continuous at the point $(x^0, t^0) \in S_1$, and by virtue of (2.10) and (2.21) we have $(T_1 g)(x^0, t^0) = 0$, $(x^0, t^0) \in S_1$. The continuity of the function $T_1 g$ at the points of the set S_2 and $(T_1 g)(x^0, t^0) = 0$, $(x^0, t^0) \in S_2$, and hence the equality (2.9) are proved analogously.

The same reasoning allows us to show that the operator T_1 acting from the space $C(S_1)$ into $C(\overline{D})$ is continuous. Let X be an arbitrary compact subset from S_1, D_X be the set of points (x, t) from $\overline{D}$ for which $S^1_{x,t} \subset X$, and let $D_X \neq \varnothing$. Then there exists a positive constant $c = c(X)$ such that for every $g \in C(S_1)$ the inequality $\|T_1 g\|_{C(D_X)} \leq c(X)\|g\|_{C(X)}$ is valid. In the same sense the operator T_0 acting from the space $C(S_1)$ into $C(\overline{D})$, is continuous. Thus the lemma is proved. □

Remark 2.2. Note that Lemma 2.1 remains valid if instead of the surface $S^1_{x,t}$ we consider $S^2_{x,t}$ for the corresponding operators T_0 and T_1.

2.3. Some properties of generalized solutions of the wave equation. We have the following

Lemma 2.2. *The following conditions are equivalent:*

(i) *the function u is a weak generalized solution of the equation* (2.1) *of the class $C(\overline{D})$;*

(ii) *the function u is a strong generalized solution of the equation* (2.1) *of the class $C(\overline{D})$;*

(iii) *the function u belongs to the class $C(\overline{D})$ and for every $(x, t) \in D$ the equality* (2.7) *is valid.*

Proof. The condition (ii) follows from (i). Indeed, let u be a weak generalized solution of the equation (2.1) of the class $C(\overline{D})$. Denote by $\omega_\varepsilon(x, t) = \varepsilon^{-3}\omega^0(x/\varepsilon, t/\varepsilon)$, $\varepsilon > 0$, an averaging function, where $\omega^0 \in C^\infty_0(R^3)$, $\int \omega^0(x, t)\, dx\, dt = 1$, $\omega^0 \geq 0$, $\operatorname{supp}\omega^0 = \{(x, t) \in R^3 : |x|^2 + t^2 \leq 1\}$ [48, p. 9]. Let $D_1 \subset D$ be a subdomain compactly embedded in D, and let ε be less than the distance $\delta > 0$ between the sets $\overline{D}_1$ and ∂D. Then by properties of convolution [48, pp. 9, 23], the function $u_\varepsilon = u * \omega_\varepsilon$ belongs to the class $C^\infty(\overline{D}_1)$, is a classical solution of the equation (2.1) in $\overline{D}_1$, and converges as $\varepsilon \to 0$ to u in the norm of the space $C(\overline{D}_1)$, i.e. the condition (ii) holds.

If the condition (ii) is fulfilled, then, as is easily seen, the function $u_\varepsilon = u * \omega_\varepsilon \in C^\infty(\overline{D}_\varepsilon)$ is a classical solution of the equation (2.1) in the closed domain $\overline{D}_\varepsilon$, where $D_\varepsilon :\ -t + \sqrt{2}\,\varepsilon < x_2 < t - \sqrt{2}\,\varepsilon$, $\sqrt{2}\,\varepsilon < t < +\infty$. Let $S_{x,t,\varepsilon}$, $(x, t) \in D_\varepsilon$ be a part of the boundary ∂D_ε lying inside the characteristic cone $K_{x,t}$ from (2.5). Then for the solution u_ε of the equation (2.1) the integral equality (2.7) is valid, in which instead of D and $S_{x,t}$ we take D_ε and $S_{x,t,\varepsilon}$ and denote it by (2.7_ε). By Lemma 2.1, since the linear operators represented by the left- and right-hand sides of the equality (2.7_ε) are continuous in the corresponding spaces of functions

$u \in C(\overline{D})$ and for every subdomain $D_1 \subset D$ compactly embedded in D we have $\lim\limits_{\varepsilon \to 0} \|u_\varepsilon - u\|_{C(\overline{D}_1)}$, passing in the integral equality (2.7_ε) to the limit as $\varepsilon \to 0$ we obtain (2.7). Consequently, (iii) follows from (ii).

Thus it remains only to show that (i) follows from (iii). Indeed, let $u \in C(\overline{D})$ and for every $(x,t) \in D$ the integral equality (2.7) be valid. We take an arbitrary function $\omega \in C_0^\infty(D)$ and introduce the set $S_\omega^i = \bigcup\limits_{(x,t)\in \operatorname{supp}\omega} S_{x,t}^i$, $i = 1,2$. It is obvious that $S_\omega^i \subset S_i$, $i = 1,2$. By the Weierstrass theorem, there exists a sequence of functions $f_n^i \in C^\infty(S_i)$, $\operatorname{diam}\operatorname{supp} f_n^i < +\infty$, $i = 1,2$, such that $\|f_n^i - u|_{S_i}\|_{C(S_\omega^i)} \to 0$ as $n \to \infty$. In [64, p. 98] it is proved that there exists a unique solution $u_n \in C^\infty(\overline{D})$ of the equation (2.1) satisfying the boundary conditions $u_n|_{S_i} = f_n^i$, $i = 1,2$, for which the integral equality (2.7) is valid. Since $\|f_n^i - u|_{S_i}\|_{C(S_\omega^i)} \to 0$ as $n \to \infty$, by Lemma 2.1 there exists a limit in the right-hand side of the equality (2.7), and in this case a limit exists in the left-hand side of the same equality, i.e. the sequence $v_n(x_1,x_2,t) = \int\limits_{x_2}^{t} u_n(x_1,x_2,\tau)\,d\tau$ tends to some continuous function $v(x_1,x_2,t)$ in the topology of the space $C(\overline{D})$. But $\partial v_n(x_,x_2,t)/\partial t = u_n(x_1,x_2,t)$ is a solution of the equation (2.1). Next, by the condition (iii) and Lemma 2.1, we have the equality $v(x_1,x_2,t) = \int\limits_{x_2}^{t} u(x_1,x_2,\tau)\,d\tau$. Therefore the function $u = \partial v/\partial t$ is a weak generalized solution of the equation (2.1) because

$$
\begin{aligned}
(u,\Box_2\omega)_{L_2(D)} &= \Big(\frac{\partial v}{\partial t},\Box_2\omega\Big)_{L_2(D)} = -\Big(v,\Big(\frac{\partial}{\partial t}\Big)\Box_2\omega\Big)_{L_2(D)} = \\
&= -\lim_{n\to\infty}\Big(v_n,\Big(\frac{\partial}{\partial t}\Big)\Box_2\omega\Big)_{L_2(D)} = \lim_{n\to\infty}\Big(\frac{\partial v_n}{\partial t},\Box_2\omega\Big)_{L_2(D)} = \\
&= \lim_{n\to\infty}(u_n,\Box_2\omega)_{L_2(D)} = \lim_{n\to\infty} 0 = 0,
\end{aligned}
$$

which was to be demonstrated. □

Remark 2.3. Without restriction of generality, we assume that in the problem (2.1)–(2.3)

$$
u\big|_{S_1\cap S_2} = \varphi\big|_{S_1\cap S_2} = \mu\big|_{S_1\cap S_2} = 0 \tag{2.22}
$$

since otherwise, if $\varphi|_{S_1\cap S_2} = \widetilde{\lambda}(x_1) \not\equiv 0$, the function $\widetilde{\lambda}(t+x_1)$ is likewise a solution of the equation (2.1), and the new unknown function $u_1(x_1,x_2,t) = u(x_1,x_2,t) - \widetilde{\lambda}(t+x_1)$ satisfies the equation (2.1) and the boundary conditions

$$
u_1(P) = \varphi_1(P), \quad P \in S_2; \qquad \int\limits_{J_-(P)J_+(P)} u_1\,ds = u_1(P) + \mu_1(P), \quad P \in S_1,
$$

in which by the consistency condition $(\mu+\varphi)|_{S_1\cap S_2} = 0$ we have $u_1|_{S_1\cap S_2} = \varphi_1|_{S_1\cap S_2} = \mu_1|_{S_1\cap S_2} = 0$.

2.4. Reduction of the problem (2.1), (2.2), (2.3) to an integral Volterra type equation with a singular kernel, and its investigation. A solution of the problem (2.1)–(2.3) will be sought in the class of generalized solutions of the equation (2.1) of the class $C(\overline{D})$. Then according to Remark 2.1, by Lemma 2.2 the boundary condition (2.3) for $P=(x_1,t/\sqrt{2},t/\sqrt{2})\in S_1$, i.e. $J(P)=(x_1,0,\sqrt{2}\,t)$, with regard for (2.2), (2.22) and the integral equality (2.7) can be written in the form of the equation

$$\psi(x_1,t)-\int\limits_{S^1_{J_+(P)}} B(x_1,t;y_1,y_2')\psi(y_1,y_2')\,dy_1\,dy_2'=f(x_1,t) \tag{2.23}$$

with respect to the unknown function $\psi(x_1,t):=u|_{S^1_{J_+(P)}}=u(x_1,t/\sqrt{2},t/\sqrt{2})$. Here, by (2.11) and (2.22),

$$f(x_1,t):=2\int\limits_{S^2_{J_+(P)}}\varphi\,\frac{\partial E(r,\sqrt{2}\,t,\tau)}{\partial N}\,ds-\mu(x_1,t),\quad f\big|_{S_1\cap S_2}=0, \tag{2.24}$$

$$\begin{aligned}2\sqrt{2}\,\pi B(x_1,t;y_1,y_2') &:= -4\sqrt{2}\,\pi\,\frac{\partial E(r,\sqrt{2}\,t,\tau)}{\partial N}\Big|_{S^1_{J_+(P)}}=\\
&=-2\big((\sqrt{2}\,t-\tau)^2-(x_1-y_1)^2-\tau^2\big)^{-\frac12}\Big|_{\tau=y_2'/\sqrt{2}}-2\tau(\sqrt{2}\,t-\tau)\times\\
&\times\big[(x_1-y_1)^2+\tau^2\big]^{-1}\big((\sqrt{2}\,t-\tau)^2-(x_1-y_1)^2-\tau^2\big)^{-\frac12}\Big|_{\tau=y_2'/\sqrt{2}}=\\
&=2\big(2t\big[t-y_2'-(2t)^{-1}(x_1-y_1)^2\big]\big)^{-\frac12}-\\
&-y_2'(2t-y_2')\Big[(x_1-y_1)^2+\frac{{y_2'}^2}{2}\Big]^{-1}\big(2t\big[t-y_2'-(2t)^{-1}(x_1-y_1)^2\big]\big)^{-\frac12}.\end{aligned} \tag{2.25}$$

To estimate the integral term in the left-hand side of the equation (2.23), we will use the following reasoning. Under the transformation $y_1=x_1+tz_1$, $y_2'=tz_2$ the domain $S^1_{J_+}(P)$ transforms into the domain $\Omega_3:\ -\sqrt{2}\le z_1\le\sqrt{2}$, $0\le z_2\le(-1/2)z_1^2+1$ of the plane of the variables z_1, z_2. In its turn, the domain Ω_3 under the transformation $z_1=\sigma$, $z_2=-(2\tau)^{-1}\sigma^2+\tau$ transforms into the triangle $\widetilde{\Omega}_3:\ 0\le\tau\le1$, $-\sqrt{2}\,\tau\le\sigma\le\sqrt{2}\,\tau$ of the plane of the variables σ, τ (the parabola $z_2=-(2\tau_0)^{-1}z_1^2+\tau_0$ for a fixed $\tau_0\in(0,1]$ transforms into the segment $\tau=\tau_0$, $-\sqrt{2}\,\tau_0\le\sigma\le\sqrt{2}\,\tau_0$). In addition, it is easy to see that

$$\frac{\partial(y_1,y_2)}{\partial(z_1,z_2)}=t^2,\ \ (z_1,z_2)\in\Omega_3, \tag{2.26}$$

$$1\le\frac{\partial(z_1,z_2)}{\partial(\sigma,\tau)}=1+\frac{\sigma^2}{(2\tau)^2}\le2,\quad \frac{\sigma^2}{(2\tau)^2}\le1,\ \ (\sigma,\tau)\in\widetilde{\Omega}_3. \tag{2.27}$$

In the plane of the variables z_1, z_2 we introduce the domains Ω_4 : $-\sqrt{2}/4\le z_1\le\sqrt{2}/4$, $0\le z_2\le-2z_1^2+1/4$ and $\Omega_5:\ z_1^2+z_2^2\le1/4$, $z_2\ge0$. It is not difficult to verify that $\Omega_4 \subset \Omega_5 \subset \Omega_3$ and hence $\Omega_3 \setminus \Omega_5 \subset \Omega_3 \setminus \Omega_4$ since $\sqrt{1/4 - z_1^2} \le 1/2 < 7/8 = -(1/2)(1/2)^2 + 1 \le -(1/2)z_1^2 + 1$ for $|z_1| \le 1/2$ and $-2z_1^2 + 1/4 \le 1/4 < 1/(2\sqrt{2}) = \sqrt{1/4 - 2(1/4)^2} \le \sqrt{1/4 - z_1^2}$ for $|z_1| \le \sqrt{2}/4$. Moreover, taking into account (2.27), we see that there take place the following inequalities:

$$\Big(1 + \frac{\sigma^2}{2\tau}\Big)^{-\frac{1}{2}} \frac{\partial(z_1, z_2)}{\partial(\sigma, \tau)} \le 2, \quad (\sigma, \tau) \in \widetilde{\Omega}_3,$$

$$1 - z_2 - \Big(\frac{1}{2}\Big)z_1^2 \ge 1 - \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{4} = \frac{3}{8}\,, \quad (z_1, z_2) \in \Omega_5,$$

$$\frac{z_2(2-z_2)}{z_1^2 + z_2^2/2} \le \frac{2z_2}{z_1^2 + z_2^2}\,(2 - z_2) \le 2\,\frac{2 - z_2}{(z_1^2 + z_2^2)^{1/2}} \le \frac{4}{(z_1^2 + z_2^2)^{1/2}}\,, \quad (z_1, z_2) \in \Omega_5,$$

$$\frac{z_2(2 - z_2)}{[z_1^2 + z_2^2/2](2(1 - z_2 - z_1^2/2))^{1/2}} \le$$

$$\le \frac{4}{(2(3/8))^{1/2}(z_1^2 + z_2^2)^{1/2}} = \frac{8}{3^{1/2}(z_1^2 + z_2^2)^{1/2}}\,, \quad (z_1, z_2) \in \Omega_5,$$

$$\frac{z_2(2 - z_2)}{z_1^2 + z_2^2/2} \le 2\,\frac{z_2(2 - z_2)}{z_1^2 + z_2^2} \le 2\,\frac{(1/2)\cdot 2}{1/4} = 8, \quad (z_1, z_2) \in \Omega_3 \setminus \Omega_5.$$

Therefore by (2.25)–(2.27) we have

$$\int\limits_{S^1_{J_+(P)}} \big|B(x_1, t; y_1, y_2')\big|\, dy_1\, dy_2' = \int\limits_{\Omega_3} \big|B(x_1, t; x_1 + tz_1, tz_2)\big| t^2\, dz_1\, dz_2 =$$

$$= \frac{t}{\sqrt{2}\,\pi} \int\limits_{\Omega_3} \frac{1}{(2(1 - z_2^2 - z_1^2/2))^{1/2}}\, dz_1\, dz_2 +$$

$$+ \frac{t}{2\sqrt{2}\,\pi} \int\limits_{\Omega_3} \frac{z_2(2 - z_2)}{[z_1^2 + z_2^2/2](2(1 - z_2 - z_1^2/2))^{1/2}}\, dz_1\, dz_2 =$$

$$= \frac{t}{\sqrt{2}\,\pi} \int\limits_0^1 d\tau \int\limits_{-\sqrt{2}\,\tau}^{\sqrt{2}\,\tau} \frac{1}{(2(1 - \tau)(1 + \sigma^2/(2\tau)))^{1/2}}\, \frac{\partial(z_1, z_2)}{\partial(\sigma, \tau)}\, d\sigma +$$

$$+ \frac{t}{2\sqrt{2}\,\pi} \int\limits_{\Omega_5} \frac{z_2(2 - z_2)}{[z_1^2 + z_2^2/2](2(1 - z_2 - z_1^2/2))^{1/2}}\, dz_1\, dz_2 +$$

$$+ \frac{t}{2\sqrt{2}\,\pi} \int\limits_{\Omega_3 \setminus \Omega_5} \frac{z_2(2 - z_2)}{[z_1^2 + z_2^2/2](2(1 - z_2 - z_1^2/2))^{1/2}}\, dz_1\, dz_2 \le$$

$$\le \frac{t}{\sqrt{2}\,\pi} \int\limits_0^1 d\tau \int\limits_{-\sqrt{2}\,\tau}^{\sqrt{2}\,\tau} \frac{2}{(2(1 - \tau))^{1/2}}\, d\sigma + \frac{t}{2\sqrt{2}\,\pi}\, \frac{8}{\sqrt{3}} \int\limits_{\Omega_5} \frac{dz_1\, dz_2}{(z_1^2 + z_2^2)^{1/2}} +$$

$$+\frac{t}{2\sqrt{2}\,\pi}\int\limits_{\Omega_3\setminus\Omega_5}\frac{8}{(2(1-z_2-z_1^2/2))^{1/2}}\,dz_1\,dz_2\le\frac{2\sqrt{2}}{\pi}\,t\int\limits_0^1\tau(1-\tau)^{-\frac{1}{2}}\,d\tau+$$

$$+\frac{4t}{\sqrt{6}\,\pi}\,t\int\limits_0^{1/2}dr\int\limits_0^{\pi}\frac{r}{r}\,d\theta+\frac{t}{2\sqrt{2}\,\pi}\int\limits_{\Omega_3\setminus\Omega_4}8\Big(2\Big(1-z_2-\frac{z_1^2}{2}\Big)\Big)^{-\frac{1}{2}}\,dz_1\,dz_2\le$$

$$\le\frac{2\sqrt{2}}{\pi}\,t\int\limits_0^1\tau(1-\tau)^{-\frac{1}{2}}\,d\tau+\frac{4t}{\sqrt{6}\,\pi}\cdot\frac{\pi}{2}+$$

$$+\frac{2t}{\pi}\int\limits_{1/4}^{1}d\tau\int\limits_{-\sqrt{2}\,\tau}^{\sqrt{2}\,\tau}\Big((1-\tau)\Big(1+\frac{\sigma^2}{2\tau}\Big)\Big)^{-\frac{1}{2}}\,\frac{\partial(z_1,z_2}{\partial(\sigma,\tau)}\,d\sigma\le$$

$$\le\frac{2\sqrt{2}}{\pi}\,t\int\limits_0^1\tau(1-\tau)^{-\frac{1}{2}}\,d\tau+\frac{2t}{\sqrt{6}}+\frac{2t}{\pi}\int\limits_{1/4}^{1}4\sqrt{2}\,\tau(1-\tau)^{-\frac{1}{2}}\,d\tau\le$$

$$\le\frac{10\sqrt{2}}{\pi}\,t\int\limits_0^1\tau(1-\tau)^{-\frac{1}{2}}\,d\tau+\frac{2}{\sqrt{6}}\,t\le\Big(\frac{20\sqrt{2}}{\pi}+\frac{2}{\sqrt{6}}\Big)t. \tag{2.28}$$

Remark 2.4. Since the function $B(x_1,t;y_1,y_2')$ has weak singularities, the operator K acting by the formula

$$(K\psi)(x_1,t):=\int\limits_{S^1_{J_+}(P)}B(x_1,t;y_1,y_2')\psi(y_1,y_2')\psi(y_1,y_2')\,dy_1\,dy_2' \tag{2.29}$$

is a linear continuous operator acting from the Banach space $C(\Sigma_\delta)$ of continuous bounded functions defined in the closed strip $\Sigma_\delta=\big\{(y_1,y_2')\in R^2:\ -\infty<y_1<\infty,\ 0\le y_2'\le\delta\big\}$ into itself, and owing to (2.29) for its norm the estimate

$$\|K\|_{C(\Sigma_\delta)\to C(\Sigma_\delta)}\le c\delta,\ \ c=\frac{20\sqrt{2}}{\pi}+\frac{2}{\sqrt{6}} \tag{2.30}$$

holds.

For $0<\tau<t$, $0<\tau<\delta$, $P=(x_1,t/\sqrt{2},t/\sqrt{2})\in S_1$ we introduce the sets $\Omega_{P,\tau}=\{(y_1,y_2')\in S^1_{J_+(P)}:\ y_2'\ge\tau\}$, $\Sigma_{\delta,\tau}=\{(y_1,y_2')\in\Sigma_\delta:\ y_2'\ge\tau\}$.

Remark 2.5. Analogously we can show that the operator K_τ acting by the formula

$$(K_\tau\psi)(x_1,t):=\int\limits_{\Omega_{P,\tau}}B(x_1,t;y_1,y_2')\psi(y_1,y_2')\,dy_1\,dy_2' \tag{2.31}$$

is a linear continuous operator acting from the space $C(\Sigma_{\delta,\tau})$ of continuous bounded functions with the domain of definition $\Sigma_{\delta,\tau}$ into itself, and for its norm the estimate

$$\|K_\tau\|_{C(\Sigma_{\delta,\tau})\to C(\Sigma_{\delta,\tau})} \le c_1(\delta-\tau) \tag{2.32}$$

is valid with a positive constant c_1 ($c_1 \ge c$) independent of δ and τ.

Lemma 2.3. *In the class $C(\Sigma_\delta)$ the equation* (2.23) *cannot have more than one solution.*

Proof. Indeed, let $\psi(x_1,t) \in C(\Sigma_\delta)$ be a solution of the homogeneous equation corresponding to (2.23), i.e. with regard for (2.29),

$$\psi(x_1,t) - (K\psi)(x_1,t) = 0, \quad (x_1,t) \in \Sigma_\delta. \tag{2.33}$$

It immediately follows from the above equality and (2.30) that for $\delta < c^{-1}$ the solution $\psi(x_1,t)$ of the equation (2.33) is identically equal to zero in Σ_δ.

Let now $\delta \ge c^{-1}$. Then there exists a natural number k such that $\delta/k < c_1^{-1}$. By the above-said, the solution $\psi(x_1,t)$ of the equation (2.33) is equal to zero identically in the strip $\Sigma_{\delta/k}$. Therefore this equation in the strip $\Sigma_{2\delta/k,\delta/k}$ with regard for the structure of the set $S^1_{J_+(P)}$ and (2.31) can be rewritten in the form $\psi(x_1,t) - (K_{\delta/k}\psi)(x_1,t) = 0$, $(x_1,t) \in \Sigma_{2\delta/k,\delta/k}$, whence by virtue of (2.32) we obtain $\psi(x_1,t) = 0$, $(x_1,t) \in \Sigma_{2\delta/k,\delta/k}$.

Continuing this process, step by step we obtain that the solution $\psi(x_1,t)$ of the equation (2.33) is equal to zero in each of the strips $\Sigma_{\delta/k}, \Sigma_{2\delta/k,\delta/k}, \dots, \Sigma_{i\delta/k,((i-1)/k)\delta}, \dots, \Sigma_{\delta,((k-1)/k)\delta}$, i.e. $\psi(x_1,t)$ is equal to zero in the entire strip Σ_δ. Thus Lemma 2.3 is proved. □

Lemma 2.4. *For every $f \in C(\Sigma_\delta)$ the equation* (2.23) *is uniquely solvable in the class $C(\Sigma_\delta)$. In addition, if $f(y_1,0) = 0$, $-\infty < y_1 < +\infty$, then $\psi(y_1,0) = 0$, $-\infty < y_1 < +\infty$.*

Proof. Indeed, by the estimate (2.30) and the principle of contracted maps the equation (2.23) is uniquely solvable in the space $C(\Sigma_{\delta/k})$, where a natural number k is chosen in such a way that $\delta/k < c_1^{-1} \le c^{-1}$, and the solution ψ of the equation (2.23) in the strip $\Sigma_{\delta/k}$ is representable in the form $\psi = \sum\limits_{i=0}^{\infty} K^i f$.

Having denoted this solution in the strip $\Sigma_{\delta/k}$ by $\psi_0 \in C(\Sigma_{\delta/k})$, for finding a solution of the equation (2.23) in the strip $\Sigma_{2\delta/k,\delta/k}$ we obtain the equality

$$\psi(x_1,t) - (K_{\delta/k})(x_1,t) =$$
$$= f(x_1,t) + \int\limits_{S^1_{J_+(P)}\setminus\Omega_{P,\delta/k}} B(x_1,t;y_1,y_2')\psi_0(y_1,y_2')\,dy_1\,dy_2',$$

which by virtue of the estimate (2.32) is likewise uniquely solvable in the space $C(\Sigma_{2\delta/k,\delta/k})$. In addition, the given solution is a continuous extension of the solution ψ_0 from the strip $\Sigma_{\delta/k}$ to the strip $\Sigma_{2\delta/k,\delta/k}$. Continuing this process in the strips $\Sigma_{3\delta/k,2\delta/k},\dots,\Sigma_{\delta,((k-1)/k)\delta}$, we can construct in the space $C(\Sigma_\delta)$ a solution ψ of the equation (2.23) whose uniqueness follows from Lemma 2.3. It can be easily seen that if $f(y_1,0)=0$, $-\infty<y_1<+\infty$, then $\psi(y_1,0)=0$, $-\infty<y_1<+\infty$, as well. Thus according to Remark 2.3 Lemma 2.4 is proved. □

Below under $C^k(\Sigma_\delta)$ it is meant the Banach space of k times continuously differentiable functions in the closed strip Σ_δ with the finite norm

$$\|\psi\|_{C^k(\Sigma_\delta)} := \sum_{\alpha_1+\alpha_2\le k}\ \sup_{(y_1,y_2')\in\Sigma_\delta}\Big|\frac{\partial^{\alpha_1+\alpha_2}\psi(y_1,y_2')}{\partial y_1^{\alpha_1}\partial y_2'^{\alpha_2}}\Big| < +\infty,\ \ k\ge 0.$$

Lemma 2.5. *Under the conditions of Lemma* 2.4*, if* $C^k(\Sigma_\delta)$*, then a continuous bounded solution* ψ *of the equation* (2.23) *belongs to the space* $C^k(\Sigma_\delta)$*.*

Proof. For the sake of simplicity, we restrict ourselves to the consideration of the case $k=1$. By (2.25), (2.26) and (2.29), we have

$$\begin{aligned}(K\psi)(x_1,t) &= \int\limits_{S^1_{J_+(P)}} B(x_1,t;y_1,y_2')\psi(y_1,y_2')\,dy_1\,dy_2' = \\ &= \int\limits_{\Omega_3} B(x_1,t;x_3+tz_1,tz_2)\psi(x_1+tz_1,tz_2)t^2\,dz_1\,dz_2 = \\ &= t\int\limits_{\Omega_3} G(z_1,z_2)\psi(x_1+tz_1,tz_2)\,dz_1\,dz_2, \end{aligned} \tag{2.34}$$

where $\Omega_3:\ -\sqrt2\le z_1\le\sqrt2,\ 0\le z_2\le -(1/2)z_1^2+1,\ 4\pi G(z_1,z_2):= -2(1-z_2-z_1^2/2)^{-\frac12}-z_2(2-z_2)[z_1^2+z_2^2/2]^{-1}(1-z_2-z_1^2/2)^{-\frac12}$. From (2.34), for $\psi\in C^1(\Sigma_\delta)$ we obtain

$$\begin{aligned}\frac{\partial(K\psi)(x_1,t)}{\partial t} &= \int\limits_{\Omega_3} G(z_1,z_2)\psi(x_1+tz_1,tz_2)\,dz_1\,dz_2+ \\ &+t\int\limits_{\Omega_3} G(z_1,z_2)z_1\,\frac{\psi(x_1+tz_1,tz_2)}{\partial y_1}\,dz_1\,dz_2+ \\ &+t\int\limits_{\Omega_3} G(z_1,z_2)z_2\,\frac{\psi(x_1+tz_1,tz_2)}{\partial y_2'}\,dz_1\,dz_2 = J_1+tJ_2+tJ_3,\end{aligned} \tag{2.35}$$

$$\frac{\partial(K\psi)(x_1,t)}{\partial x_1} = tJ_2. \tag{2.36}$$

Comparing with (2.28) and taking into account that $|z_1|\le\sqrt2$, $0\le z_2\le 1$ when $(z_1,z_2)\in\Omega_3$, we can see that

$$J_1,\ J_2\le c,\quad J_3\le\sqrt2\,c,\quad c=\frac{20\sqrt2}{\pi}+\frac{2}{\sqrt6}\,. \tag{2.37}$$

By (2.22), (2.24) and Lemma 2.4 we have

$$f(y_1,0)=\psi(y_1,0)=0,\ \ -\infty<y_1<\infty, \tag{2.38}$$

and hence for $\psi\in C^1(\Sigma_\delta)$ we obtain

$$\psi(y_1,y_2')=\int_0^{y_2'}\frac{\partial\psi(y_1,\xi)}{\partial y_2'}\,d\xi,$$
$$|\psi(y_1,y_2')|\le y_2'\Big\|\frac{\partial\psi}{\partial y_2'}\Big\|_{C(\Sigma_{y_2'})},\quad \big|\psi(x_1+tz_1,tz_2)\big|\le t\Big\|\frac{\partial\psi}{\partial y_2'}\Big\|_{C(\Sigma_t)}. \tag{2.39}$$

From (2.35)–(2.39) we find that

$$\big|(K\psi)(x_1,t)\big|+\Big|\frac{\partial(K\psi)(x_1,t)}{\partial t}\Big|+\Big|\frac{\partial(K\psi)(x_1,t)}{\partial x_1}\Big|\le t\|\psi\|_{C(\Sigma_t)}+t\Big\|\frac{\partial\psi}{\partial y_2'}\Big\|_{C(\Sigma_t)}+$$
$$+\sqrt{2}\,t\Big\|\frac{\partial\psi}{\partial y_1}\Big\|_{C(\Sigma_t)}+t\Big\|\frac{\partial\psi}{\partial y_2'}\Big\|_{C(\Sigma_t)}+t\Big\|\frac{\partial\psi}{\partial y_1}\Big\|_{C(\Sigma_t)}\int_{\Omega_3}|G(z_1,z_2)|\,dz_1\,dz_2\le$$
$$\le 3ct\Big[\|\psi\|_{C(\Sigma_t)}+\Big\|\frac{\partial\psi}{\partial y_1}\Big\|_{C(\Sigma_t)}+\Big\|\frac{\partial\psi}{\partial y_2'}\Big\|_{C(\Sigma_t)}\Big]=$$
$$=3ct\|\psi\|_{C^1(\Sigma_t)},\ \ (x_1,t)\in\Sigma_\delta. \tag{2.40}$$

Due to (2.40), for the norm of the operator $K:C^1(\Sigma_\delta)\to C^1(\Sigma_\delta)$ the estimate

$$\|K\|_{C^1(\Sigma_\delta)\to C^1(\Sigma_\delta)}\le 3c\delta$$

holds, from which it follows that Lemma 2.5 is valid for $\delta<1/(3c)$. If $\delta\ge 1/(3c)$, then our reasoning is the same as in proving Lemmas 2.3 and 2.4. The case $k>1$ is considered analogously. Thus the proof of Lemma 2.5 is complete. □

Let $D_\tau:=\{(x_1,x_2,t)\in D;\ t<\tau\}$, $\tau=\text{const}>0$. Under $C^k(\overline{D})$ we mean the space of k times continuously differentiable in $\overline{D}$ functions for which the norm

$$\|u\|_{C^k(\overline{D}_\tau)}:=\sum_{|\alpha|\le k}\ \sup_{(x,t)\in\overline{D}_\tau}\Big|\frac{\partial^{|\alpha|}u(x,t)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial t^{\alpha_3}}\Big|<\infty$$

is finite for every $\tau>0$; here $x=(x_1,x_2)$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. For $k=0$, instead of $C^0(\overline{D})$ we write $C(\overline{D})$.

In accordance with Remarks 2.1, if y_1 and y_2' are rectangular coordinates on S_1, then we assume that $S_{1\tau}:=\{(y_1,y_2')\in S_1:\ y_2'\le\tau\}$, $\tau=\text{const}>0$. Denote by $C^k(S_1)$ the space of k times continuously differentiable in S_1 functions for which the norm

$$\|\psi\|_{C^k(S_{1\tau})}:=\sum_{|\alpha|\le k}\ \sup_{(y_1,y_2')\in S_{1\tau}}\Big|\frac{\partial^{\alpha_1+\alpha_2}\psi(y_1,y_2')}{\partial y_1^{\alpha_1}\partial {y_2'}^{\alpha_2}}\Big|<\infty$$

is finite for $\tau>0$. For $k=0$, instead of $C^0(S_1)$ we write $C(S_1)$. Analogously we introduce the space $C^k(S_1)$.

From Lemmas 2.2 and 2.3 immediately follows

Lemma 2.6. *The problem* (2.1)–(2.3) *cannot have more than one generalized solution of the class* $C(\overline{D})$.

To construct a generalized solution of the class $C(\overline{D})$ of the problem (2.1)–(2.3), of the functions φ and μ in the boundary conditions (2.2) and (2.3) it is required that $\varphi \in C^1(S_2)$, $\mu \in C^1(S_1)$, and as is mentioned above, without restriction of generality we assume that these functions satisfy the equalities (2.22). In this case $f \in C^1(S_1)$, and by Lemma 2.5 the solution ψ of the equation (2.23) belongs to the space $C^1(S_1)$ and satisfies the equality (2.38).

Let us show that the function

$$U_1(x,t) := 2 \int\limits_{S^1_{x,t}} \psi \, \frac{\partial E(r,t,\tau)}{\partial N} \, ds \tag{2.41}$$

and its derivative $\partial U_1/\partial t$, belong to the space $C(\overline{D})$, where the surfaces $S^i_{x,t}$, $i=1,2$, $S_{x,t}$ and the function $E(r,t,\tau)$ are defined above (see the equalities (2.4)–(2.7)). Indeed, the fact that $U_1 \in C(\overline{D})$ follows from Lemma 2.1. Taking into account (2.38) and integrating the right-hand side of (2.41) by parts, we obtain

$$\begin{aligned} U_1(x,t) &= -2 \int\limits_{S^1_{x,t}} \frac{\partial \psi}{\partial N} \, E(r,t,\tau) \, ds = \\ &= -2 \int\limits_{x_1-\sqrt{t^2-x_2^2}}^{x_1+\sqrt{t^2-x_2^2}} dy_1 \int\limits_0^{\sigma(x,y_1,t)} \frac{\partial \psi}{\partial N} \, E(r,t,\tau) \, dy_2', \end{aligned} \tag{2.42}$$

where $\sigma(x,y_1,t) := [-(y_1-x_1)^2/(t-x_2)+t+x_2]/\sqrt{2}$. Denote by $\gamma^1_{x,t}$ the part of the parabola $y_2' - \sigma(x,y_1,t) = 0$ lying on $S^1_{x,t}$. Since $\sigma(x,y_1,t)$ for $y_1 = x_1 \pm \sqrt{t^2-x_2^2}$ and $E(r,t,\tau)|_{\gamma^1_{x,t}} = 0$, from (2.42) it follows that $\partial U_1/\partial t = -2 \int\limits_{S^1_{x,t}} (\partial\psi/\partial N)(\partial E(r,t,\tau)/\partial t) \, ds = 2(T_0 \partial\psi/\partial N)(x,t)$, where T_0 is the operator from Lemma 2.1. Therefore by Lemma 2.1 the function $\partial U_1/\partial t$ belongs to $C(\overline{D})$, and by the equalities (2.10) and (2.38) we have

$$\begin{gathered} \frac{\partial U_1(x,t)}{\partial t} = \int\limits_0^{\sqrt{2}\,t} \frac{\partial \psi(x_1,y_2')}{\partial N} \, dy_2' = \\ = \int\limits_0^{\sqrt{2}\,t} \frac{\partial \psi(x_1,y_2')}{\partial y_2'} \, dy_2' = \psi(x_1,\sqrt{2}\,t), \ \ (x,t) \in S_1, \\ \frac{\partial U_1(x,t)}{\partial t} = 0, \ \ (x,t) \in S_2. \end{gathered} \tag{2.43}$$

Similarly, by Remark 2.2, the function

$$U_2(x,t) := 2 \int\limits_{S^2_{x,t}} \varphi \, \frac{\partial E(r,t,\tau)}{\partial N} \, ds \tag{2.44}$$

and its derivative $\partial U_2/\partial t$ belong to the space $C(\overline{D})$, where

$$\frac{\partial U_2(x,t)}{\partial t} = \varphi(x_1, \sqrt{2}\,t), \ \ (x,t) \in S_2; \quad \frac{\partial U_2(x,t)}{\partial t} = 0, \ \ (x,t) \in S_1. \tag{2.45}$$

Consider now the function

$$u(x,t) := \Big(\frac{\partial}{\partial t}\Big)\big(U_1(x,t) + U_2(x,t)\big) \tag{2.46}$$

which by virtue of the above-said belongs to the space $C(\overline{D})$. By (2.43)–(2.46) and Remark 2.1, we have

$$u\big|_{S_1} = \psi, \quad u\big|_{S_2} = \varphi, \tag{2.47}$$

$$\int\limits_{x_2}^{t} u(x_1, x_2, \tau)\, d\tau = 2 \int\limits_{S^1_{x,t}} \psi \, \frac{\partial E(r,t,\tau)}{\partial N} \, ds + 2 \int\limits_{S^2_{x,t}} \varphi \, \frac{\partial E(r,t,\tau)}{\partial N} \, ds =$$

$$= 2 \int\limits_{S_{x,t}} u \, \frac{\partial E(r,t,\tau)}{\partial N} \, ds. \tag{2.48}$$

It follows from (2.22), (2.48) and Lemma 2.2 that the function $u(x,t)$ defined by the formula (2.46) is a generalized solution of the equation (2.1) of the class $C(\overline{D})$, and from the equalities (2.23) and (2.47) it follows that this function satisfies the conditions (2.2) and (2.3). Consequently, the function $u(x,t)$ constructed by the formula (2.46) is a generalized solution of the problem (2.1)–(2.3) of the class $C(\overline{D})$. Thus with regard for Lemma 2.6 and Remark 2.3 we have the following

Theorem 2.1. *For every* $\mu \in C^1(S_1)$ *and* $\varphi \in C^1(S_2)$ *the problem* (2.1)–(2.3) *has a unique generalized solution of the class* $C(\overline{D})$.

Remark 2.6. On the basis of Lemma 2.5 we can show that if $\varphi \in C^{k+1}(S_2)$ and $\mu \in C^{k+1}(S_1)$, $k \geq 1$, then the solution of the problem (2.1)–(2.3) whose existence is stated in Theorem 2.1 will belong to the class $C^k(\overline{D})$, and hence for $k \geq 2$ it will be a classical one.

CHAPTER 4

The Characteristic Cauchy Problem for Multi-Dimensional Wave Equations with Power Nonlinearity

1. Nonexistence of Global Solutions of the Characteristic Cauchy Problem for the Wave Equation with Power Nonlinearity of Type $\lambda|u|^\alpha$

1.1. Statement of the problem. For a nonlinear wave equation of the type

$$\square u := u_{tt} - \Delta u = \lambda |u|^\alpha + F, \tag{1.1}$$

where λ and α are given positive constants, F is a given and u is an unknown real function, we consider the characteristic Cauchy problem on finding in the light cone of the future $D:\ t > |x|$, $x = (x_1, \dots, x_n)$, $n > 1$, a solution $u(x,t)$ of the equation (1.1) by the boundary condition

$$u\big|_{\partial D} = f. \tag{1.2}$$

Here f is a given real function on the characteristic conic surface $\partial D:\ t = |x|$, Δ is the Laplace operator with respect to the variables $x_1, \dots, x_n$.

Below it will be shown that under certain conditions imposed on the nonlinearity exponent α and on the functions F and f, the problem (1.1), (1.2) has no global solution, although, as it will be proved, this problem is locally solvable.

To introduce the definition of weak generalized solution of the problem (1.1), (1.2), it should be noted that if $u \in C^2(\overline{D})$ is a classical solution of that problem, then multiplying both parts of the equation (1.1) by an arbitrary function $\varphi \in C^1(\overline{D})$ with a bounded support with respect to the variable $r = (t^2 + |x|^2)^{1/2}$, i.e. equal to zero for sufficiently large r, after integration by parts we obtain

$$\int\limits_{\partial D} \frac{\partial u}{\partial N}\, \varphi\, ds - \int\limits_{D} u_t \varphi_t\, dx\, dt + \int\limits_{D} \nabla u \nabla \varphi\, dx\, dt =$$
$$= \lambda \int\limits_{D} |u|^\alpha \varphi\, dx\, dt + \int\limits_{D} F \varphi\, dx\, dt, \tag{1.3}$$

where $\frac{\partial}{\partial N} = \nu_{n+1}\frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal, $\nu = (\nu_1, \dots, \nu_n, \nu_{n+1})$ is the unit vector of the outer normal to ∂D, $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

Taking into account that on the conic surface $\partial D: \ t = |x|$ the derivative with respect to the conormal $\frac{\partial}{\partial N} = \nu_{n+1}\frac{\partial}{\partial t} - \sum\limits_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}$ is an inner differential operator, the equality (1.3) by virtue of (1.2) can be written as

$$-\int\limits_D u_t\varphi_t \, dx\, dt + \int\limits_D \nabla u \nabla\varphi \, dx\, dt = \\ = \lambda \int\limits_D |u|^\alpha \varphi \, dx\, dt + \int\limits_D F\varphi \, dx\, dt - \int\limits_{\partial D} \frac{\partial f}{\partial N}\, \varphi \, ds. \tag{1.4}$$

The equality (1.4) is basic in the definition of generalized solution of the problem (1.1), (1.2).

Definition 1.1. For $F \in \widetilde{L}_{2,\mathrm{loc}}(D)$ and $f \in \widetilde{W}^1_{2,\mathrm{loc}}(\partial D)$, the function $u \in \widetilde{L}_{\alpha,\mathrm{loc}}(D) \cap \widetilde{W}^1_{2,\mathrm{loc}}(D)$ is said to be a weak generalized solution of the problem (1.1), (1.2) if for every function $\varphi \in C^1(\overline{D})$ with a bounded support with respect to the variable $r = (t^2 + |x|^2)^{1/2}$ the integral equality (1.4) is fulfilled. Such a solution will be also called a global solution of the problem (1.1), (1.2).

Here the space $\widetilde{L}_{2,\mathrm{loc}}(D)$ $(\widetilde{W}^1_{2,\mathrm{loc}}(\partial D))$ consists of the functions F (f) whose restriction to the set $D \cap \{t < \tau\}$ $(\partial D \cap \{t < \tau\})$ for every $\tau > 0$ belongs to the space $L_2(D \cap \{t < \tau\})$ $(W^1_2(\partial D \cap \{t < \tau\}))$. The spaces $\widetilde{L}_{\alpha,\mathrm{loc}}(D)$ and $\widetilde{W}^1_{2,\mathrm{loc}}(D)$ are determined analogously. The space $W^1_2(\Omega)$ is the well-known Sobolev space [84, p. 56].

For the equation (1.1), the characteristic problem in the conic domain $D_\tau = D \cap \{t < \tau\}$, $\tau = \mathrm{const} > 0$, i.e. $D_\tau : \ |x| < t < \tau$, is formulated analogously. Assume $S_\tau = \partial D \cap \partial D_\tau$, i.e. $S_\tau :; t = |x|$, $t \le \tau$.

Definition 1.2. Let $F \in L_2(D_\tau)$ and $f \in W^1_2(S_\tau)$. Then the function $u \in L_\alpha(D_\tau) \cap W^1_2(D_\tau)$ is said to be a weak generalized solution of the equation (1.1) in the domain D_τ satisfying instead of (1.2) the boundary condition $u|_{S_\tau} = f$, if for every function $\varphi \in C^1(\overline{D}_\tau)$ such that $\varphi|_{\partial D_\tau \setminus S_\tau} = 0$ the integral equality

$$-\int\limits_{\partial D_\tau} u_t\varphi_t \, dx\, dt + \int\limits_{D_\tau} \nabla u \nabla\varphi \, dx\, dt = \\ = \lambda \int\limits_{D_\tau} |u|^\alpha \varphi \, dx\, dt + \int\limits_{D_\tau} F\varphi \, dx\, dt - \int\limits_{S_\tau} \frac{\partial f}{\partial N}\, \varphi \, ds \tag{1.5}$$

is fulfilled.

1.2. The nonexistence of a global solution of the problem (1.1), (1.2).

Theorem 1.1. *Let*

$$f \in \widetilde{L}_{2,loc}(D), \quad F\big|_D \geq 0 \tag{1.6}$$

and

$$f \in \widetilde{W}^1_{2,loc}(\partial D), \quad f\big|_{\partial D} \geq 0, \quad \frac{\partial f}{\partial r}\bigg|_{\partial D} \geq 0. \tag{1.7}$$

Then if the nonlinearity exponent α *in the equation* (1.1) *satisfies the inequalities*

$$1 < \alpha \leq \frac{n+1}{n-1}, \tag{1.8}$$

then the problem (1.1), (1.2) *cannot have a global* (*if* $F = 0$ *and* $f = 0$ *nontrivial*) *weak generalized solution* $u \in \widetilde{L}_{\alpha,\mathrm{loc}}(D) \cap \widetilde{W}^1_{2,\mathrm{loc}}(D)$.

Proof. It should be noted that the inequality $\frac{\partial f}{\partial r}\big|_{\partial D} \geq 0$ in the condition (1.7) is understood in a generalized sense, i.e. by the assumption $f \in \widetilde{W}^1_{2,\mathrm{loc}}(\partial D)$ there exists the nonnegative generalized derivative $\frac{\partial f}{\partial r} \in \widetilde{L}_{2,\mathrm{loc}}(D)$, and hence for every function $\psi \in C(\partial D)$, $\psi \geq 0$, with a bounded support with respect to the variable r the inequality

$$\int\limits_{\partial D} \frac{\partial f}{\partial r}\, \psi\, ds \geq 0 \tag{1.9}$$

holds.

Here we apply the method of test functions [101, pp. 10–12]. Assume that under the conditions of Theorem 1.1 there exists a nontrivial global weak generalized solution $u \in \widetilde{L}_{\alpha,\mathrm{loc}}(D) \cap \widetilde{W}^1_{2,\mathrm{loc}}(D)$ of the problem (1.1), (1.2).

Assuming that in the integral equality (1.4) $\varphi \in C^2(\overline{D})$ and $\operatorname{diam}\operatorname{supp}\varphi < +\infty$, and integrating the left-hand side of that equality by parts, with regard for the boundary condition (1.2) we obtain

$$-\int\limits_D u_t \varphi_t\, dx\, dt + \int\limits_D \nabla u \nabla \varphi\, dx\, dt =$$

$$= \int\limits_D u \Box \varphi\, dx\, dt - \int\limits_{\partial D} u\, \frac{\partial \varphi}{\partial N}\, ds = \int\limits_D u \Box \varphi\, dx\, dt - \int\limits_{\partial D} f\, \frac{\partial \varphi}{\partial N}\, ds. \tag{1.10}$$

Taking now into account that the derivative with respect to the conormal $\partial/\partial N$ coincides on ∂D with that with respect to the spherical variable $r = (t^2 + |x|^2)^{1/2}$ with the minus sign and taking as the test function in (1.4) the function $\varphi(x,t) = \varphi_0[R^{-2}(t^2 + |x|^2)]$, where $\varphi_0 \in C^2((-\infty, +\infty))$, $\varphi_0 \geq 0$, $\varphi_0' \leq 0$, $\varphi_0(\sigma) = 1$ for $0 \leq \sigma \leq 1$ and $\varphi_0(\sigma) = 0$ for $\sigma \geq 2$,

$R = \text{const} \geq 0$ [101, p. 22], by virtue of (1.6), (1.7) and (1.9) we have

$$\int\limits_D F\varphi \, dx \, dt \geq 0, \quad \int\limits_{\partial D} f \frac{\partial \varphi}{\partial N} \, ds \geq 0, \quad \int\limits_{\partial D} \frac{\partial f}{\partial N} \varphi \, ds \leq 0. \tag{1.11}$$

By (1.10) and (1.11), from (1.4) it follows that

$$\int\limits_D u\Box\varphi \, dx \, dt \geq \lambda \int\limits_D |u|^\alpha \varphi \, dx \, dt. \tag{1.12}$$

Using the Hölder inequality

$$\int\limits_D g_1 g_2 \, dx \, dt \leq \Big(\int\limits_D |g_1|^\alpha \, dx \, dt \Big)^{1/\alpha} \Big(\int\limits_D |g_2|^\alpha \, dx \, dt \Big)^{1/\alpha'}, \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1,$$

we find that

$$\int\limits_D u\Box\varphi \, dx \, dt \leq \int\limits_D \big(|u| \varphi^{\frac{1}{\alpha}} \big) \big(\varphi^{-\frac{1}{\alpha}} |\Box\varphi| \big) \, dx \, dt \leq$$

$$\leq \Big(\int\limits_D |u|^\alpha \varphi \, dx \, dt \Big)^{1/\alpha} \Big(\int\limits_D \varphi^{-\frac{\alpha'}{\alpha}} |\Box\varphi|^{\alpha'} \, dx \, dt \Big)^{1/\alpha'} =$$

$$= \Big(\int\limits_D |u|^\alpha \varphi \, dx \, dt \Big)^{1/\alpha} \Big(\int\limits_D \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt \Big)^{1/\alpha'}. \tag{1.13}$$

It follows from (1.12) and (1.13) that

$$\lambda \int\limits_D |u|^\alpha \varphi \, dx \, dt \leq \Big(\int\limits_D |u|^\alpha \varphi \, dx \, dt \Big)^{1/\alpha} \Big(\int\limits_D \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt \Big)^{1/\alpha'},$$

whence

$$\int\limits_D |u|^\alpha \varphi \, dx \, dt \leq \lambda^{-\alpha'} \int\limits_D \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt. \tag{1.14}$$

After the change of variables $t = R\xi_0$, $x = R\xi$, we obtain $\varphi(x,t) = \varphi_0(\xi_0^2 + |\xi|^2)$ and

$$\int\limits_D \frac{|\Box\varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt = \int\limits_D \frac{\big| 2(1-n)\varphi_0' + 4R^{-2}(t^2 - |x|^2)\varphi_0'' \big|^{\alpha'}}{R^{2\alpha'} \varphi^{\alpha'-1}} \, dx \, dt =$$

$$= R^{n+1-2\alpha'} \int\limits_{\substack{1 \leq |\xi_0|^2 + |\xi|^2 \leq 2, \\ \xi_0 < |\xi|}} \frac{\big| 2(1-n)\varphi_0' + 4(\xi_0^2 - |\xi|^2)\varphi_0'' \big|^{\alpha'}}{\varphi_0^{\alpha'-1}} \, d\xi \, d\xi_0. \tag{1.15}$$

As is known [101, p. 22], the test function $\varphi(x,t) = \varphi_0[R^{-2}(t^2 + |x|^2)]$ with the above-mentioned properties exists, and its integrals in the right-hand side of (1.14) and (1.15) are finite.

From (1.14) and (1.15) it follows that

$$\int\limits_D |u|^\alpha \varphi\, dx\, dt \leq C R^{n+1-2\alpha'} \tag{1.16}$$

with a positive constant C independent of R. Passing in (1.16) to limit as $R \to \infty$, when $n+1-2\alpha' < 0$ which for $n > 1$ is equivalent to the condition $\alpha < \frac{n+1}{n-1}$ we obtain

$$\int\limits_D |u|^\alpha\, dx\, dt = 0,$$

but this contradicts our assumption. The limiting case in the condition (1.8) when $n+1-2\alpha' = 0$, i.e. for $\alpha = \frac{n+1}{n-1}$, is similar to that considered in [101, p. 23]. Thus Theorem 1.1 is proved completely. □

Remark 1.1. Despite the fact that under the conditions of Theorem 1.1 the problem (1.1), (1.2) cannot have a global solution, there may exist a local solution of the characteristic problem in the domain D_τ in the sense of Definition 1.2, i.e. of the problem

$$\Box u(x,t) = \lambda |u(x,t)|^\alpha + F(x,t), \quad (x,t) \in D_\tau, \tag{1.17}$$

$$u(x,t) = f(x,t), \quad (x,t) \in S_\tau. \tag{1.18}$$

Therefore there naturally arises the question on the estimation of the value $\tau = T$ such that for $\tau < T$ a solution of the problem (1.17), (1.18) exists in the domain D_τ, while for $\tau \geq T$ no solution of that problem exists in the space $L_\alpha(D_\tau) \cap W_2^1(D_\tau)$.

Assume that $u \in L_\alpha(D_\tau) \cap W_2^1(D_\tau)$ is a solution of the problem (1.17), (1.18) in the domain D_τ in the sense of the integral equality (1.5). As a test function in the equality (1.5) we take the function $\varphi(x,t) = \varphi_0\big[\frac{2}{\tau^2}(t^2 + |x|^2)\big]$, where the function $\varphi_0 \in C^2((-\infty,+\infty))$ is introduced in proving Theorem 1.1. Obviously this function satisfies all the conditions quoted in Definition 1.2. Integrating the left-hand side (1.5) by parts, just as in (1.10) we obtain

$$\int\limits_{D_\tau} u \Box \varphi\, dx\, dt =$$

$$= \lambda \int\limits_{D_\tau} |u|^\alpha \varphi\, dx\, dt + \int\limits_{D_\tau} F\varphi\, dx\, dt + \int\limits_{S_\tau} f \frac{\partial \varphi}{\partial N}\, ds - \int\limits_{S_\tau} \frac{\partial f}{\partial N}\, \varphi\, ds. \tag{1.19}$$

By (1.6) and (1.7), similarly to (1.11) the inequalities

$$\int\limits_{D_\tau} F\varphi\, dx\, dt \geq 0, \quad \int\limits_{S_\tau} f \frac{\partial \varphi}{\partial N}\, ds \geq 0, \quad \int\limits_{S_\tau} \frac{\partial f}{\partial N}\, \varphi\, ds \leq 0 \tag{1.20}$$

are valid.

Assuming that the functions F, f and φ are fixed, we introduce a function of one variable τ,

$$\gamma(\tau) = \int\limits_{D_\tau} F\varphi \, dx \, dt + \int\limits_{S_\tau} f \, \frac{\partial \varphi}{\partial N} \, ds - \int\limits_{S_\tau} \frac{\partial f}{\partial N} \, \varphi \, ds, \quad \tau > 0. \tag{1.21}$$

Due to the absolute continuity of the integral and the inequalities (1.20), the function $\gamma(\tau)$ from (1.21) is nonnegative, continuous and nondecreasing, and

$$\lim_{\tau \to 0} \gamma(\tau) = 0. \tag{1.22}$$

Taking into account (1.21), we rewrite the equality (1.19) in the form

$$\lambda \int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt = \int\limits_{D_\tau} u \Box \varphi \, dx \, dt - \gamma(\tau). \tag{1.23}$$

If in the Young inequality with the parameter $\varepsilon > 0$

$$ab \le \frac{\varepsilon}{\alpha} \, a^\alpha + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \, b^{\alpha'}, \quad ab \ge 0, \quad \alpha' = \frac{\alpha}{\alpha - 1}$$

we take $a = |u| \varphi^{1/\alpha}$, $b = \frac{|\Box \varphi|}{\varphi^{1/\alpha}}$, then with regard for $\frac{\alpha'}{\alpha} = \alpha - 1$ we obtain

$$|u \Box \varphi| = |u| \varphi^{1/\alpha} \cdot \frac{|\Box \varphi|}{\varphi^{1/\alpha}} \le \frac{\varepsilon}{\alpha} \, |u|^\alpha \varphi + \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \, \frac{|\Box \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, . \tag{1.24}$$

By (1.24), from (1.23) we have

$$\Big(\lambda - \frac{\varepsilon}{\alpha} \Big) \int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt \le \frac{1}{\alpha' \varepsilon^{\alpha'-1}} \int\limits_{D_\tau} \frac{|\Box \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt - \gamma(\tau),$$

whence for $\varepsilon < \lambda \alpha$ it follows

$$\int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt \le \frac{\alpha}{(\lambda \alpha - \varepsilon) \alpha' \varepsilon^{\alpha'-1}} \int\limits_{D_\tau} \frac{|\Box \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt - \frac{\alpha}{\lambda \alpha - \varepsilon} \, \gamma(\tau). \tag{1.25}$$

Since $\alpha' = \frac{\alpha}{\alpha - 1}$, $\alpha = \frac{\alpha'}{\alpha' - 1}$ and $\min\limits_{0 < \varepsilon < \lambda \alpha} \frac{\alpha}{(\lambda \alpha - \varepsilon) \alpha' \varepsilon^{\alpha'-1}} = \frac{1}{\lambda^{\alpha'}}$ is achieved for $\varepsilon = \lambda$, from (1.25) it follows that

$$\int\limits_{D_\tau} |u|^\alpha \varphi \, dx \, dt \le \frac{1}{\lambda^{\alpha'}} \int\limits_{D_\tau} \frac{|\Box \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt - \frac{\alpha'}{\lambda} \, \gamma(\tau). \tag{1.26}$$

According to the properties of the function φ_0, the test function $\varphi(x,t) = \varphi_0 \big[\frac{2}{\tau^2} \, (t^2 + |x|^2) \big]$ equals to 0 for $r = (t^2 + |x|^2)^{1/2} \ge \tau$. Therefore after the change of variables $t = \sqrt{2} \, \tau \xi_0$, $x = \sqrt{2} \, \tau \xi$, just as when obtaining (1.15) we can easily verify that

$$\int\limits_{D_\tau} \frac{|\Box \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt = \int\limits_{r = (t^2 + |x|^2)^{1/2} \le \tau} \frac{|\Box \varphi|^{\alpha'}}{\varphi^{\alpha'-1}} \, dx \, dt = (\sqrt{2} \, \tau)^{n+1-2\alpha'} \varkappa_0, \tag{1.27}$$

where

$$\varkappa_0 = \int\limits_{1\le|\xi_0|^2+\xi|^2\le 2} \frac{\big|2(1-n)\varphi_0' + 4(\xi_0^2-|\xi|^2)\varphi_0''\big|^{\alpha'}}{\varphi^{\alpha'-1}}\, d\xi\, d\xi_0 < +\infty.$$

By virtue of (1.27), from the inequality (1.26) and the fact that $\varphi_0(\sigma)=1$ for $0\le\sigma\le 1$ we obtain

$$\int\limits_{r\le\frac{\tau}{\sqrt{2}}} |u|^\alpha\, dx\, dt \le \int\limits_{D_\tau} |u|^\alpha \varphi\, dx\, dt \le \frac{(\sqrt{2}\,\tau)^{n+1-2\alpha'}}{\lambda^{\alpha'}}\,\varkappa_0 - \frac{\alpha'}{\lambda}\,\gamma(\tau). \tag{1.28}$$

In the case $\alpha < \frac{n+1}{n-1}$, i.e. for $n+1-2\alpha'<0$, the equation

$$g(\tau) = \frac{(\sqrt{2}\,\tau)^{n+1-2\alpha'}}{\lambda^{\alpha'}}\,\varkappa_0 - \frac{\alpha'}{\lambda}\,\gamma(\tau) = 0 \tag{1.29}$$

has a unique positive root $\tau=\tau_0>0$ since the function $g_1(\tau) = \frac{(\sqrt{2}\,\tau)^{n+1-2\alpha'}}{\lambda^{\alpha'}}\,\varkappa_0$ is positive, continuous and strictly decreasing on the interval $(0,+\infty)$ with $\lim\limits_{\tau\to 0} g_1(\tau)=+\infty$ and $\lim\limits_{\tau\to+\infty} g_1(\tau)=0$, while the function $\gamma(\tau)$ is, as is mentioned above, nonnegative, continuous and nondecreasing. In addition, since we assume that at least one of the functions F and f is nontrivial, we have $\lim\limits_{\tau\to+\infty}\gamma(\tau)>0$. Moreover, $g(\tau)<0$ for $\tau>\tau_0$ and $g(\tau)>0$ for $0<\tau<\tau_0$. Consequently, for $\tau>\tau_0$ the right-hand side of (1.28) is negative, but this is impossible. Therefore if a solution of the problem (1.17), (1.18) exists in the domain D_τ, then $\tau\le\tau_0$ without fail, and hence for the value $\tau=T$ from Remark 1.1 the estimate

$$T\le\tau_0, \tag{1.30}$$

is valid, where τ_0 is the unique positive root of the equation (1.29).

In the limiting case $\alpha=\frac{n+1}{n-1}$, i.e. for $n+1-2\alpha'=0$, if

$$\lim_{\tau\to+\infty}\gamma(\tau) > \frac{\varkappa_0}{\alpha'\lambda^{\alpha'-1}}\,, \tag{1.31}$$

then arguing word for word as in the case $\alpha<\frac{n+1}{n-1}$, we again arrive at the estimate (1.30) in which τ_0 is the least positive root of the equation (1.29) which by (1.31) does exist.

Remark 1.2. Under the conditions (1.6) and (1.7) of Theorem 1.1 the right-hand sides in the equation (1.1) and in the boundary conditions (1.2), as well as the derivative $\frac{\partial f}{\partial r}$ are nonnegative. Therefore for $n=2$ and $n=3$, by the well-known properties of solutions of the linear characteristic problem [24, p. 745], [10, p. 84] a solution $u(x,t)$ of the nonlinear problem (1.1), (1.2) will likewise be nonnegative. But in this case, for $\alpha=1$ this solution will satisfy the following linear problem:

$$\Box u = \lambda u + F,$$

$$u\big|_{\partial D} = f,$$

which is globally solvable in the corresponding function spaces.

Remark 1.3. In case $0 < \alpha < 1$ the problem (1.1), (1.2) may have more than one global solution. For example, for $F = 0$ and $f = 0$ the conditions (1.6) and (1.7) are fulfilled, but the problem (1.1), (1.2) has, besides the trivial solution, an infinite set of global linearly independent solutions $u_\sigma(x,t)$ depending on a parameter $\sigma \geq 0$ and given by the formula

$$u_\sigma(x,t) = \begin{cases} \beta\big[(t-\sigma)^2 - |x|^2\big]^{\frac{1}{1-\alpha}}, & t > \sigma + |x|, \\ 0, & |x| \leq t \leq \sigma + |x|, \end{cases}$$

where $\beta = \lambda^{\frac{1}{1-\alpha}}\big[\frac{4\alpha}{(1-\alpha)^2} + \frac{2(n+1)}{1-\alpha}\big]^{-\frac{1}{1-\alpha}}$. It can be easily seen that $u_\sigma(x,t) \in \widetilde{L}_{2,\mathrm{loc}}(D) \cap \widetilde{W}^1_{2,\mathrm{loc}}(D)$ and, moreover, $u_\sigma(x,t) \in C^1(\overline{D})$, but for $1/2 < \alpha < 1$ the function $u_\sigma(x,t)$ belongs to $C^2(\overline{D})$.

Remark 1.4. The conclusion of Theorem 1.1 ceases to be valid if instead of (1.8) the inequality $\alpha > \frac{n+1}{n-1}$ is fulfilled and simultaneously only the first of the conditions (1.7) violates, i.e. $f|_{\partial D} \geq 0$. Indeed, the function $u(x,t) = -\varepsilon(1+t^2-|x|^2)^{\frac{1}{1-\alpha}}$, $\varepsilon = \mathrm{const} > 0$, is a global classical and hence a generalized solution of the problem (1.1), (1.2) for $f = -\varepsilon\ \big(\frac{\partial f}{\partial r}\big|_{\partial D} = 0\big)$ and $F = \big[2\varepsilon\,\frac{n+1}{\alpha-1} - 4\varepsilon\,\frac{\alpha}{(\alpha-1)^2}\,\frac{t^2-|x|^2}{1+t^2-|x|^2} - \lambda\varepsilon^\alpha\big](1+t^2-|x|^2)^{\frac{1}{1-\alpha}}$, where, as it can be easily verified, $F|_D \geq 0$ if $\alpha > \frac{n+1}{n-1}$ and $0 < \varepsilon \leq \big\{\frac{2}{\lambda}\,\big[\frac{n+1-\frac{2\alpha}{\alpha-1}}{\alpha-1}\big]\big\}^{\frac{1}{\alpha-1}}$. Note that the inequality $n+1-\frac{2\alpha}{\alpha-1} > 0$ is equivalent to $\alpha > \frac{n+1}{n-1}$.

Remark 1.5. The conclusion of Theorem 1.1 ceases likewise to be valid if violates only the second of the conditions (1.7), i.e. the condition $\frac{\partial f}{\partial r}\big|_{\partial D} \geq 0$. Indeed, the function $u(x,t) = \beta[(t+1)^2 - |x|^2]^{\frac{1}{1-\alpha}}$, where $\beta = \lambda^{\frac{1}{1-\alpha}}\big[\frac{4\alpha}{(1-\alpha)^2} + \frac{2(n+1)}{1-\alpha}\big]^{-\frac{1}{1-\alpha}}$, is a global classical solution of the problem (1.1), (1.2) for $F = 0$ and $f = u|_{\partial D:\ t=|x|} = \beta[(t+1)^2 - t^2]^{\frac{1}{1-\alpha}} > 0$.

1.3. Local solvability of the characteristic Cauchy problem. Below we will restrict ourselves to the consideration of the problem (1.17), (1.18) in the domain D_τ under the homogeneous boundary condition (1.18), i.e.

$$u\big|_{S_\tau} = 0. \tag{1.32}$$

First of all, we consider the linear case, when in the equation (1.17) the parameter $\lambda = 0$, i.e. we consider the linear boundary value problem

$$Lu(x,t) = F(x,t), \quad (x,t) \in D_\tau, \tag{1.33}$$

$$u(x,t) = 0, \quad (x,t) \in S_\tau, \tag{1.34}$$

where for convenience we introduce the notation $L = \Box\big(= \frac{\partial^2}{\partial t^2} - \Delta\big)$.

Definition 1.3. Let $F \in L_2(D_\tau)$. The function $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) = \{u \in W_2^1(D_\tau) : \ u|_{S_\tau} = 0\}$ is said to be a strong generalized solution of the problem (1.33), (1.34) if there exists a sequence of functions $u_m \in W_2^2(D_\tau) \cap \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ such that

$$\lim_{m\to\infty} \|u_m - u\|_{W_2^1(D_\tau)} = 0, \quad \lim_{m\to\infty} \|Lu_m - F\|_{L_2(D_\tau)} = 0.$$

To obtain the required a priori estimate for the solution $u \in W_2^2(D_\tau)$ of the problem (1.33), (1.34), we will use the considerations from [61]. Multiplying both parts of the equation (1.33) by $2u_t$ and integrating over the domain D_δ, $0 < \delta \le \tau$, after simple transformations together with integration by parts and conditions (1.34), we obtain the equality

$$\int\limits_{\Omega_\delta} \Big[u_t^2 + \sum_{i=1}^n u_{x_i}^2\Big]\, dx = 2 \int\limits_{D_\delta} F u_t \, dx\, dt, \tag{1.35}$$

where $\Omega_\delta = D_\tau \cap \{t = \delta\}$. Under the notation $w(\delta) = \int\limits_{\Omega_\delta} \big[u_t^2 + \sum\limits_{i=1}^n u_{x_i}^2\big]\, dx$, with regard for the inequality $2Fu_t \le \varepsilon u_t^2 + \frac{1}{\varepsilon} F^2$ for every $\varepsilon = \text{const} > 0$, from (1.35) we have

$$w(\delta) \le \varepsilon \int\limits_0^\delta w(\sigma)\, d\sigma + \frac{1}{\varepsilon} \|F\|^2_{L_2(D_\delta)}, \quad 0 < \delta \le \tau. \tag{1.36}$$

From (1.36), taking into account that $\|F\|^2_{L_2(D_\delta)}$ as a function of δ is nondecreasing, by Gronwall's lemma [44, p. 13], we have

$$w(\delta) \le \frac{1}{\varepsilon} \|F\|^2_{L_2(D_\delta)} \exp \delta\varepsilon,$$

whence with regard for the fact that $\inf\limits_{\varepsilon>0} \frac{\exp \delta\varepsilon}{\varepsilon} = e\delta$ is achieved for $\varepsilon = \frac{1}{\delta}$ we find that

$$w(\delta) \le e\delta \|F\|^2_{L_2(D_\delta)}.$$

In its turn, it follows that

$$\int\limits_{D_\tau} \Big[u_t^2 + \sum_{i=1}^n u_{x_i}^2\Big]\, dx\, dt = \int\limits_0^\tau w(\sigma)\, d\sigma \le e\tau^2 \|F\|^2_{L_2(D_\delta)}$$

and hence

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} \le \sqrt{e}\, \tau \|F\|_{L_2(D_\tau)}. \tag{1.37}$$

Here we have used the fact that in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ the norm

$$\|u\|_{W_2^1(D_\tau)} \le \Big\{ \int\limits_{D_\tau} \Big[u^2 + u_t^2 + \sum_{i=1}^n u_{x_i}^2\Big]\, dx\, dt \Big\}^{1/2}$$

is equivalent to the norm

$$\|u\| = \left\{ \int\limits_{D_\tau} \left[u_t^2 + \sum_{i=1}^{n} u_{x_i}^2 \right] dx\, dt \right\}^{1/2}.$$

Since the space $C_0^\infty(D_\tau)$ is dense in $L_2(D_\tau)$, for the given $F \in L_2(D_\tau)$ there exists a sequence of functions $F_m \in C_0^\infty(D_\tau)$ such that $\lim\limits_{m\to\infty} \|F_m - F\|_{L_2(D_\tau)} = 0$. For fixed m, extending the function F_m by zero outside the domain D_τ and leaving for it the same notation, we have $F_m \in C^\infty(R_+^{n+1})$, for which $\operatorname{supp} F_m \subset D$, where $R_+^{n+1} = R^{n+1} \cap \{t \geq 0\}$. Denote by u_m a solution of the Cauchy problem $Lu_m = F_m$, $u_m|_{t=0} = 0$, $\frac{\partial u_m}{\partial t}\big|_{t=0} = 0$. As is known, the solution of that problem exists, is unique and belongs to the space $C^\infty(R_+^{n+1})$, and since $\operatorname{supp} F_m \subset D$, $u_m|_{t=0} = 0$, $\frac{\partial u_m}{\partial t}\big|_{t=0} = 0$, according to the geometry of the domain of dependence of a solution of the wave equation we have $\operatorname{supp} u_m \subset D:\ t > |x|$ [48, p. 191]. Leaving for the restriction of the function u_m on the domain D_τ the same notation, we can see that $u_m \in W_2^2(D_\tau) \cap \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$, and by (1.37) we find that

$$\|u_m - u_{m_1}\|_{\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)} \leq \sqrt{e}\,\tau \|F_m - F_{m_1}\|_{L_2(D_\tau)}. \tag{1.38}$$

Since the sequence $\{F_m\}$ is fundamental in $L_2(D_\tau)$, by virtue of (1.38) the sequence $\{u_m\}$ is likewise fundamental in the complete space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$. Therefore there exists a function $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ such that $\lim\limits_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)} = 0$, and as far as $Lu_m = F_m \to F$ in the space $L_2(D_\tau)$, this function is, according to Definition 1.3, a strong generalized solution of the problem (1.33), (1.34). The uniqueness of a strong generalized solution of the problem (1.33), (1.34) from the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ follows from the a priori estimate (1.37). Consequently, for a solution u of the problem (1.33), (1.34) we can write $u = L^{-1}F$, where $L^{-1} : L_2(D_\tau) \to \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ is a linear continuous operator whose norm by virtue of (1.37) admits the estimate

$$\|L^{-1}\|_{L_2(D_\tau)\to \overset{\circ}{W}{}_2^1(D_\tau,S_\tau)} \leq \sqrt{e}\,\tau. \tag{1.39}$$

Remark 1.6. The embedding operator $I : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to L_q(D_\tau)$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$ when $n > 1$ [84, p. 81]. At the same time, the Nemytski operator $T : L_q(D_\tau) \to L_2(D_\tau)$, acting by the formula $Tu = \lambda |u|^\alpha$ is continuous and bounded if $q \geq 2\alpha$ [79, p. 349], [82, pp. 66, 67]. Thus if $\alpha < \frac{n+1}{n-1}$, i.e. $2\alpha < \frac{2(n+1)}{n-1}$, there exists a number q such that $1 < 2\alpha \leq q < \frac{2(n+1)}{n-1}$ and hence the operator

$$T_0 = TI : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to L_q(D_\tau) \tag{1.40}$$

is continuous and compact. In addition, from $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ it follows $u \in L_\alpha(D_\tau)$. Everywhere above we assumed that $\alpha > 1$.

Definition 1.4. Let $F \in L_2(D_\tau)$ and $1 < \alpha < \frac{n+1}{n-1}$. The function $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ is said to be a strong generalized solution of the nonlinear problem (1.17), (1.32) if there exists a sequence of functions $u_m \in W_2^2(D_\tau) \cap \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ such that $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ and $[Lu_m - \lambda |u_m|^\alpha] \to F$ in the space $L_2(D_\tau)$. Moreover, the convergence of the sequence $\{\lambda |u_m|^\alpha\}$ to the function $\lambda |u|^\alpha$ in the space $L_2(D_\tau)$ as $u_m \to u$ in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ follows from Remark 1.6, and since $|u|^\alpha \in L_2(D_\tau)$, by the boundedness of the domain D_τ the function u belongs to $L_\alpha(D_\tau)$ all the more.

Remark 1.7. It can be easily verified that by Remark 1.6, for $1 < \alpha < \frac{n+1}{n-1}$, if u is a strong generalized solution of the problem (1.17), (1.32) in the sense of Definition 1.4, then this solution is a weak generalized solution of that problem for $f = 0$ in the sense of Definition 1.2, i.e. in the sense of the integral identity (1.5).

Remark 1.8. Note that for $F \in L_2(D_\tau)$, $1 < \alpha < \frac{n+1}{n-1}$, the function $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ is a strong generalized solution of the problem (1.17), (1.32) if and only if u is a solution of the following functional equation

$$u = L^{-1}\big(\lambda |u|^\alpha + F\big) \tag{1.41}$$

in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$.

We rewrite the equation (1.41) as follows:

$$u = Au + u_0, \tag{1.42}$$

where $A = L^{-1}T_0 : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ by virtue of (1.39), (1.40) and Remark 1.6 is a continuous and compact operator acting in the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$, and $u_0 = L^{-1}F \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$.

Remark 1.9. Let $B(0, z_2) := \big\{u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) : \ \|u\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} \le z_2\big\}$ be a closed (convex) ball in the Hilbert space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ of radius $z_2 > 0$, with center in the zero. Since the operator $A : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ for $1 < \alpha < \frac{n+1}{n-1}$ is continuous and compact, by the Schauder principle for the equation (1.42) to be solvable it is sufficient to prove that the operator A_1, acting by the formula $A_1 u = A_0 u + u_0$, transforms the ball $B(0, z_2)$ into itself for some $z_2 > 0$ [120, p. 370]. Towards this end, below we will indicate the needed estimate for $\|Au\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)}$.

If $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$, then we denote by $\widetilde{u}$ the function which is the even continuation of the function u through the plane $t = \tau$ into the domain $D_\tau^*: \ \tau < t < 2\tau - |x|$, i.e.

$$\widetilde{u}(x,t) = \begin{cases} u(x,t), & (x,t) \in D_\tau, \\ u(x, 2\tau - t), & (x,t) \in D_\tau^* \end{cases}$$

and $\widetilde{u}(x,t) = u(x,t)$ for $t = \tau$, $|x| < \tau$ in the sense of the trace theory. Obviously $\widetilde{u} \in \overset{\circ}{W}{}_2^1(\widetilde{D}_\tau)$, where $\widetilde{D}_\tau : \ |x| < t < 2\tau - |x|$. It is clear that $\widetilde{D}_\tau = D_\tau \cup \{(x,t) : \ t = \tau, \ |x| < \tau\} \cup D_\tau^*$.

Using the inequality [127, p. 258]

$$\int\limits_\Omega |v| \, d\Omega \leq (\operatorname{mes} \Omega)^{1-\frac{1}{p}} \|v\|_{p,\Omega}, \ \ p \geq 1,$$

and taking into account the equalities

$$\|\widetilde{u}\|^p_{L_p(\widetilde{D}_\tau)} = 2\|u\|^p_{L_p(D_\tau)}, \quad \|\widetilde{u}\|^2_{\overset{\circ}{W}{}_2^1(\widetilde{D}_\tau)} = 2\|u\|^2_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)},$$

from the well-known multiplicative inequality [84, p. 78]

$$\|v\|_{p,\Omega} \leq \beta \|v_x\|^{\widetilde{\alpha}}_{m,\Omega} \|v\|^{1-\widetilde{\alpha}}_{r,\Omega} \quad \forall\, v \in \overset{\circ}{W}{}_2^1(\Omega), \ \ \Omega \subset R^{n+1},$$
$$\widetilde{\alpha} = \Big(\frac{1}{r} - \frac{1}{p}\Big)\Big(\frac{1}{r} - \frac{1}{\widetilde{m}}\Big)^{-1}, \ \ \widetilde{m} = \frac{(n+1)m}{n+1-m}\,,$$

for $\Omega = \widetilde{D}_\tau \subset R^{n+1}$, $v = \widetilde{u}$, $r = 1$, $m = 2$ and $1 < p \leq \frac{2(n+1)}{n-1}$, where $\beta = \text{const} > 0$ does not depend on v and τ, we obtain the inequality

$$\|u\|_{L_p(D_\tau)} \leq c_0 (\operatorname{mes} D_\tau)^{\frac{1}{p} + \frac{1}{n+1} - \frac{1}{2}} \|u\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} \quad \forall\, u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau), \tag{1.43}$$

where $c_0 = \text{const} > 0$ is independent of u.

Taking into account that $\operatorname{mes} D_\tau = \frac{\omega_n}{n+1}\,\tau^{n+1}$, where ω_n is the volume of the unit ball in R^n, from (1.43) for $p = 2\alpha$ we obtain

$$\|u\|_{L_{2\alpha}(D_\tau)} \leq$$
$$\leq c_0 \widetilde{\ell}_{\alpha,n} \tau^{(n+1)\left(\frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}\right)} \|u\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} \quad \forall\, u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau), \tag{1.44}$$

where $\widetilde{\ell}_{\alpha,n} = \big(\frac{\omega_n}{n+1}\big)^{\left(\frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}\right)}$.

For $\|T_0 u\|_{L_2(D_\tau)}$, where $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ and the operator T_0 acts by the formula (1.40), by virtue of (1.44) we have the estimate

$$\|T_0 u\|_{L_2(D_\tau)} \leq \lambda \Big[\int\limits_{D_\tau} |u|^{2\alpha} \, dx \, dt \Big]^{1/2} = \lambda \|u\|^\alpha_{L_{2\alpha}(D_\tau)} \leq$$
$$\leq \lambda \ell_{\alpha,n} \tau^{\alpha(n+1)\left(\frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2}\right)} \|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)}, \tag{1.45}$$

where $\ell_{\alpha,n} = [c_0\widetilde{\ell}_{\alpha,n}]^\alpha$.

Now from (1.39) and (1.45) for $\|Au\|_{\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)}$, where $Au = L^{-1}T_0u$, the estimate

$$\|Au\|_{\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)} \leq \|L^{-1}\|_{L_2(D_\tau)\to\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)} \|T_0u\|_{L_2(D_\tau)} \leq$$
$$\leq \sqrt{e}\,\lambda\ell_{\alpha,n}\tau^{1+\alpha(n+1)\left(\frac{1}{2\alpha}+\frac{1}{n+1}-\frac{1}{2}\right)}\|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)} \quad \forall\, u \in \overset{\circ}{W}{}_2^1(D_\tau,S_\tau) \quad (1.46)$$

is valid. Note that $\frac{1}{2\alpha} + \frac{1}{n+1} - \frac{1}{2} > 0$ for $\alpha < \frac{n+1}{n-1}$.

Consider the equation

$$az^\alpha + b = z \quad (1.47)$$

with respect to the unknown z, where

$$a = \sqrt{e}\,\lambda\ell_{\alpha,n}\tau^{1+\alpha(n+1)\left(\frac{1}{2\alpha}+\frac{1}{n+1}-\frac{1}{2}\right)}, \quad b = \sqrt{e}\,\tau\|F\|_{L_2(D_\tau)}. \quad (1.48)$$

For $\tau > 0$ it is obvious that $a > 0$ and $b \geq 0$. A simple analysis, similar to that carried out for $\alpha = 3$ in [120, pp. 373, 374], shows that:

(1) for $b = 0$, along with the zero root $z_1 = 0$ the equation (1.47) has only one positive root $z_2 = a^{-\frac{1}{\alpha-1}}$;

(2) if $b > 0$, then for $0 < b < b_0$, where

$$b_0 = \left[\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right]a^{-\frac{1}{\alpha-1}}, \quad (1.49)$$

the equation (1.47) has two positive roots z_1 and z_2, $0 < z_1 < z_2$. For $b = b_0$ these roots get equal, and we have one positive root

$$z_1 = z_2 = z_0 = (\alpha a)^{-\frac{1}{\alpha-1}};$$

(3) for $b > b_0$, the equation (1.47) has no nonnegative roots.

Note that for $0 < b < b_0$ the inequalities

$$z_1 < z_0 = (\alpha a)^{-\frac{1}{\alpha-1}} < z_2$$

hold.

Owing to (1.48) and (1.49), the condition $b \leq b_0$ is equivalent to the condition

$$\sqrt{e}\,\tau\|F\|_{L_2(D_\tau)} \leq \left[\sqrt{e}\lambda\ell_{\alpha,n}\tau^{1+\alpha(n+1)\left(\frac{1}{2\alpha}+\frac{1}{n+1}-\frac{1}{2}\right)}\right]^{-\frac{1}{\alpha-1}}\left[\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right],$$

that is,

$$\|F\|_{L_2(D_\tau)} \leq \gamma_{n,\lambda,\alpha}\tau^{-\alpha_n}, \ \ \alpha_n > 0, \quad (1.50)$$

where

$$\gamma_{n,\lambda,\alpha} = \left[\alpha^{-\frac{1}{\alpha-1}} - \alpha^{-\frac{\alpha}{\alpha-1}}\right](\lambda\ell_{\alpha,n})^{-\frac{1}{\alpha-1}}\exp\left[-\frac{1}{2}\left(1+\frac{1}{\alpha-1}\right)\right],$$
$$\alpha_n = 1 + \frac{1}{\alpha-1}\left[1+\alpha(n+1)\left(\frac{1}{2\alpha}+\frac{1}{n+1}-\frac{1}{2}\right)\right].$$

Because of the absolute continuity of the Lebesgue integral, we have $\lim\limits_{\tau\to 0}\|F\|_{L_2(D_\tau)} = 0$. At the same time $\lim\limits_{\tau\to 0}\tau^{-\alpha_n} = +\infty$. Therefore there exists a number $\tau_1 = \tau_1(F)$, $0 < \tau_1 < +\infty$, such that the inequality (1.50) holds for

$$0 < \tau \leq \tau_1(F). \tag{1.51}$$

Let us now show that if the condition (1.51) is fulfilled, the operator $A_1 u = Au + u_0 : \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) \to \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ transforms the ball $B(0, z_2)$ mentioned in Remark 1.9 into itself, where z_2 is the maximal positive root of the equation (1.47). Indeed, if $u \in B(0, z_2)$, then by (1.46)–(1.48) we have

$$\|A_1 u\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} \leq a\|u\|^\alpha_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} + b \leq a z_2^\alpha + b = z_2.$$

Thus by Remarks 1.7–1.9, the following theorem is valid.

Theorem 1.2. *Let* $F \in \widetilde{L}_{2,\mathrm{loc}}(D)$, $1 < \alpha < \frac{n+1}{n-1}$, *and for the value* τ *the condition* (1.51) *be fulfilled. Then the problem* (1.17), (1.32) *in the domain* D_τ *has at least one strong generalized solution* $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ *in the sense of Definition* 1.4*, which is at the same time a weak generalized solution of that problem in the sense of Definition* 1.2*.*

Remark 1.10. Note that for $1 < \alpha < \frac{n+1}{n-1}$ the uniqueness of a solution of the problem (1.17), (1.32) in the domain D_τ can be proved in a more narrow than $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ space of functions

$$\overset{\circ}{E}{}_2^1 = \left\{ u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau) : \operatorname*{ess\,supp}_{0<\sigma\leq\tau} \int\limits_{\Omega_\sigma = D\cap\{t=\sigma\}} \Big[u_t^2 + \sum_{i=1}^n u_{x_i}^2\Big]\, dx < +\infty \right\}.$$

Remark 1.11. It is easily seen that the value $t = T$ considered in Remark 1.1 is contained in the interval $[\tau_1, \tau_0]$, by virtue of the estimates (1.30) and (1.51).

2. The Existence or Nonexistence of Global Solutions of the Characteristic Cauchy Problem for the Wave Equation with Power Nonlinearity of Type $-\lambda|u|^p u$

2.1. Statement of the problem. Consider a nonlinear wave equation of the type

$$\Box u := \frac{\partial^2 u}{\partial t^2} - \Delta u = f(u) + F, \tag{2.1}$$

where f and F are given real functions, f is a nonlinear function, and u is an unknown real function, $\Delta = \sum\limits_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

For the equation (2.1) we consider the characteristic Cauchy problem on finding in the truncated light cone of the future $D_T: \ |x| < t < T$, $x = (x_1, \dots, x_n)$, $n > 1$, $T = \mathrm{const} > 0$, a solution $u(x,t)$ of that equation by the boundary condition

$$u\big|_{S_T} = g, \tag{2.2}$$

where g is a given real function on the characteristic surface $S_T: \ t=|x|$, $t\leq T$. Considering the case $T=+\infty$, we assume that $D_\infty: \ t>|x|$ and $S_\infty=\partial D: \ t=|x|$.

Below we will distinguish particular cases for the nonlinear function $f=f(u)$, when in some cases the problem (2.1), (2.2) is globally solvable, while in other cases such solvability does not take place.

2.2. The global solvability of the problem. Consider the case $f(u)=-\lambda|u|^p u$, where $\lambda\neq 0$ and $p>0$ are given real numbers. In this case the equation (2.1) takes the form

$$Lu:=\frac{\partial^2 u}{\partial t^2}-\Delta u=-\lambda|u|^p u+F, \tag{2.3}$$

where for convenience we introduce the notation $L=\Box$. As is known, the equation (2.3) arises in the relativistic quantum mechanics [87], [110], [112], [114].

For the sake of simplicity, we assume that the boundary condition (2.2) is homogeneous, i.e.

$$u\big|_{S_T}=0. \tag{2.4}$$

Assume $\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)=\{u\in W_2^1(D_T): \ u|_{S_T}=0\}$, where $W_2^1(D_T)$ is the known Sobolev space.

Remark 2.1. The embedding operator $I:\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)\to L_q(D_T)$ is linear continuous compact for $1<q<\frac{2(n+1)}{n-1}$, when $n>1$ [84, p. 81]. At same time, the Nemytski operator $K: L_q(D_T)\to L_2(D_T)$, acting by the formula $Ku=-\lambda|u|^p u$ is continuous and bounded if $q\geq 2(p+1)$ [79, p. 349], [82, p. 66, 67]. Thus if $p<\frac{2}{n-1}$, i.e. $2(p+1)<\frac{2(n+1)}{n-1}$, then there exists a number q such that $1<2(p+1)\leq q<\frac{2(n+1)}{n-1}$, and hence the operator

$$K_0=KI:\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)\to L_2(D_T) \tag{2.5}$$

is continuous and compact. In addition, from $u\in\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)$ it all the more follows that $u\in L_{p+1}(D)$. As is mentioned above, here and in what follows we assume $p>0$.

Definition 2.1. Let $F\in L_2(D_T)$ and $0<p<\frac{2}{n-1}$. The function $u\in\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)$ is said to be a strong generalized solution of the nonlinear problem (2.3), (2.4) in the domain D_T, if there exists a sequence of functions $u_m\in\overset{\circ}{C}{}^2(\overline{D}_\tau,S_\tau)=\{u\in C^2(\overline{D}_T): \ u|_{S_T}=0\}$ such that $u_m\to u$ in the space $\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)$ and $[Lu_m+\lambda|u_m|^p u_m]\to F$ in the space $L_2(D_T)$. In addition, the convergence of the sequence $\{\lambda|u_m|^p u_m\}$ to the function $\lambda|u|^p u$ in the space $L_2(D_T)$, as $u_m\to u$ in the space $\overset{\circ}{W}{}_2^1(D_\tau,S_\tau)$, follows from Remark 2.1, and since $|u|^{p+1} \in L_2(D_T)$, due to the boundedness of the domain D_T, the function u belongs to $L_{p+1}(D_T)$ all the more.

Definition 2.2. Let $0 < p < \frac{2}{n-1}$, $F \in L_{2,\text{loc}}(D_\infty)$ and $F \in L_2(D_T)$ for every $T > 0$. We say that the problem (2.3), (2.4) is globally solvable, if for every $T > 0$ this problem has a strong generalized solution from the space $\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ in the domain D_T.

Lemma 2.1. *Let $\lambda > 0$, $0 < p < \frac{2}{n-1}$ and $F \in L_2(D_T)$. Then for every strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ of the problem* (2.3), (2.4) *in the domain D_T the estimate*

$$\|u\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} \leq \sqrt{e}T\|F\|_{L_2(D_T)} \tag{2.6}$$

is valid.

Proof. Let $u \in \overset{\circ}{W}{}_2^1(D_\tau, S_\tau)$ be a strong generalized solution of the problem (2.3), (2.4). By Definition 2.1 and Remark 2.1, there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_\tau, S_T)$ such that

$$\lim_{n\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_\tau, S_\tau)} = 0, \quad \lim_{m\to\infty} \|Lu_m + \lambda |u_m|^p u_m - F\|_{L_2(D_T)} = 0. \tag{2.7}$$

We can consider the function $u_m \in \overset{\circ}{C}{}^2(\overline{D}_\tau, S_\tau)$ as a solution of the problem

$$Lu_m + \lambda |u_m|^p u_m = F_m, \tag{2.8}$$

$$u_m\big|_{S_T} = 0. \tag{2.9}$$

Here

$$F_m = Lu_m + \lambda |u_m|^p u_m. \tag{2.10}$$

Multiplying both parts of the equation (2.8) by $\frac{\partial u_m}{\partial t}$ and integrating over the domain D_τ, $0 < \tau \leq T$, we obtain

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2 \, dx\, dt - \int\limits_{D_\tau} \Delta u_m \frac{\partial u_m}{\partial t} \, dx\, dt + \frac{\lambda}{p+2}\int\limits_{D_\tau} \frac{\partial}{\partial t} |u_m|^{p+2} \, dx\, dt =$$

$$= \int\limits_{D_\tau} F_m \frac{\partial u_m}{\partial t} \, dx\, dt. \tag{2.11}$$

Assume $\Omega_\tau := D_T \cap \{t = \tau\}$ and denote by $\nu = (\nu_1, \dots, \nu_n, \nu_0)$ the unit vector of the outer normal to $S_T \setminus \{(0, \dots, 0, 0)\}$. Taking into account the equalities (2.9) and $\nu|_{\Omega_\tau} = (0, \dots, 0, 1)$ and integrating by parts, we get

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial t}\Big)^2 \, dx\, dt = \int\limits_{\partial D_\tau} \Big(\frac{\partial u_m}{\partial t}\Big)^2 \nu_0 \, ds =$$

$$= \int\limits_{\Omega_\tau} \Big(\frac{\partial u_m}{\partial t}\Big)^2\, dx + \int\limits_{S_\tau} \Big(\frac{\partial u_m}{\partial t}\Big)^2 \nu_0\, ds,$$

$$\int\limits_{D_\tau} \frac{\partial}{\partial t}\, |u_m|^{p+2}\, dx\, dt = \int\limits_{\partial D_\tau} |u_m|^{p+2}\nu_0\, ds = \int\limits_{\Omega_\tau} |u_m|^{p+2}\, dx,$$

$$\int\limits_{D_\tau} \frac{\partial^2 u_m}{\partial x_i^2}\, \frac{\partial u_m}{\partial t}\, dx\, dt = \int\limits_{\partial D_\tau} \frac{\partial u_m}{\partial x_i}\, \frac{\partial u_m}{\partial t}\, \nu_i\, ds - \frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u_m}{\partial x_i}\Big)^2 dx\, dt =$$

$$= \int\limits_{\partial D_\tau} \frac{\partial u_m}{\partial x_i}\, \frac{\partial u_m}{\partial t}\, \nu_i\, ds - \frac{1}{2}\int\limits_{\partial D_\tau} \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 \nu_0\, ds =$$

$$= \int\limits_{\partial D_\tau} \frac{\partial u_m}{\partial x_i}\, \frac{\partial u_m}{\partial t}\, \nu_i\, ds - \frac{1}{2}\int\limits_{S_\tau} \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 \nu_0\, ds - \frac{1}{2}\int\limits_{\Omega_\tau} \Big(\frac{\partial u_m}{\partial x_i}\Big)^2 dx,$$

whence by virtue of (2.11) we obtain

$$\int\limits_{D_\tau} F_m\, \frac{\partial u_m}{\partial t}\, dx\, dt =$$

$$= \int\limits_{S_\tau} \frac{1}{2\nu_0}\Big[\sum_{i=1}^n \Big(\frac{\partial u_m}{\partial x_i}\, \nu_0 - \frac{\partial u_m}{\partial t}\, \nu_i\Big)^2 + \Big(\frac{\partial u_m}{\partial t}\Big)^2 \Big(\nu_0^2 - \sum_{j=1}^n \nu_j^2\Big)\Big]\, ds +$$

$$+ \frac{1}{2}\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial u_m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_m}{\partial x_i}\Big)^2\Big]\, dx + \frac{\lambda}{p+2}\int\limits_{\Omega_\tau} |u_m|^{p+2}\, dx. \tag{2.12}$$

Since S_τ is a characteristic surface, we have

$$\Big(\nu_0^2 - \sum_{j=1}^n \nu_j^2\Big)\Big|_{S_\tau} = 0. \tag{2.13}$$

Taking into account that $\big(\nu_0\, \frac{\partial}{\partial x_i} - \nu_i\, \frac{\partial}{\partial t}\big)$, $i = 1, \dots, n$, is an inner differential operator on S_τ, owing to (2.9) we have

$$\Big(\frac{\partial u_m}{\partial x_i}\, \nu_0 - \frac{\partial u_m}{\partial t}\, \nu_i\Big)\Big|_{S_\tau} = 0, \ \ i = 1, \dots, n. \tag{2.14}$$

By (2.13) and (2.14), from (2.12) we obtain

$$\int\limits_{\Omega_\tau} \Big[\Big(\frac{\partial u_m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_m}{\partial x_i}\Big)^2\Big]\, dx + \frac{2\lambda}{p+2}\int\limits_{\Omega_\tau} |u_m|^{p+2}\, dx =$$

$$= 2\int\limits_{D_\tau} F_m\, \frac{\partial u_m}{\partial t}\, dx\, dt. \tag{2.15}$$

Under the notation $w(\delta) = \int\limits_{\Omega_\delta} \big[\big(\frac{\partial u_m}{\partial t}\big)^2 + \sum\limits_{i=1}^n \big(\frac{\partial \nu_m}{\partial x_i}\big)^2\big]\, dx$, taking into account that $\frac{\lambda}{p+2} > 0$ and the inequality $2F_m\, \frac{\partial u_m}{\partial t} \le \varepsilon\big(\frac{\partial u_m}{\partial t}\big)^2 + \frac{1}{\varepsilon}\, F_m^2$, which is valid for every $\varepsilon = \text{const} > 0$, we have

$$w(\delta) \le \varepsilon \int\limits_0^\delta w(\sigma)\,d\sigma + \frac{1}{\varepsilon}\,\|F_m\|^2_{L_2(D_\delta)}, \quad 0 < \delta \le T. \tag{2.16}$$

From (2.16), bearing in mind that $\|F_m\|^2_{L_2(D_\delta)}$ as a function of δ is nondecreasing, by the Gronwall lemma [44, p. 13] we find that

$$w(\delta) \le \frac{1}{\varepsilon}\,\|F_m\|^2_{L_2(D_\delta)} \exp \delta\varepsilon,$$

whence with regard for the fact that $\inf\limits_{\varepsilon>0} \frac{\exp \delta\varepsilon}{\varepsilon} = e\delta$ which is achieved for $\varepsilon = \frac{1}{\delta}$, we obtain

$$w(\delta) \le e\delta \|F_m\|^2_{L_2(D_\delta)}, \quad 0 < \delta \le T. \tag{2.17}$$

From (2.17) in its turn it follows that

$$\begin{aligned}\|u_m\|^2_{\overset{\circ}{W}{}^1_2(D_T,S_T)} &= \int\limits_{D_T} \left[\Big(\frac{\partial u_m}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u_m}{\partial x_i}\Big)^2\right] dx\,dt = \\ &= \int\limits_0^T w(\delta)\,d\delta \le eT^2 \|F_m\|^2_{L_2(D_T)},\end{aligned}$$

and thus

$$\|u_m\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \le \sqrt{e}\,T \|F_m\|_{L_2(D_T)}. \tag{2.18}$$

Here we have used the fact that in the space $\overset{\circ}{W}{}^1_2(D_T, S_T)$ the norm

$$\|u\|_{W^1_2(D_T)} = \left\{ \int\limits_{D_T} \left[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\right] dx\,dt \right\}^{1/2}$$

is equivalent to the norm

$$\|u\| = \left\{ \int\limits_{D_T} \left[\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2\right] dx\,dt \right\}^{1/2}$$

since from the equalities $u|_{S_T} = 0$ and $u(x,t) = \int\limits_{|x|}^t \frac{\partial u(x,t)}{\partial t}\,d\tau$, $(x,t) \in \overline{D}_T$, valid for every function $u \in \overset{\circ}{C}{}^2(\overline{D}_\tau, S_\tau)$, we obtain the inequality [84, p. 63]

$$\int\limits_{D_T} u^2(x,t)\,dx\,dt \le T^2 \int\limits_{D_T} \Big(\frac{\partial u}{\partial t}\Big)^2\,dx\,dt.$$

In view of (2.7) and (2.10), passing in the inequality (2.18) to limit as $m \to \infty$, we obtain (2.6), which proves the lemma. □

Remark 2.2. Before we proceed to considering the question on the solvability of the nonlinear problem (2.3),(2.4), we have to consider the same question for the linear case, when in the equation (2.3) the parameter $\lambda = 0$, i.e. for the problem

$$\begin{aligned} Lu(x,t) &= F(x,t), \quad (x,t) \in D_T, \\ u(x,t) &= 0, \quad (x,t) \in S_T. \end{aligned} \tag{2.19}$$

In this case, for $F \in L_2(D_T)$ we analogously introduce the notion of a strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ of the problem (2.19) for which there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_\tau, S_\tau)$ such that $\lim\limits_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0$, $\lim\limits_{m\to\infty} \|Lu_m - F\|_{L_2(D_T)} = 0$. Moreover, it should be noted that as is seen from the proof of Lemma 2.1, the a priori estimate (2.6) is likewise valid for a strong generalized solution of the problem (2.19).

Since the space $C_0^\infty(D_T)$ of infinitely differentiable in D_T functions with bounded support is dense in $L_2(D_T)$, for the given $F \in L_2(D_T)$ there exists a sequence of functions $F_m \in C_0^\infty(D_T)$ such that $\lim\limits_{m\to\infty} \|F_m - F\|_{L_2(D_T)} = 0$. For m fixed, extending the function F_m by zero outside the domain D_T and preserving for it the same notation, we will have $F_m \in C^\infty(R_+^{n+1})$ and $\operatorname{supp} F_m \subset D_\infty$, where $R_+^{n+1} = R^{n+1} \cap \{t \geq 0\}$. Denote by u_m the solution of the Cauchy problem $Lu_m = F_m$, $u_m|_{t=0} = 0$, $\frac{\partial u_m}{\partial t}\big|_{t=0} = 0$, which, as is known, exists, is unique and belongs to the space $C^\infty(R_+^{n+1})$ [48, p. 192]. In addition, since $\operatorname{supp} F_m \subset D_\infty$, $u_m|_{t=0} = 0$, $\frac{\partial u_m}{\partial t}\big|_{t=0} = 0$, in view of the geometry of the domain of dependence of a solution of the linear wave equation we have $\operatorname{supp} u_m \subset D_\infty$ [48, p. 191]. Preserving for the restriction of the function u_m to the domain D_T the same notation, we easily see that $u_m \in \overset{\circ}{C}{}^2(\overline{D}_\tau, S_\tau)$, and by (2.6) and Remark 2.2 we have

$$\|u_m - u_k\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{e}\,T \|F_m - F_k\|_{L_2(D_T)}. \tag{2.20}$$

Since the sequence $\{F_m\}$ is fundamental in $L_2(D_T)$, by virtue of (2.20) the sequence $\{u_m\}$ is likewise fundamental in the complete space $\overset{\circ}{W}{}_2^1(D_T, S_T)$. Therefore there exists a function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ such that $\lim\limits_{m\to\infty} \|u_m - u\|_{\overset{\circ}{W}{}_2^1(D_T,S_T)} = 0$, and as far as $Lu_m = F_m \to F$ in the space $L_2(D_T)$, this function will, by Remark 2.2, be a strong generalized solution of the problem (2.19). The uniqueness of that solution in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$ follows from the a priori estimate (2.6). Consequently, for the solution $u = L^{-1}F$ of the problem (2.19) we can write $u = L^{-1}F$, where $L^{-1} : L_2(D_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a linear continuous operator whose norm, by virtue of (2.6), admits the estimate

$$\|L^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}_2^1(D_T,S_T)} \leq \sqrt{e}\,T. \tag{2.21}$$

Remark 2.3. In view of (2.21) for $F \in L_2(D_T)$, $0 < p < \frac{2}{n-1}$ and Remark 2.1 we can see that the function $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ is a strong generalized solution of the problem (2.3), (2.4) if and only if u is a solution of the following functional equation

$$u = L^{-1}(-\lambda|u|^p u + F) \tag{2.22}$$

in the space $\overset{\circ}{W}{}_2^1(D_T, S_T)$.

We rewrite the equation (2.22) in the form

$$u = Au := L^{-1}(K_0 u + F), \tag{2.23}$$

where the operator $K_0 : \overset{\circ}{W}{}_2^1(D_T, S_T) \to L_2(D_T)$ from (2.5) is, by Remark 2.1, continuous and compact. Consequently, by (2.21) the operator $A : \overset{\circ}{W}{}_2^1(D_T, S_T) \to \overset{\circ}{W}{}_2^1(D_T, S_T)$ is likewise continuous and compact. At the same time, by Lemma 2.1, for an arbitrary parameter $\tau \in [0, 1]$ and for every solution of the equation with the parameter $u = \tau Au$ the a priori estimate $\|u\|_{\overset{\circ}{W}{}_2^1(D_T, S_T)} \leq c\|F\|_{L_2(D_T)}$ is valid with a positive constant c independent of u, τ and F. Therefore by the Leray–Schauder theorem [120, p. 375] the equation (2.23) and hence the problem (2.3), (2.4) has at least one solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$.

Thus the following theorem is valid.

Theorem 2.1. *Let $\lambda > 0$, $0 < p < \frac{2}{n-1}$, $F \in L_{2,\mathrm{loc}}(D_\infty)$ and $F \in L_2(D_T)$ for every $T > 0$. Then the problem* (2.3), (2.4) *is globally solvable, i.e. for every $T > 0$ this problem has a strong generalized solution $u \in \overset{\circ}{W}{}_2^1(D_T, S_T)$ in the domain D_T.*

2.3. Absence of the global solvability. Below we will restrict ourselves to the consideration of the case where in the equation (2.3) the parameter $\lambda < 0$ and the spatial dimension $n = 2$.

Definition 2.3. Let $F \in C(\overline{D}_T)$. A function u is said to be a strong generalized continuous solution of the problem (1.19) if $u \in \overset{\circ}{C}{}^2(\overline{D}_\tau, S_\tau) = \{u \in C^2(\overline{D}) : u|_{S_T} = 0\}$ and there exists a sequence of functions $u_m \in \overset{\circ}{C}{}^2(\overline{D}_\tau, S_\tau)$ such that $\lim\limits_{m\to\infty} \|u_m - u\|_{C(\overline{D}_T)} = 0$ and $\lim\limits_{m\to\infty} \|Lu_m - F\|_{C(\overline{D}_T)} = 0$.

Introduce into consideration the domain $D_{x^0,t^0} = \{(x,t) \in R^3 : |x| < t < t^0 - |x - x^0|\}$ which for $(x^0, t^0) \in D_T$ is bounded below by the light cone of the future S_∞ with the vertex at the origin and above by the light cone of the past $S^-_{x^0,t^0} : t = t^0 - |x - x_0|$ with the vertex at the point (x^0, t^0).

Lemma 2.2. *Let $n = 2$, $F \in \overset{\circ}{C}(\overline{D}_\tau, S_\tau)$. Then there exists a unique strong generalized continuous solution of the problem* (2.19) *for which the* *integral representation*

$$u(x,t)=\frac{1}{2\pi}\int\limits_{D_{x,t}}\frac{F(\xi,\tau)}{\sqrt{(t-\tau)^2-|x-\xi|^2}}\,d\xi\,d\tau,\ \ (x,t)\in D_T, \tag{2.24}$$

and the estimate

$$\|u\|_{C(\overline{D}_T)}\le c\|F\|_{C(\overline{D}_T)} \tag{2.25}$$

are valid with a positive constant c independent of F.

Proof. Without restriction of generality, we can assume that the function $F\in \overset{\circ}{C}(\overline{D}_\tau,S_\tau)$ is extended into the domain D_∞ so that $F\in\overset{\circ}{C}(\overline{D}_\infty,S_\infty)$. Indeed, if $(x,t)\in\overline{D}_\infty\setminus\overline{D}_T$, we can take $F(x,t)=F(\frac{T}{t}\,x,T)$. Assume $D_{T,\delta}:|x|+\delta<t<T$, where $0<\delta=\text{const}<\frac{1}{2}\,T$. Obviously, $D_{T,\delta}\subset D_T$. Since $F\in C(\overline{D}_T)$ and $F|_{S_T}=0$ for some vanishing, strictly monotonically decreasing sequence of positive numbers $\{\delta_k\}$, there exists a sequence of functions $\{F_k\}$ such that

$$\begin{gathered}F_k\in C^\infty(\overline{D}_T),\quad \operatorname{supp}F_k\subset\overline{D}_{T,\delta_k},\ \ k=1,2,\dots,\\ \lim_{k\to\infty}\|F_k-F\|_{C(\overline{D}_T)}=0.\end{gathered} \tag{2.26}$$

Indeed, let $\varphi_\delta\in C([0,+\infty))$ be a nondecreasing continuous function of one variable such that $\varphi_\delta(\tau)=0$ for $0\le\tau\le2\delta$ and $\varphi_\delta(\tau)=1$ for $t\ge3\delta$. Assume $\widetilde{F}_\delta(x,t)=\varphi_\delta(t-|x|)F(x,t)$, $(x,t)\in\overline{D}_T$. Since $F\in C(\overline{D}_T)$ and $F|_{S_T}=0$, as it can be easily verified,

$$\widetilde{F}_\delta\in C(\overline{D}_T),\quad \operatorname{supp}\widetilde{F}_\delta\subset\overline{D}_{T,2\delta},\quad \lim_{\delta\to\infty}\|\widetilde{F}_\delta-F\|_{C(\overline{D}_T)}=0. \tag{2.27}$$

Now we apply the averaging operation and assume

$$G_\delta(x,t)=\varepsilon^{-n}\int\limits_{R^3}\widetilde{F}_\delta(\xi,\tau)\rho\Big(\frac{x-\xi}{\varepsilon},\frac{\tau}{\varepsilon}\Big)\,d\xi\,d\tau,\ \ \varepsilon=(\sqrt{2}-1)\delta,$$

where

$$\rho\in C_0^\infty(R^3),\ \ \int\limits_{R^3}\rho\,dx\,dt=1,\ \ \rho\ge0,\ \ \operatorname{supp}\rho=\big\{(x,t)\in R^3:\ x^2+t^2\le1\big\}.$$

From (2.27) and averaging properties [48, p. 9] it follows that the sequence $F_k=G_{\delta_k}$, $k=1,2,\dots$, satisfies (2.26). Extending the function F_k by zero into the layer $\Lambda_T:0<t<T$ and preserving for it the same notation, we have $F_k\in C^\infty(\overline{\Lambda}_T)$, where $\operatorname{supp}F_k\subset\overline{D}_{T,\delta_k}\subset\overline{D}_T$, $k=1,2,\dots$. Therefore analogously as in proving Lemma 2.1, for the solution of the Cauchy problem $Lu_k=F_k$, $u_k|_{t=0}=0$, $\frac{\partial u_k}{\partial t}\big|_{t=0}=0$ in the layer Λ_T which exists, is unique and belongs to the space $C^\infty(\overline{\Lambda}_T)$ we have $\operatorname{supp}u_k\subset D_T$, and all the more $u_k\in\overset{\circ}{C}{}^2(\overline{D}_\tau,S_\tau)$, $k=1,2,\dots$.

On the other hand, since $\operatorname{supp} F_k \subset \overline{D}_T$ and $F_k \in C^\infty(\overline{\Lambda}_T)$, for the solution u_k of the Cauchy problem by the Poisson formula the integral representation [124, p. 227]

$$u_k(x,t) = \frac{1}{2\pi} \int\limits_{D_{x,t}} \frac{F_k(\xi,\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau, \quad (x,t) \in D_T, \tag{2.28}$$

is valid and the estimate [124, p. 215]

$$\|u_k\|_{C(\overline{D}_T)} \leq \frac{T^2}{2} \|F_k\|_{C(\overline{D}_T)} \tag{2.29}$$

holds.

In view of (2.27) and (2.29), the sequence $\{u_k\} \subset \overset{\circ}{C}{}^2(\overline{D}_\tau, S_\tau)$ is fundamental in the space $\overset{\circ}{C}(\overline{D}_\tau, S_\tau)$. Therefore it tends to some function $u \in \overset{\circ}{C}(\overline{D}_\tau, S_\tau)$ in that space. For the function u by (2.28) the representation (2.24) is valid and the estimate (2.25) holds. Thus we have proved the solvability of the problem (2.19) in the space $\overset{\circ}{C}(\overline{D}_\tau, S_\tau)$.

As for the uniqueness of a strong generalized continuous solution of the problem (2.19), it follows from the following considerations. Let $u \in \overset{\circ}{C}(\overline{D}_\tau, S_\tau)$, $F = 0$, and there exist a sequence of functions $u_k \in \overset{\circ}{C}{}^2(\overline{D}_\tau, S_\tau)$ such that $\lim\limits_{k\to\infty} \|u_k - u\|_{C(\overline{D}_T)} = 0$, $\lim\limits_{k\to\infty} \|Lu_k\|_{C(\overline{D}_T)} = 0$. This implies that $\lim\limits_{k\to\infty} \|u_k - u\|_{L_2(D_T)} = 0$ and $\lim\limits_{k\to\infty} \|Lu_k\|_{L_2(D_T)} = 0$. Since we can consider the function $u_k \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ as a strong generalized solution of the problem (2.19) for $F_k = Lu_k$ from the space $\overset{\circ}{W}{}^1_2(D_T, S_T)$, according to Remark 2.2 the estimate $\|u_k\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} \leq \sqrt{e}\, T \|Lu_k\|_{L_2(D_T)}$ is valid. Therefore $\lim\limits_{k\to\infty} \|Lu_k\|_{L_2(D_T)} = 0$ implies $\lim\limits_{k\to\infty} \|u_k\|_{\overset{\circ}{W}{}^1_2(D_T,S_T)} = 0$, and hence $\lim\limits_{k\to\infty} \|u_k\|_{L_2(D_T)} = 0$. Taking into account that $\lim\limits_{k\to\infty} \|u_k - u\|_{L_2(D_T)} = 0$, we obtain $u = 0$. Thus Lemma 2.2 is proved completely. □

Lemma 2.3. *Let $n = 2$, $\lambda < 0$, $F \in \overset{\circ}{C}(\overline{D}_T, S_T)$ and $F \geq 0$. Then if $u \in C^2(\overline{D}_T)$ is a classical solution of the problem* (2.3), (2.4), *then $u \geq 0$ in the domain D.*

Proof. If $u \in C^2(\overline{D}_T)$ is a classical solution of the problem (2.3), (2.4), then $u \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$, and since $F \in \overset{\circ}{C}(\overline{D}_T, S_T)$, the right-hand side $G = -\lambda |u|^p u + F$ of the equation (2.3) belongs to the space $\overset{\circ}{C}(\overline{D}_T, S_T)$. Consider the function $u \in \overset{\circ}{C}{}^2(\overline{D}_T, S_T)$ as a classical solution of the problem (2.19) for $F = G$, i.e.

$$Lu = G, \quad u\big|_{S_T} = 0. \tag{2.30}$$

This function will, all the more, be a strong generalized continuous solution of the problem (2.30). Therefore taking into account that $G \in \overset{\circ}{C}(\overline{D}_T, S_T)$, according to Lemma 2.2 for the function u the integral representation

$$u(x,t) = -\frac{1}{2\pi} \int\limits_{D_{x,t}} \frac{|u|^p u}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau + F_0(x,t) \tag{2.31}$$

holds. Here

$$F_0(x,t) = \frac{1}{2\pi} \int\limits_{D_{x,t}} \frac{F(\xi,\eta)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau. \tag{2.32}$$

Consider now the integral equation

$$v(x,t) = \int\limits_{D_{x,t}} \frac{g_0 v}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau + F_0(x,t), \quad (x,t) \in \overline{D}_T, \tag{2.33}$$

with respect to the unknown function v, where $g_0 = -\frac{\lambda}{2\pi}|u|^p$. As far as g_0, $F \in \overset{\circ}{C}(\overline{D}_T, S_T)$ and the operator in the right-hand side of (2.33) is a Volterra type integral operator with weak singularity, the equation (2.33) is uniquely solvable in the space $C(\overline{D}_T)$. In addition, a solution v of the equation (2.33) can be obtained by the Picard method of successive approximations:

$$v_0 = 0, \quad v_{k+1}(x,t) = \int\limits_{D_{x,t}} \frac{g_0 v_k}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\tau + F_0(x,t), \tag{2.34}$$
$$k = 1, 2, \dots .$$

Indeed, let $\Omega_\tau = D_T \cap \{t = \tau\}$, $w_m|_{\overline{D}_T} = v_{m+1} - v_m$ $(w_0|_{\overline{D}_T} = F_0)$, $w_m|_{\{0\le t\le T\}\setminus \overline{D}_T} = 0$, $\lambda_m(t) = \max\limits_{x\in\overline{\Omega}_t} w_m(x,t)|$, $m = 0, 1, \dots$, $b = \int\limits_{|\eta<1} \frac{d\eta_1\, d\eta_2}{\sqrt{1-|\eta|^2}} \|g_0\|_{C(\overline{D}_T)} = 2\pi \|g_0\|_{C(\overline{D}_T)}$. Then if $B_\beta \varphi(t) = b\int\limits_0^t (t-\tau)^{\beta-1}\varphi(\tau)d\tau$, $\beta > 0$, then taking into account the equality $B_\beta^m \varphi(t) = \frac{1}{\Gamma(m\beta)} \int\limits_0^t (b\Gamma(\beta))^m (t-\tau)^{m\beta-1}\varphi(\tau)\, d\tau$ [44, p. 206], by virtue of (2.34) we have

$$|w_m(x,t)| = \left| \int\limits_{D_{x,t}} \frac{g_0 w_{m-1}}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi\, d\eta \right| \le$$
$$\le \int\limits_0^t d\tau \int\limits_{|x-\xi|<t-\tau} \frac{|g_0|\, |w_{m-1}|}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi \le$$
$$\le \|g_0\|_{C(\overline{D}_T)} \int\limits_0^t d\tau \int\limits_{|x-\xi|<t-\tau} \frac{\lambda_{m-1}(\tau)}{\sqrt{(t-\tau)^2 - |x-\xi|^2}}\, d\xi =$$

$$= \|g_0\|_{C(\overline{D}_T)} \int_0^t (t-\tau)\lambda_{m-1}(\tau)\,d\tau \int_{|\eta|<1} \frac{d\eta_1\,d\eta_2}{\sqrt{1-|\eta|^2}} =$$
$$= B_2\lambda_{m-1}(t), \quad (x,t) \in D_T,$$

whence

$$\lambda_m(t) \le B_2\lambda_{m-1}(t) \le \cdots \le B_2^m \lambda_0(t) =$$
$$= \frac{1}{\Gamma(2m)} \int_0^t (b\Gamma_2))^m (t-\tau)^{2m-1} \lambda_0(\tau)\,d\tau \le$$
$$\le \frac{b^m}{\Gamma(2m)} \int_0^t (t-\tau)^{2m-1} \|w_0\|_{C(\overline{D}_T)}\,d\tau = \frac{(bT^2)^m}{\Gamma(2m)2m} \|F_0\|_{C(\overline{D}_T)} =$$
$$= \frac{(bT^2)^m}{(2m)!} \|F_0\|_{C(\overline{D}_T)},$$

and hence

$$\|w_m\|_{C(\overline{D}_T} = \|\lambda_m\|_{C([0,T])} \le \frac{(bT^2)^m}{(2m)!} \|F_0\|_{C(\overline{D}_T)}.$$

Therefore the series $v = \lim\limits_{m\to\infty} v_m = v_0 + \sum\limits_{m=0}^{\infty} w_m$ converges in the class $C(\overline{D}_T)$, and its sum is a solution of the equation (2.33). The uniqueness of the solution of the equation (2.33) in the space $C(\overline{D}_T)$ is proved analogously.

Since $\lambda < 0$, we have $g_0 = -\frac{\lambda}{2\pi}|u|^p \ge 0$, and by (2.32) the function $F_0 \ge 0$ because by the condition we have $F \ge 0$. Therefore the successive approximations v_k from (2.34) are nonnegative, and as far as $\lim\limits_{k\to\infty} \|v_k - v\|_{C(\overline{D}_T)} = 0$, the solution $v \ge 0$ in the domain D_T. Now it remains only to note that the function u by virtue of (2.31) is a solution of the equation (2.33), and due to the unique solvability of that equation, $u = v \ge 0$ in the domain D_T. Thus Lemma 2.3 is proved. □

Remark 2.4. As is seen from the proof, Lemma 2.3 is likewise valid if instead of the condition $F \ge 0$ we require fulfilment of the weaker condition $F_0 \ge 0$, where the function F_0 is given by the formula (2.32).

Lemma 2.4. *Let* $n = 2$, $F \in \overset{\circ}{C}(\overline{D}_T, S_T)$ *and* $u \in C^2(\overline{D}_T)$ *be a classical solution of the problem* (2.3), (2.4). *Then if for some point* $(x^0, t^0) \in D_T$ *the function* $F|_{D_{x^0,t^0}} = 0$, *then also* $u|_{D_{x^0,t^0}} = 0$, *where* $D_{x^0,t^0} = \big\{(x,t) \in R^3 : |x| < t < t^0 - |x - x^0|\big\}$.

Proof. Since $F|_{D_{x^0,t^0}} = 0$, by the representation (2.24) and Lemma 2.2 a solution u of the problem (2.3), (2.4) in the domain D_{x^0,t^0} satisfies the integral equation

$$u(x,t)=\frac{1}{2\pi}\int\limits_{D_{x,t}}\frac{\widetilde{g}_0(\xi,\tau)u(\xi,\tau)}{\sqrt{(t-\tau)^2-|x-\xi|^2}}\,d\xi\,d\tau,\ \ (x,t)\in D_{x^0,t^0},\tag{2.35}$$

where $\widetilde{g}_0=-\lambda|u|^p$. Taking into account that

$$\frac{1}{2\pi}\int\limits_{D_{x,t}}\frac{\tau^m}{\sqrt{(t-\tau)^2-|x-\xi|^2}}\,d\xi\,d\tau\le$$

$$\le\frac{1}{2\pi}\int\limits_0^t d\tau\int\limits_{|x-\xi|<t-\tau}\frac{\tau^m}{\sqrt{(t-\tau)^2-|x-\xi|^2}}\,d\xi=$$

$$=\frac{1}{2\pi}\int\limits_0^t\tau^m(t-\tau)\,d\tau\int\limits_{|\eta|<1}\frac{d\eta}{\sqrt{1-|\eta|^2}}=\frac{t^{m+2}}{(m+1)(m+2)}\,,$$

from (2.34) using the method of mathematical induction we easily get

$$|u(x,t)|\le MM_1^k\,\frac{t^{2k}}{(2k)!}\,,\ \ (x,t)\in D_{x^0,t^0},\ \ k=1,2,\dots,$$

where $M=\max\limits_{\overline{D}_T}|u(x,t)|=\|u\|_{C(\overline{D}_T)}$, $M_1=\max\limits_{\overline{D}_T}|\widetilde{g}_0(x,t)|$. Therefore, as $k\to\infty$ this implies that $u|_{D_{x^0,t^0}}=0$. Thus Lemma 2.4 is proved. □

Let c_R and $\varphi_R(x)$ be respectively the first eigenvalue and eigenfunction of the Dirichlet problem in the circle $\Omega_R:\ x_1^2+x_2^2<R^2$. Consequently,

$$(\Delta\varphi_R+c_R\varphi_R)\big|_{\Omega_R}=0,\quad \varphi_R\big|_{\partial\Omega_R}=0.\tag{2.36}$$

In addition, as is known, $c_R>0$, and changing the sign and making the corresponding normalization we can assume that [111, p. 25]

$$\varphi_R\big|_{\Omega_R}>0,\quad \int\limits_{\Omega_R}\varphi_R\,dx=1.\tag{2.37}$$

Theorem 2.2. *Let $n=2$, $\lambda<0$, $p>0$, $F\in C(\overline{D}_\infty)$, $\operatorname{supp}F\cap S_\infty=\varnothing$ and $F\ge 0$. Then if the condition*

$$\overline{\lim_{T\to\infty}}\,T^{\frac{p+2}{p}}\int\limits_0^T dt\int\limits_{\Omega_1}F(2T\xi,t)\varphi_1(\xi)\,d\xi=+\infty\tag{2.38}$$

is fulfilled, then there exists a number $T_0=T_0(F)>0$ such that for $T\ge T_0$ the problem (2.3), (2.4) cannot have a classical solution $u\in C^2(\overline{D}_T)$ in the domain D_T.

Proof. Assume that the problem (2.3), (2.4) has a classical solution $u\in C(\overline{D}_T)$ in the domain D_T. Since $\operatorname{supp}F\cap S_\infty=\varnothing$, there exists a positive number $\delta < T/2$ such that $F|_{U_\delta(S_T)} = 0$, where $U_\delta(S_T):\ |x| \le t \le |x| + \delta$, $t \le T$, whence, according to Lemma 2.4 we get

$$u|_{U_\delta(S_T)} = 0. \tag{2.39}$$

Next, as far as by the condition we have $F \ge 0$, by Lemma 2.3 we find

$$u|_{\overline{D}_T} \ge 0. \tag{2.40}$$

Therefore extending the functions F and u by zero outside the domain D_T into the layer $\Lambda_T:\ 0 < t < T$ and preserving for them the same notation, we find that $u \in C^2(\overline{\Lambda}_T)$ is a classical solution of the equation (2.3) in the layer Λ_T, which by virtue of $\lambda < 0$ and (2.40) can be written as

$$u_{tt} - \Delta u = |\lambda| u^{p+1} + F(x,t), \ \ (x,t) \in \Lambda_T. \tag{2.41}$$

By (2.39),

$$\operatorname{supp} u \subset \overline{D}_{T,\delta}, \quad D_{T,\delta} = \big\{(x,t) \in R^3:\ |x| + \delta < t < T\big\}. \tag{2.42}$$

Below without restriction of generality we assume that $\lambda = -1$ and, consequently, $|\lambda| = 1$, since the case $\lambda < 0$, $\lambda \ne -1$, owing to $p > 0$ is reduced to the case $\lambda = -1$ by introducing a new unknown function $v = |\lambda|^{1/p} u$. In addition, the function v will satisfy the condition

$$v_{tt} - \Delta v = v^{p+1} + |\lambda|^{1/p} F(x,t), \ \ (x,t) \in \Lambda_T. \tag{2.43}$$

In accordance with (2.43), below instead of (2.3) we will consider the equation

$$u_{tt} - \Delta u = u^{p+1} + F(x,t), \ \ (x,t) \in \Lambda_T. \tag{2.44}$$

We take $R \ge T$ and introduce into consideration the functions

$$E(t) = \int\limits_{\Omega_R} u(x,t)\varphi_R(x)\,dx, \quad f_R(t) = \int\limits_{\Omega_R} F(x,t)\varphi_R(x)\,dx, \ \ 0 \le t \le T. \tag{2.45}$$

It is clear that $E \in C^2([0,T])$, $f_R \in C([0,T])$, and the function $E \ge 0$, by (2.40).

In view of (2.36), (2.42) and (2.45), integration by parts yields

$$\int\limits_{\Omega_R} \Delta u \varphi_R\,dx = \int\limits_{\Omega_R} u \Delta\varphi_R\,dx = -c_R \int\limits_{\Omega_R} u\varphi_R\,dx = -c_R E. \tag{2.46}$$

By virtue of (2.37), (2.40) and $p > 0$, using Jensen's inequality [111, p. 26] we obtain

$$\int\limits_{\Omega_R} u^{p+1}\varphi_R\,dx \ge \Big(\int\limits_{\Omega_R} u\varphi_R\,dx\Big)^{p+1} = E^{p+1}. \tag{2.47}$$

It easily follows from (2.42) and (2.44)–(2.47) that

$$E'' + c_R E \ge E^{p+1} + f_R, \ \ 0 \le t \le T, \tag{2.48}$$

$$E(0) = 0, \quad E'(0) = 0. \tag{2.49}$$

To investigate the problem (2.48), (2.49) we use the method of test functions [101, pp. 10–12]. Towards this end, we take T_1, $0 < T_1 < T$, and consider a nonnegative test function $\psi \in C^2([0,T])$ such that

$$0 \le \psi \le 1, \ \ \psi(t) = 1, \ \ 0 \le t \le T_1, \ \ \psi^{(k)}(T) = 0, \ \ k = 0,1,2. \tag{2.50}$$

It follows from (2.48)–(2.50) that

$$\int\limits_0^T E^{p+1}(t)\psi(t)\,dt \le \int\limits_0^T E(t)\big[\psi''(t) + c_R\psi(t)\big]\,dt - \int\limits_0^T f_R(t)\psi(t)\,dt. \tag{2.51}$$

If in the Young inequality with the parameter $\varepsilon > 0$, $ab \le \frac{\varepsilon}{a}\,a^\alpha + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\,b^{\alpha'}$, $a,b \ge 0$, $\alpha' = \frac{\alpha}{\alpha-1}$, we take $\alpha = p+1$, $\alpha' = \frac{p+1}{p}$, $a = E\psi^{\frac{1}{p+1}}$, $b = \frac{|\psi''+c_R\psi|}{\psi^{\frac{1}{p+1}}}$, and take into account that $\frac{\alpha'}{\alpha} = \frac{1}{\alpha-1} = \alpha' - 1$, then we get

$$\begin{gathered} E|\psi'' + c_R\psi| = \\ = E\psi^{1/\alpha}\,\frac{|\psi''+c_R\psi|}{\psi^{1/\alpha}} \le \frac{\varepsilon}{\alpha}\,E^\alpha\psi + \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\,\frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}\,. \end{gathered} \tag{2.52}$$

By (2.52), from (2.51) we have

$$\Big(1 - \frac{\varepsilon}{\alpha}\Big)\int\limits_0^T E^\alpha\psi\,dt \le \frac{1}{\alpha'\varepsilon^{\alpha'-1}}\int\limits_0^T \frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}\,dt - \int\limits_0^T f_R(t)\psi(t)\,dt. \tag{2.53}$$

Bearing in mind that $\min\limits_{0<\varepsilon<\alpha}\big[\frac{\alpha-1}{\alpha-\varepsilon}\,\frac{1}{\varepsilon^{\alpha'-1}}\big] = 1$, which is achieved for $\varepsilon = 1$, from (2.53) and (2.50) we find that

$$\int\limits_0^{T_1} E^\alpha\,dt \le \int\limits_0^T \frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}\,dt - \alpha'\int\limits_0^T f_R(t)\psi(t)\,dt. \tag{2.54}$$

Now as the test function ψ we take the function of the type

$$\psi(t) = \psi_0(\tau), \ \ \tau = \frac{t}{T_1}\,, \ \ 0 \le \tau \le \tau_1 = \frac{T}{T_1}\,, \tag{2.55}$$

where

$$\begin{gathered} \psi_0 \in C^2([0,\tau_1]), \ \ 0 \le \psi_0 \le 1, \quad \psi_0(\tau) = 1, \ \ 0 \le \tau \le 1, \\ \psi_0^{(k)}(\tau_1) = 0, \ \ k = 0,1,2. \end{gathered} \tag{2.56}$$

It can be easily seen that

$$c_R = \frac{c_1}{R^2} \le \frac{c_1}{T^2} \le \frac{c_1}{T_1^2}\,, \quad \varphi_R(x) = \frac{1}{R^2}\,\varphi_1\Big(\frac{x}{R}\Big). \tag{2.57}$$

Owing to (2.55), (2.56), (2.57) and taking into account that $\psi''(t) = 0$ for $0 \le t \le T_1$ and $f_R \ge 0$ since $F \ge 0$, as well as the well-known inequality

$|a+b|^{\alpha'} \leq 2^{\alpha'-1}(|a|^{\alpha'}+|b|^{\alpha'})$, from (2.54) we obtain

$$\int_0^{T_1} E^\alpha\,dt \leq \int_0^{T_1} \frac{c_R^{\alpha'}\psi^{\alpha'}}{\psi^{\alpha'-1}}\,dt + \int_{T_1}^{T} \frac{|\psi''+c_R\psi|^{\alpha'}}{\psi^{\alpha'-1}}\,dt - \alpha'\int_0^T f_R(t)\psi(t)\,dt \leq$$

$$\leq c_R^{\alpha'}\int_0^{T_1}\psi\,dt + T_1\int_1^{\tau_1} \frac{\left|\frac{1}{T_1^2}\,\psi_0''(\tau)+c_R\psi_0(\tau)\right|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\,d\tau - \alpha'\int_0^{T_1} f_R(t)\,dt \leq$$

$$\leq c_R^{\alpha'}T_1 + \frac{2^{\alpha'-1}}{T_1^{2\alpha'-1}}\int_1^{\tau_1}\frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\,d\tau + T_1 2^{\alpha'-1}c_R^{\alpha'}\int_1^{\tau_1}\psi_0(\tau)\,d\tau -$$

$$-\alpha'\int_0^{T_1} f_R(t)\,dt \leq \frac{c_1^{\alpha'}}{T_1^{2\alpha'-1}} + \frac{2^{\alpha'-1}}{T_1^{2\alpha'-1}}\int_1^{\tau_1}\frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\,d\tau +$$

$$+\frac{2^{\alpha'-1}c_1^{\alpha'}}{T_1^{2\alpha'-1}}\,(\tau_1-1) - \alpha'\int_0^{T_1} f_R(t)\,dt. \tag{2.58}$$

Now we put $R=T$, $\tau_1=2$, i.e. $T_1=\frac{1}{2}T$. Then the inequality (2.58) takes the form

$$\int_0^{\frac{1}{2}T} E^\alpha\,dt \leq \Big(\frac{1}{2}\,T\Big)^{1-2\alpha'}\bigg[c_1^{\alpha'}(1+2^{\alpha'-1}) + 2^{\alpha'-1}\int_1^2\frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\,d\tau -$$

$$-\alpha'\Big(\frac{1}{2}\,T\Big)^{2\alpha'-1}\int_0^{\frac{1}{2}T} f_T(t)\,dt,\quad 2\alpha'-1=\frac{p+2}{p}\,. \tag{2.59}$$

As is known, the function ψ_0 with the properties (2.56) for which the integral

$$\varkappa(\psi_0) = \int_1^2 \frac{|\psi_0''(\tau)|^{\alpha'}}{(\psi_0(\tau))^{\alpha'-1}}\,d\tau < +\infty \tag{2.60}$$

is finite does exist.

With regard for (2.45) and (2.57), we have

$$\beta(T) = \int_0^{\frac{1}{2}T} f_T(t)\,dt = \int_0^{\frac{1}{2}T} dt\int_{\Omega_T} F(x,t)\,\varphi_T(x)\,dx =$$

$$= \int_0^{\frac{1}{2}T} dt\int_{\Omega_T} F(x,t)\,\frac{1}{T_2}\,\varphi_1\Big(\frac{x}{T}\Big)\,dx = \int_0^{\frac{1}{2}T} dt\int_{\Omega_1} F(T\xi,t)\varphi_1(\xi)\,d\xi. \tag{2.61}$$

If the condition (2.38) is fulfilled, then due to (2.59), (2.60) and (2.61) there exists a number $T=T_0>0$ such that the right-hand side of the inequality (2.59) is negative, but this is impossible because the left-hand side of the inequality (2.59) is nonnegative. Thus for $T = T_0$, and hence for $T \geq T_0$, the problem (2.3), (2.4) cannot have a classical solution $u \in C^2(\overline{D}_T)$ in the domain D_T. Thus Theorem 2.2 is proved completely. □

Corollary 2.1. *Let* $n=2$, $\lambda<0$, $F\in C(\overline{D}_\infty)$, $\operatorname{supp} F \cap S_\infty = \varnothing$, $F \not\equiv 0$ *and* $F \geq 0$. *Then if* $0 < p < 2$, *then there exists a number* $T_0 = T_0(F) > 0$ *such that for* $T \geq T_0$ *the problem* (2.3), (2.4) *cannot have a classical solution* $u \in C^2(\overline{D}_T)$ *in the domain* D_T.

Indeed, since $F \not\equiv 0$ and $F \geq 0$, there exists a point $P_0(x^0, t^0) \in D_\infty$ such that $F(x^0, t^0) > 0$. Without restriction of generality, we can assume that the point P_0 lies on the t-axis, i.e. $x^0 = 0$; otherwise, this can be achieved by the Lorentz transformation under which the equation (2.3) is invariant and the characteristic cone $S_\infty :\ t = |x|$ remains unchanged [24, p. 744]. Since $F(0, t^0) > 0$ and $F \in C(\overline{D}_\infty)$, there exist numbers $t^0 > 0$, $\varepsilon_0 > 0$ and $\sigma > 0$ such that $F(x,t) \geq \sigma$ for $|x| < \varepsilon_0$, $|t - t^0| < \varepsilon_0$. We take $T > 2(t^0 + \varepsilon_0)$. Then for $|x| < \varepsilon_0$ it is obvious that $|x/T| < 1/2$. If we introduce the notation $m_0 = \inf\limits_{|\eta|<1/2} \varphi_1(\eta)$, then since $\varphi_1(x) > 0$ in the unit circle $\Omega_1 :\ |x| < 1$, we have $m_0 > 0$, and by (2.61) we get

$$\beta(T) = \frac{1}{T^2}\int\limits_0^{\frac{1}{2}T} dt \int\limits_{\Omega_T} F(x,t)\,\varphi_1\Big(\frac{x}{T}\Big)\,dx \geq \frac{1}{T^2}\int\limits_{t^0-\varepsilon}^{t^0+\varepsilon} dt \int\limits_{|x|<\varepsilon_0} F(x,t)\,\varphi_1\Big(\frac{x}{T}\Big)\,dx \geq$$

$$\geq \frac{1}{T^2}\int\limits_{t^0-\varepsilon}^{t^0+\varepsilon} dt \int\limits_{|x|<\varepsilon_0} \sigma m_0\,dx = \frac{2\pi\varepsilon_0^3\sigma m_0}{T^2}\,.$$

Consequently,

$$T^{\frac{p+2}{p}} \int\limits_0^T dt \int\limits_{\Omega_1} F(2T\xi, t)(\xi)\,d\xi = T^{\frac{p+2}{2}}\beta(2T) \geq \frac{1}{2}\,\pi\varepsilon_0^3\sigma m_0 T^{\frac{2-p}{p}}\,.$$

The above expression for $0 < p < 2$ immediately results in (2.38), and by Theorem 2.2 the problem (2.3), (2.4) cannot have a classical solution $u \in C^2(\overline{D}_T)$ in the domain D_T for $T \geq T_0$.

Corollary 2.2. *Let* $n = 2$, $\lambda < 0$, $F \in C(\overline{D}_\infty)$, $\operatorname{supp} F \cap S_\infty = \varnothing$ *and* $F \geq 0$. *Assume that* $F(x,t) \geq \gamma(t) \geq 0$ *for* $|x| < \varepsilon(t) < t$, $t > \delta$, *and* $\sup\limits_{t>\delta} \frac{\varepsilon(t)}{t} = \varepsilon_0 < 1$, *where* $\gamma(t)$ *and* $\varepsilon(t)$ *are given continuous functions with* $\gamma(t) \geq 0$ *and* $\varepsilon(t) > 0$. *Then if the condition*

$$\varlimsup_{T\to+\infty} T^{\frac{2-p}{p}} \int\limits_\delta^T \varepsilon^2(t)\gamma(t)\,dt = +\infty \tag{2.62}$$

is fulfilled, then there exists a number $T_0 = T_0(F) > 0$ *such that for* $T \geq T_0$ *the problem* (2.3), (2.4) *cannot have a classical solution* $u \in C^2(\overline{D}_T)$ *in the domain* D_T.

Indeed, for $|x| < \varepsilon(t)$, $t \leq \frac{1}{2}T$ we have $\left|\frac{x}{T}\right| < \frac{\varepsilon(t)}{T} = \frac{\varepsilon(t)}{t}\,\frac{t}{T} \leq \frac{1}{2}\,\varepsilon_0$. Since $\inf\limits_{|\eta|<\frac{1}{2}\varepsilon_0} \varphi_1(\eta) = m_0 > 0$, by virtue of (2.61) we have

$$\beta(T) = \frac{1}{T^2}\int\limits_0^{\frac{1}{2}T} dt \int\limits_{\Omega_T} F(x,t)\,\varphi_1\Big(\frac{x}{T}\Big)\,dx \geq \frac{1}{T^2}\int\limits_\delta^{\frac{1}{2}T} dt \int\limits_{|x|<\varepsilon(t)} \gamma(t)\varphi_1\Big(\frac{x}{T}\Big)\,dx \geq$$

$$\geq \frac{m_0}{T^2}\int\limits_\delta^{\frac{1}{2}T} dt \int\limits_{|x|<\varepsilon(t)} \gamma(t)\,dx = \frac{\pi m_0}{T^2}\int\limits_\delta^{\frac{1}{2}T} \varepsilon^2(t)\gamma(t)\,dt.$$

Therefore

$$T^{\frac{p+2}{p}}\int\limits_0^T dt \int\limits_{\Omega_1} F(2T\xi,t)\varphi_1(\xi)\,d\xi = T^{\frac{p+2}{p}}\beta(2T) \geq \frac{\pi m_0}{4}\,T^{\frac{2-p}{p}}\int\limits_\delta^T \varepsilon^2(t)\gamma(t)\,dt,$$

whence by (2.62) it follows (2.38), and the conclusion of Theorem 2.2 is valid.

Remark 2.5. The inequality (2.59) allows one to estimate the time interval after which a solution ceases to exist. Indeed, assume that

$$\chi(T) = \sup_{0<t<T} \alpha'\Big(\frac{1}{2}\,t\Big)^{2\alpha'-1}\int\limits_0^{\frac{1}{2}t} f_t(\tau)\,d\tau, \quad \chi_0 = c_1^{\alpha'}(1+2^{\alpha'-1}) + 2^{\alpha'-1}\varkappa(\psi_0),$$

where $\alpha' = \frac{p+1}{p}$, and the finite positive number $\varkappa(\psi_0)$ is given by the equality (2.60). Since $F \in C(\overline{D}_\infty)$, the function $\chi(T)$ on the interval $0 < T < +\infty$ is continuous and nondecreasing, and owing to (2.38) and (2.61), we have $\lim\limits_{T\to+\infty}\chi(T) = +\infty$. Therefore since $\lim\limits_{T\to 0}\chi(T) = 0$, the equation $\chi(T) = \chi_0$ is solvable. Denote by $T = T_1$ that root of the above equation for which $\chi(T) > \chi(T_1)$ for $T_1 < T < T_1 + \varepsilon$, where ε is a sufficiently small positive number. It becomes now clear that the problem (2.3), (2.4) cannot have a classical solution in the domain D_T for $T > T_1$, since in that case the right-hand side of the inequality (2.59) would be negative.

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Bibliography

1. M. S. Agranovich, Positive mixed problems for certain hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* **167**(1966), 1215–1218; English transl.: *Sov. Math., Dokl.* **7**(1966), 539–542.
2. M. S. Agranovich, On the theory of boundary value problems for symmetrizable first-order systems. (Russian) *Mat. Sb.* (*N.S.*) **73(115)**(1967), 161–197.
3. M. S. Agranovich, Boundary value problems for systems of first order pseudodifferential operators. (Russian) *Uspekhi Mat. Nauk* **24**(1969), No. 1(145), 61–125.
4. G. Avalishvili and D. Gordeziani, On one class of spatial nonlocal problems for some hyperbolic equations. *Georgian Math. J.* **7**(2000), No. 3, 417–425.
5. G. A. Avalishvili, Nonlocal in time problems for evolution equations of second order. *J. Appl. Anal.* **8**(2002), No. 2, 245–259.
6. Yu. M. Berezans'kiĭ, Expansions in eigenfunctions of selfadjoint operators. (Russian) *Naukova Dumka, Kiev*, 1965.
7. A. V. Bitsadze, Equations of mixed type in three-dimensional regions. (Russian) *Dokl. Akad. Nauk SSSR* **143**(1962), 1017–1019.
8. A. V. Bitsadze, On the question of formulating the characteristic problem for second order hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* **223**(1975), No. 6, 1289–1292.
9. A. V. Bitsadze, Influence of the lower terms on the correctness of the formulation of characteristic problems for second order hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* **225**(1975), No. 1, 31–34.
10. A. V. Bitsadze, Some classes of partial differential equations. (Russian) *Nauka, Moscow*, 1981.
11. A. V. Bitsadze and A. M. Nakhushev, On the theory of degenerate hyperbolic equations in multidimensional domains. (Russian) *Dokl. Akad. Nauk SSSR* **204**(1972), 1289–1291; English transl.: *Sov. Math., Dokl.* **13**(1972), 807–810.
12. A. V. Bitsadze and A. M. Nakhushev, Correctly posed problems for equations of mixed type in multidimensional domains. (Russian) *Dokl. Akad. Nauk SSSR* **205**(1972), 9–12; English transl.: *Sov. Math., Dokl.* **13**(1972), 857–860.
13. A. V. Bitsadze and A. M. Nakhushev, On the theory of equations of mixed type in multidimensional domains. (Russian) *Differentsial'nye Uravneniya* **10**(1974), 2184–2191.
14. A. V. Bitsadze and A. A. Samarskiĭ, Some elementary generalizations of linear elliptic boundary value problems. (Russian) *Dokl. Akad. Nauk SSSR* **185**(1969), 739–740; English transl.: *Sov. Math., Dokl.* **10**(1969), 398–400.
15. A. V. Bitsadze, On the theory of nonlocal boundary value problems. (Russian) *Dokl. Akad. Nauk SSSR* **277**(1984), No. 1, 17–19.
16. A. V. Bitsadze, A class of conditionally solvable nonlocal boundary value problems for harmonic functions. (Russian) *Dokl. Akad. Nauk SSSR* **280**(1985), No. 3, 521–524.

17. G. Bogveradze and S. Kharibegashvili, On some nonlocal problems for a hyperbolic equation of second order on a plane. *Proc. A. Razmadze Math. Inst.* **136**(2004), 1–36.
18. A. Bouziani, On a class of parabolic equations with a nonlocal boundary condition. *Acad. Roy. Belg. Bull. Cl. Sci. (6)* **10**(1999), No. 1–6, 61–77.
19. A. Bouziani, On a class of nonclassical hyperbolic equations with nonlocal conditions. *J. Appl. Math. Stochastic Anal.* **15**(2002), No. 2, 135–153.
20. F. Cagnac, Problème de Cauchy sur un conoïde caractéristique. *Ann. Mat. Pura Appl. (4)* **104**(1975), 355–393.
21. J. R. Cannon, The solution of the heat equation subject to the specification of energy. *Quart. Appl. Math.* **21**(1963), 155–160.
22. R. W. Carrol and R. E. Showalter, Singular and degenerate Cauchy problems. *Mathematics in Science and Engineering, Vol.* 127*, Academic Press, New York–London*, 1976.
23. F. J. S. A. Corrêa and S. D. B. Menezes, Existence of solutions to nonlocal and singular elliptic problems via Galerkin method. Electron. *J. Differential Equations*, 2004, No. 19, 10 pp. (electronic).
24. R. Courant, Partial differential equations. (Russian) *Mir, Moscow*, 1964.
25. A. A. Dezin, Mixed problems for certain symmetric hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* (*N.S.*) **107**(1956), 13–16.
26. A. A. Dezin, Boundary value problems for some symmetric linear systems of the first order. (Russian) *Mat. Sb.* **49**(1959), No. 4, 459–484.
27. A. A. Dezin, Invariant hyperbolic systems and the Goursat problem. (Russian) *Dokl. Akad. Nauk SSSR* **135**(1961), 1042–1045; English transl.: *Soviet Math. Dokl.* **1**(1961), 1340–1343.
28. M. Dehghan, On the solution of an initial-boundary value problem that combines Neumann and integral condition for the wave equation. *Numer. Methods Partial Differential Equations* **21**(2005), No. 1, 24–40.
29. C. M. Fichtengolz, Course of differential and integral calculus. I. (Russian) *Nauka, Moscow*, 1969.
30. K. O. Friedrichs, Symmetric hyperbolic linear differential equations. *Comm. Pure Appl. Math.* **7**(1954), 345–392.
31. K. O. Friedrichs, Symmetric positive linear differential equations. *Comm. Pure Appl. Math.* **11**(1958), 333–418.
32. K. O. Friedrichs and P. D. Lax, Boundary value problems for first order operators. *Comm. Pure Appl. Math.* **18**(1965), 355–388.
33. K. O. Friedrichs and P. D. Lax, On symmetrizable differential operators. 1967 *Singular Integrals* (*Proc. Sympos. Pure Math., Chicago, Ill.*, 1966, pp. 128–137, *Amer. Math. Soc., Providence, R.I.*
34. Z. Gan and J. Zhang, Instability of standing waves for Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions. *J. Math. Anal. Appl.* **307**(2005), No. 1, 219–231.
35. V. Georgiev, H. Lindblad, and C. D. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equations. *Amer. J. Math.* **119**(1997), No. 6, 1291–1319.
36. D. G. Gordeziani, Methods for solving a class of nonlocal boundary value problems. (Russian) *Tbilis. Gos. Univ., Inst. Prikl. Mat., Tbilisi*, 1981.
37. D. G. Gordeziani and G. A. Avalishvili, Solution of nonlocal problems for one-dimensional oscillations of a medium. (Russian) *Mat. Model.* **12**(2000), No. 1, 94–103.
38. D. Gordeziani and G. Avalishvili, Investigation of the nonlocal initial boundary value problems for some hyperbolic equations. *Hiroshima Math. J.* **31**(2001), No. 3, 345–366.

39. A. K. Gushchin and V. P. Mikhaĭlov, Continuity of solutions of a class of nonlocal problems for an elliptic equation. (Russian) *Mat. Sb.* **186**(1995), No. 2, 37–58; English transl.: *Sb. Math.* **186**(1995), No. 2, 197–219.
40. J. Gvazava and S. Kharibegashvili, To the theory of boundary value problems for hyperbolic type equations and systems. *Mem. Differential Equations Math. Phys.* **12**(1997), 68–75.
41. J. Gvazava, The mean value property for nonstrictly hyperbolic second order quasilinear equations and the nonlocal problems. *Proc. A. Razmadze Math. Inst.* **135**(2004), 79–96.
42. J. Gvazava, Nonlocal and initial problems for quasilinear, nonstrictly hyperbolic equations with general solutions represented by superposition of arbitrary functions. *Georgian Math. J.* **10**(2003), No. 4, 687–707.
43. J. Hadamard, Lectures on Cauchy's Problem on Partial Differential Equations. *New Haven, Yale University Press*, 1933.
44. D. Henry, Geometric theory of semilinear parabolic equations. (Russian) *Mir, Moscow*, 1985.
45. K. Hidano, The global existence theorem for quasi-linear wave equations with multiple speeds. *Hokkaido Math. J.* **33**(2004), No. 3, 607–636.
46. R. Horn and Ch. Johnson, Matrix analysis. (Russian) *Mir, Moscow*, 1989.
47. L. Hörmander, Lectures on nonlinear hyperbolic differential equations. *Mathematics & Applications*, 26, *Springer–Verlag, Berlin*, 1997.
48. L. Hörmander, Linear partial differential operators. (Russian) *Mir, Moscow*, 1965.
49. V. A. Il'in and E. I. Moiseev, A two-dimensional nonlocal boundary value problem for the Poisson operator in the differential and the difference interpretation. (Russian) *Mat. Model.* **2**(1990), No. 8, 139–156.
50. V. A. Il'in and E. I. Moiseev, On the uniqueness of the solution of a mixed problem for the wave equation with nonlocal boundary conditions. (Russian) *Differentsial'nye Uravneniya* **36**(2000), No. 5, 656–661, 719; English transl.: *Differ. Equ.* **36**(2000), No. 5, 728–733.
51. N. I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition. (Russian) *Differentsial'nye Uravneniya* **13**(1977), No. 2, 294–304.
52. F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Math.* **28**(1979), No. 1–3, 235–268.
53. F. John, Blow-up for quasilinear wave equations in three space dimensions. *Comm. Pure Appl. Math.* **34**(1981), No. 1, 29–51.
54. F. John and S. Klainerman, Almost global existence to nonlinear wave equations in three space dimensions. *Comm. Pure Appl. Math.* **37**(1984), No. 4, 443–455.
55. O. Jokhadze, On the boundary value problems for normally hyperbolic systems of first order equations in the space. *Rend. Mat. Appl. (7)* **18**(1998), No. 3, 497–528.
56. O. Jokhadze, On the boundary value problem in a dihedral angle for normally hyperbolic systems of first order. *Georgian Math. J.* **5**(1998), No. 2, 121–138.
57. K. Jörgens, Konrad. Das Anfangswertproblem im Grossen fur eine Klasse nichtlinearer Wellengleichungen. *Math. Z.* **77**(1961), 295–308.
58. L. I. Kamynin, A boundary-value problem in the theory of heat conduction with non-classical boundary conditions. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **4**(1964), 1006–1024; English transl.: *U.S.S.R. Comput. Math. Math. Phys.* **4**(1964), No. 6, 33–59.
59. T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations. *Comm. Pure Appl. Math.* **33**(1980), No. 4, 501–505.
60. M. Keel, H. F. Smith, and C. D. Sogge, Almost global existence for quasilinear wave equations in three space dimensions. *J. Amer. Math. Soc.* **17**(2004), No. 1, 109–153 (electronic).

61. S. S. Kharibegashvili, A problem of Goursat type for a class of hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* **17**(1981), No. 1, 157–164, 199.
62. S. S. Kharibegashvili, A boundary value problem for second-order normally hyperbolic systems with variable coefficients. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 1, 149–155; English transl.: *Differ. Equations* **21**(1985), 121–126.
63. S. S. Kharibegashvili, Solvability of a boundary value problem for second-order hyperbolic equations. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **19**(1987), 122–173, 208–209.
64. S. Kharibegashvili, Goursat and Darboux type problems for linear hyperbolic partial differential equations and systems. *Mem. Differential Equations Math. Phys.* **4**(1995), 1–127.
65. S. Kharibegashvili, On a Darboux type multidimensional problem for second-order hyperbolic systems. *Georgian Math. J.* **3**(1996), No. 4, 363–378.
66. S. Kharibegashvili, On the solvability of the multidimensional version of the first Darboux problem for a model second-order degenerating hyperbolic equation. *Georgian Math. J.* **4**(1997), No. 4, 341–354.
67. S. Kharibegashvili, On the solvability of a multidimensional version of the Goursat problem for a second order hyperbolic equation with characteristic degeneration. Dedicated to Revaz Gamkrelidze. *Mem. Differential Equations Math. Phys.* **11**(1997), 89–103.
68. S. Kharibegashvili, On some multidimensional versions of a characteristic problem for second-order degenerating hyperbolic equations. *Georgian Math. J.* **5**(1998), No. 2, 139–156.
69. S. S. Kharibegashvili, Boundary value problems for a class of systems of second-order partial differential equations. (Russian) *Differentsial'nye Uravneniya* **34**(1998), No. 1, 114–122; English transl.: *Differential Equations* **34**(1998), No. 1, 117–125.
70. S. Kharibegashvili, On some boundary value problems for an ultrahyperbolic equation. *Georgian Math. J.* **5**(1998), No. 4, 341–360.
71. S. S. Kharibegashvili, On the solvability of some multidimensional versions of the Goursat problem for a first-order hyperbolic system. (Russian) *Differentsial'nye Uravneniya* **35**(1999), No. 8, 1112–1121; English transl.: *Differential Equations* **35**(1999), No. 8, 1125–1135.
72. S. S. Kharibegashvili, On the formulation of characteristic problems for first-order symmetric hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* **36**(2000), No. 8, 1113–1122; English transl.: *Differential Equations* **36**(2000), No. 8, 1233–1243.
73. S. Kharibegashvili, On the solvability of some boundary value problems for symmetric first order hyperbolic systems in a dihedral angle. *Mem. Differential Equations Math. Phys.* **27**(2002), 115–144.
74. S. Kharibegashvili, On the correct formulation of some boundary value problems for symmetric hyperbolic systems of first order in a dihedral angle. *Mem. Differential Equations Math. Phys.* **29**(2003), 1–30.
75. S. S. Kharibegashvili, On the well-posedness of some nonlocal problems for the wave equation. (Russian) *Differentsial'nye Uravneniya* **39**(2003), No. 4, 539–553; English transl.: *Differential Equations* **39**(2003), No. 4, 577–592.
76. S. S. Kharibegashvili, A multidimensional version of the Darboux problem for a model second-order degenerate hyperbolic equation. (Russian) *Differentsial'nye Uravneniya* **40**(2004), No. 4, 565–573; English transl.: *Differential Equations* **40**(2004), No. 4, 610–619.
77. T. Kiguradze, Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Phys.* **1**(1994), 1-144.

78. T. I. Kiguradze, Some nonlocal problems for linear hyperbolic systems. (Russian) *Dokl. Akad. Nauk* **345**(1995), No. 3, 300–302; English transl.: *Dokl. Math.* **52**(1995), No. 3, 376–378.
79. M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik, and P. E. Sobolevskiĭ, Integral operators in spaces of summable functions. (Russian) *Nauka, Moscow*, 1966.
80. K. Kubota and T. Ohkubo, On well posedness of mixed problems for Maxwell's equations. *Hokkaido Math. J.* **7**(1978), No. 1, 142–168.
81. K. Kubota and T. Ohkubo, On well-posedness of mixed problems for Maxwell's equations. II. *Hokkaido Math. J.* **7**(1978), No. 2, 308–327.
82. A. Kufner and S. Fuchic, Nonlinear differential equations. (Russian) *Nauka, Moscow*, 1988.
83. A. G. Kurosh, Course of Higher Algebra. (Russian) *Nauka, Moscow*, 1968.
84. O. A. Ladyzhenskaya, Boundary value problems of mathematical physics. (Russian) *Nauka, Moscow*, 1973.
85. H. A. Levin, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$. *Trans. Amer. Math. Soc.* **192**(1974), 1–24.
86. I. I. Ljashko, V. P. Didenko, and P. E. Tsitritskiĭ, Filtering of noise. (Russian) *Naukova Dumka, Kiev*, 1979.
87. J. L. Lions, Quelcues méthods de résolution des problèmes aux limites non linéaires. *Dunod, Gauthier–Villars, Paris*, 1969.
88. L. Lundberg, The Klein–Gordon equation with light-cone data. *Comm. Math. Phys.* **62**(1978), No. 2, 107–118.
89. W. McLean, Strongly elliptic systems and boundary integral equations. *Cambridge University Press, Cambridge*, 2000.
90. Z. O. Mel'nik, A problem with integral constraints for general two-dimensional hyperbolic equations and systems. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No. 2, 246–253.
91. Z. O. Mel'nik and V.M. Kirilich, Problems without initial conditions with integral constraints for hyperbolic equations and systems on the line. (Russian) *Ukrain. Mat. Zh.* **35**(1983), No. 6, 722–727.
92. F. Merle and H. Zaag, Determination of the blow-up rate for a critical semilinear wave equation. *Math. Ann.* **331**(2005), No. 2, 395–416.
93. S. Mesloub and A Bouziani, On a class of singular hyperbolic equation with a weighted integral condition. *Int. J. Math. Math. Sci.* **22**(1999), No. 3, 511–519.
94. S. Mesloub and A. Bouziani, Mixed problem with a weighted integral condition for a parabolic equation with the Bessel operator. *J. Appl. Math. Stochastic Anal.* **15**(2002), No. 3, 291–300.
95. S. Mesloub and N. Lekrine, On a nonlocal hyperbolic mixed problem. *Acta Sci. Math.* (*Szeged*) **70**(2004), No. 1–2, 65–75.
96. C. Miao, B. Zhang, and D. Fang, Global well-posedness for the Klein–Gordon equation below the energy norm. *J. Partial Differential Equations* **17**(2004), No. 2, 97–121.
97. B. Midodashvili, A nonlocal problem for fourth order hyperbolic equations with multiple characteristics. Electron. *J. Differential Equations*, 2002, No. 85, 7 pp. (electronic).
98. B. Midodashvili, Generalized Goursat problem for a spatial fourth order hyperbolic equation with dominated low terms. *Proc. A. Razmadze Math. Inst.* **138**(2005), 43–54.
99. J. W. Milnor and J. Stasheff, Characteristic classes. (Russian) *Mir, Moscow*, 1979.
100. S. G. Mikhlin, A course in mathematical physics. (Russian) *Nauka, Moscow*, 1968.
101. È. Mitidieri and S. I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. (Russian) *Tr. Mat.*

Inst. Steklova **234**(2001), 1–384; English transl.: *Proc. Steklov Inst. Math.*, 2001, No. 3(234), 1–362.

102. E. Moiseev, Spectral characteristics of some nonlocal boundary-value problems. Computational tools of complex systems, II. *Comput. Math. Appl.* **34**(1997), No. 5–6, 649–655.
103. A. M. Nakhushev, A multidimensional analogue of the problem of Darboux for hyperbolic equations. (Russian) *Dokl. Akad. Nauk SSSR* **194**(1970), 31–34.
104. A. M. Nakhushev, Equations of mathematical biology. (Russian) *Vysshaya Shkola, Moscow*, 1995.
105. E. I. Obolashvili, Some boundary value problems for a metaparabolic equation. (Russian) *Reports of the extended sessions of a seminar of the I. N. Vekua Institute of Applied Mathematics, Vol. I, No.* 1 (*Russian*) (*Tbilisi*, 1985), 161–164, *Tbilis. Gos. Univ., Tbilisi*, 1985.
106. T. Ohkubo, Regularity of solutions to hyperbolic mixed problems with uniformly characteristic boundary. *Hokkaido Math. J.* **10**(1981), No. 1, 93–123.
107. N. I. Popivanov and M. F. Schneider, The Darboux problems in R^3 for a class of degenerating hyperbolic equations. *J. Math. Anal. Appl.* **175**(1993), No. 2, 537–578.
108. L. S. Pul'kina, A mixed problem with an integral condition for a hyperbolic equation. (Russian) *Mat. Zametki* **74**(2003), No. 3, 435–445; English transl.: *Math. Notes* **74**(2003), No. 3-4, 411–421.
109. J. M. Rassias, On a Goursat type problem. *C. R. Math. Rep. Acad. Sci. Canada* **2**(1980/81), No. 1, 49–51.
110. M. Reed and B. Simon, Methods of modern mathematical physics. II. Fourier analysis, self-adjointness. *Academic Press, New York-London*, 1975.
111. A. A. Samarskĭ, V. A. Galaktionov, S. P. Kurdjumov, and A. P. Mikhaĭlov, Peaking modes in problems for quasilinear parabolic equations. (Russian) *Nauka, Moscow*, 1987.
112. L. I. Schiff, Nonlinear meson theory of nuclear forces. I. Neutral scalar mesons with point-contact repulsion. *Phys. Rev.,* II. *Ser.* **84**(1951), 1–9.
113. P. Secchi, Full regularity of solutions to a nonuniformly characteristic boundary value problem for symmetric positive systems. *Adv. Math. Sci. Appl.* **10**(2000), No. 1, 39–55.
114. I. E. Segal, The global Cauchy problem for a relativistic scalar field with power interaction. *Bull. Soc. Math. France* **91**(1963), 129–135.
115. A. L. Skubachevskiĭ, Nonlocal elliptic problems and multidimensional diffusion processes. *Russian J. Math. Phys.* **3**(1995), No. 3, 327–360.
116. M. M. Smirnov, Degenerating elliptic and hyperbolic equations. (Russian) *Nauka, Moscow*, 1966.
117. Y. Tanaka, The existence of solutions to the mixed problem for symmetric hyperbolic systems. *Japan. J. Math.* (*N.S.*) **30**(2004), No. 2, 349–386.
118. E. Titchmarsh, The theory of functions. (Russian) *Nauka, Moscow*, 1980.
119. G. Todorova and E. Vitillaro, Blow-up for nonlinear dissipative wave equations in $\mathbb{R}^n$. *J. Math. Anal. Appl.* **303**(2005), No. 1, 242–257.
120. V. A. Trenogin, Functional analysis. (Russian) *Nauka, Moscow*, 1993.
121. H. Triebel, Interpolation theory, function spaces, differential operators. (Russian) *Mir, Moscow*, 1980.
122. V. S. Vladimirov, Linear passive systems. (Russian) *Teoret. Mat. Fiz.* **1**(1969), No. 1, 67–94. (loose errata).
123. V. S. Vladimirov, Generalized functions in mathematical physics. (Russian) *Nauka, Moscow*, 1979.
124. V. S. Vladimirov, Equations of mathematical physics. (Russian) *Nauka, Moscow*, 1971.
125. V. N. Vragov, The Goursat and Darboux problems for a certain class of hyperbolic equations. (Russian) *Differentsial'nye Uravneniya* **8**(1972), 7–16.

126. V. N. Vragov, Boundary value problems for nonclassical equations of mathematical physics. (Russian) *Novosibirsk: NSU*, 1983.
127. B. Z. Vulikh, Concise course of the theory of functions of a real variable. (Russian) *Nauka, Moscow*, 1973.
128. Z. Yin, Well-posedness, global solutions and blowup phenomena for a nonlinearly dispersive wave equation. *J. Evol. Equ.* **4**(2004), No. 3, 391–419.
129. N. I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations. (Russian) *Differentsial'nye Uravneniya* **22**(1986), No. 12, 2117-2126.
130. Y. Zhou, A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in $\mathbb{R}^N$. *Appl. Math. Lett.* **18**(2005), No. 3, 281–286.

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