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ON THE GENERAL AND MULTIPOINT BOUNDARY VALUE PROBLEMS FOR LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS, LINEAR IMPULSE AND LINEAR DIFFERENCE SYSTEMS

Abstract. The system of the generalized linear ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t)$$

is considered with general $\ell(x) = c_0$, multipoint $\sum_{j=1}^{n_0} L_j x(t_j) = c_0$, and Cauchy–Nicoletti type $x_i(t_i) = \ell_i(x_1, \dots, x_n) + c_{0i}$ $(i = 1, \dots, n)$ boundary value conditions, where $A : [a, b] \to \mathbb{R}^{n\times n}$ and $f : [a, b] \to \mathbb{R}^n$ are, respectively, matrix- and vector-functions with bounded total variation components on the closed interval $[a, b]$, $c_0 = (c_{0i})_{i=1}^n \in \mathbb{R}^n$, $t_i \in [a, b]$ $(i = 1, \dots, n(n_0))$, n_0 is a fixed natural number, $L_j \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, n_0)$, x_i is the i-th component of x, and ℓ and ℓ_i $(i = 1, \dots, n)$ are linear operators.

Effective sufficient, among them spectral, conditions are obtained for the unique solvability of these problems. The obtained results are realized for the linear impulsive system

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_k+) - x(\tau_k-) = G_k x(\tau_k) + g_k \ \ (k = 1, 2, \dots),$$

where $P \in L([a, b], \mathbb{R}^{n\times n})$, $q \in L([a, b], \mathbb{R}^n)$, $G_k \in \mathbb{R}^{n\times n}$, $g_k \in \mathbb{R}^n$ and $\tau_k \in [a, b]$ $(k = 1, 2, \dots)$, and linear difference system

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + G_3(k)y(k+1) + g_0(k)$$

$(k = 1, \dots, m_0)$, where $G_j(k) \in \mathbb{R}^{n\times n}$, $g_0(k) \in \mathbb{R}^n$ $(j = 1, 2, 3;\ k = 1, \dots, m_0)$.

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რეზიუმე. განზოგადებულ ჩვეულებრივ წრფივ დიფერენციალურ განტოლებათა

$$dx(t) = dA(t) \cdot x(t) + df(t)$$

სისტემისათვის განხილულია ზოგადი $\ell(x) = c_0$, მრავალწერტილოვანი $\sum_{j=1}^{n_0} L_j x(t_j) = c_0$ და კოში–ნიკოლეტის ტიპის $x_i(t_i) = \ell_i(x_1, \dots, x_n) + c_{0i}$ $(i = 1, \dots, n)$ სასაზღვრო ამოცანები, სადაც $A : [a, b] \to \mathbb{R}^{n\times n}$ და $f : [a, b] \to \mathbb{R}^n$, შესაბამისად, სასრული ვარიაციის კომპონენტებიანი მატრიცული და ვექტორული ფუნქციებია, $c_0 = (c_{0i})_{i=1}^n \in \mathbb{R}^n$, $t_i \in [a, b]$ $(i = 1, \dots, n(n_0))$, n_0 ფიქსირებული ნატურალური რიცხვია, $L_j \in \mathbb{R}^{n\times n}$ $(j = 1, \dots, n_0)$, x_i x-ის i-ური კომპონენტაა, ხოლო ℓ და ℓ_i $(i = 1, \dots, n)$ კი წრფივი ოპერატორებია.

მიღებულია ამ ამოცანების ცალსახად ამოხსნადობის ეფექტური საკმარისი, მათ შორის სპექტრალური, პირობები.

მიღებული შედეგები კონკრეტიზებულია წრფივი იმპულსური

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_k+) - x(\tau_k-) = G_k x(\tau_k) + g_k \ \ (k = 1, 2, \dots)$$

სისტემისათვის, სადაც $P \in L([a, b], \mathbb{R}^{n\times n})$, $q \in L([a, b], \mathbb{R}^n)$, $G_k \in \mathbb{R}^{n\times n}$, $g_k \in \mathbb{R}^n$, $\tau_k \in [a, b]$ $(k = 1, 2, \dots)$, და წრფივი სხვაობიანი

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + G_3(k)y(k+1) + g_0(k)$$

$(k = 1, \dots, m_0)$ სისტემისათვის, სადაც $G_j(k) \in \mathbb{R}^{n\times n}$, $g_0(k) \in \mathbb{R}^n$ $(j = 1, 2, 3;\ k = 1, \dots, m_0)$.

1. Statement of the Problem and Basic Notation

Let $A = (a_{ik})_{i,k=1}^n : [a,b] \to \mathbb{R}^{n\times n}$ and $f = (f_i)_{i=1}^n : [a,b] \to \mathbb{R}^n$ be, respectively, matrix- and vector-functions with bounded total variation components on the closed interval $[a,b]$.

Consider a linear system of generalized ordinary differential equations of the form

$$dx(t) = dA(t)\cdot x(t) + df(t) \quad \text{for} \quad t\in[a,b]. \tag{1.1}$$

We investigate the problem on existence of the solutions of the system (1.1) satisfying the multipoint boundary condition

$$\sum_{j=1}^{n_0} L_j x(t_j) = c_0, \tag{1.2}$$

where $t_j \in [a,b]$ $(j=1,\dots,n_0)$, $L_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,n_0)$, $c_0\in\mathbb{R}^n$, and n_0 is a fixed natural number.

A particular case of the condition (1.2) is the Cauchy–Nicoletti problem

$$x_i(t_i) = c_{0i} \quad (i=1,\dots,n), \tag{1.3}$$

where $c_{0i}\in\mathbb{R}$ and x_i is the i-th component of the solution x.

Along with the problem (1.1), (1.2), we also consider the problem with the boundary condition

$$x_i(t_i) = \ell_i(x_1,\dots,x_n) + c_{0i} \quad (i=1,\dots,n), \tag{1.4}$$

where $\ell_i \in \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$ $(i=1,\dots,n)$ are linear bounded functionals satisfying some conditions of smallness, as well as with the boundary condition

$$\ell(x) = c_0, \tag{1.5}$$

where

$$\ell(x) \equiv \int_a^b dL(t)\cdot x(t)$$

and $L : [a,b] \to \mathbb{R}^{n\times n}$, $L(b) = O_{n\times n}$, is a matrix-function with bounded total variation components on $[a,b]$.

We also consider the differential system

$$dx(t) = \varepsilon dA(t)\cdot x(t) + df(t) \tag{1.6}$$

which depends on a small positive parameter ε.

Along with the system (1.1) and the boundary conditions (1.2)–(1.5), we consider the corresponding homogeneous system

$$dx(t) = dA(t)\cdot x(t) \tag{1.1$_0$}$$

and the corresponding homogeneous conditions

$$\sum_{j=1}^{n_0} L_j x(t_j) = 0, \tag{1.2$_0$}$$

$$x_i(t_i) = 0 \quad (i=1,\dots,n), \tag{1.3$_0$}$$

$$x_i(t_i) = \ell_i(x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{1.4$_0$}$$

and

$$\ell(x) = 0. \tag{1.5$_0$}$$

In the present paper, we establish effective necessary and sufficient conditions for unique solvability of the general problem (1.1), (1.5) (of the problem (1.1), (1.2)). Such conditions differ from those given in [27], [7]. This result for linear systems of ordinary differential equations belongs to T. Kiguradze [20], [18].

The boundary value problems with the condition (1.3) have, for the first time, been considered by O. Nicoletti [24] for systems of ordinary differential equations. The optimal conditions for the solvability and unique solvability of the problem with the boundary condition (1.4) are established in [16], [17], [18], [19], [23] for the linear and nonlinear cases.

The multipoint boundary value problems for functional differential equations are investigated in [13], and for systems of generalized ordinary differential equations in [2]–[6].

The results presented in the paper generalize the concrete definition for the linear case of the results from [2]–[6].

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view. In particular, the following systems can be rewritten in the form (1.1):

(a) the impulsive system

$$\begin{aligned} &\frac{dx}{dt} = P(t)x + q(t) \quad \text{for} \quad t \in [a, b], \\ &x(\tau_k+) - x(\tau_k-) = G_k x(\tau_k) + g_k \quad (k = 1, 2, \dots), \end{aligned} \tag{1.7}$$

where $P = (p_{il})_{i,l=1}^n : [a, b] \to \mathbb{R}^{n\times n}$ and $q = (q_i)_{i=1}^n : [a, b] \to \mathbb{R}^n$ are, respectively, matrix- and vector-functions with Lebesgue integrable components on $[a, b]$, $\tau_k \in [a, b]$ $(k = 1, 2, \dots)$, and $G_k = (g_{kil})_{i,l=1}^n \in \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$ and $g_k = (g_{ki})_{i=1}^n \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ are constant matrices and vectors such that

$$\sum_{k=1}^{\infty} (\|G_k\| + \|g_k\|) < \infty; \tag{1.8}$$

(b) the difference system

$$\begin{aligned} \Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + \\ + G_3(k)y(k+1) + g_0(k) \quad (k = 1, \dots, m_0), \end{aligned} \tag{1.9}$$

where m_0 is a fixed natural number, and $C_j = (g_{jil})_{i,l=1}^n : \{0, \dots, m_0\} \to \mathbb{R}^{n\times n}$ $(j = 1, 2, 3)$ and $g_0 = (g_{0i})_{i=1}^n : \{0, \dots, m_0\} \to \mathbb{R}^n$ are, respectively, matrix- and vector-functions.

We consider both the impulse system (1.7) with each of the boundary conditions (1.2)–(1.5) and the difference system (1.9) with each of the boundary conditions

$$\sum_{j=1}^{n_{0m}} L_{mj} y(k_j + m - 1) = c_{0m} \quad (m = 1, 2), \tag{1.10}$$

where $k_j + m - 1 \in \{0, \dots, m_0\}$, $L_{mj} \in \mathbb{R}^{n \times n}$ $(m = 1, 2;\ j = 1, \dots, n_{0m})$, $c_{0m} = (c_{0mi})_{i=1}^n \in \mathbb{R}^n$ $(m = 1, 2)$, and n_{01} and n_{02} are fixed natural numbers;

$$y_i(k_i + m - 1) = c_{0mi} \quad (m = 1, 2;\ \ i = 1, \dots, n) \tag{1.11}$$

and

$$y_i(k_i + m - 1) = \ell_{mi}(y_1, \dots, y_n) + c_{0mi} \quad (m = 1, 2;\ \ i = 1, \dots, n), \tag{1.12}$$

where ℓ_{mi} $(m = 1, 2;\ i = 1, \dots, n)$ are linear bounded functionals, and y_i is the i-th component of the solution y.

Along with the systems (1.7) and (1.9) and the boundary conditions (1.10)–(1.12), we consider the corresponding homogeneous systems

$$\begin{gathered} \frac{dx}{dt} = P(t)x + q(t) \quad \text{for} \quad t \in [a, b], \\ x(\tau_k+) - x(\tau_k-) = G_k x(\tau_k) + g_k \quad (k = 1, 2, \dots), \end{gathered} \tag{1.7_0}$$

and

$$\begin{aligned} \Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + {} \\ + G_3(k)y(k+1) \quad (k = 1, \dots, m_0), \end{aligned} \tag{1.9_0}$$

and the homogeneous boundary conditions

$$\sum_{i=1}^{n_{0m}} L_{mi} y(k_i + m - 1) = 0 \quad (m = 1, 2), \tag{1.10_0}$$

$$y_i(k_j + m - 1) = 0 \quad (m = 1, 2;\ \ i = 1, \dots, n) \tag{1.11_0}$$

and

$$y_i(k_j + m - 1) = \ell_{mi}(y_1, \dots, y_n) \quad (m = 1, 2;\ \ i = 1, \dots, n). \tag{1.12_0}$$

Remark 1.1. Note that the results obtained for the difference problem (1.9), (1.10) (see Section 4) do not permit to extend them automatically to the particular case when $G_3(k) \equiv O_{n\times n}$. We note that this fact is natural and hence we investigate separately the system

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + g_0(k) \quad (k = 1, \dots, m_0) \tag{1.13}$$

with each of the boundary conditions

$$\sum_{j=1}^{n_0} L_{1j} y(k_j) = c_{01}, \tag{1.14}$$

$$y_i(k_i) = c_{01i} \quad (i = 1, \dots, n) \tag{1.15}$$

and

$$y_i(k_i) = \ell_{1i}(y_1, \dots, y_n) + c_{01i} \quad (i = 1, \dots, n). \tag{1.16}$$

Along with the system (1.13) and the boundary conditions (1.14)–(1.16), we consider the homogeneous system

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) \quad (k = 1, \dots, m_0) \tag{1.13_0}$$

and the homogeneous boundary conditions

$$\sum_{j=1}^{n_0} L_{1j} y(k_j) = 0, \tag{1.14_0}$$

$$y_i(k_i) = 0 \quad (i = 1, \dots, n) \tag{1.15_0}$$

and

$$y_i(k_i) = \ell_{1i}(y_1, \dots, y_n) \quad (i = 1, \dots, n). \tag{1.16_0}$$

In the paper the use will be made of the following notation and definitions.

$\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, \dots\}$, $\mathbb{Z}$ is the set of all integers.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

I is an arbitrary closed or open interval from $\mathbb{R}$.

$[t]$ is the integer part of $t \in \mathbb{R}$. χ_M is the characteristic function of the set $M \subset \mathbb{R}$, i.e.

$$\chi_M(t) = \begin{cases} 1 & \text{for } t \in M, \\ 0 & \text{for } t \notin M. \end{cases}$$

$\mathbb{R}^{n \times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$\mathbb{R}_+^{n \times m} = \big\{(x_{ij})_{i,j=1}^{n,m} : x_{ij} \geq 0 \ (i = 1, \dots, n; \ \ j = 1, \dots, m)\big\}$.

$O_{n \times m}$ (or O) is the zero $n \times m$ matrix.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n \times m}$, then

$$|X| = (|x_{ij}|)_{i,j=1}^{n,m}.$$

If $X \in \mathbb{R}^{2n \times 2n}$, then by X_{lm} $(l, m = 1, 2)$ we denote $n \times n$ matrices such that

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Sometimes, by $[X]_{ij}$ we denote the element x_{ij} in the i-th row and in the j-th column of the matrix $X = (x_{ij})_{i,j=1}^{n,m}$, i.e. $x_{ij} = [X]_{ij}$ $(i = 1, \dots, n;$ $j = 1, \dots, m)$.

$\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$.

If $X \in \mathbb{R}^{n \times m}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix; $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$; δ_{ij} is the Kroneker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, \dots)$.

The inequalities between the real matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

If $X : [a, b] \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\overset{b}{\underset{a}{\mathrm{V}}}(X)$ is the sum of total variations on $[a, b]$ of its components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$; $V(X)(t) = (V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a) = 0$, $V(x_{ij})(t) = \overset{t}{\underset{a}{\mathrm{V}}}(x_{ij})$ for $a < t \leq b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-) = X(a),\ X(b+) = X(b))$.

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\|X\|_s = \sup\left\{\|X(t)\| : t \in [a, b]\right\}$, $|X|_s = (\|x_{ij}\|_s)_{i,j=1}^{n,m}$.

$\mathrm{BV}([a, b], \mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X : [a, b] \to \mathbb{R}^{n\times m}$ (i.e. such that $\overset{b}{\underset{a}{\mathrm{V}}}(X) < \infty$) with the norm $\|X\|_s$.

$\mathrm{BV}([a, b], \mathbb{R}^{n\times m}_+) = \big\{X \in \mathrm{BV}([a, b], \mathbb{R}^{n\times m}) : X(t) \geq O_{n\times m} \text{ for } t \in [a, b]\big\}$.

$C(I, \mathbb{R}^{n\times m})$ is the space of all continuous and bounded matrix-functions $X : [a, b] \to \mathbb{R}^{n\times m}$ with the norm $\|X\|_{s,I} = \sup\{\|X(t)\| : t \in I\}$.

$C(I, D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all continuous and bounded matrix-functions $X : I \to D$.

$C_{loc}(I, D)$ is the set of all continuous matrix-functions $X : I \to D$.

$\widetilde{C}([a, b], D)$ is the set of all absolutely continuous matrix-functions $X : [a, b] \to D$.

$\widetilde{C}_{loc}(I, D)$ is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from I belong to $\widetilde{C}([a, b], D)$.

$\widetilde{C}_{loc}(I \setminus \{\tau_k\}_{k=1}^{\infty}, D)$ is the set of all matrix-functions $X : I \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I \setminus \{\tau_k\}_{k=1}^{\infty}$ belong to $\widetilde{C}([a, b], D)$.

If B_1 and B_2 are normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if

$$g(\lambda x) = \lambda g(x)$$

for every $\lambda \in \mathbb{R}_+$ and $x \in B_1$.

An operator $\varphi : \mathrm{BV}([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is called nondecreasing if for every $x, y \in \mathrm{BV}([a, b], \mathbb{R}^n)$ such that $x(t) \leq y(t)$ for $t \in [a, b]$ the inequality $\varphi(x)(t) \leq \varphi(y)(t)$ holds for $t \in [a, b]$.

If $\alpha \in \mathrm{BV}([a, b], \mathbb{R})$ has no more than a finite number of points of discontinuity, and $m \in \{1, 2\}$, then $D_{\alpha m} = \{t_{\alpha m 1}, \dots, t_{\alpha m n_{\alpha m}}\}$ $(t_{\alpha m 1} < \dots < t_{\alpha m n_{\alpha m}})$ is the set of all points from $[a, b]$ for which $d_m\alpha(t) \neq 0$.

$\mu_{\alpha m} = \max\{d_m\alpha(t) : t \in D_{\alpha m}\}$ $(m = 1, 2)$.

If $\beta \in \mathrm{BV}([a,b],\mathbb{R})$, then

$$\nu_{\alpha m\beta j} = \max\left\{d_j\beta(t_{\alpha ml}) + \sum_{t_{\alpha m\, l+1-m}<\tau<t_{\alpha m\, l+2-m}} d_j\beta(\tau) : l = 1,\dots,n_{\alpha m}\right\}$$

$(j,m = 1,2)$; here $t_{\alpha 20} = a-1$, $t_{\alpha 1 n_{\alpha 1}+1} = b+1$.

$S_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$S_1(x)(a) = S_2(x)(a) = 0,$$

$$S_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad \text{and} \quad S_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad \text{for} \quad a<t\le b,$$

and

$$S_0(x)(t) = x(t) - S_1(x)(t) - S_2(x)(t) \quad \text{for} \quad t\in[a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,dS_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,dS_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(S_0(g))$ corresponding to the function $S_0(g)$.

If $a = b$, then we assume

$$\int_a^b x(t)\,dg(t) = 0,$$

and if $a > b$, then we assume

$$\int_a^b x(t)\,dg(t) = -\int_b^a x(t)\,dg(t).$$

$L^p([a,b],\mathbb{R};g)$ $(1 \le p < +\infty)$ is the space of all $\mu(g)$-measurable functions $x : [a,b] \to \mathbb{R}$ such that $\int_a^b |x(t)|^p dg(t) < +\infty$ with the norm

$$\|x\|_{p,g} = \left(\int_a^b |x(t)|^p dg(t)\right)^{\frac{1}{p}}.$$

$L^{+\infty}([a,b],\mathbb{R};g)$ is the space of all $\mu(g)$-measurable and $\mu(g)$-essentially bounded functions $x : [a,b] \to \mathbb{R}$ with the norm

$$\|x\|_{+\infty,g} = \operatorname{ess\,sup}\{|x(t)| : t\in[a,b]\}.$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_s^t x(\tau)\, dg_1(\tau) - \int_s^t x(\tau)\, dg_2(\tau) \quad \text{for} \quad s \le t.$$

$L([a,b],\mathbb{R};g)$ is the set of all functions $x : [a,b] \to \mathbb{R}$, measurable and integrable with respect to the measures $\mu(g_i)$ $(i = 1,2)$, i.e. such that

$$\int_a^b |x(t)|\, dg_i(t) < +\infty \quad (i = 1,2).$$

If $G = (g_{ik})_{i,k=1}^{l,n} \in \mathrm{BV}([a,b],\mathbb{R}^{l\times n})$ and $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to \mathbb{R}^{n\times m}$, then

$$S_j(G)(t) \equiv (S_j(g_{ik})(t))_{i,k=1}^{l,n} \quad (j = 0,1,2)$$

and

$$\int_a^b dG(\tau) \cdot X(\tau) = \bigg(\sum_{k=1}^n \int_a^b x_{kj}(\tau)\, dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

$L^p([a,b],\mathbb{R}^{n\times m};G)$ is the space of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to \mathbb{R}^{n\times m}$ sitisfying $x_{kj} \in L^p([a,b],\mathbb{R}; g_{ik})$ with the norm

$$\|X\|_{p,G} = \sum_{i,k,j=1}^n \|x_{kj}\|_{p,g_{ik}}.$$

If $G(t) \equiv \mathrm{diag}(t,\dots,t)$, then we assume $\|X\|_{L^p} = \|X\|_{p,G}$ and omit G in the notations containing G.

$L^p([a,b],D;G)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all matrix-functions $X \in L^P([a,b],\mathbb{R}^{n\times m};G)$ such that $X(t) \in D$ for $t \in [a,b]$.

For every matrix-function $X \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ such that $\det(I_n - d_1X(t)) \ne 0$ for $t \in [a,b]$ we put

$$[X(t)]_0 = (I_n - d_1X(t))^{-1},$$

$$[X(t)]_i = (I_n - d_1X(t))^{-1} \int_a^t dX_-(\tau) \cdot [X(\tau)]_{i-1}$$
$$\text{for} \quad t \in [a,b] \quad (i = 1,2,\dots), \tag{1.17$_1$}$$

$$(X(t))_0 = O_{n\times n}, \quad (X(t))_1 = X(t), \quad (X(t))_{i+1} = \int_a^t dX_-(\tau) \cdot (X(\tau))_i$$
$$\text{for} \quad t \in [a,b] \quad (i = 1,2,\dots), \tag{1.18$_1$}$$

and

$$V_1(X)(t) = |(I_n - d_1X(t))^{-1}|V(X_-)(t),$$

$$V_{i+1}(X)(t) = |(I_n - d_1X(t))^{-1}| \int\limits_a^t dV(X_-)(\tau)\cdot V_i(X)(\tau)$$
$$\text{for } \ t\in[a,b] \ \ (i=1,2,\dots), \tag{1.19$_1$}$$

where $X_-(t) \equiv X(t-)$; and for every $X \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ such that $\det(I_n + d_2X(t)) \neq 0$ for $t\in[a,b]$ we put

$$[X(t)]_0 = (I_n + d_2X(t))^{-1},$$
$$[X(t)]_i = (I_n + d_2X(t))^{-1} \int\limits_b^t dX_+(\tau)\cdot [X(\tau)]_{i-1}$$
$$\text{for} \quad t\in[a,b] \quad (i=1,2,\dots), \tag{1.17$_2$}$$

$$(X(t))_0 = O_{n\times n}, \quad (X(t))_1 = X(t), \quad (X(t))_{i+1} = \int\limits_a^t dX_+(\tau)\cdot (X(\tau))_i$$
$$\text{for} \quad t\in[a,b] \quad (i=1,2,\dots) \tag{1.18$_2$}$$

and

$$V_1(X)(t) = |(I_n + d_2X(t))^{-1}|(V(X_+)(t)(b) - V(X_+)(t)|,$$
$$V_{i+1}(X)(t) = |(I_n + d_2X(t))^{-1}| \bigg| \int\limits_b^t dV(X_+)(\tau)\cdot V_i(X)(\tau) \bigg|$$
$$\text{for } \ t\in[a,b] \ \ (i=1,2,\dots), \tag{1.19$_2$}$$

where $X_+(t) \equiv X(t+)$.

If $l\in\mathbb{N}$, then $\mathbb{N}_l = \{1,\dots,l\}$, $\widetilde{\mathbb{N}}_l = \{0,1,\dots,l\}$.

$E(J,\mathbb{R}^{n\times m})$, where $J\subset\mathbb{Z}$, is the space of all matrix-functions $Y = (y_{ij})_{i,j=1}^{n,m} : J \to \mathbb{R}^{n\times m}$ with the norm

$$\|Y\|_J = \max\left\{\|Y(k)\| : k\in J\right\}, \quad |Y|_J = (\|y_{ij}\|_J)_{i,j=1}^{n,m}.$$

If $\alpha\in E(J,\mathbb{R}_+)$, then

$$\|Y\|_{\nu,\alpha} = \bigg(\sum_{k\in J}\alpha(k)\|Y(k)\|^{\nu}\bigg)^{\frac{1}{\nu}} \ \text{ if } \ 1\le\nu<+\infty, \ \text{ and } \ \|Y\|_{+\infty,\alpha} = \|Y\|_J$$

(if $\alpha(k)\equiv 1$, then we omit α in these notations).

Δ is the difference operator of the first order, i.e.

$$\Delta Y(k-1) = Y(k) - Y(k-1) \quad \text{for} \quad Y\in E(\widetilde{\mathbb{N}}_l,\mathbb{R}^{n\times m}), \ \ k\in\mathbb{N}_l.$$

If a function Y is defined on $\mathbb{N}_l$ or $\widetilde{\mathbb{N}}_{l-1}$, then we assume $Y(0) = O_{n\times m}$, or $Y(l) = O_{n\times m}$, respectively, if it is necessary.

We say that the matrix-function $X\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition if the matrices $S_0(X)(t)$, $S_1(X)(t)$ and $S_2(X)(t)$

are pairwise permutable for every $t \in [a,b]$, and there exists $t_0 \in [a,b]$ such that

$$\int\limits_{t_0}^{t} S_0(X)(\tau)\, dS_0(X)(\tau) = \int\limits_{t_0}^{t} dS_0(X)(\tau) \cdot S_0(X)(\tau) \quad \text{for } \ t \in [a,b]. \tag{1.20}$$

A vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is said to be a solution of the system (1.1) if

$$x(t) = x(s) + \int\limits_{s}^{t} dA(\tau) \cdot x(\tau) + f(t) - f(s) \quad \text{for } \ a \le s \le t \le b.$$

By a solution of the system of generalized ordinary differential inequalities

$$dx(t) \le dA(t) \cdot x(t) + f(t) \quad (\ge)$$

we mean a vector-function $x \in \mathrm{BV}([a,b],\mathbb{R}^n)$ such that

$$x(t) \le x(s) + \int\limits_{s}^{t} dA(\tau) \cdot x(\tau) + f(t) - f(s) \ \ (\ge) \quad \text{for } \ a \le s \le t \le b.$$

We assume that $A(0) = O_{n\times n}$, $f(0) = 0$ and

$$\det(I_n + (-1)^j d_j A(t)) \ne 0 \quad \text{for} \quad t \in [a,b] \quad (j=1,2). \tag{1.21}$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding systems (see [28, Theorem III.1.4]).

If $s \in \mathbb{R}$ and $\beta \in \mathrm{BV}[a,b],\mathbb{R})$ are such that

$$1 + (-1)^j d_j\beta(t) \ne 0 \quad \text{for} \quad (-1)^j(t-s) < 0 \quad (j=1,2),$$

then by $\gamma_\beta(\cdot,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t)\, d\beta(t), \quad \gamma(s) = 1.$$

It is known (see [14], [15]) that

$$\gamma_\beta(t,s) = \begin{cases} \exp(S_0(\beta)(t) - S_0(\beta)(s)) \prod\limits_{s<\tau\le t} (1 - d_1\beta(\tau))^{-1} \times \\ \qquad \times \prod\limits_{s\le\tau<t} (1 + d_2\beta(\tau)) \ \text{ for } \ t > s, \\ \exp(S_0(\beta)(t) - S_0(\beta)(s)) \prod\limits_{t<\tau\le s} (1 - d_1\beta(\tau)) \times \\ \qquad \prod\limits_{t\le\tau<s} (1 + d_2\beta(\tau))^{-1} \ \text{ for } \ t < s, \\ \qquad\qquad 1 \qquad\qquad\qquad \text{for } \ t = s. \end{cases} \tag{1.22}$$

Definition 1.1. Let $t_1,\dots,t_n \in [a,b]$. We say that a pair (C,ℓ_0) consisting of a matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and a positive homogeneous nondecreasing continuous operator $\ell_0 = (\ell_{0i})_{i=1}^n$:

$\mathrm{BV}([a,b],\mathbb{R}_+^{n\times n})\to\mathbb{R}_+^n$ belongs to the set $U(t_1,\dots,t_n)$ if the functions c_{il} $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing on $[a,b]$, and the system

$$\begin{gathered}\operatorname{sgn}(t-t_i)dx_i(t)\le\sum_{l=1}^n x_l(t)\,dc_{il}(t)\\ \text{for}\quad t\in[a,b],\quad t\neq t_i\quad (i=1,\dots,n),\\ (-1)^j d_jx_i(t_i)\le\sum_{l=1}^n x_l(t_i)\,d_jc_{il}(t_i)\quad (j=1,2;\ \ i=1,\dots,n)\end{gathered}\tag{1.23}$$

has no nontrivial, nonnegative solution satisfying the condition

$$x_i(t_i)\le\ell_{0i}(x_1,\dots,x_n)\quad (i=1,\dots,n).\tag{1.24}$$

The above definition of the set $U(t_1,\dots,t_n)$ differs from that given in [3], [6], where it is, in particular, required that the functions c_{il} $(i\neq l;$ $i,l=1,\dots,n)$ be continuous at the point t_i and the condition

$$d_jc_{ii}(t)\ge 0\quad\text{for}\quad t\in[a,b]\quad (j=1,2;\ \ i=1,\dots,n)$$

be satisfied.

The set $U(t_1,\dots,t_n)$ has been introduced by I. Kiguradze for ordinary differential equations (see [17], [18]).

Here we quote some general results from [7], [27] on the solvability of the problem (1.1), (1.5).

Let $Y\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ be a fundamental matrix of the system (1.1_0) under the condition

$$Y(a)=I_n.$$

Definition 1.2. A matrix-function $\mathcal{G}:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ is said to be the Green matrix of the problem (1.1_0), (1.5_0) if:

(a) for every $s\in]a,b[$ the matrix-function $\mathcal{G}(\cdot,s)$ satisfies the matrix equation

$$dX(t)=dA(t)\cdot X(t)$$

both on $[a,s[$ and $]s,b]$;

(b) $$\mathcal{G}(t,t+)-\mathcal{G}(t,t-)=Y(t)D^{-1}\bigg\{\int_a^t dL(\tau)\cdot X(\tau)X^{-1}(t)(I_n-d_1A(t))^{-1}+\\ +\int_t^b dL(\tau)\cdot X(\tau)X^{-1}(t)(I_n+d_2A(t))^{-1}-\\ -d_1L(t)\cdot(I_n-d_1A(t))^{-1}-d_2L(t)\cdot(I_n+d_2A(t))^{-1}\bigg\}\quad\text{for }\ t\in]a,b[,$$

where

$$D=\int_a^b dL(t)\cdot Y(t);\tag{1.25}$$

(c) the vector-function $x(t) = \int\limits_a^b d_s\mathcal{G}(t,s)\cdot f(s)$ satisfies the condition (1.5_0).

The Green matrix of the problem (1.1_0), (1.5_0) exists, and

$$\mathcal{G}(t,s) = \begin{cases} -X(t)D^{-1}\int\limits_a^s dL(t)\cdot X(\tau)X^{-1}(s) & \text{for } a \le s < t \le b, \\ X(t)D^{-1}\int\limits_s^b dL(t)\cdot X(\tau)X^{-1}(s) & \text{for } a \le t < s \le b, \\ O_{n\times n} & \text{for } a \le t = s \le b. \end{cases} \tag{1.26}$$

Note that we can choose an arbitrary value for $\mathcal{G}(t,t)$ $(a \le t \le b)$ instead of that given above.

The Green matrix is unique in the following sense. If $\mathcal{G}_1(t,s)$ is a matrix-function satisfying the conditions (a)–(c), then

$$\mathcal{G}(t,s) - \mathcal{G}_1(t,s) \equiv Y(t)H(s),$$

where $H \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ is a matrix-function such that

$$H(s+) = H(s-) = C = const \quad \text{for} \quad s \in [a,b],$$

where $C \in \mathbb{R}^{n\times n}$ is a constant matrix.

Theorem 1.1. *The boundary value problem* (1.1), (1.5) *has a unique solution if and only if the corresponding homogeneous problem* (1.1_0), (1.5_0) *has only the trivial solution. If the latter condition holds, then the solution* x *of the problem* (1.1), (1.5) *admits the representation*

$$x(t) = x_0(t) + \int_a^b d_s\mathcal{G}(t,s)\cdot f(s) \quad \textit{for} \quad t \in [a,b], \tag{1.27}$$

where x_0 *is a solution of the problem* (1.1_0), (1.5), *and* $\mathcal{G}$ *is the Green matrix of the problem* (1.1_0), (1.5_0).

We note that the problem (1.1), (1.5) is uniquely solvable if and only if

$$\det D \ne 0, \tag{1.28}$$

where the matrix D is defined by (1.25).

Corollary 1.1. *Let the matrix-function* A *satisfy the Lappo–Danilevskiĭ condition. Then the problem* (1.1), (1.5) *is uniquely solvable if and only if*

$$\det\bigg(\int_a^b dK(t)\cdot\exp(S_0(A)(t))\prod_{a\le\tau<t}(I_n + d_2A(\tau))\times \\ \times\prod_{a<\tau\le t}(I_n - d_1A(\tau))^{-1}\bigg) \ne 0. \tag{1.29}$$

This corollary follows from Theorem 1.1 and (1.28), since by Lemma 2.3 from [8] the matrix-function Y defined by $Y(a) = I_n$ and

$$Y(t) = \exp(S_0(A)(t)) \prod_{a\le\tau<t} (I_n + d_2A(\tau))\times \\ \times \prod_{a<\tau\le t} (I_n - d_1A(\tau))^{-1} \quad \text{for } \ a < t \le b \tag{1.30}$$

is the fundamental matrix of the system (1.1_0).

Remark 1.2. If the homogeneous problem (1.1_0) (1.5_0) has a nontrivial solution, then for every $f \in \mathrm{BV}([a,b],\mathbb{R}^n)$ there exists a vector $c_0 \in \mathbb{R}^n$ such that the problem (1.1), (1.5) has no solution.

In general, it is quite difficult to verify the condition (1.28) directly even in the case when one is able to write out the fundamental matrix of the system (1.1_0) explicitly. Therefore it is important to seek for effective conditions which would guarantee the absence of nontrivial solutions of the homogeneous problem (1.1_0), (1.5_0). Such results can be found in Section 2. Analogous results have been obtained by T. Kiguradze for ordinary differential equations [18], [20].

Here the use will be made of the following formulas:

$$\begin{gathered} \int_a^b f(t)\,dg(t) = \int_a^b f(t)\,dg(t-) + f(b)d_1g(b), \\ \int_a^b f(t)\,dg(t) = \int_a^b f(t)\,dg(t+) + f(a)d_2g(a), \end{gathered} \tag{1.31}$$

$$\begin{gathered} \int_a^b f(t)\,dg(t) + \int_a^b g(t)\,df(t) = f(b)g(b) - f(a)g(a) + \\ + \sum_{a<t\le b} d_1f(t)\cdot d_1g(t) - \sum_{a\le t<b} d_2f(t)\cdot d_2g(t) \\ \text{(integration-by-parts formula),} \end{gathered} \tag{1.32}$$

$$\begin{gathered} \int_a^b f(t)\,dS_1(g)(t) = \sum_{a<t\le b} f(t)\,d_1g(t), \\ \int_a^b f(t)\,dS_2(g)(t) = \sum_{a\le t<b} f(t)\,d_2g(t) \end{gathered} \tag{1.33}$$

(see [28, Theorems I.4.25, I.4.33, Lemma I.4.23]) and

$$\int\limits_a^b f^k(t)\,df(t) = \frac{1}{k+1}\bigg[f^{k+1}(b) - f^{k+1}(a) + \\ + \sum_{m=0}^{k-1}\bigg(\sum_{a<t\le b} f^m(t)\,d_1f(t)\cdot d_1f^{k-m}(t) - \\ - \sum_{a\le t<b} f^m(t)\,d_2f(t)\cdot d_2f^{k-m}(t)\bigg)\bigg] \quad (k=1,2,\dots) \tag{1.34}$$

(see [7, Lemma 1.1]) for $f,g\in \mathrm{BV}([a,b],\mathbb{R})$.

If (1.21) holds and X ($X(a)=I_n$) is the fundamental matrix of the system (1.1), then

$$X^{-1}(t) = I + A(0) - X^{-1}(t)A(t) + \int\limits_a^t dX^{-1}(\tau)\cdot A(\tau) \ \text{ for } \ t\in[a,b] \tag{1.35}$$

and

$$x(t) = f(t) - f(t_0) + X(t)\bigg\{X^{-1}(t_0)c_0 - \\ - \int\limits_{t_0}^t dX^{-1}(s)\cdot (f(s)-f(t_0))\bigg\} \ \text{ for } \ t\in[a,b] \\ \text{(variation-of-constants formula)}, \tag{1.36}$$

where $t_0\in[a,b]$ and $c_0\in\mathbb{R}^n$ are arbitrary, and x is the solution of the system (1.1) satisfying the Cauchy condition $x(a_0)=c_0$ (see [28, p. 120]).

2. Formulation of the Results

2.1. **Theorems on the Solvability of the General Linear Problem (1.1), (1.5).**

Theorem 2.1. *The boundary value problem* (1.1), (1.5) *has a unique solution if and only if there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\int\limits_a^b dL(t)\cdot [A(t)]_i \tag{2.1}$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{2.2}$$

where

$$M_{k,m} = V_m(A)(c) + \bigg(\sum_{i=0}^{m-1} |[A(\cdot)]_i|_s\bigg)\cdot \int\limits_a^b dV(M_k^{-1}L)(t)\cdot V_k(A)(t), \tag{2.3}$$

$[A(t)]_i$ $(i=0,1,\dots)$ and $V_i(A)(t)$ $(i=0,1,\dots)$ are defined, respectively, by (1.17_l) and (1.19_l) for some $l\in\{1,2\}$, and $c=b+(a-b)(l-1)$.

Theorem 2.1$'$. *Let there exist natural numbers k and m such that the matrix*

$$M_k=L(a)-\sum_{i=0}^{k-1}\int_a^b dL(t)\cdot(A(t))_i \tag{2.4}$$

is nonsingular and the inequality (2.2) holds, where

$$M_{k,m}=(V(A)(c))_m+\bigg(I_n+\sum_{i=0}^{m-1}|(A(\cdot))_i|_s\bigg)\times \\ \times\int_a^b dV(M_k^{-1}L)(t)\cdot(V(A)(t))_k, \tag{2.5}$$

$(A(t))_i$ $(i=0,1,\dots)$ and $(V(A)(t))_i$ $(i=0,1,\dots)$ are defined by (1.18_l) for some $l\in\{1,2\}$, and $c=b+(a-b)(l-1)$. Then the problem (1.1), (1.5) has one and only one solution.

Corollary 2.1. *Let either*

$$\det(L(a))\neq 0 \tag{2.6}$$

or

$$L(a)=O_{n\times n}, \tag{2.7}$$

and the conditions

$$\int_a^b dL(t)\cdot(A(t))_i=O_{n\times n}\quad (i=0,\dots,j-1) \tag{2.8}$$

and

$$\det\bigg(\int_a^b dL(t)\cdot(A(t))_j\bigg)\neq 0 \tag{2.9}$$

hold for some natural j, where $(A(t))_i$ $(i=0,\dots,l)$ are defined by (1.18_1) or (1.18_2). Then there exists $\varepsilon_0>0$ such that the problem (1.6), (1.5) has one and only one solution for every $\varepsilon\in]0,\varepsilon_0[$.

Theorem 2.2. *Let a matrix-function $A_0\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ be such that the homogeneous system*

$$dx(t)=dA_0(t)\cdot x(t) \tag{2.10}$$

has only the trivial solution satisfying the boundary condition (1.5_0), and let the matrix-function $A\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ admit the estimate

$$\int_a^b|\mathcal{G}_0(t,\tau)|dV(S_0(A-A_0))(\tau)+\sum_{a<\tau\le b}|\mathcal{G}_0(t,\tau-)\cdot d_1(A(\tau)-A_0(\tau))|+$$

$$+\sum_{a\le\tau<b}|\mathcal{G}_0(t,\tau+)\cdot d_2(A(\tau)-A_0(\tau))|\le M \ \ \textit{for} \ \ t\in[a,b], \quad (2.11)$$

where $\mathcal{G}_0(t,\tau)$ *is the Green matrix of the problem* (2.10), (1.5_0), *and* $M\in\mathbb{R}_+^{n\times n}$ *is a constant matrix such that*

$$r(M)<1. \quad (2.12)$$

Then the problem (1.1), (1.5) *has one and only one solution.*

2.2. **Theorems on the Solvability of the General Multi-Point Boundary Value Problem (1.1), (1.2).**

Theorem 2.3. *The boundary value problem* (1.1), (1.2) *has a unique solution if and only if the corresponding homogeneous problem* (1.1_0), (1.2_0) *has only the trivial solution, i.e. if and only if*

$$\det\bigg(\sum_{j=1}^{n_0}L_j\,Y(t_j)\bigg)\neq 0, \quad (2.13)$$

where Y *is a fundamental matrix of the system* (1.1_0). *If the latter condition holds, then the solution* x *of the problem* (1.1), (1.2) *admits the representation* (1.27), *where* x_0 *is a solution of the problem* (1.1_0), (1.2), *and* $\mathcal{G}$ *is the Green matrix of the problem* (1.1_0), (1.2_0).

It is not difficult to verify that the Green matrix of the problem (1.1_0), (1.2_0) has the following form:

$$\mathcal{G}(t,s)=\begin{cases} Y(t)\sum\limits_{j=1}^{n_0}(1-\chi_{[a,t_j]}(s))Z_jY^{-1}(s) & \text{for } \ a\le s<t\le b,\\ -Y(t)\sum\limits_{j=1}^{n_0}\chi_{[a,t_j]}(s)Z_jY^{-1}(s) & \text{for } \ a\le t<s\le b,\\ O_{n\times n} & \text{for } \ a\le t=s\le b,\end{cases} \quad (2.14)$$

where

$$Z_j=\bigg(\sum_{i=1}^{n_0}L_i\,Y(t_i)\bigg)^{-1}L_j\,Y(t_j) \quad (j=1,\dots,n_0),$$

and $\chi_{[a,t_j]}$ is the characteristic function of the closed interval $[a,t_j]$.

Corollary 2.2. *Let the matrix-function* A *satisfy the Lappo–Danilevskiĭ condition. Then the problem* (1.1), (1.2) *is uniquely solvable if and only if*

$$\det\bigg(\sum_{j=1}^{n_0}L_j\exp(S_0(A)(t_j))\cdot\prod_{a\le\tau<t_j}(I_n+d_2A(\tau))\times \\ \times\prod_{a<\tau\le t_j}(I_n-d_1A(\tau))^{-1}\bigg)\neq 0.$$

Theorem 2.4. *The boundary value problem* (1.1), (1.2) *is uniquely solvable if and only if there exist natural numbers k and m such that the matrix*
$$M_k=\sum_{j=1}^{n_0}\sum_{i=0}^{k-1}L_j[A(t_j)]_i$$
is nonsingular and the inequality (2.2) *holds, where*
$$M_{k,m}=V_m(A)(c)+\bigg(\sum_{i=0}^{m-1}|[A(\cdot)]_i|_s\bigg)\sum_{j=1}^{n_0}|M_k^{-1}L_j|V_k(A)(t_j),$$
$[A(t)]_i$ $(i=0,1,\dots)$ *and* $V_i(A)(t)$ $(i=0,1,\dots)$ *are defined, respectively, by* (1.17_l) *and* (1.19_l) *for some* $l\in\{1,2\}$, *and* $c=b+(a-b)(l-1)$.

Theorem 2.4′. *Let there exist natural numbers k and m such that the matrix*
$$M_k=\sum_{j=1}^{n_0}L_j\bigg(\sum_{i=0}^{k-1}(A(t_j))_i-1\bigg)$$
is nonsingular and the inequality (2.2) *holds, where*
$$M_{k,m}=(V(A)(c))_m+\bigg(I_n+\sum_{i=0}^{m-1}|(A(\cdot))_i|_s\bigg)\sum_{j=1}^{n_0}|M_k^{-1}L_j|(V(A)(t_j))_k,$$
$(A(t))_i$ $(i=0,1\dots)$ *and* $(V(A)(t))_i$ $(i=0,1,\dots)$ *are defined by* (1.18_l) *for some* $l\in\{1,2\}$, *and* $c=b+(a-b)(l-1)$. *Then the problem* (1.1), (1.2) *has one and only one solution.*

Corollary 2.3. *Let*
$$\det\bigg(\sum_{j=1}^{n_0}L_j\bigg)\neq 0 \tag{2.15}$$
and
$$r\left(L_0V(A)(b)\right)<1,$$
where
$$L_0=I_n+\bigg|\bigg(\sum_{j=1}^{n_0}L_j\bigg)^{-1}\bigg|\sum_{j=1}^{n_0}|L_j|.$$
Then the problem (1.1), (1.2) *has one and only one solution.*

Corollary 2.4. *Let either the condition* (2.15) *hold, or there exist a natural number k such that the conditions*
$$\sum_{j=1}^{n_0}L_j=O_{n\times n},\quad \det\bigg(\sum_{j=1}^{n_0}L_j(A(t_j))_i\bigg)=0\quad (i=0,\dots,k-1)$$
and
$$\det\bigg(\sum_{j=1}^{n_0}L_j(A(t_j))_k\bigg)\neq 0$$

hold. Then there exists $\varepsilon_0 > 0$ *such that the problem* (1.6), (1.2) *has one and only one solution for every* $\varepsilon \in]0, \varepsilon_0[$.

2.3. **Theorems on the Solvability of the Problems (1.1), (1.3) and (1.1), (1.4).**

Theorem 2.5. *Let there exist a matrix-function* $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b], \mathbb{R}^{n\times n})$ *and an operator* $\ell_0 = (\ell_{0i})_{i=1}^n$ *satisfying the condition*

$$(C, \ell_0) \in U(t_1, \dots, t_n) \tag{2.16}$$

such that

$$S_0(a_{ii})(t) - S_0(a_{ii})(s) \le (S_0(c_{ii})(t) - S_0(c_{ii})(s)) \operatorname{sgn}(t-s) \quad \text{for } (t-s)(s-t_i) > 0, \ \ s,t \in [a,b] \ \ (i = 1, \dots, n), \tag{2.17}$$

$$(-1)^{j+m} \left(|1 + (-1)^m d_m a_{ii}(t)| - 1\right) \le d_m c_{ii}(t) \quad \text{for } (-1)^j (t - t_i) \ge 0 \ \ (j,m = 1,2; \ \ i = 1, \dots, n), \tag{2.18}$$

$$|S_0(a_{il})(t) - S_0(a_{il})(s)| \le S_0(c_{il})(t) - S_0(c_{il})(s) \quad \text{for } a \le s < t \le b \ \ (i \ne l; \ \ i,l = 1, \dots, n) \tag{2.19}$$

and

$$|d_j a_{il}(t)| \le d_j c_{il}(t) \ \ \text{for } t \in [a,b] \ \ (j = 1,2; \ \ i \ne l; \ \ i,l = 1, \dots, n). \tag{2.20}$$

Let, moreover,

$$|\ell_i(x_1, \dots, x_n)| \le \ell_{0i}(|x_1|, \dots, |x_n|) \quad \text{for } (x_l)_{l=1}^n \in \mathrm{BV}([a,b], \mathbb{R}^n) \ \ (i = 1, \dots, n). \tag{2.21}$$

Then the problem (1.1), (1.4) *has one and only one solution.*

Theorem 2.6. *Let the conditions*

$$S_0(a_{ii})(t) - S_0(a_{ii})(s) \le \operatorname{sgn}(t-s) \int_s^t h_{ii}(\tau)\, dS_0(\alpha_i)(\tau) \quad \text{for } (t-s)(s-t_i) > 0, \ \ s,t \in [a,b] \ \ (i = 1, \dots, n), \tag{2.22}$$

$$(-1)^{j+m} \left(|1 + (-1)^m d_m a_{ii}(t)| - 1\right) \le h_{ii}(t)\, d_m \alpha_i(t) \quad \text{for } (-1)^j (t - t_i) \ge 0 \ \ (j,m = 1,2; \ \ i = 1, \dots, n), \tag{2.23}$$

$$|S_0(a_{il})(t) - S_0(a_{il})(s)| \le \int_s^t h_{il}(\tau)\, dS_0(\alpha_l)(\tau) \quad \text{for } a \le s < t \le b \ \ (i \ne l; \ \ i,l = 1, \dots, n) \tag{2.24}$$

and

$$|d_j a_{il}(t)| \le h_{il}(t) d_j \alpha_l(t) \quad \text{for } t \in [a,b] \ \ (j = 1,2; \ \ i \ne l; \ \ i,l = 1, \dots, n) \tag{2.25}$$

hold, where α_l $(l = 1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and having not more than a finite number of points of discontinuity,* $h_{ii} \in L^\mu([a,b],\mathbb{R};\alpha_i)$, $h_{il} \in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i \neq l;\ l = 1,\dots,n)$, $1 \le \mu \le +\infty$. *Let, moreover,*

$$|\ell_i(x_1,\dots,x_n)| \le \sum_{m=0}^{2}\sum_{k=1}^{n} \ell_{mik}\|x_k\|_{\nu,S_m(\alpha_k)}$$
$$\text{for}\quad (x_k)_{k=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^n) \quad (i = 1,\dots,n) \tag{2.26}$$

and

$$r(\mathcal{H}) < 1, \tag{2.27}$$

where $\ell_{mik} \in \mathbb{R}_+$ $(m = 0,1,2;\ i,k = 1,\dots,n)$, $\frac{1}{\mu} + \frac{2}{\nu} = 1$, *and the* $3n \times 3n$*-matrix* $\mathcal{H} = (\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$ *is defined by*

$$\mathcal{H}_{j+1\,m+1} = \big(\xi_{ij}\ell_{mik} + \lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_i)}\big)_{i,k=1}^n \quad (j,m = 0,1,2),$$

$$\xi_{ij} = (S_j(\alpha_i)(b) - S_j(\alpha_i)(a))^{\frac{1}{\nu}} \quad (j = 0,1,2,;\ \ i = 1,\dots,n);$$

$$\lambda_{k0i0} = \begin{cases} \big(\frac{4}{\pi^2}\big)^{\frac{1}{\nu}}\,\xi_{k0}^2 & \text{if } S_0(\alpha_i)(t) \equiv S_0(\alpha_k)(t), \\ \xi_{k0}\xi_{i0} & \text{if } S_0(\alpha_i)(t) \not\equiv S_0(\alpha_k)(t) \ \ (i,k = 1,\dots,n); \end{cases}$$

$$\lambda_{kmij} = \xi_{km}\xi_{ij} \quad \text{if} \quad m^2 + j^2 > 0,\ \ mj = 0 \ \ (j,m = 0,1,2;\ \ i,k = 1,\dots,n),$$

$$\lambda_{kmij} = \left(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\right)^{\frac{1}{\nu}} \quad (j,m=1,2;\ \ i,k = 1,\dots,n).$$

Then the problem (1.1), (1.4) *has one and only one solution.*

Remark 2.1. The $3n \times 3n$-matrix $\mathcal{H}$ appearing in Theorem 2.6 can be replaced by the $n \times n$-matrix

$$\left(\max\left\{\sum_{j=0}^{2}\big(\xi_{ij}\ell_{mik} + \lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_k)}\big)\ :\ m = 0,1,2\right\}\right)_{i,k=1}^n.$$

Corollary 2.5. *Let the conditions* (2.22)–(2.25) *hold, where* α_l $(l = 1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and having not more than a finite number of point of discontinuity,* $h_{ii} \in L^\mu([a,b],\mathbb{R};\alpha_i)$, $h_{il} \in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i \neq l;\ i,l = 1,\dots,n)$, $1 \le \mu \le +\infty$. *Let, moreover,*

$$r(\mathcal{H}_0) < 1,$$

where $\mathcal{H}_0 = \Big((\lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_i)})_{i,k=1}^n\Big)_{m,j=0}^2$ *is a* $3n \times 3n$*-matrix, and* λ_{kmij}, ξ_{ij} $(j,m = 0,1,2;\ i,k = 1,\dots,n)$ *and* ν *are defined as in Theorem* 2.6. *Then the problem* (1.1), (1.3) *has one and only one solution.*

Remark 2.2. The $3n \times 3n$-matrix $\mathcal{H}_0$ appearing in Corollary 2.5 can be replaced by the $n \times n$-matrix

$$\left(\max\left\{\sum_{j=0}^{2}\lambda_{kmij}\|h_{ik}\|_{\mu,S_m(\alpha_k)}\ :\ m = 0,1,2\right\}\right)_{i,k=1}^n.$$

By Remark 2.2, Corollary 2.5 has the following form for $h_{il}(t) \equiv h_{il} = const$ $(i, l = 1, \dots, n)$ and $\mu = +\infty$.

Corollary 2.6. *Let the conditions*

$$S_0(a_{ii})(t) - S_0(a_{ii})(s) \le \operatorname{sgn}(t-s) h_{ii} |S_0(\alpha)(t) - S_0(\alpha)(s)| \quad \text{for } (t-s)(s-t_i) > 0, \ s,t \in [a,b] \ (i = 1, \dots, n), \tag{2.28}$$

$$(-1)^{j+m} \left(|1 + (-1)^m d_m a_{ii}(t)| - 1\right) \le h_{ii}\, d_m \alpha(t) \quad \text{for } (-1)^j (t - t_i) \ge 0 \ (j, m = 1, 2; \ \ i = 1, \dots, n), \tag{2.29}$$

$$|S_0(a_{il})(t) - S_0(a_{il})(s)| \le h_{il}(S_0(\alpha)(t) - S_0(\alpha)(s)) \quad \text{for } a \le s < t \le b \ (i \ne l; \ \ i, l = 1, \dots,) \tag{2.30}$$

and

$$|d_j a_{il}(t)| \le h_{il} d_j \alpha(t) \ \text{ for } t \in [a,b] \ (j = 1,2; \ i \ne l; \ i, l = 1, \dots, n) \tag{2.31}$$

hold, where α a is function nondecreasing on $[a,b]$ and having not more than a finite number of points of discontinuity, $h_{ii} \in \mathbb{R}$, $h_{il} \in \mathbb{R}_+$ $(i \ne l;$ $i, l = 1, \dots, n)$. Let, moreover,

$$\rho_0\, r(\mathcal{H}) < 1, \tag{2.32}$$

where

$$\mathcal{H} = (h_{ik})_{i,k=1}^n, \quad \rho_0 = \max\left\{ \sum_{j=0}^{2} \lambda_{mj} : m = 0,1,2 \right\},$$

$$\lambda_{00} = \frac{2}{\pi} \left(S_0(\alpha)(b) - S_0(\alpha)(a)\right),$$

$$\lambda_{0j} = \lambda_{j0} = (S_0(\alpha)(b) - S_0(\alpha)(a))^{\frac{1}{2}} \, (S_j(\alpha)(b) - S_j(\alpha)(a))^{\frac{1}{2}} \quad (j = 1,2),$$

$$\lambda_{mj} = \frac{1}{2} \left(\mu_{\alpha m} \nu_{\alpha m \alpha j}\right)^{\frac{1}{2}} \sin^{-1} \frac{\pi}{4 n_{\alpha_m + 2} + 2} \quad (m, j = 1, 2).$$

Then the problem (1.1), (1.3) *has one and only one solution.*

Remark 2.3. The condition (2.32) is optimal in the sense that it cannot be replaced by the nonstrict inequality

$$\rho_0 r(\mathcal{H}) \le 1.$$

The corresponding example is constructed for ordinary differential equations in [18]. For the sake of completeness we present here this example.

Consider the problem

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -\frac{\pi^2}{4(b-a)^2}\, x_1, \tag{2.33}$$

$$x_1(a) = 0, \quad x_2(b) = 0, \tag{2.34}$$

In this case

$$n = 2, \ \ t_1 = a, \ \ t_2 = b, \ \ a_{11}(t) = a_{22}(t) \equiv 0,$$

$$a_{12}(t) \equiv t, \quad a_{21}(t) \equiv -\frac{\pi}{4(b-a)^2}\, t,$$

and the conditions (2.28)–(2.31) are fulfilled for

$$h_{11} = h_{22} = 0, \quad h_{12} = 1, \quad h_{21} = \frac{\pi^2}{4(b-a)^2}, \quad \alpha(t) \equiv t.$$

Moreover,

$$\rho_0 = \frac{2(b-a)}{\pi}\,,$$

and

$$\lambda_1 = \frac{\pi}{2(b-a)} \quad \text{and} \quad \lambda_2 = -\frac{\pi}{2(b-a)}$$

are the eigenvalues of the matrix

$$H = \begin{pmatrix} 0 & 1 \\ \frac{\pi^2}{4(b-a)^2} & 0 \end{pmatrix}.$$

Therefore

$$\rho_0 r(H) = 1. \tag{2.35}$$

Thus for the problem (2.33) and (2.34) all the conditions of Corollary 2.6 are fulfilled, with exclusion of the condition (2.32) instead of which the equality (2.35) holds. On the other hand, the problem (2.33), (2.34) is not uniquely solvable because it has a nontrivial solution

$$x_1(t) = \sin\frac{\pi(t-a)}{2(b-a)}\,, \quad x_2(t) = \frac{\pi}{2(b-a)}\,\cos\frac{\pi(t-a)}{2(b-a)}$$

along with the trivial one.

Below, we will give a general theorem (see Theorem 2.8) on the unsolvability of the problem (1.1), (1.4) in the case where the condition (2.16) is violated.

Theorem 2.7. *Let the conditions*

$$\begin{aligned}
&S_0(a_{ii})(t) - S_0(a_{ii})(s) \le [h_{ii}(S_0(\alpha_i)(t) - S_0(\alpha_i)(s)) + \\
&\qquad\qquad + S_0(\alpha_{ii})(t) - S_0(\alpha_{ii})(s)]\,\mathrm{sgn}(t-s) \\
&\qquad\quad \textit{for } \ (t-s)(s-t_i) > 0, \ \ s,t \in [a,b] \ \ (i = 1,\dots,n); \\
&(-1)^{j+m}\,(|1 + (-1)^m d_m a_{ii}(t)| - 1) \le h_{ii}\, d_m\alpha_i(t) + d_m\alpha_{ii}(t) \\
&\qquad\quad \textit{for } \ (-1)^j(t-t_i) \ge 0 \ \ (j,m = 1,2; \ \ i = 1,\dots,n); \\
&|S_0(a_{il})(t) - S_0(a_{il})(s)| \le h_{il}(S_0(\alpha_i)(t) - S_0(\alpha_i)(s)) + \\
&\qquad\qquad + S_0(\alpha_{il})(t) - S_0(\alpha_{il})(s) \\
&\qquad\quad \textit{for } \ a \le s < t \le b \ \ (i \ne l; \ \ i,l = 1,\dots,n), \\
&|d_j a_{il}(t)| \le h_{il} d_j\alpha_i(t) + d_j\alpha_{il}(t) \\
&\qquad\quad \textit{for } \ t \in [a,b] \ \ (j = 1,2; \ \ i \ne l; \ \ i,l = 1,\dots,n); \\
&|\ell_i(x_1,\dots,x_n)| \le |\mu_i|\,|x_i(s_i)|
\end{aligned}$$

$$\text{for } x=(x_l)_{l=1}^n\in \mathrm{BV}([a,b],\mathbb{R}^n) \ \ (i=1,\dots,n); \tag{2.36}$$

$$d_j\alpha_i(t_i)\le 0, \quad 0\le d_j\alpha_i(t)<|\eta_i|^{-1}$$
$$\text{for } (-1)^j(t-t_i)<0 \ \ (j=1,2;\ \ i=1,\dots,n) \tag{2.37}$$

and

$$|\mu_i|\gamma_i(s_i,t_i)<1 \quad (i=1,\dots,n) \tag{2.38}$$

hold, where $h_{ii}<0$, $h_{il}\ge 0$ $(i\ne l;\ i,l=1,\dots,n)$, $\mu_i\in\mathbb{R}$, $\eta_i<0$, $s_i\in[a,b]$, $s_i\ne t_i$ $(i=1,\dots,n)$; α_{ii} $(i=1,\dots,n)$ *are functions nondecreasing on* $[a,b]$; $\alpha_{il},\alpha_i\in \mathrm{BV}([a,b],\mathbb{R})$ $(i\ne l;\ i,l=1,\dots,n)$ *are functions nondecreasing on every interval* $[a,t_i[$ *and* $]t_i,b]$; $\gamma_i(t,s)\equiv\gamma_{a_i}(t,s)$ $(i=1,\dots,n)$, *the function* γ_{a_i} *is defined according to* (1.22), *and* $a_i(t)\equiv \eta_i(\alpha_i(t)-\alpha_i(t_i))\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$. *Let, moreover,*

$$g_{ii}<1 \quad (i=1,\dots,n) \tag{2.39}$$

and the real part of every characteristic value of the matrix $(\xi_{il})_{i,l=1}^n$ *be negative, where*

$$\xi_{il}=h_{il}\left(\delta_{il}+(1-\delta_{il})h_i\right)-h_{ii}g_{il} \quad (i,l=1,\dots,n),$$
$$g_{il}=|\mu_i|\left(1-|\mu_i|\gamma_i(s_i,t_i)\right)^{-1}\gamma_{il}(s_i)+\max\{\gamma_{il}(a),\gamma_{il}(b)\} \ \ (i,l=1,\dots,n),$$
$$\gamma_{il}(t_i)=0, \quad \gamma_{il}(t)=|\alpha_{il}(t)-\alpha_{il}(t_i)|-(1-\delta_{il})d_j\alpha_{il}(t_i)$$
$$\text{for } (-1)^j(t-t_i)>0 \ \ (j=1,2;\ \ i,l=1,\dots,n),$$
$$h_i=1 \ \text{ if } |\mu_i|\le 1, \quad \text{and} \quad h_i=1+(|\mu_i|-1)(1-|\mu_i|\gamma_i(s_i,t_i))^{-1}$$
$$\text{if } |\mu_i|>1 \ \ (i=1,\dots,n).$$

Then the problem (1.1), (1.5) *has one and only one solution.*

Theorem 2.8. *Let* $\ell_{0i}:\mathrm{BV}([a,b],\mathbb{R}_+^n)\to\mathbb{R}_+$ $(i=1,\dots,n)$ *be linear continuous functionals, the matrix-function* $C=(c_{il})_{i,l=1}^n\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ *be such that the functions* c_{il} $(i\ne l;\ i,l=1,\dots,n)$ *are nondecreasing on* $[a,b]$ *and the problem* (1.23), (1.24) *has a nontrivial nonnegative solution* $x=(x_i)_{i=1}^n$, *i.e. the condition* (2.16) *is violated. Let, moreover,*

$$d_jc_{ii}(t)\ge 0 \ \text{ for } t\in[a,b] \ \ (j=1,2;\ \ i=1,\dots,n). \tag{2.40}$$

Then there exist a matrix-function $A=(a_{il})_{i,l=1}^n\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, *linear continuous functionals* $\ell_i:\mathrm{BV}([a,b],\mathbb{R}^n)\to\mathbb{R}$ $(i=1,\dots,n)$ *and numbers* $c_{0i}\in\mathbb{R}$ $(i=1,\dots,n)$ *such that the conditions* (2.17)–(2.21) *are fulfilled,but the problem* (1.1_0), (1.4) *is unsolvable. In addition, if the matrix-function* $C=(c_{il})_{i,l=1}^n$ *is such that*

$$\det\Big(I_n+(-1)^j\operatorname{diag}(\operatorname{sgn}(t-t_1),\dots,\operatorname{sgn}(t-t_n))d_jC(t)\times$$
$$\times\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_n)\Big)\ne 0 \quad \text{for} \quad t\in[a,b] \ \ (j=1,2), \tag{2.41}$$

where $\varepsilon_i \in [0,1]$ $(i = 1,\dots,n)$, *then the matrix-function* $A = (a_{il})_{i,l=1}^n$ *satisfies the condition* (1.21).

Remark 2.4. The condition (2.41) holds, for example, if either

$$\sum_{l=1}^n |d_j c_{il}(t)| < 1 \quad \text{for} \quad t \in [a,b] \ \ (j=1,2; \ \ i=1,\dots,n) \tag{2.42}$$

or

$$d_j c_{ii}(t) \le 1 \quad \text{for} \quad (-1)^j (t - t_i) < 0 \ \ (j=1,2; \ \ i=1,\dots,n) \tag{2.43}$$

and

$$\sum_{l=1,\, l\neq i}^n |d_j c_{il}(t)| < \big|1 + (-1)^j \operatorname{sgn}(t-t_i) d_j c_{ii}(t)\big| \quad \text{for} \quad t \in [a,b] \ \ (j=1,2; \ \ i=1,\dots,n) \tag{2.44}$$

$$\Bigg(\sum_{l=1,\, l\neq i}^n |d_j c_{li}(t)| < \big|1 + (-1)^j \operatorname{sgn}(t-t_i) d_j c_{ii}(t)\big| \quad \text{for} \quad t \in [a,b] \ \ (j=1,2; \ \ i=1,\dots,n) \Bigg).$$

3. Boundary Value Problems for Impulsive Systems

In this section we will realize the results of Section 2 for the impulsive systems (1.7), (1.2)-(1.7), (1.5).

We will assume that $P \in L([a,b],\mathbb{R}^{n\times n})$, $q \in L([a,b],\mathbb{R}^n)$, $G_k \in \mathbb{R}^{n\times n}$, $g_k \in \mathbb{R}^n$, $\tau_k \in [a,b]$ $(k = 1,2,\dots)$.

By a solution of the impulsive system (1.7) we understand a continuous from the left vector-function $x \in \widetilde{C}_{loc}([a,b] \setminus \{\tau_k\}_{k=1}^\infty, \mathbb{R}^n) \cap \mathrm{BV}([a,b]\mathbb{R}^n)$ satisfying both the system

$$\frac{dx(t)}{dt} = P(t)x(t) + q(t) \ \text{ for a.e. } \ t \in [a,b] \setminus \{\tau_k\}_{k=1}^\infty$$

and the relation

$$x(\tau_k+) - x(\tau_k-) = G_k x(\tau_k) + g_k$$

for every $k \in \{1,2,\dots\}$.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for survey of the results on impulsive systems see, e.g., [21], [25], [26], [29], [30], and references therein). But the above-mentioned works do not contain the results analogous to those obtained in [17], [18] for ordinary differential equations. Using the theory of generalized ordinary differential equations, we extend these results to the systems of impulsive equations.

Here we assume that the conditions (1.8) and

$$\det(I_n + G_k) \neq 0 \ \ (k = 1,2,\dots) \tag{3.1}$$

hold.

By $\nu(t)$ $(a < t \le b)$ we denote the number of the points τ_k $(k = 1, 2, \dots)$ belonging to $[a, t[$, $\nu(a) = 1$.

To establish the results dealing with the boundary value problems for the impulsive system (1.7), we use the following concept.

It is easy to show that the vector-function x is a solution of the impulsive system (1.7) if and only if it is a solution of the system (1.1), where

$$A(a) = O_{n\times n}, \quad f(a) = 0,$$
$$A(t) = \int\limits_a^t P(\tau)\,d\tau + \sum_{a\le\tau_k<t} G_k,$$
$$f(t) = \int\limits_a^t q(\tau)\,d\tau + \sum_{a\le\tau_k<t} g_k \ \text{ for } \ a < t \le b$$

(by (1.8), we have $A \in \mathrm{BV}([a, b], \mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}([a, b], \mathbb{R}^n)$). Therefore the system (1.7) is a particular case of the system (1.1). In addition, in this case the condition (3.1) is equivalent to the condition (1.21), since A and f are continuous from the left.

We will need the forms of operators defined by means of (1.17_j)–(1.19_j) $(j = 1, 2)$. First of all, we note that the operators defined by (1.17_1) $((1.17_2))$ and (1.18_1) $((1.18_2))$ coincide among themselves if X is a continuous from the left (from the right) matrix-function.

For every matrix-function $X \in L([a, b], \mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 0, 1, \dots)$ we put

$$\big[(X, \{Y_k\}_{k=1}^{\infty})(t)\big]_0 = I_n \ \text{ for } \ a \le t \le b,$$
$$\big[(X, \{Y_k\}_{k=1}^{\infty})(a)\big]_i = O_{n\times n} \ \ (i = 1, 2, \dots),$$
$$\big[(X, \{Y_k\}_{k=1}^{\infty})(t)\big]_{i+1} = \int\limits_a^t X(\tau)\cdot\big[(X, \{Y_k\}_{k=1}^{\infty})(\tau)\big]_i\,d\tau +$$
$$+ \sum_{a\le\tau_k<t} Y_k\cdot\big[(X, \{Y_k\}_{k=1}^{\infty})(\tau_k)\big]_i \ \text{ for } \ a < t \le b \ \ (i = 1, 2, \dots). \tag{3.2}$$

Note that in this case for the operators V_i $(i = 1, 2, \dots)$ defined by (1.19_1), we have

$$V_i(X, \{Y_k\}_{k=1}^{\infty})(t) = \big[\big(|X(\cdot)|, \{|Y_k|\}_{k=1}^{\infty}\big)(t)\big]_i$$
$$\text{for } \ a \le t \le b \ \ (i = 1, 2, \dots).$$

The definition of the set $U(t_1, \dots, t_n)$ has in this case the following form.

Definition 3.1. Let $t_1, \dots, t_n \in [a, b]$ and $\tau_k \in [a, b]$ $(k = 1, 2, \dots)$. We say that the triple $(Q, \{H_k\}_{k=1}^{\infty}, \ell_0)$ consisting of a matrix-function $Q = (q_{il})_{i,l=1}^n \in L([a, b], \mathbb{R}^{n\times n})$, a sequence of constant matrices $H_k = (h_{kil})_{i,l=1}^n \in \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$ and a positive homogeneous nondecreasing continuous operator $\ell_0 = (\ell_{0i})_{i=1}^n : \mathrm{BV}([a,b],\mathbb{R}_+^n) \to \mathbb{R}_+^n$ belongs to the set $U(t_1,\dots,t_n;\tau_1,\tau_2,\dots)$ if $q_{il}(t) \geq 0$ $(i \neq l;\ i,l = 1,\dots,n)$ for a.e. $t \in [a,b]$, $h_{kil} \geq 0$ $(i \neq l;\ i,l = 1,\dots,n;\ k = 1,2,\dots)$, and the system

$$\begin{aligned} x_i'(t)\operatorname{sgn}(t-t_i) &\leq \sum_{l=1}^n q_{il}(t)x_l(t) \quad (i = 1,\dots,n), \\ x_i(\tau_k+) - x_i(\tau_k-) &\leq \sum_{l=1}^n h_{kil}x_l(\tau_k) \quad (i = 1,\dots,n;\quad k = 1,2,\dots) \end{aligned} \tag{3.3}$$

has no nontrivial nonnegative solution satisfying the condition

$$x_i(t_i) \leq \ell_{0i}(x_1,\dots,x_n) \quad (i = 1,\dots,n). \tag{3.4}$$

3.1. **Solvability of the Problem (1.7), (1.5).**

Theorem 3.1. *The boundary value problem* (1.7), (1.5) *has a unique solution if and only if the corresponding homogeneous problem* $(1.7_0), (1.5_0)$ *has only the trivial solution. If the latter condition holds, then the solution* x *of the problem* (1.7), (1.5) *admits the representation*

$$x(t) = x_0(t) - \int_a^b \mathcal{G}(t,\tau)q(\tau)\,d\tau - \sum_{a\leq\tau_k<b} \mathcal{G}(t,\tau_k)g_k,$$

where x_0 *is a solution of the problem* $(1.7_0), (1.5)$*, and* $\mathcal{G}(t,\tau)$ *defined by* (1.26) *is the Green matrix of the problem* $(1.7_0), (1.5_0)$.

Theorem 3.2. *The boundary value problem* (1.7), (1.5) *has a unique solution if and only if there exist natural numbers* k *and* m *such that the matrix*

$$M_k = -\sum_{i=0}^{k-1} \int_a^b dL(t) \cdot \big[(P,\{G_k\}_{k=1}^\infty)(t)\big]_i$$

is nonsingular and the inequality (2.2) *holds, where the operators* $[(P,\{G_k\}_{k=1}^\infty)(t)]_i$ $(i = 0,1,\dots)$ *are defined by* (3.2),

$$\begin{aligned} M_{k,m} = &\big[\big(|P|,\{|G_k|\}_{k=1}^\infty\big)(b)\big]_m + \\ &+ \sum_{i=0}^{m-1} \big[\big(|P|,\{|G_k|\}_{k=1}^\infty\big)(b)\big]_i \cdot \int_a^b dV(M_k^{-1}L)(t) \cdot \big[\big(|P|,\{|G_k|\}_{k=1}^\infty\big)(t)\big]_k. \end{aligned}$$

Corollary 3.1. *Let either the condition* (2.6), *or the conditions* (2.7),

$$\int_a^b dL(t) \cdot \big[(P,\{G_k\}_{k=1}^\infty)(t)\big]_i = O_{n\times n} \quad (i = 0,\dots,j-1)$$

and

$$\det\Big(\int_a^b dL(t) \cdot \big[(P,\{G_k\}_{k=1}^\infty)(t)\big]_j\Big) \neq 0$$

hold for some natural j. Then there exists $\varepsilon_0 > 0$ such that the impulsive system

$$\begin{gathered} \frac{dx}{dt} = \varepsilon P(t)x + q(t), \\ x(\tau_k+) - x(\tau_k-) = \varepsilon G_k x(\tau_k) + g_k \;\; (k = 1, 2, \dots) \end{gathered} \tag{3.5}$$

has one and only one solution satisfying the condition (1.5) *for every $\varepsilon \in]0, \varepsilon_0[$.*

Theorem 3.3. *Let a matrix-function $P_0 \in L([a, b], \mathbb{R}^{n\times n})$ and constant matrices $G_{0k} \in \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$ be such that*

$$\sum_{k=1}^{\infty} \|G_{0k}\| < \infty$$

and the homogeneous system

$$\begin{gathered} \frac{dx}{dt} = P_0(t)x, \\ x(\tau_k+) - x(\tau_k-) = G_{0k} x(\tau_k) \;\; (k = 1, 2, \dots) \end{gathered} \tag{3.6}$$

has only the trivial solution satisfying the condition (1.5_0). *Let, moreover, the matrix-function $P \in L([a, b], \mathbb{R}^{n\times n})$ and the constant matrices $G_k \in \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$ admit the estimate*

$$\int\limits_a^b |\mathcal{G}_0(t,\tau)| \cdot \big|P(\tau) - P_0(\tau)\big|\, d\tau + \sum_{k=1}^{\infty} \big|\mathcal{G}_0(t, \tau_k+) \cdot (G_k - G_{0k})\big| < M,$$

where $\mathcal{G}_0$ is the Green matrix of the problem $(3.6), (1.5_0)$, *and $M \in \mathbb{R}_+^{n\times n}$ is a constant matrix satisfying the inequality* (2.12). *Then the problem* (1.7), (1.5) *has one and only one solution.*

3.2. **Solvability of the Problem (1.7), (1.2).**

Theorem 3.4. *Let the function $P \in L([a, b], \mathbb{R}^{n\times n})$ satisfy the Lappo–Danilevskiĭ condition, and*

$$P(t)G_k = G_k P(t) \;\; \text{for a.e. } t \in [a, b] \;\; (k = 1, 2, \dots).$$

Then the problem (1.7), (1.2) *is uniquely solvable if and only if*

$$\det\left(\sum_{j=1}^{n_0} L_j \exp(P(t_j)) \cdot \prod_{a\le\tau_k<t_j} (I_n + G_k)\right) \neq 0.$$

Theorem 3.5. *The boundary value problem* (1.7), (1.2) *is uniquely solvable if and only if there exist natural numbers k and m such that the matrix*

$$M_k = \sum_{j=1}^{n_0}\sum_{i=0}^{k-1} L_j \big[(P, \{G_l\}_{l=1}^{\infty})(t_j)\big]_i$$

is nonsingular and the inequality (2.2) *holds, where*

$$M_{k,m} = \big[\big(|P|, \{|G_l|\}_{l=1}^{\infty}\big)(b)\big]_m +$$

$$+\Big(\sum_{i=0}^{m-1}\big[\big(|P|,\{|G_l|\}_{l=1}^{\infty}\big)(b)\big]_i\Big)\cdot\sum_{j=1}^{n_0}|M_k^{-1}L_j|\cdot\big[\big(|P|,\{|G_l|\}_{l=1}^{\infty}\big)(t_j)\big]_k.$$

Corollary 3.2. *Let the condition* (2.15) *hold and*

$$r(L_0A_0)<1,$$

where

$$L_0=I_n+\Big|\Big(\sum_{j=1}^{n_0}L_j\Big)^{-1}\Big|\sum_{j=1}^{n_0}|L_j|$$

and

$$A_0=\int_a^b|P(t)|\,dt+\sum_{k=1}^{\infty}|G_k|.$$

Then the problem (1.7), (1.2) *has one and only one solution.*

Corollary 3.3. *Let either the condition* (2.15) *hold, or there exist a natural number* m *such that*

$$\sum_{j=1}^{n_0}L_j=O_{n\times n},$$

$$\det\bigg(\sum_{j=1}^{n_0}L_j\Big[\int_a^{t_j}P(\tau)\,d\tau+\sum_{a\le\tau_k<t_j}G_k\Big]_i\bigg)=0\ \ (i=0,\dots,m-1)$$

and

$$\det\bigg(\sum_{j=1}^{n_0}L_j\Big[\int_a^{t_j}P(\tau)\,d\tau+\sum_{a\le\tau_k<t_j}G_k\Big]_m\bigg)\neq0.$$

Then there exists $\varepsilon_0>0$ *such that the problem* (3.5), (1.2) *has one and only one solution for every* $\varepsilon\in\,]0,\varepsilon_0[\,$.

3.3. **Solvability of the problems (1.7), (1.3) and (1.7), (1.4).**

Theorem 3.6. *Let there exist a matrix-function* $Q=(q_{il})_{i,l=1}^n\in L([a,b],\mathbb{R}^{n\times n})$, *a sequence of constant matrices* $H_k=(h_{kil})_{i,l=1}^n\in\mathbb{R}^{n\times n}$ $(k=1,2,\dots)$ *and a positive homogeneous nondecreasing continuous operator* $\ell_0=(\ell_{0i})_{i=1}^n:\mathrm{BV}([a,b],\mathbb{R}_+^n)\to\mathbb{R}_+^n$ *satisfying the condition*

$$(Q,\{H_k\}_{k=1}^{\infty},\ell_0)\in U(t_1,\dots,t_n;\tau_1,\tau_2,\dots)\ \text{ for a.e. } t\in[a,b] \tag{3.7}$$

such that

$$p_{ii}(t)\,\mathrm{sgn}(t-t_i)\le q_{ii}(t)\ \ (i=1,\dots,n), \tag{3.8}$$

$$(-1)^j\big(|1+g_{kii}|-1\big)\le h_{kii}$$

$$\text{for } (-1)^j(\tau_k-t_i)\ge0\ \ (j=1,2;\ \ i=1,\dots,n;\ \ k=1,2,\dots), \tag{3.9}$$

$$|p_{il}(t)|\le q_{il}(t)\ \text{ for a.e. } t\in[a,b]\ \ (i\ne l;\ \ i,l=1,\dots,n) \tag{3.10}$$

and

$$|g_{kil}| \le h_{kil} \ \ (i \ne l; \ \ i,l = 1,\dots,n; \ \ k = 1,2,\dots). \tag{3.11}$$

Let, moreover, the condition (2.21) *hold. Then the problem* (1.7), (1.4) *has one and only one solution.*

Theorem 3.7. *Let the impulsive system* (1.7) *have a finite number of jump points* (*i.e.* $\tau_{m_0} = \tau_{m_0+1} = \cdots$ *for some* $m_0 \in \{1,2,\dots\}$), *and the conditions*

$$p_{ii}(t)\operatorname{sgn}(t - t_i) \le h_{ii}(t) \ \ \text{for a.e. } t \in [a,b] \ \ (i = 1,\dots,n), \tag{3.12}$$

$$(-1)^j\big(|1 + g_{kii}| - 1\big) \le h_{ii}(\tau_k)\alpha_{ki}$$
$$\text{for } \ (-1)^j(\tau_k - t_i) \ge 0 \ \ (j = 1,2; \ \ i = 1,\dots,n; \ \ k = 1,\dots,m_0), \tag{3.13}$$

$$|p_{il}(t)| \le h_{il}(t) \ \ \text{for a.e. } t \in [a,b] \ \ (i \ne l; \ \ i,l = 1,\dots,n) \tag{3.14}$$

and

$$|g_{kil}| \le h_{il}(\tau_k)\alpha_{kl} \ \ (i \ne l; \ \ i,l = 1,\dots,n; \ \ k = 1,\dots,m_0) \tag{3.15}$$

hold, where $h_{ii} \in L^\mu([a,b],\mathbb{R})$, $h_{il} \in L^\mu([a,b],\mathbb{R}_+)$ ($i \ne l$; $i,l = 1,\dots,n$), $1 \le \mu \le +\infty$, $\alpha_{ik} \in \mathbb{R}_+$ ($i = 1,\dots,n$; $k = 1,\dots,m_0$). *Let, moreover,*

$$\big|\ell_i(x_1,\dots,x_n)\big| \le \sum_{k=1}^n \Big(\gamma_{1ik}\|x_k\|_{L^\nu} + \gamma_{2ik}\Big[\sum_{l=1}^{m_0} |x_k(\tau_l)|^\nu\Big]^{\frac{1}{\nu}}\Big)$$
$$\text{for } \ x = (x_k)_{k=1}^n \in \mathrm{BV}[a,b],\mathbb{R}^n) \ \ (i = 1,\dots,n)$$

and

$$r(\mathcal{H}_0) < 1, \tag{3.16}$$

where γ_{1ik}, $\gamma_{2ik} \in \mathbb{R}_+$ ($i,k = 1,\dots,n$), $\frac{1}{\mu} + \frac{2}{\nu} = 1$, *and the* $2n \times 2n$*-matrix* $\mathcal{H}_0 = (\mathcal{H}_{0jm})_{j,m=1}^2$ *is defined by*

$$\mathcal{H}_{011} = \bigg((b-a)^{\frac{1}{\nu}}\gamma_{1ik} + \Big[\frac{2}{\pi}\,(b-a)\Big]^{\frac{1}{\nu}}\|h_{ik}\|_{L^\mu}\bigg)_{i,k=1}^n,$$
$$\mathcal{H}_{012} = \bigg((b-a)^{\frac{1}{\nu}}\gamma_{2ik} + \Big[(b-a)\sum_{l=1}^{m_0}\alpha_{li}\Big]^{\frac{1}{\nu}}\Big(\sum_{l=1}^{m_0}|h_{ik}(\tau_l)|^\mu\Big)^{\frac{1}{\mu}}\bigg)_{i,k=1}^n,$$
$$\mathcal{H}_{021} = \bigg(\Big(\sum_{l=1}^{m_0}\alpha_{li}\Big)^{\frac{1}{\nu}}\gamma_{1ik} + \Big[(b-a)\sum_{l=1}^{m_0}\alpha_{li}\Big]^{\frac{1}{\nu}}\|h_{ik}\|_{L^\mu}\bigg)_{i,k=1}^n,$$
$$\mathcal{H}_{022} = \bigg(\Big(\sum_{l=1}^{m_0}\alpha_{li}\Big)^{\frac{1}{\nu}}\gamma_{2ik} + \Big(\frac{1}{4}\,\mu_i\mu_k\sin^{-2}\frac{\pi}{4n_k+2}\Big)^{\frac{1}{\nu}}\Big(\sum_{l=1}^{m_0}|h_{ik}(\tau_l)|^\mu\Big)^{\frac{1}{\mu}}\bigg)_{i,k=1}^n;$$

here $\mu_i = \max\{\alpha_{li} : l = 1,\dots,m_0\}$, *and* $n_k = n_{\alpha_k 2}$ *is the quantity of nonzero numbers from the sequence* $\alpha_{1k},\dots,\alpha_{m_0k}$. *Then the problem* (1.7), (1.4) *has one and only one solution.*

Remark 3.1. The $2n\times 2n$-matrix $\mathcal{H}$ appearing in Theorem 3.7 can be replaced by the $n\times n$-matrix

$$\bigg(\max\bigg\{\Big[(b-a)^{\frac{1}{\nu}}+\Big(\sum_{l=1}^{m_0}\alpha_{li}\Big)^{\frac{1}{\nu}}\Big]\gamma_{1ik}+$$

$$+\Big(\Big[\frac{2}{\pi}(b-a)\Big]^{\frac{2}{\nu}}+\Big[(b-a)\sum_{l=1}^{m_0}\alpha_{li}\Big]^{\frac{1}{\nu}}\Big)\|h_{ik}\|_{L^\mu},\ \Big[(b-a)^{\frac{1}{\nu}}+\Big(\sum_{l=1}^{m_0}\alpha_{li}\Big)^{\frac{1}{\nu}}\Big]\gamma_{2ik}+$$

$$+\Big(\Big[(b-a)\sum_{l=1}^{m_0}\alpha_{li}\Big]^{\frac{1}{\nu}}+\Big(\frac{1}{4}\,\mu_i\mu_k\sin^{-2}\frac{\pi}{4n_k+2}\Big)^{\frac{1}{\nu}}\Big)\Big(\sum_{l=1}^{m_0}|h_{lk}(\tau_l)|^{\mu}\Big)^{\frac{1}{\mu}}\bigg\}\bigg)_{i,k=1}^{n}.$$

Corollary 3.4. *Let the impulsive system* (1.7) *have a finite number of jump points* (*i.e.* $\tau_{m_0}=\tau_{m_0+1}=\cdots$ *for some* $m_0\in\{1,2,\dots\}$) *and the conditions* (3.12)–(3.15) *hold, where* $h_{ii}\in L^\mu([a,b],\mathbb{R})$, $h_{il}\in L^\mu([a,b],\mathbb{R}_+)$ ($i\neq l$; $i,l=1,\dots,n$), $1\le\mu\le+\infty$, $\alpha_{ki}\in\mathbb{R}_+$ ($i=1,\dots,n$; $k=1,\dots,m_0$). *Let, moreover, the inequality* (3.16) *hold, where the* $2n\times 2n$-*matrix* $\mathcal{H}_0=(\mathcal{H}_{0jm})_{j,m=1}^2$ *is defined by*

$$\mathcal{H}_{011}=\bigg(\Big[\frac{2}{\pi}(b-a)\Big]^{\frac{1}{\nu}}\|h_{ik}\|_{L^\mu}\bigg)_{i,k=1}^{n},$$

$$\mathcal{H}_{012}=\bigg(\Big[(b-a)\sum_{l=1}^{m_0}\alpha_{li}\Big]^{\frac{1}{\nu}}\Big(\sum_{l=1}^{m_0}|h_{ik}(\tau_l)|^{\mu}\Big)^{\frac{1}{\mu}}\bigg)_{i,k=1}^{n},$$

$$\mathcal{H}_{021}=\bigg(\Big[(b-a)\sum_{l=1}^{m_0}\alpha_{li}\Big]^{\frac{1}{\nu}}\|h_{ik}\|_{L^\mu}\bigg)_{i,k=1}^{n},$$

$$\mathcal{H}_{022}=\bigg(\Big(\frac{1}{4}\,\mu_i\mu_k\sin^{-2}\frac{\pi}{4n_k+2}\Big)^{\frac{1}{\nu}}\Big(\sum_{l=1}^{m_0}|h_{ik}(\tau_l)|^{\mu}\Big)^{\frac{1}{\mu}}\bigg)_{i,k=1}^{n};$$

here $\frac{1}{\mu}+\frac{2}{\nu}=1$, $\mu_i=\max\{\alpha_{li}:l=1,\dots,m_0\}$, *and* $n_k=n_{\alpha_k 2}$ *is the quantity of nonzero numbers from the sequence* $\alpha_{1k},\dots,\alpha_{m_0k}$. *Then the problem* (1.7), (1.3) *has one and only one solution.*

Remark 3.2. The $2n\times 2n$-matrix $\mathcal{H}_0$ appearing in Corollary 3.4 can be replaced by the $n\times n$-matrix

$$\bigg(\max\bigg\{\Big(\Big[\frac{2}{\pi}(b-a)\Big]^{\frac{2}{\nu}}+\Big[(b-a)\sum_{l=1}^{m_0}\alpha_{li}\Big]^{\frac{1}{\nu}}\Big)\|h_{ik}\|_{L^\mu},$$

$$\Big(\Big[(b-a)\sum_{l=1}^{m_0}\alpha_{li}\Big]^{\frac{1}{\nu}}+\Big(\frac{1}{4}\,\mu_i\mu_k\sin^{-2}\frac{\pi}{4n_k+2}\Big)^{\frac{1}{\nu}}\Big)\Big(\sum_{l=1}^{m_0}|h_{lk}(\tau_l)|^{\mu}\Big)^{\frac{1}{\mu}}\bigg\}\bigg)_{i,k=1}^{n}.$$

By Remark 3.2, Corollary 3.4 has the following form for $h_{il}(t)\equiv h_{il}=\mathrm{const}$ $(i,l=1,\dots,n)$ and $\mu=+\infty$.

Corollary 3.5. *Let the impulsive system* (1.7) *have a finite number of jump points* (*i.e.* $\tau_{m_0}=\tau_{m_0+1}=\cdots$ *for some* $m_0\in\{0,1,\dots\}$) *and the* *conditions*

$$p_{ii}(t)\operatorname{sgn}(t-t_i) \le h_{ii} \ \text{ for a.e. } \ t \in [a,b] \ \ (i = 1,\dots,n),$$
$$(-1)^j\big(|1+g_{kii}|-1\big) \le h_{ii}\alpha_k \ \ \text{ for } \ (-1)^j(\tau_k - t_i) \ge 0$$
$$(j = 1,2; \ \ i = 1,\dots,n; \ \ k = 1,\dots,m_0),$$
$$|p_{il}(t)| \le h_{il} \ \text{ for a.e. } \ t \in [a,b] \ \ (i \ne l; \ \ i,l = 1,\dots,n)$$

and

$$|g_{kil}| \le h_{il}\alpha_k \ \ (i \ne l; \ \ i,l = 1,\dots,n; \ \ k = 1,\dots,m_0)$$

hold, where $h_{ii} \in \mathbb{R}$, $h_{il} \in \mathbb{R}_+$ $(i \ne l;\ i,l = 1,\dots,n)$, $\alpha_k \in \mathbb{R}_+$ $(k = 1,\dots,m_0)$*. Let, moreover,*

$$\rho_0 r(\mathcal{H}_0) < 1,$$

where $\mathcal{H}_0 = (h_{ik})_{i,k=1}^n$,

$$\rho_0 = \Big((b-a)\sum_{l=1}^{m_0}\alpha_l\Big)^{\frac12} + \max\Big\{\frac{2}{\pi}\,(b-a), \frac{1}{2}\,\mu_\alpha \sin^{-1}\frac{\pi}{4n_\alpha+2}\Big\},$$

$\mu_\alpha = \max\{\alpha_l : l = 1,\dots,m_0\}$, n_α *is the quantity of nonzero numbers from the sequence* $\alpha_1,\dots,\alpha_{m_0}$*. Then the problem* (1.7), (1.3) *has one and only one solution.*

Theorem 3.8. *Let the conditions* (2.36), (2.38) *and*

$$p_{ii}(t)\operatorname{sgn}(t-t_i) \le h_{ii}\beta_i(t) + \beta_{ii}(t) \ \text{ for a.e. } \ t \in [a,b] \ \ (i = 1,\dots,n),$$
$$(-1)^j\big(|1+g_{kii}|-1\big) \le h_{ii}\beta_{ki} + \beta_{kii} \ \text{ for } \ (-1)^j(\tau_k - t_i) \ge 0$$
$$(j = 1,2; \ \ i = 1,\dots,n; \ \ k = 1,\dots,m_0),$$
$$|p_{il}(t)| \le h_{il}\beta_i(t) + \beta_{il}(t) \ \text{ for a.e. } \ t \in [a,b] \ \ (i \ne l; \ \ i,l = 1,\dots,n),$$
$$|g_{kil}| \le h_{il}\beta_{ki} + \beta_{kil} \ \ (i \ne l; \ \ i,l = 1,\dots,n; \ \ k = 1,\dots,m_0)$$

hold, where $h_{ii} < 0$, $h_{il} \ge 0$ $(i \ne l;\ i,l = 1,\dots,n)$; $\mu_i \in \mathbb{R}$, $s_i \in [a,b]$, $s_i \ne t_i$ $(i = 1,\dots,n)$; $\beta_{ii} \in L([a,b],\mathbb{R}_+)$ $(i = 1,\dots,n)$; β_{il}, $\beta_i \in L([a,b];\mathbb{R})$ $(i \ne l;\ i,l = 1,\dots,n)$ *are such that* $\beta_{il}(t) \ge 0$ $(i \ne l)$ *and* $\beta_i(t) \ge 0$ *for a.e.* $t \in [a,t_i[\cup]t_i,b]$; $\beta_{kii} \in \mathbb{R}_+$ $(i = 1,\dots,n;\ k = 1,\dots,m_0)$; β_{kil}, $\beta_{ki} \in \mathbb{R}$ $(i \ne l;\ i,l = 1,\dots,n;\ k = 1,\dots,m_0)$ *are such that* $\beta_{kil} \ge 0$ *and* $\beta_{ki} \ge 0$ *if* $\tau_k \ne t_i$, $\beta_{ki} \le 0$ *if* $\tau_k = t_i$*, and* $0 \le \beta_{ki} < |\eta_{ii}|^{-1}$ *if* $\tau_k > t_i$; $\gamma_i(t,t) = 1$, $\gamma_i(t,s) = \gamma_i^{-1}(s,t)$ *for* $t < s$ *and*

$$\gamma_i(t,s) = \exp\Big(\eta_{ii}\int_s^t \beta_i(\tau)\,d\tau\Big)\prod_{s\le\tau_k<t}(1+\eta_{ii}\beta_{ki}) \ \text{ for } \ t > s \ \ (i = 1,\dots,n).$$

Let, moreover, the condition (2.39) *hold and the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^n$ *be negative, where*

$$\xi_{il} = h_{il}\big(\delta_{il} + (1-\delta_{il})h_i\big) - h_{ii}g_{il} \ \ (i,l = 1,\dots,n),$$
$$g_{il} = |\mu_i|\big(1-|\mu_i|\gamma_i(s_i,t_i)\big)^{-1}\gamma_{il}(\tau_i) + \max\big\{\gamma_{il}(a), \gamma_{il}(b)\big\} \ \ (i,l = 1,\dots,n),$$

$$\gamma_{il}(t) = \big|\alpha_{il}(t) - \alpha_{il}(t_i)\big| \ \ for\ \ t > t_i \ \ if\ \ t_i \notin \{\tau_1, \dots, \tau_m\},$$
$$or\ for\ \ t \le t_i \ \ (i, l = 1, \dots, n),$$
$$\gamma_{il}(t) = \big|\alpha_{il}(t) - \alpha_{il}(t_i)\big| - (1 - \delta_{il})\beta_{ki}$$
$$for\ \ t > t_i \ \ if\ \ t_i = \tau_k \ \ (i, l = 1, \dots, n;\ \ k = 1, \dots, m_0),$$
$$h_i = 1 \ \ if\ \ |\mu_i| \le 1, \ \ and$$
$$h_i = 1 + (|\mu_i| - 1)\big(1 - |\mu_i|\lambda_i(s_i, t_i)\big)^{-1} \ \ if\ \ |\mu_i| > 1 \ \ (i = 1, \dots, n),$$
$$\alpha_{il}(t) \equiv \int_a^t \beta_{il}(\tau)\, d\tau + \sum_{a \le \tau_k < t} \beta_{kil} \ \ (i, l = 1, \dots, n).$$

Then the problem (1.7), (1.4) *has one and only one solution.*

Corollary 3.6. *Let the conditions* (2.36),

$$p_{ii}(t)\,\mathrm{sgn}(t - t_i) \le h_{ii} \ \ for\ a.e.\ \ t \in [a, b] \ \ (i = 1, \dots, n), \tag{3.17}$$
$$(-1)^j\big(|1 + g_{kii}| - 1\big) \le \beta_{kii} \ \ for\ \ (-1)^j(\tau_k - t_i) \ge 0$$
$$(j = 1, 2;\ \ i = 1, \dots, n;\ \ k = 1, \dots, m_0),$$
$$|p_{il}(t)| \le h_{il} \ \ for\ a.e.\ \ t \in [a, b] \ \ (i \ne l;\ \ i, l = 1, \dots, n), \tag{3.18}$$
$$|g_{kl}| \le \beta_{kil} \ \ (i \ne l;\ \ i, l = 1, \dots, n;\ \ k = 1, \dots, m_0)$$

and

$$|\mu_i| \exp\big(h_{ii}|s_i - t_i|\big) < 1 \ \ (i = 1, \dots, n) \tag{3.19}$$

hold, where $h_{ii} < 0$, $h_{il} \ge 0$ $(i \ne l;\ i, l = 1, \dots, n)$; $\mu_i \in \mathbb{R}$, $s_i \in [a, b]$, $s_i \ne t_i$ $(i = 1, \dots, n)$; $\beta_{kii} \ge 0$ $(i = 1, \dots, n;\ k = 1, \dots, m_0)$, $\beta_{kil} \in \mathbb{R}$ $(i \ne l;\ i, l = 1, \dots, n)$ *are such that* $\beta_{kil} \ge 0$ *if* $t_i \ne \tau_k$. *Let, moreover, the condition* (2.39) *hold and the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^n$ *be negative, where*

$$\xi_{il} = h_{il}\big(\delta_{il} + (1 - \delta_{il})h_i\big) - h_{ii}g_{il} \ \ (i, l = 1, \dots, n),$$
$$g_{il} = |\mu_i|\big(1 - |\mu_i| \exp(h_{ii}|s_i - t_i|)\big)^{-1}\gamma_{il}(\tau_i) + \max\big\{\gamma_{il}(a), \gamma_{il}(b)\big\}$$
$$(i, l = 1, \dots, n),$$
$$\gamma_{il}(t) \equiv \big|\alpha_{il}(t) - \alpha_{il}(t_i)\big| \ \ (i, l = 1, \dots, n),$$
$$h_i = 1 \ \ if\ \ |\mu_i| \le 1, \ \ and$$
$$h_i = 1 + (|\mu_i| - 1)\big(1 - |\mu_i| \exp(h_{ii}|s_i - t_i|)\big)^{-1} \ \ if\ \ |\mu_i| \ge 1 \ \ (i = 1, \dots, n),$$
$$\alpha_{il}(t) \equiv \sum_{a \le \tau_k < t} \beta_{kil} \ \ (i, l = 1, \dots, n).$$

Then the problem (1.7), (1.4) *has one and only one solution.*

The following corollary is a generalization of Theorem 4.5 from [18].

Corollary 3.7. *Let the conditions* (2.36), (3.17)–(3.19) *hold, where* $h_{ii} \in \mathbb{R}$, $h_{il} \ge 0$ $(i \ne l;\ i, l = 1, \dots, n)$; $\mu_i \in \mathbb{R}$, $s_i \in [a, b]$, $s_i \ne t_i$

$(i = 1, \dots, n)$. *Let, moreover, the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^n$ *be negative, where*

$$\xi_{il} = h_{il}\big(\delta_{il} + (1-\delta_{il})h_i\big) \ \ (i,l = 1, \dots, n),$$
$$h_i = 1 \ \ \text{if} \ \ |\mu_i| \le 1, \ \ \text{and}$$
$$h_i = 1 + (|\mu_i| - 1)\big(1 - |\mu_i| \exp(h_{ii}|s_i - t_i|)\big)^{-1} \ \ \text{if} \ \ |\mu_i| > 1 \ \ (i = 1, \dots, n).$$

Then the system

$$\frac{dx}{dt} = P(t)x + q(t) \tag{3.20}$$

has one and only one solution satisfying (1.4).

Remark 3.3. In Corollary 3.7, unlike Theorem 3.8 and Corollary 3.6, we do not require the condition

$$h_{ii} < 0 \ \ (i = 1, \dots, n). \tag{3.21}$$

This condition holds if and only if the real part of every characteristic value of the matrix $(\xi_{il})_{i,l=1}^n$ is negative (see Theorems 1.13 and 1.18 from [18]). Thus the condition (3.21) holds automatically. Moreover, the inequality (3.19) also holds automatically if $|\mu_i| \le 1$. Therefore Corollary 3.7 is a generalization of Theorem 4.5 from [18] because in this theorem the case $|\mu_i| \le 1$ $(i = 1, \dots, n)$ is considered. Note also that the condition (3.21) is optimal and we cannot reject it. For the sake of completeness, we give here an example from [18].

Let

$$\mu_1 = 1, \quad |\mu_i| < 1 \ \ (i = 2, \dots, n), \quad t_i = a < s_i \le b \ \ (i = 1, \dots, n),$$
$$p_{ik}(t) \equiv (1-i)\delta_{ik} \ \ (i,k = 1, \dots, n)$$

and

$$c_{01} = 1, \ \ q_1(1) \equiv 0.$$

Then the system (3.20) has no solution satisfying the condition

$$x_i(t_i) = \mu_i x_i(s_i) + c_{0i} \ \ (i = 1, \dots, n).$$

On the other hand, the conditions of Corollary 3.7 hold for

$$h_{ik} = (1-i)\delta_{ik} \ \ (i,k = 1, \dots, n).$$

Theorem 3.9. *Let* $\ell_{0i} : \mathrm{BV}([a,b], \mathbb{R}_+^n) \to \mathbb{R}_+$ $(i = 1, \dots, n)$ *be linear continuous functionals, a matrix-function* $Q = (q_{il})_{i,l=1}^n \in L([a,b], \mathbb{R}^{n\times n})$ *and a sequence of constant matrices* $H_k = (h_{kil})_{i,l=1}^n \in \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$ *be such that* $q_{il}(t) \ge 0$ $(i \ne l;\ i,l = 1, \dots, n)$ *for a.e.* $t \in [a,b]$ *and* $h_{kil} \ge 0$ $(i \ne l;\ i,l = 1, \dots, n;\ k = 1, 2, \dots)$*, respectively, and let the problem* (3.3), (3.4) *have a nontrivial nonnegative solution* $x = (x_i)_{i=1}^n$*, i.e. the condition* (3.7) *be violated. Let, moreover,*

$$h_{kii} \ge 0 \ \ (i = 1, \dots, n; \ \ k = 1, 2, \dots).$$

Then there exist a matrix-function $P=(p_{il})_{i,l=1}^n \in L([a,b],\mathbb{R}^n)$, *a sequence of matrices* $G_k=(g_{kil})_{i,l=1}^n$ $(k=1,2,\dots)$ *and linear continuous functionals* $\ell_i : \mathrm{BV}([a,b],\mathbb{R}^n)\to\mathbb{R}$ $(i=1,\dots,n)$ *for which the conditions* (2.21), (3.8)–(3.11) *are fulfilled, but the problem* $(1.7_0),(1.4)$ *is unsolvable. In addition, if the matrices* H_k $(k=1,2,\dots)$ *are such that*

$$\det\Big(I_n+\operatorname{diag}\big(\operatorname{sgn}(\tau_k-t_1),\dots,\operatorname{sgn}(\tau_k-t_k)\big)H_k\cdot\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_n)\Big)\neq 0 \quad (k=1,2,\dots), \tag{3.22}$$

where $\varepsilon_i\in[0,1]$ $(i=1,\dots,n)$, *then the matrices* G_k $(k=1,2,\dots)$ *satisfy the condition* (3.1).

Remark 3.4. The condition (3.22) holds, for example, if either

$$\sum_{l=1}^n |h_{kil}|<1 \quad (i=1,\dots,n;\ \ k=1,2,\dots)$$

or

$$h_{kii}\leq 1 \ \text{ if } \ \tau_k<t_i \quad (i=1,\dots,n;\ \ k=1,2,\dots)$$

and

$$\sum_{l=1,\,l\neq i}^n |h_{kil}|<\big|1+\operatorname{sgn}(\tau_k-t_i)h_{kii}\big| \quad (i=1,\dots,n;\ \ k=1,2,\dots)$$

$$\bigg(\sum_{l=1,\,l\neq i}^n |h_{kli}|<\big|1+\operatorname{sgn}(\tau_k-t_i)h_{kii}\big| \quad (i=1,\dots,n;\ \ k=1,2,\dots)\bigg).$$

4. Boundary Value Problems for the Difference System (1.9)

In this section we realize the results obtained in Section 2 for the difference problems (1.9), (1.10)–(1.9), (1.12) and (1.13), (1.14)–(1.13), (1.16).

Investigation of the theory of difference equations has been continuing since long time. Many interesting and profound results have been obtained recently (see, e.g., [1], [5], [9], [10], [11], [13], [22] and the references therein). Some of the results obtained in the present paper are analogous to ones for ordinary differential equations, but some of them differ. Therefore, to explain this difference, it is important to investigate the equations from a unified point of view.

In this direction, a unified concept has been used for investigation based on the invariance principle of some quadratic forms (see, e.g., [10]). Unlike this method, we use the theory of generalized ordinary differential equations for investigation, from a unified point of view, of continuous and discrete processes. In this way, some results, analogous to those for differential equations, have been extended to the difference equations (see, e.g., [3], [7], [8]). Moreover, the convergence conditions for the difference schemes corresponding to the boundary value problems for systems of ordinary differential equations are obtained on the basis of the results of the appropriate boundary value problems for systems of generalized ordinary differential equations ([3], [5]).

First, we consider the problems $(1.9), (1.10)$–$(1.9), (1.12)$. To prove the results dealing with the difference system (1.9), we construct a system of the form (1.1) which corresponds to the system (1.9), in order to apply the results of Section 2.

In this section we assume that $G_l = (g_{lij})_{i,j=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_0, \mathbb{R}^{n\times n})$ $(l = 1,2,3)$, $g_0 = (g_{0i})_{i=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ and

$$\det(I_n + G_1(k)) \neq 0, \ \ \det G_3(k) \neq 0 \ \ (k = 0,\dots,m_0). \tag{4.1}$$

Let $y \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ be a solution of the difference system (1.9). Then the vector-function $z = (z_l)_{l=1}^2 \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{2n})$, where

$$\begin{gathered} z_1(k) = (I_n + G_1(k))y(k) \ \ (k = 0,\dots,m_0), \\ z_2(k) = y(k+1) \ \ (k = 0,\dots,m_0 - 1), \quad z_2(m_0) = z_2(m_0 - 1), \end{gathered} \tag{4.2}$$

is a solution of the $2n \times 2n$-difference system

$$\Delta z(k-1) = G(k)z(k) + g(k) \ \ (k = 1,\dots,m_0), \tag{4.3}$$

where $G(k) = (G_{lm}(k))_{l,m=1}^2$ $(k = 0,\dots,m_0)$ is defined by

$$\begin{gathered} G_{lm}(0) = O_{n\times n} \ \ (l,m = 1,2), \\ G_{11}(k) = \big(G_1(k) + G_2(k)\big)(I_n + G_1(k))^{-1}, \quad G_{12}(k) = G_3(k), \\ G_{21}(k) = -(I_n + G_1(k))^{-1}, \quad G_{22}(k) = I_n \ \text{ for } \ k = 1,\dots,m_0, \end{gathered} \tag{4.4}$$

and $g(k) = (g_l(k))_{l=1}^2$ $(k = 0,\dots,m_0)$ is defined by

$$g_1(0) = 0, \quad g_1(k) = g_0(k), \quad g_2(k) = 0 \ \text{ for } \ k = 1,\dots,m_0. \tag{4.5}$$

Conversely, if $z(k) = (z_l(k))_l^2$ is a solution of the $2n \times 2n$-system (4.3), then due to (4.1) the vector-function

$$y(k) = (I_n + G_1(k))^{-1} z_1(k) \ \ (k = 0,\dots,m_0)$$

is a solution of the system (1.9).

Indeed, by (4.3) we have

$$z_2(k) = (I_n + G_1(k+1))^{-1} z_1(k+1) = y(k+1) \ \ (k = 0,\dots,m_0)$$

and

$$\begin{gathered} (I_n + G_1(k))y(k) - (I_n + G_1(k-1))y(k-1) = \\ = \big(G_1(k) + G_2(k)\big)y(k) + G_3(k)z_2(k) + y_1(k) \ \ (k = 0,\dots,m_0) \end{gathered}$$

i.e., y satisfies the system (1.9).

On the other hand, the vector-function $z \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ is a solution of the difference system (4.3) if and only if the vector-function

$$x(t) = z\Big(\Big[t + \frac{1}{2}\Big]\Big) = \Big(z_l\Big(\Big[t + \frac{1}{2}\Big]\Big)\Big)_{l=1}^2 \ \text{ for } \ t \in [0, m_0] \tag{4.6}$$

($[t]$ is the integer part of t) is a solution of the system (1.1), where

$$A(t)=\sum_{i=0}^{[t+\frac{1}{2}]}G(i),\quad f(t)=\sum_{i=0}^{[t+\frac{1}{2}]}g(i)\ \text{ for }\ t\in[0,m_0],\tag{4.7}$$

the matrix-function $G(k)=(G_{lm}(k))_{l,m=1}^{2}$ and the vector-function $g(k)=(g_l(k))_{l=1}^{2}$ are defined by (4.4) and (4.5), respectively. Note that the condition (1.21) is equivalent for that case to the condition (4.1).

Consider now the boundary value problems.

If $y\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$ is a solution of the problem (1.9), (1.10), then the vector-function $x\in\mathrm{BV}([0,m_0],\mathbb{R}^{2n})$ defined by (4.2), (4.6) is a solution of the problem (1.1), (1.2), where

$$\begin{gathered}n_0=n_{01}+n_{02},\quad c_0=(c_{0l})_{l=1}^{2},\\ t_j=k_j\ \ (j=1,\dots,n_{01}),\quad t_j=k_j+1\ \ (j=n_{01}+1,\dots,n_0),\end{gathered}\tag{4.8}$$

$$\begin{gathered}L_j=\begin{pmatrix}L_{1j}(I_n+G_1(k_j))^{-1} & O_{n\times n}\\ O_{n\times n} & O_{n\times n}\end{pmatrix}\ \ (j=1,\dots,n_{01}),\\ L_j=\begin{pmatrix}O_{n\times n} & O_{n\times n}\\ O_{n\times n} & L_{1\,j-n_{01}}\end{pmatrix}\ \ (j=n_{01}+1,\dots,n_0).\end{gathered}\tag{4.9}$$

Conversely, if the vector-function $x=(x_l)_{l=1}^{2}$, $x_l\in\mathrm{BV}([0,m_0],\mathbb{R}^n)$ $(l=1,2)$ is a solution of the problem (1.1), (1.2), where $A(t)$, $g(t)$, n_0, c_0, t_j $(j=1,\dots,n_0)$ and L_j $(j=1,\dots,n_0)$ are defined by (4.7)–(4.9), respectively, then the vector-function

$$y(k)=(I_n+G_1(k))^{-1}x_1(k)\ \ (k=0,\dots,m_0)\tag{4.10}$$

will be a solution of the problem (1.9), (1.10).

Let now $y\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$ be a solution of the problem (1.9), (1.12), and

$$\big[(I_n+G_1(k_i))^j\big]_{ii}\neq 0\ \ (j=-1,1;\ \ i=1,\dots,n)\,^{1)}.\tag{4.11}$$

Moreover, let $z(k)=(z_l(k))_{l=1}^{2}$, $z_l(k)=(z_{li}(k))_{i=1}^{n}$ $(l=1,2)$ be the vector-functions defined by (4.2). Then

$$z_{1i}(k_i)=\sum_{l=1}^{n}[I_n+G_1(k_i)]_{il}y_l(k_i)=$$

$$=[I_n+G_1(k_i)]_{ii}\big(\ell_{1i}(y_1,\dots,y_n)+c_{01i}\big)+\sum_{l=1,\,l\neq i}^{n}\big[I_n+G_1(k_i)\big]_{il}y_l(k_i)=$$

$$=[I_n+G_1(k_i)]_{ii}\big(\ell_{1i}(y_1,\dots,y_n)+c_{01i}\big)+$$

$$+\sum_{l=1,\,l\neq i}^{n}\big[I_n+G_1(k_i)\big]_{il}\big[(I_n+G_1(k_i))^{-1}\big]_{li}z_{1i}(k_i)+$$

[1)] Under $[X]_{il}$ we mean the element in the i-th row and in the l-th column of the matrix X.

$$+\sum_{l,j=1,\,l\neq i,\,j\neq i}^{n}\big[I_n+G_1(k_i)\big]_{il}\big[(I_n+G_1(k_i))^{-1}\big]_{lj}z_{1l}(k_i)=$$
$$=[I_n+G_1(k_i)]_{ii}\big(\ell_{1i}(y_1,\dots,y_n)+c_{01i}\big)+z_{1i}(k_i)-$$
$$-[I_n+G_1(k_i)]_{ii}\big[(I_n+G_1(k_i))^{-1}\big]_{ii}z_{1i}(k_i)+$$
$$+\sum_{l,j=1,\,l\neq i,\,j\neq i}^{n}\big[I_n+G_1(k_i)\big]_{il}\big[(I_n+G_1(k_i))^{-1}\big]_{lj}z_{1l}(k_i)\ \ (i=1,\dots,n),$$

whence, by (4.11), we conclude that

$$z_{1i}(k_i)=\Big(\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\Big)^{-1}\big(\ell_{1i}(y_1,\dots,y_n)+c_{01i}\big)+$$
$$+\Big((1+g_{1ii}(k_i))\cdot\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\Big)^{-1}\sum_{l,j=1,\,l\neq i,\,j\neq i}^{n}g_{1il}(k_i)\times$$
$$\times\big[(I_n+G_1(k_i))^{-1}\big]_{lj}z_{1l}(k_i)\ \ (i=1,\dots,n).\tag{4.12}$$

In view of (4.12), the vector-function $x=(x_l)_{l=1}^{2}$, $x_l=(x_{li})_{i=1}^{n}\in \mathrm{BV}([0,m_0],\mathbb{R}^n)$ $(l=1,2)$, is a solution of the $2n$-problem (1.1), (1.4), where

$$\begin{gathered}t_i=k_i,\ \ t_{n+i}=k_i\ \ (i=1,\dots,n),\\ c_{0i}=(1+g_{1ii}(k_i))c_{01i},\quad c_{0\,n+i}=c_{02i}\ \ (i=1,\dots,n);\end{gathered}\tag{4.13}$$

$$\ell_i(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})=\Big(\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\Big)^{-1}\ell_{1i}(y_1,\dots,y_n)+$$
$$+\Big((1+g_{1ii}(k_i))\cdot\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\Big)^{-1}\times$$
$$\times\sum_{l,j=1,\,l\neq i,\,j\neq i}^{n}g_{1il}(k_i)\cdot\big[(I_n+G_1(k_i))^{-1}\big]_{lj}x_{1l}(k_i)\ \ (i=1,\dots,n)\tag{4.14}$$

(the vector-function $y(k)=(y_i(k))_{i=1}^{n}$ is defined by (4.10)) and

$$\ell_{n+i}(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})=\ell_{2i}(y_1,\dots,y_n)\ \ (i=1,\dots,n);\tag{4.15}$$

here

$$y_i(k)\equiv x_{2i}(k-1)\ \ (i=1,\dots,n).$$

If we take

$$\ell_i(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})\equiv x_{1i}(k_i)\ \ (i=1,\dots,n),$$

and

$$\ell_{1i}(y_1,\dots,y_n)\equiv y_i(k_i)\ \ (i=1,\dots,n),$$

then from (4.14) we will get

$$x_{1i}(k_i)=\Big(\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\Big)^{-1}y_i(k_i)+$$
$$+\Big((1+g_{1ii}(k_i))\cdot\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\Big)^{-1}\times$$
$$\times\sum_{l,j=1,\,l\neq i,\,j\neq i}^{n}g_{1il}(k_i)\cdot\big[(I_n+G_1(k_i))^{-1}\big]_{lj}x_{1l}(k_i)\ \ (i=1,\dots,n)\tag{4.16}$$

and

$$\ell_i(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})-x_{1i}(k_i)=$$

$$=\left(\left[(I_n+G_1(k_i))^{-1}\right]_{ii}\right)^{-1}\cdot\left(\ell_{1i}(y_1,\dots,y_n)-y_i(k_i)\right)\ \ (i=1,\dots,n),\quad (4.17)$$

where $y(k)=(y_i(k))_{i=1}^n$ is defined by (4.10).

On the other hand, if the vector-function $x=(x_l)_{l=1}^2$, $x_l=(x_{li})_{i=1}^n\in\mathrm{BV}([0,m_0],\mathbb{R}^n)$ $(l=1,2)$ is a solution of the problem (1.1), (1.4), where $A(t)$, $f(t)$, t_i $(i=1,\dots,2n)$, c_{0i} $(i=1,\dots,2n)$ and ℓ_i $(i=1,\dots,2n)$ are defined by (4.7), (4.13)–(4.15), respectively, then in view of (4.17), the vector-function y defined by (4.10) will be a solution of the problem (1.9), (1.12).

As to the problem (1.9), (1.11), it is a particular case of the problem (1.9), (1.12) and is equivalent to the problem (1.1), (1.3) in the above-described sense.

Along with the difference system (4.3), we consider the corresponding homogeneous difference system

$$\Delta z(k-1)=G(k)z(k)\ \ (k=1,\dots,m_0).\quad (4.3_0)$$

By (1.30), (4.1) and the definitions of the matrix-functions $G(k)$ and $A(t)$, the matrix-function $Y(k)=(Y_{lm}(k))_{l,m=1}^2$, where $Y_{lm}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ $(l,m=1,2)$, are such that

$$\begin{pmatrix}Y_{11}(k) & Y_{12}(k)\\ Y_{21}(k) & Y_{22}(k)\end{pmatrix}=$$

$$=\prod_{i=k}^{0}\begin{pmatrix}O_{n\times n} & (I_n+G_1(i))\\ -G_3^{-1}(i) & G_3^{-1}(i)(I_n-G_2(i))\end{pmatrix}\ \ (k=0,\dots,m_0)\quad (4.18)$$

is a fundamental matrix of the system (4.3) satisfying the condition $Y(0)=I_{2n}$.

From (4.18), we have the following formulas for calculation of $Y_{lm}(k)$ $(l,m=1,2)$:

$$\begin{cases} Y_{lm}(0)=I_n\ \ (l,m=1,2),\\ Y_{1m}(k)=(I_n+G_1(k))Y_{2m}(k-1)\ \ (m=1,2;\ \ k=1,\dots,m_0),\\ Y_{2m}(k)=-G_3^{-1}(k)Y_{1m}(k-1)+\\ \quad +G_3^{-1}(k)(I_n-G_2(k))Y_{2m}(k-1)\ \ (m=1,2;\ \ k=1,\dots,m_0).\end{cases}\quad (4.19)$$

Relying now on (2.14), we construct the Green matrix for the problem $(1.9_0), (1.10_0)$.

Let $\mathcal{G}_{lm}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0}\times\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ $(l,m=1,2)$ be such that $\mathcal{G}(k,i)\equiv(\mathcal{G}_{lm}(k,i))_{l,m=1}^2$, where in view of (2.14),

$$\mathcal{G}(k,i)=$$

$$
= \begin{cases} Y(k)\Big(\sum\limits_{m=1}^{2}\sum\limits_{j\in\{1,\dots,n_{0m}:\ k_j<i-m+1\}} Z_j\Big)Y^{-1}(i) & \text{for } 0\le i<k\le m_0, \\ -Y(k)\Big(\sum\limits_{m=1}^{2}\sum\limits_{j\in\{1,\dots,n_{0m}:\ k_j\ge i-m+1\}} Z_j\Big)Y^{-1}(i) & \text{for } 0\le k<i\le m_0, \\ O_{2n\times 2n} & \text{for } 0\le k=i\le m_0 \end{cases} \tag{4.20}
$$

and $Z_j=(Z_{jlm})_{l,m=1}^{2}$ $(j=1,\dots,n_0)$ are defined by

$$
Z_j=\Big(\sum_{i=1}^{n_{01}}L_{1i}Y(k_i)+\sum_{i=1}^{n_{02}}L_{2i}Y(k_i+1)^{-1}\cdot L_{mj}Y(k_j+m-1)
$$
$$
(m=1,2;\ \ j=1+n_{01}(m-1),\dots,n_{01}+n_{02}(m-1)). \tag{4.21}
$$

The matrix-function $\mathcal{G}(k,i)$ is called the augmented Green matrix of the problem $(1.9_0),(1.10_0)$.

Under the Green matrix of the problem $(1.9_0),(1.10_0)$ we understand the matrix-function

$$
\mathcal{G}_*(k,i)\equiv-(I_n+G_1(k))^{-1}\mathcal{G}_{11}(k,i-1). \tag{4.22}
$$

Introduce the operator

$$
\begin{gathered} \big[(G_1,G_2,G_3)(k)\big]_0\equiv I_n,\quad \big[(G_1,G_2,G_3)(k)\big]_i\equiv \\ \equiv-\sum_{j=k+1}^{m_0}G(j)\big[(G_1,G_2,G_3)(j)\big]_{i-1}\ \ (i=1,2,\dots), \end{gathered} \tag{4.23}
$$

and the operators

$$
\begin{gathered} V_1(G_1,G_2,G_3)(k)\equiv\sum_{j=k+1}^{m_0}|G(j)|,\quad V_{i+1}(G_1,G_2,G_3)(k)\equiv \\ \equiv\sum_{j=k+1}^{m_0}|G(j)|\cdot V_i(G_1,G_2,G_3)(j)\ \ (i=1,2,\dots), \end{gathered} \tag{4.24}
$$

where the matrix-function $G(k)=(G_{ij}(k))_{i,j=1}^{2}$ is defined by (4.4).

Definition 4.1. Let $k_i,\,k_{i+1}\in\widetilde{\mathbb{N}}_{m_0}$ $(i=1,\dots,n)$, and let $G_1=(g_{1il})_{i,l=1}^{n}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ be a matrix-function satisfying (4.1). We say that the quadruple $(C_1,C_2,\ell_{01},\ell_{02})$ consisting of matrix-functions $C_j=(c_{jil})_{i,l=1}^{n}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ $(j=1,2)$ and positive homogeneous nondecreasing continuous operators $\ell_{0j}=(\ell_{0ji})_{i=1}^{n}:\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+^n)\to\mathbb{R}_+^n$ $(j=1,2)$ belongs to the set $U_{G_1}(k_1,\dots,k_n)$, if $c_{1il}(k)\ge0$ for $k\in\widetilde{\mathbb{N}}_{m_0}$ $(i\ne l;\ i,l=1,\dots,n)$, $c_{2il}(k)\ge0$ for $k\in\widetilde{\mathbb{N}}_{m_0}$ $(i,l=1,\dots,n)$, and the problem

$$
\Big(\Delta y_i(k-1)+\sum_{l=1}^{n}\big(g_{1il}(k)y_l(k)-g_{1il}(k-1)y_l(k-1)\big)\Big)\operatorname{sgn}\Big(k-k_i-\frac12\Big)\le
$$

$$\leq \sum_{l,j=1}^{n} c_{1il}(k)\big(\delta_{jl}+g_{1il}(k)\big)y_l(k)+$$

$$+\sum_{l=1}^{n} c_{2il}(k)y_l(k+1) \text{ for } k\in\{1,\dots,m_0-1\} \ (i=1,\dots,n) \tag{4.25}$$

has no nontrivial nonnegative solution satisfying the conditions

$$y_i(k_i+m-1)\leq \ell_{0mi}(y_1,\dots,y_n) \ (m=1,2;\ i=1,\dots,n). \tag{4.26}$$

The definition of the set $U_{G_1}(k_1,\dots,k_n)$ is based on that of the set $U(t_1,\dots,t_n)$ (see Definition 1.1) for the corresponding generalized ordinary differential system. In this case, i.e. under the definitions (4.2)–(4.7), (4.13)–(4.15), the system (1.9) contains $2n$ inequalities, and as to the matrix-function $C=(C_{jm})_{j,m=1}^2$, we take

$$\begin{gathered}
C_{11}(t)=\sum_{k=0}^{[t+\frac12]}C_1(k),\ \ C_{12}(t)=\sum_{k=0}^{[t+\frac12]}C_2(k),\ \ C_{21}(t)=\sum_{k=0}^{[t+\frac12]}(I_n+G_1(k))^{-1},\\
C_{22}(t)=\sum_{i=k}^{[t+\frac12]}\operatorname{diag}\big(\operatorname{sgn}(k-k_1-1),\dots,\operatorname{sgn}(i-k_n-1)\big).
\end{gathered} \tag{4.27}$$

The definitions of the matrix-functions G_{21} and G_{22} differ from those given in the proof of Theorem 4.7 (see 7.32).

4.1. **Solvability of the Problem (1.9), (1.10).**

Theorem 4.1. *The boundary value problem* (1.9), (1.10) *has a unique solution if and only if the corresponding homogeneous problem* $(1.9_0), (1.10_0)$ *has only the trivial solution. If the latter condition holds, then the solution* y *of the problem* (1.9), (1.10) *admits the representation*

$$y(k)=y_0(k)+\sum_{i=1}^{m_0}\mathcal{G}_*(k,i)g_0(i) \ \ (k=0,\dots,m_0), \tag{4.28}$$

where y_0 *is a solution of the problem* $(1.9_0), (1.10)$*, and the matrix-function* $\mathcal{G}_*(k,i)$ *defined by* (4.20)–(4.22) *is the Green matrix of the problem* (1.9_0), (1.10_0).

Remark 4.1. If the homogeneous problem $(1.9_0), (1.10_0)$ has a nontrivial solution, then for every $g_0\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$ there exists a vector $c_0=(c_{0m})_{m=1}^2$ such that the problem (1.9), (1.10) has no solution.

Corollary 4.1. *The boundary value problem* (1.9), (1.10) *is uniquely solvable if and only if*

$$\det\begin{pmatrix} \sum\limits_{j=1}^{n_{01}} L_{1j}(I_n+G_1(k_j))^{-1}Y_{11}(k_j) & \sum\limits_{j=1}^{n_{01}} L_{1j}(I_n+G_1(k_j))^{-1}Y_{12}(k_j) \\ \sum\limits_{j=1}^{n_{02}} L_{2j}Y_{21}(k_j+1) & \sum\limits_{j=1}^{n_{02}} L_{2j}Y_{22}(k_j+1) \end{pmatrix} \neq$$
$$\neq 0, \tag{4.29}$$

where the $n\times n$-matrix-functions $Y_{lm}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0}\mathbb{R}^{n\times n})$ $(l,m=1,2)$ are defined by (4.19).

Theorem 4.2. *The boundary value problem* (1.9), (1.10) *is uniquely solvable if and only if there exist natural numbers k and m such that the matrix $M_k=(M_{klj})_{l,j=1}^2$ is nonsingular, and*

$$r(M_{k,m})<1, \tag{4.30}$$

where

$$M_{k11}=\sum_{j=1}^{n_{01}}\sum_{i=0}^{k-1} L_{1j}(I_n+G_1(k_j))^{-1}\big[(G_1,G_2,G_3)(k_j)\big]_{i11},$$
$$M_{k12}=\sum_{j=1}^{n_{01}}\sum_{i=0}^{k-1} L_{1j}(I_n+G_1(k_j))^{-1}\big[(G_1,G_2,G_3)(k_j)\big]_{i12},$$
$$M_{k21}=\sum_{j=1}^{n_{02}}\sum_{i=0}^{k-1} L_{2j}\big[(G_1,G_2,G_3)(k_j+1)\big]_{i21},$$
$$M_{k22}=\sum_{j=1}^{n_{02}}\sum_{i=0}^{k-1} L_{2j}\big[(G_1,G_2,G_3)(k_j+1)\big]_{i22},$$

the $n\times n$-matrix-functions $\big[(G_1,G_2,G_3)(\cdot)\big]_{ilj}$ $(l,j=1,2;\ i=0,\dots,k-1)$ are such that $\big[(G_1,G_2,G_3)(\varkappa)\big]_i\equiv(\big[(G_1,G_2,G_3)(\varkappa)\big]_{ilj})_{l,j=1}^2$,

$$M_{k,m}=V_m(G_1,G_2,G_3)(0)+$$
$$+\Big(\sum_{i=0}^{m-1}\big|\big[(G_1,G_2,G_3)(\cdot)\big]_i\big|_{\widetilde{\mathbb{N}}_{m_0}}\Big)\Big(\sum_{j=1}^{n_{01}}|M_k^{-1}L_j|V_k(G_1,G_2,G_3)(k_j)+$$
$$+\sum_{j=n_{01}+1}^{n_{01}+n_{02}}|M_k^{-1}L_j|V_k(G_1,G_2,G_3)(k_j+1)\Big),$$

$L_j=(L_{jil})_{i,l=1}^2$ $(j=1,\dots,n_{01}+n_{02})$; $L_{j11}=L_{1j}(I_n+G_1(k_j))^{-1}$, $L_{j12}=L_{j21}=L_{j22}=O_{n\times n}$ $(j=1,\dots,n_{01})$; $L_{j11}=L_{j12}=L_{j21}=O_{n\times n}$, $L_{j22}=L_{2\,j-n_{01}}$ $(j=n_{01}+1,\dots,n_{01}+n_{02})$; and the matrix-functions $[(G_1,G_2,G_3)(\varkappa)]_i$ and $V_i(G_1,G_2,G_3)(\varkappa)$ are defined by (4.23) *and* (4.24), *respectively.*

Corollary 4.2. *Let*

$$\det\Big(\sum_{j=1}^{n_{01}} L_{1j}(I_n+G_1(k_j))^{-1}\Big)\neq 0, \quad \det\Big(\sum_{j=1}^{n_{02}} L_{2j}\Big)\neq 0 \tag{4.31}$$

and

$$r(L_0M_0)<1,$$

where

$$L_0=\begin{pmatrix} L_{01} & O_{n\times n} \\ O_{n\times n} & L_{02}\end{pmatrix},$$

$$L_{01}=I_n+\Big|\Big(\sum_{j=1}^{n_{01}} L_{1j}(I_n+G_1(k_j))^{-1}\Big)^{-1}\Big|\cdot\sum_{j=1}^{n_{01}}\big|L_{1j}(I_n+G_1(k_j))^{-1}\big|,$$

$$L_{02}=I_n+\Big|\Big(\sum_{j=1}^{n_{02}} L_{2j}\Big)^{-1}\Big|\cdot\sum_{j=1}^{n_{02}}|L_{2j}|$$

and

$$M_0=\sum_{i=1}^{m_0}\begin{pmatrix} \big|(G_1(i)+G_2(i))(I_n+G_1(i))^{-1}\big| & |G_3(i)| \\ \big|(I_n+G_1(i))^{-1}\big| & I_n\end{pmatrix}.$$

Then the problem (1.9), (1.10) *has one and only one solution.*

Corollary 4.3. *Let either the condition* (4.31) *hold, or there exist a natural number k such that the conditions*

$$\sum_{j=1}^{n_{01}} L_{1j}(I_n+G_1(k_j))^{-1}=O_{n\times n}, \quad \sum_{j=1}^{n_{02}} L_{2j}=O_{n\times n},$$

$$\det M_i=0 \quad (i=0,\dots,k-1)$$

and

$$\det M_k\neq 0$$

hold, where $M_i=(M_{ilj})_{l.j=1}^{2}$ $(i=0,\dots,k)$,

$$M_{i11}=\sum_{j=1}^{n_{01}} L_{1j}(I_n+G_1(k_j))^{-1}\big[(G_1,G_2,G_3)(k_j)\big]_{i11},$$

$$M_{i12}=\sum_{j=1}^{n_{01}} L_{1j}(I_n+G_1(k_j))^{-1}\big[(G_1,G_2,G_3)(k_j)\big]_{i12},$$

$$M_{i21}=\sum_{j=1}^{n_{02}} L_{2j}\big[(G_1,G_2,G_3)(k_j+1)\big]_{i21},$$

$$M_{i22}=\sum_{j=1}^{n_{02}} L_{2j}\big[(G_1,G_2,G_3)(k_j+1)\big]_{i22}$$

and the $n\times n$ *matrix-functions* $[(G_1,G_2,G_3)(\varkappa)]_{ilj}$ $(l,j=1,2;\ i=0,\dots,k)$ *are defined in Theorem* 4.2*. Then there exists* $\varepsilon_0>0$ *such that the system*

$$\Delta y(k-1)=\varepsilon\big(G_1(k-1)y(k-1)+G_2(k)y(k)+G_3(k)y(k+1)\big)+g_0(k)\quad (k=1,\dots,m_0)$$

has a solution satisfying the condition (1.10) *for every* $\varepsilon\in\,]0,\varepsilon_0[$.

Theorem 4.3. *Let the matrix-functions* $G_{0j}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ $(j=1,2,3)$ *be such that*

$$\det(I_n+G_{01}(k-1))\neq 0,\ \ \det G_{03}(k)\neq 0\ \ (k=1,\dots,m_0),$$

the homogeneous system

$$\begin{aligned}\Delta y(k-1)=G_{01}(k-1)y(k-1)+G_{02}(k)y(k)+\\ +G_{03}(k)y(k+1)\ \ (k=1,\dots,m_0)\end{aligned}\tag{4.32}$$

have only the trivial solution satisfying the boundary condition (1.10_0)*, and let the matrix-functions* $G_j\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ $(j=1,2,3)$ *admit the estimates*

$$\begin{aligned}\sum_{i=1}^{m_0}\Big|\big[\mathcal{G}_{0j1}(k,i-1)(G_1(i)+G_2(i))-\mathcal{G}_{0j2}(k,i-1)\big](I_n+G_1(i))^{-1}-\\ -\big[\mathcal{G}_{0j1}(k,i-1)(G_{01}(i)+G_{02}(i))-\mathcal{G}_{0j2}(k,i-1)\big](I_n+G_{01}(i))^{-1}\Big|\leq M_{j1}\\ (j=1,2),\end{aligned}$$

$$\sum_{i=1}^{m_0}\big|\mathcal{G}_{0j1}(k,i-1)(G_3(i)-G_{03}(i))\big|\leq M_{j2}\ \ (j=1,2),$$

where $\mathcal{G}_0(k,i)=(\mathcal{G}_{0jl}(k,i))_{j,l=1}^2$ *is the augmented Green matrix of the problem* (4.32)*,* (1.10_0)*, and* $M_{jl}\in\mathbb{R}_+^{n\times n}$ $(j,l=1,2)$ *are constant matrices such that*

$$r\big((M_{jl})_{j,l=1}^2\big)<1.$$

Then the problem (1.9)*,* (1.10) *has one and only one solution.*

4.2. **Solvability of the Problems (1.9), (1.11) and (1.9), (1.12).**

Theorem 4.4. *Let the matrix-function* G_1 *be such that*

$$\begin{aligned}(I_n+G_1(k))^{-1}>O_{n\times n}\ \ \textit{and}\ \ g_{1ii}(k)\neq -1\\ \textit{for}\ \ k\in\{0,\dots,m_0\}\ \ (i=1,\dots,n),\end{aligned}\tag{4.33}$$

and let there exist matrix-functions $C_j=(c_{jil})_{i,l=1}^n$ $(j=1,2)$ *and positive homogeneous nondecreasing continuous operators* $\ell_{0j}=(\ell_{0ji})_{i=1}^n$ $(j=1,2)$ *satisfying the condition*

$$(C_1,C_2,\ell_{01},\ell_{02})\in U_{G_1}(k_1,\dots,k_n)\tag{4.34}$$

such that

$$(-1)^{j+1}\Big(\Big|\big[(I_n-G_2(k))(I_n+G_1(k))^{-1}\big]_{ii}\Big|-1\Big)\leq c_{1ii}(k)$$

$$\text{for } (-1)^j(k-k_i)\geq 0 \ \ (j=1,2;\ \ i=1,\dots,n), \tag{4.35}$$

$$\Big|\big[(G_1(k)+G_2(k))(I_n+G_1(k))^{-1}\big]_{il}\Big|\leq c_{1il}(k)$$
$$\text{for } k\in\{0,\dots,m_0\} \ \ (i\neq l;\ \ i,l=1,\dots,n) \tag{4.36}$$

and

$$|g_{3il}(k)|\leq c_{2il}(k) \ \ \text{for } k\in\{1,\dots,m_0\} \ \ (i,l=1,\dots,n). \tag{4.37}$$

Let, moreover,

$$\big|\ell_{ji}(y_1,\dots,y_n)\big|\leq \ell_{0ji}\big(|y_1|,\dots,|y_n|\big)$$
$$\text{for } (y_l)_{l=1}^n\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n) \ \ (j=1,2;\ \ i=1,\dots,n). \tag{4.38}$$

Then the problem (1.9), (1.12) *has one and only one solution.*

Theorem 4.5. *Let the conditions* (4.33),

$$(-1)^{j+1}\Big(\Big|\big[(I_n-G_2(k))(I_n+G_1(k))^{-1}\big]_{ii}\Big|-1\Big)\leq h_{1ii}(k)$$
$$\text{for } (-1)^j(k-k_i)\geq 0 \ \ (j=1,2;\ \ i=1,\dots,n), \tag{4.39}$$

$$\Big|\big[\big(G_1(k)+G_2(k)\big)(I_n+G_1(k))^{-1}\big]_{il}\Big|\leq h_{1il}(k)$$
$$\text{for } k\in\{0,\dots,m_0\} \ \ (i\neq l;\ \ i,l=1,\dots,n) \tag{4.40}$$

and

$$|g_{3il}(k)|\leq h_{2il}(k) \ \ \text{for } k\in\{1,\dots,m_0\} \ \ (i,l=1,\dots,n) \tag{4.41}$$

hold, where $h_{1ii}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R})$, $h_{1il}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+)$ $(i\neq l)$, $h_{2il}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+)$ $(i,l=1,\dots,n)$. *Let, moreover,*

$$\Big|\ell_{1i}(y_1,\dots,y_n)+\big[I_n+G_1(k_i)\big]_{ii}\sum_{l=1}^n\big[I_n+G_1(k_i)\big]_{il}y_l(k_i)-y_i(k_i)\Big|\leq$$
$$\leq\sum_{l=1}^n\gamma_{1il}\Big\|\sum_{j=1}^n\big[I_n+G_1(\cdot)\big]_{lj}y_j(\cdot)\Big\|_\nu^{1)}$$
$$\text{for } (y_l)_{l=1}^n\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n) \ \ (i=1,\dots,n), \tag{4.42}$$

$$\big|\ell_{2i}(y_1,\dots,y_n)\big|\leq\sum_{l=1}^n\gamma_{2il}\|y_l\|_\nu$$
$$\text{for } (y_l)_{l=1}^n\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n) \ \ (i=1,\dots,n), \tag{4.43}$$

and

$$r(\mathcal{H}^*)<1, \tag{4.44}$$

---

1) Here $\|y\|_\nu=\big(\sum\limits_{k=0}^{m_0}\|y(k)\|^\nu\big)^{\frac{1}{\nu}}$ for $y\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$.

where γ_{1il}, $\gamma_{2il} \in \mathbb{R}_+$ $(i,l = 1,\dots,n)$, $\nu \geq 2$, *and the* $2n \times 2n$*-matrix* $\mathcal{H}^* = (\mathcal{H}^*_{jm})^2_{j,m=1}$ *is defined by*

$$\mathcal{H}^*_{11} = \Big(\xi_0\big(\big[(I_n + G_1(k_i))^{-1}\big]_{ii}\big)^{-1}\gamma_{1il} + \lambda_0\|h_{1il}\|_\mu\Big)^n_{i,l=1},$$

$$\mathcal{H}^*_{12} = \big(\lambda_0\|h_{2il}\|_\mu\big)^n_{i,l=1}, \quad \mathcal{H}^*_{21} = \big(\lambda_0\big\|\big[(I_n + G_1(\cdot))^{-1}\big]_{il}\big\|_\mu\big)^n_{i,l=1},$$

$$\mathcal{H}^*_{22} = (\xi_0\gamma_{2il})^n_{i,l=1} + \lambda_0 m_0^{\frac{1}{\mu}} I_n,$$

$\frac{1}{\mu} + \frac{2}{\nu} = 1$, $\xi_0 = m_0^{\frac{1}{\nu}}$, $\lambda_0 = \big(\frac{1}{2}\sin^{-1}\frac{\pi}{4m_0+2}\big)^{\frac{2}{\nu}}$. *Then the problem* (1.9), (1.12) *has one and only one solution.*

Corollary 4.4. *Let* $G_1(k) \equiv \operatorname{diag}(g_{11}(k),\dots,g_{1n}(k))$ *be the diagonal matrix-function such that the conditions*

$$g_{1i}(k) > -1 \quad \textit{for} \;\; k \in \{0,\dots,m_0\} \;\; (i = 1,\dots,n), \tag{4.45}$$

$$(-1)^{j+1}\Big((1+g_{1i}(k))^{-1}|1 - g_{2ii}(k)| - 1\Big) \leq h_{1ii}(k)$$
$$\textit{for} \;\; (-1)^j(k - k_i) \geq 0 \;\; (j = 1,2; \;\; i = 1,\dots,n),$$
$$(1+g_{1l}(k))^{-1}\big|g_{1il}(k) + g_{2il}(k)\big| \leq h_{1il}(k)$$
$$\textit{for} \;\; k \in \{0,\dots,m_0\} \;\; (i \neq l; \;\; i,l = 1,\dots,n)$$

and (4.41) *hold, where* $h_{1ii} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R})$, $h_{1il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+)$ $(i \neq l)$, $h_{2il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+)$ $(i,l = 1,\dots,n)$. *Let, moreover,*

$$\big|\ell_{ji}(y_1,\dots,y_n)\big| \leq \sum_{l=1}^n \gamma_{1il}\big\|(1+g_{1l}(\cdot))^{2-j}y_l(\cdot)\big\|_\nu$$
$$\textit{for} \;\; (y_l)^n_{l=1} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n) \;\; (j = 1,2; \;\; i = 1,\dots,n)$$

and the inequality (4.44) *hold, where* γ_{1il}, $\gamma_{2il} \in \mathbb{R}_+$ $(i,l = 1,\dots,n)$, $\nu \geq 2$, $\mathcal{H}^* = (\mathcal{H}^*_{jm})^2_{j,m=1}$, *the matrices* $\mathcal{H}^*_{11}$, $\mathcal{H}^*_{12}$, $\mathcal{H}^*_{22}$ *and numbers* μ, ξ_0, λ_0 *are defined as in Theorem* 4.5, *and*

$$\mathcal{H}^*_{21} = \lambda_0 \operatorname{diag}\Big(\big\|(1+g_{11}(\cdot))^{-1}\big\|_\mu, \dots, \big\|(1+g_{1n}(\cdot))^{-1}\big\|_\mu\Big).$$

Then the conclusion of Theorem 4.5 *is true.*

Corollary 4.5. *Let the conditions* (4.33), (4.39)–(4.41),

$$\big[(I_n + G_1(k_i))^{-1}\big]_{il} = 0 \;\; (i \neq l; \;\; i,l = 1,\dots,n) \tag{4.46}$$

and

$$r(\mathcal{H}_0) < \Big(2\sin\frac{\pi}{4m_0+2}\Big)^{\frac{2}{\nu}} \tag{4.47}$$

hold, where $h_{1ii} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R})$, $h_{1il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+)$ $(i \neq l)$, $h_{2il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+)$ $(i,l = 1,\dots,n)$, $\nu \geq 2$,

$$\mathcal{H}_0 = \begin{pmatrix} \big(\|h_{1il}\|_\mu\big)^n_{i,l=1} & \big(\|h_{2il}\|_\mu\big)^n_{i,l=1} \\ \big(\big\|\big[(I_n + G_1(\cdot))^{-1}\big]_{il}\big\|_\mu\big)^n_{i,l=1} & m_0^{\frac{1}{\nu}} I_n \end{pmatrix},$$

and $\frac{1}{\mu}+\frac{2}{\nu}=1$. *Then the problem* (1.9), (1.11) *has one and only one solution.*

Corollary 4.6. *Let the conditions* (4.33), (4.46),

$$(-1)^{j+1}\Big(\Big|\big[(I_n-G_2(k))(I_n+G_1(k))^{-1}\big]_{ii}-1\Big|\Big)\leq h_{1ii}$$
$$\text{for } (-1)^j(k-k_i)\geq 0 \ \ (j=1,2;\ \ i=1,\dots,n),$$
$$\Big|\big[(G_1(k)+G_2(k))(I_n+G_1(k))^{-1}\big]_{il}\Big|\leq h_{1il}$$
$$\text{for } k\in\{0,\dots,m_0\} \ \ (i\neq l;\ \ i,l=1,\dots,n),$$
$$|g_{3il}(k)|\leq h_{2il} \ \ \text{for } k\in\{0,\dots,m_0\} \ \ (i,l=1,\dots,n)$$

and

$$\big[(I_n+G_1(k))^{-1}\big]_{il}\leq h_{3il} \ \ \text{for } k\in\{0,\dots,m_0\} \ \ (i,l=1,\dots,n)$$

hold, where $h_{1ii}\in\mathbb{R}$, $h_{1il}\in\mathbb{R}_+$ $(i\neq l)$, $h_{2il}\in\mathbb{R}_+$ $(i,l=1,\dots,n)$. *Let, moreover,*

$$r(\mathcal{H}_0)<m_0^{-\frac{1}{\mu}}\Big(2\sin\frac{\pi}{4m_0+2}\Big)^{\frac{2}{\nu}}, \tag{4.48}$$

where $\mu\geq 1$, $\nu\geq 2$, $\frac{1}{\mu}+\frac{2}{\nu}=1$,

$$\mathcal{H}_0=\begin{pmatrix}(h_{1il})_{i,l=1}^n & (h_{2il})_{i,l=1}^n\\ (h_{3il})_{i,l=1}^n & I_n\end{pmatrix}.$$

Then the problem (1.9), (1.11) *has one and only one solution.*

Let $a\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R})$. On the basis of Definition 1.22, we introduce the following function:

$$\gamma_a(j,k)=\begin{cases}\displaystyle\prod_{l=k+1}^{j}\big(1-\Delta a(l-1)\big)^{-1} & \text{for } j>k,\\ \displaystyle\prod_{l=j+1}^{k}\big(1-\Delta a(l-1)\big) & \text{for } j<k,\\ 1 & \text{for } j=k.\end{cases} \tag{4.49}$$

Theorem 4.6. *Let the conditions*

$$(-1)^{j+1}\Big(\Big|\big[(I_n-G_2(k))(I_n+G_1(k))^{-1}\big]_{ii}\Big|-1\Big)\leq h_{1ii}\beta_i(k)+\beta_{11ii}(k)$$
$$\text{for } (-1)^j(k-k_i)\geq 0 \ \ (j=1,2;\ \ i=1,\dots,n),$$
$$\Big|\big[(G_1(k)+G_2(k))(I_n+G_1(k))^{-1}\big]_{il}\Big|\leq h_{1il}\beta_i(k)+\beta_{11il}(k)$$
$$\text{for } k\in\{0,\dots,m_0\} \ \ (i\neq l;\ \ i,l=1,\dots,n),$$
$$|g_{3il}(k)|\leq h_{2il}\beta_i(k)+\beta_{12il}(k) \ \ \text{for } k\in\{0,\dots,m_0\} \ \ (i,l=1,\dots,n),$$
$$\big|\big[(I_n+G_1(k))^{-1}\big]_{il}\big|\leq \beta_{21il}(k) \ \ \text{for } k\in\{0,\dots,m_0\} \ \ (i,l=1,\dots,n),$$
$$\Big|\ell_{1i}(y_1,\dots,y_n)+\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\sum_{l=1}^{n}\big[I_n+G_1(k_i)\big]_{il}y_l(k_i)-y_i(k_i)\Big|\leq$$

$$\leq |\mu_i| \Big| \sum_{l=1}^{n} \big[I_n + G_1(m_i)\big]_{il} y_l(m_i) \Big|$$

$$\textit{for } (y_l)_{l=1}^n \in \mathbb{N}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n) \ (i = 1, \dots, n), \tag{4.50}$$

$$\big|\ell_{2i}(y_1, \dots, y_n)\big| \leq |\mu_{2i}| \, \big|y_i(m_i + 1)\big|$$

$$\textit{for } (y_l)_{l=1}^n \in \mathbb{N}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n) \ (i = 1, \dots, n), \tag{4.51}$$

$$\beta_i(k_i) \leq 0, \ \ 0 \leq \beta_i(k) < |\eta_i|^{-1}$$

$$\textit{for } k \in \{k_i + 1, \dots, m_0\} \ (i = 1, \dots, n)$$

and

$$|\mu_{1i}|\gamma_i(m_i, k_i) < 1, \quad |\mu_{2i}| < 1 \ (i = 1, \dots, n)$$

hold, where $h_{1ii} < 0$, $h_{1il} \geq 0$ $(i \neq l)$, $h_{2il} \geq 0$ $(i, l = 1, \dots, n)$*;* $\mu_{ji} \in \mathbb{R}$, $\eta_i < 0$, $m_i \in \widetilde{\mathbb{N}}_{m_0}$, $m_i \neq k_i$ $(i = 1, \dots, n)$*;* $\beta_{11ii}, \beta_{21il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}_+)$ $(i, l = 1, \dots, n)$*;* $\beta_{1jil}(k) \geq 0$ $(j = 1, 2;\ i \neq l)$ *and* $\beta_i(k) \geq 0$ *for* $k \in \widetilde{\mathbb{N}}_{m_0} \setminus \{k_i\}$ $(i, l = 1, \dots, n)$*;* $\gamma_i(m, k) \equiv \gamma_{a_i}(m, k)$*, the function* γ_{a_i} *is defined by* (4.49)*, and* $a_i(k) \equiv \eta_i(\alpha_i(k) - \alpha_i(k_i)) \operatorname{sgn}(k - k_i)$*,* $\alpha_i(k) \equiv \sum\limits_{l=0}^{k} \beta_i(l)$ $(i = 1, \dots, n)$*. Let, moreover,*

$$g_{ii} < 1 \ \ (i = 1, \dots, 2n) \tag{4.52}$$

and the real part of every characteristic value of the $2n \times 2n$*-matrix* $(\xi_{il})_{i,l=1}^{2n}$ *be negative, where*

$$\xi_{ii} = h_{1ii}(1 - g_{ii}), \quad \xi_{il} = h_{1+j\,il} h_i - h_{1ii} g_{il}$$

$$(j = 0, 1;\ \ i \neq l;\ \ i = 1, \dots, n;\ \ l = nj + 1, \dots, nj + n),$$

$$\xi_{il} = g_{il} - j\delta_{il} \ \ (j = 0, 1;\ \ i = n + 1, \dots, 2n;\ \ l = nj + 1, \dots, nj + n),$$

$$g_{il} = |\mu_{1+ji}|\big(1 - |\mu_{1+j\,i}|\gamma_i^{1-j}(m_i, k_i)\big)^{-1} \gamma_{il}(m_i) +$$

$$+ \max\big\{\gamma_{il}(0), \gamma_{il}(m_0)\big\} \ \ (j = 0, 1;\ \ i = nj + 1, \dots, nj + n;\ \ l = 1, \dots, 2n);$$

$$\gamma_{il}(k_i + \mu) = 0 \ \ (\mu = 0, 1;\ \ i = n\mu + 1, \dots, n\mu + n;\ \ l = 1, \dots, 2n),$$

$$\gamma_{il}(k) = \big|\alpha_{il}(k) - \alpha_{il}(k_i + \mu)\big| - (1 - \delta_{il})\chi_{[0, k_i + \mu]}(k)\beta_{1+\mu\,1+\nu\,il}(k_i + \mu)$$

$$\textit{for } k \in \widetilde{\mathbb{N}}_{m_0} \setminus \{k_i + \mu\} \ \ (\mu, \nu = 0, 1;\ \ \mu + \nu \leq 1;$$

$$i = n\mu + 1, \dots, n\mu + n;\ \ l = n\nu + 1, \dots, n\nu + n),$$

$$\gamma_{il}(k) = (k - k_i - 1)\big(1 - \chi_{[0, k_i]}(k)\big) + \delta_{il}\chi_{[0, k_i]}(k)$$

$$\textit{for } k \in \widetilde{\mathbb{N}}_{m_0} \setminus \{k_i + 1\} \ \ (i, l = n + 1, \dots, 2n);$$

$$\alpha_{il}(k) \equiv \sum_{m=0}^{k} \beta_{1+\mu\,1+\nu\,il}(m)$$

$$(\mu, \nu = 0, 1;\ \ \mu + \nu \leq 1;\ \ i = n\mu + 1, \dots, n\mu + n;\ \ l = n\nu + 1, \dots, n\nu + n);$$

$$h_i = 1 \ \textit{ if } |\mu_{1i}| \leq 1, \ \ \textit{and}$$

$$h_i = 1 + (|\mu_{1i}| - 1)\big(1 - |\mu_{1i}|\gamma_i(m_i, k_i)\big)^{-1} \ \textit{ if } |\mu_{1i}| > 1 \ \ (i = 1, \dots, n).$$

Then the problem (1.9), (1.12) *has one and only one solution.*

Theorem 4.7. *Let the condition* (4.33) *hold, and let* $\ell_{0ji} : \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n_+) \to \mathbb{R}_+$ $(j = 1, 2;\ i = 1, \dots, n)$ *be linear continuous functionals, the matrix-functions* $C_j = (c_{jil})_{i,l=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n})$ $(j = 1, 2)$ *be such that* $c_{jil}(k) \geq 0$ *for* $k \in \widetilde{\mathbb{N}}_{m_0}$ $(j = 1, 2;\ i \neq l;\ i, l = 1, \dots, n)$*, and the problem* (4.25), (4.26) *have a nontrivial nonnegative solution* $y = (y_i)_{i=1}^n$*, i.e. the condition* (4.34) *be violated. Let. moreover,*

$$c_{1ii}(k) \geq 0 \quad \textit{for} \quad k \in \{0, \dots, m_0\} \ \ (i = 1, \dots, n) \tag{4.53}$$

and

$$\det(C_2(k)) \neq 0 \quad \textit{for} \quad k \in \{0, \dots, m_0\}. \tag{4.54}$$

Then there exist matrix-functions G_2, $G_3 \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n})$*, linear continuous functionals* $\ell_{mi} : \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n) \to \mathbb{R}$ $(m = 1, 2;\ i = 1, \dots, n)$ *and numbers* $c_{0mi} \in \mathbb{R}$ $(m = 1, 2;\ i = 1, \dots, n)$ *such that the conditions* (4.1), (4.35)–(4.38) *are fulfilled, but the problem* (1.9_0), (1.12) *is unsolvable.*

5. Boundary Value Problems for the Difference System (1.13)

As we noted above (see Remark 1.1), the results obtained for the system (1.9) in Section 4 cannot be extended to the system (1.13) automatically because in these results the condition

$$\det G_3(k) \not\equiv 0 \ \text{ for } \ k \in \{1, \dots, m_0\},$$

that is, $G_3(k) \neq 0$ for $k \in \{1, \dots, m_0\}$ is always required, and we cannot consider the system (1.9) for $G_3(k) \equiv 0$.

Thus the systems (1.9) and (1.13) may have different properties. This conclusion is based on the following arguments.

The general solution of the homogeneous system (1.9_0) contains $2n$ constants. Therefore for these constants to be equal to zero, it is necessary to have boundary conditions with $2n$ equalities. In this connection, we consider the boundary conditions (1.10)-(1.12) and (1.10_0)-(1.12_0) consisting of $2n$ equalities.

Unlike the system (1.9_0), the general solution of the system (1.13_0) contains n constants. In this regard, we consider the boundary conditions (1.14)-(1.16) and (1.14_0)-(1.16_0) containing n equalities.

To apply the results of Section 2, we construct a system of the form (1.1) corresponding to the difference system (1.13).

In this section we assume that $G_l = (g_{lij})_{i,j=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^{n\times n})$ $(l = 1, 2)$, $g_0 = (g_{0i})_{i=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ and

$$\det(I_n + G_1(k)) \neq 0, \quad \det(I_n - G_2(k)) \neq 0 \quad (k = 0, \dots, m_0). \tag{5.1}$$

Let $y \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ be a solution of the difference system (1.13). Then the vector-function

$$z(k) = (I_n + G_1(k))y(k) \quad (k = 0, \dots, m_0) \tag{5.2}$$

is a solution of the difference system

$$\Delta z(k-1) = G(k)z(k) + g_0(k) \ \ (k = 1, \dots, m_0), \tag{5.3}$$

where

$$G(k) \equiv (G_1(k) + G_2(k))(I_n + G_1(k))^{-1}. \tag{5.4}$$

Conversely, if the function z is a solution of the system (5.3), then due to (5.1), the function

$$y(k) = (I_n + G_1(k))^{-1} z(k) \ \ (k = 0, \dots, m_0)$$

is a solution of the system (1.13).

On the other hand, the vector-function $z \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ is a solution of the difference system (5.3) if and only if the vector-function

$$x(t) = z\Big(\Big[t + \frac{1}{2}\Big]\Big) \ \text{ for } \ t \in [0, m_0] \tag{5.5}$$

is a solution of the system (1.1), where

$$\begin{aligned} A(t) &= \sum_{i=0}^{[t+\frac{1}{2}]} (G_1(i) + G_2(i))(I_n + G_1(i))^{-1}, \\ f(t) &= \sum_{i=0}^{[t+\frac{1}{2}]} g_0(i) \ \text{ for } \ t \in [0, m_0]. \end{aligned} \tag{5.6}$$

It should be noted that in this case the condition (1.21) is of the form (5.1).

Consider now the boundary value problems.

If $y \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ is a solution of the problem (1.13), (1.14), then the vector-function $x \in \mathrm{BV}([0, m_0], \mathbb{R}^n)$ defined by (5.2) and (5.5) is a solution of the problem (1.1), (1.2), where

$$t_j = k_j \ \ (j = 1, \dots, n), \tag{5.7}$$

$$L_j = L_{1j}(I_n + G_1(k_j))^{-1} \ \ (j = 1, \dots, n). \tag{5.8}$$

Let now $y \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n)$ be a solution of the problem (1.13), (1.16), and

$$\big[(I_n + G_1(k_i))^j\big]_{ii} \neq 0 \ \ (j = -1, 1; \ \ i = 1, \dots, n). \tag{5.9}$$

Moreover, let $z(k) = (z_i(k))_{i=1}^n$ be the vector-function defined by (5.2). Then due to (4.11), just as in Section 4, we conclude that

$$\begin{aligned} z_i(k_i) = \big(\big[(I_n + G_1(k_i))^{-1}\big]_{ii}\big)^{-1}\big(\ell_{1i}(y_1, \dots, y_n) + c_{01i}\big) + \\ + \Big((1 + g_{1ii}(k_i)) \cdot \big[(I_n + G_1(k_i))^{-1}\big]_{ii}\Big)^{-1} \times \\ \times \sum_{l,j=1,\, l\neq i,\, j\neq i}^{n} g_{1il}(k_i) \cdot \big[(I_n + G_1(k_i))^{-1}\big]_{lj} z_l(k_i) \ \ (i = 1, \dots, n). \end{aligned} \tag{5.10}$$

In view of (5.10), the vector-function $x = (x_i)_{i=1}^n \in \mathrm{BV}([0, m_0], \mathbb{R}^n)$ is a solution of the problem (1.1), (1.4), where t_j $(j = 1, \dots, n)$ are defined by (5.7),

$$c_{0i} = (1 + g_{1ii}(k_i))c_{01i} \quad (i = 1, \dots, n), \tag{5.11}$$

$$\begin{aligned}\ell_i(x_1, \dots, x_n) = \big(\big[(I_n + G_1(k_i))^{-1}\big]_{ii}\big)^{-1} \ell_{1i}(y_1, \dots, y_n) + \\ + \Big((1 + g_{1ii}(k_i)) \cdot \big[(I_n + G_1(k_i))^{-1}\big]_{ii}\Big)^{-1} \times \\ \times \sum_{l,j=1,\, l\neq i,\, j\neq i}^{n} g_{1il}(k_i) \cdot \big[(I_n + G_1(k_i))^{-1}\big]_{lj} x_l(k_i) \quad (i = 1, \dots, n);\end{aligned} \tag{5.12}$$

here $y(k) = (y_i(k))_{i=1}^n$ is defined by

$$y(k) = (I_n + G_1(k))^{-1} x(k) \quad (k = 0, \dots, m_0). \tag{5.13}$$

Using (5.12), as in Section 4 we get

$$\begin{aligned}x_i(k_i) = \big(\big[(I_n + G_1(k_i))^{-1}\big]_{ii}\big)^{-1} y_i(k_i) + \\ + \Big((1 + g_{1ii}(k_i)) \cdot \big[(I_n + G_1(k_i))^{-1}\big]_{ii}\Big)^{-1} \times \\ \times \sum_{l,j=1,\, l\neq i,\, j\neq i}^{n} g_{1il}(k_i) \cdot \big[(I_n + G_1(k_i))^{-1}\big]_{lj} x_l(k_i) \quad (i = 1, \dots, n)\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\ell_i(x_1, \dots, x_n) - x_{1i}(k_i) = \\ = \big(\big[(I_n + G_1(k_i))^{-1}\big]_{ii}\big)^{-1} \cdot \big(\ell_{1i}(y_1, \dots, y_n) - y_i(k_i)\big) \quad (i = 1, \dots, n),\end{aligned} \tag{5.15}$$

where $y(k) = (y_i(k))_{i=1}^n$ is defined by (5.13).

On the other hand, if the vector-function $x = (x_i)_{i=1}^n \in \mathrm{BV}([0, m_0], \mathbb{R}^n)$ is a solution of the problem (1.1), (1.4), where $A(t)$, $f(t)$, t_i $(i = 1, \dots, n)$, c_{0i} $(i = 1, \dots, n)$ and ℓ_i $(i = 1, \dots, n)$ are defined by (5.6), (5.7), (5.11) and (5.12), respectively, then due to (5.15), the vector-function y defined by (5.13) will be a solution of the problem (1.13), (1.16).

The problem (1.13), (1.15) is a particular case of the problem (1.13), (1.16), which is equivalent to the problem (1.1), (1.3) in the sense described above.

Along with the difference system (5.3), we consider the corresponding homogeneous difference system

$$\Delta z(k-1) = G(k) z(k) \quad (k = 1, \dots, m_0), \tag{5.3$_0$}$$

where $G(k)$ is defined by (5.4).

Note that the matrix-function

$$Y(k) = \prod_{i=0}^{k} ((I_n - G_2(i))^{-1}(I_n + G_1(i-1)) \tag{5.16}$$

(here $G_1(-1) = G_2(0) = O_{n\times n}$) is a fundamental matrix of the system (1.13$_0$) satisfying $Y(0) = I_n$.

The matrix-function

$$\mathcal{G}(k,i)=\begin{cases} Y(k)\displaystyle\sum_{j\in\{1,\dots,n_0:\;k_j<i\}} Z_jY^{-1}(i) & \text{for } 0\le i<k\le m_0,\\ -Y(k)\displaystyle\sum_{j\in\{1,\dots,n_0:\;k_j\ge i\}} Z_jY^{-1}(i) & \text{for } 0\le k<i\le m_0,\\ O_{n\times n} & \text{for } 0\le k=i\le m_0, \end{cases} \tag{5.17}$$

where $Y(k)$ is defined by (5.16) and

$$Z_j=\Big(\sum_{i=1}^{n_0}L_{1i}Y(k_i)\Big)^{-1}L_{1j}Y(k_j)\ \ (j=1,\dots,n_0), \tag{5.18}$$

is called the augmented Green matrix of the problem $(1.13_0),(1.14_0)$.

Under the Green matrix of the problem $(1.13_0),(1.14_0)$ we mean the matrix-function

$$\mathcal{G}_*(k,i)\equiv-(I_n+G_1(k))^{-1}\mathcal{G}(k,i). \tag{5.19}$$

We introduce the operators

$$\begin{gathered}\big[(G_1,G_2)(k)\big]_0\equiv I_n,\quad \big[(G_1,G_2)(k)\big]_i\equiv-\sum_{j=k+1}^{m_0}(G_1(k)+G_2(k))\times\\ \times(I_n+G_1(k))^{-1}\big[(G_1,G_2)(j)\big]_{i-1}\ \ (i=1,2,\dots),\end{gathered} \tag{5.20}$$

and

$$\begin{gathered}V_1(G_1,G_2)(k)\equiv\sum_{j=k+1}^{m_0}\big|(G_1(j)+G_2(j))(I_n+G_1(j))^{-1}\big|,\\ V_{i+1}(G_1,G_2)(k)\equiv\sum_{j=k+1}^{m_0}\big|(G_1(j)+G_2(j))(I_n+G_1(j))^{-1}\big|\times\\ \times V_i(G_1,G_2)(j)\ \ (i=1,\dots,n).\end{gathered} \tag{5.21}$$

Definition 5.1. Let $k_i\in\widetilde{\mathbb{N}}_{m_0}$ $(i=1,\dots,n)$, and let $G_1=(g_{1il})_{i,l=1}^n\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ be a matrix-function satisfying (5.1). We say that a pair (C,ℓ_0) consisting of a matrix-function $C=(c_{il})_{i,l=1}^n\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ and a positive homogeneous nondecreasing continuous operator $\ell_0=(\ell_{0i})_{i=1}^n:\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+^n)\to\mathbb{R}_+^n$ belongs to the set $U_{G_1}(k_1,\dots,k_n)$ if $c_{il}(k)\ge0$ for $k\in\widetilde{\mathbb{N}}_{m_0}$ $(i\ne l;\ i,l=1,\dots,n)$ and the system

$$\begin{gathered}\Big(\Delta y_i(k-1)+\sum_{l=1}^n\big(g_{1il}(k)y_l(k)-g_{1il}(k-1)y_l(k-1)\big)\Big)\operatorname{sgn}\Big(k-k_i-\frac12\Big)\le\\ \le\sum_{l,j=1}^n c_{il}(k)(\delta_{jl}+g_{1il}(k))y_l(k)\ \text{ for } k\in\{1,\dots,m_0-1\}\ \ (i=1,\dots,n)\end{gathered} \tag{5.22}$$

has no nontrivial nonnegative solution satisfying the condition

$$y_i(k_i)\le\ell_{0i}(y_1,\dots,y_n)\ \ (i=1,\dots,n). \tag{5.23}$$

5.1. **Solvability of the Problem (1.13), (1.14).**

Theorem 5.1. *The boundary value problem* (1.13), (1.14) *has a unique solution if and only if the corresponding homogeneous problem* $(1.13_0), (1.14_0)$ *has only the trivial solution. If the latter condition holds, the solution y of the problem* (1.13), (1.14) *admits the representation* (4.28), *where y_0 is a solution of the problem* $(1.13_0), (1.14)$, *and the matrix-function $\mathcal{G}_*(k,i)$ defined by* (5.16)–(5.19) *is the Green matrix of the problem* $(1.13_0), (1.14_0)$.

Corollary 5.1. *The boundary value problem* (1.13), (1.14) *is uniquely solvable if and only if*

$$\det\Big(\sum_{j=1}^{n_0} L_{1j}(I_n+G_1(k_j))^{-1}\prod_{i=0}^{k_j}(I_n-G_2(i))^{-1}(I_n+G_1(i-1))\Big)\neq 0. \quad (5.24)$$

Theorem 5.2. *The boundary value problem* (1.13), (1.14) *is uniquely solvable if and only if there exist natural numbers k and m such that the matrix*

$$M_k=\sum_{j=1}^{n_0}\sum_{i=0}^{k-1} L_{1j}(I_n+G_1(k_j))^{-1}\big[(G_1,G_2)(k_j)\big]_i$$

is nonsingular and the inequality (4.30) *holds, where*

$$M_{k,m}=V_m(G_1,G_2)(0)+$$
$$+\sum_{i=0}^{m-1}\big|\big[(G_1,G_2)(\cdot)\big]_i\big|_{\widetilde{\mathbb{N}}_{m_0}}\cdot\sum_{j=1}^{n_0}|M_k^{-1}L_{1j}|(I_n+G_1(k_j))^{-1}V_k(G_1,G_2)(k_j),$$

and the matrix-functions $[(G_1,G_2)(l)]_i$ and $V_i(G_1,G_2)(l)$ are defined by (5.20) *and* (5.21), *respectively.*

Corollary 5.2. *Let*

$$\det\Big(\sum_{j=1}^{n_0} L_{1j}(I_n+G_1(k_j))^{-1}\Big)\neq 0 \quad (5.25)$$

and

$$r(L_0M_0)<1,$$

where

$$L_0=I_n+\Big|\Big(\sum_{j=1}^{n_0} L_{1j}(I_n+G_1(k_j))^{-1}\Big)^{-1}\Big|\cdot\sum_{j=1}^{n_0}\big|L_{1j}(I_n+G_1(k_j))^{-1}\big|$$

and

$$M_0=\sum_{i=1}^{m_0}\big|\big(G_1(i)+G_2(i)\big)(I_n+G_1(i))^{-1}\big|.$$

Then the problem (1.13), (1.14) *has one and only one solution.*

Corollary 5.3. *Let the condition* (5.25) *hold, or there exist a natural number k such that the conditions*

$$\sum_{j=1}^{n_0} L_{1j}(I_n+G_{1kj})^{-1}=O_{n\times n},\quad \det M_i=0 \ \ (i=0,\dots,k-1)$$

and

$$\det M_k\neq 0$$

hold, where

$$M_i=\sum_{j=1}^{n_0} L_{1j}(I_n+G_1(k_j))^{-1}\big[(G_1,G_2)(k_j)\big]_i.$$

Then there exists $\varepsilon_0>0$ such that the system

$$\Delta y(k-1)=\varepsilon\big(G_1(k-1)y(k-1)+G_2(k)y(k)\big)+g_0(k) \ \ (k=1,\dots,m_0)$$

has a solution satisfying (1.14) *for every $\varepsilon\in\,]0,\varepsilon_0[$.*

Theorem 5.3. *Let matrix-functions $G_{0j}\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ $(j=1,2)$ be such that*

$$\det(I_n+G_{01}(k))\neq 0,\quad \det(I_n-G_{02}(k))\neq 0 \ \ (k=0,\dots,m_0),$$

the homogeneous system

$$\Delta y(k-1)=G_{01}(k-1)y(k-1)+G_{02}(k)y(k) \ \ (k=1,\dots,m_0) \tag{5.26}$$

has only the trivial solution satisfying the condition (1.14_0), *and let the matrix-functions $G_j\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ $(j=1,2)$ admit the estimates*

$$\sum_{i=1}^{m_0}\bigg|\mathcal{G}_0(k,i-1)\cdot\Big\{(I_n-G_{02}(i))(I_n+G_{01}(i))^{-1}- \\ -(I_n-G_2(i))(I_n+G_1(i))^{-1}\Big\}\bigg|\leq M,$$

where $\mathcal{G}_0(k,i)$ is the augmented Green matrix of the problem (5.26), (1.14_0), *and $M\in\mathbb{R}^{n\times n}$ is a constant matrix such that*

$$r(M)<1.$$

Then the problem (1.13), (1.14) *has one and only one solution.*

5.2. **Solvability of the Problems (1.13), (1.15) and (1.13), (1.16).**

Theorem 5.4. *Let the matrix-function $G_1=(g_{1il})_{i,l=1}^n$ satisfy the condition* (4.33), *and let there exist a matrix-function $C=(c_{il})_{i,l=1}^n$ and a positive homogeneous nondecreasing continuous operator $\ell_0=(\ell_{0i})_{i=1}^n$ such that*

$$(C,\ell_0)\in U_{G_1}(k_1,\dots,k_n), \tag{5.27}$$

$$(-1)^{j+1}\Big(\Big|\big[(I_n-G_2(k))(I_n+G_1(k))^{-1}\big]_{ii}\Big|-1\Big)\leq c_{ii}(k) \\ \text{for } \ (-1)^j(k-k_i)\geq 0 \ \ (j=1,2;\ \ i=1,\dots,n), \tag{5.28}$$

$$\Big|\big[\big(G_1(k)+G_2(k)\big)(I_n+G_1(k))^{-1}\big]_{il}\Big| \leq c_{il}(k)$$
$$\text{for } \ k \in \{1,\dots,m_0\} \ \ (i \neq l; \ \ i,l=1,\dots,n) \tag{5.29}$$

and

$$\big|\ell_{1i}(y_1,\dots,y_n)\big| \leq \ell_{0i}\big(|y_1|,\dots,|y_n|\big)$$
$$\text{for } \ (y_l)_{l=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n) \ \ (i=1,\dots,n). \tag{5.30}$$

Then the problem (1.13), (1.16) *has one and only one solution.*

Theorem 5.5. *Let the conditions* (4.33), (4.44),

$$(-1)^{j+1}\Big(\Big|\big[(I_n-G_2(k))(I_n+G_1(k))^{-1}\big]_{ii}\Big|-1\Big) \leq h_{ii}(k)$$
$$\text{for } \ (-1)^j(k-k_i) \geq 0 \ \ (j=1,2; \ \ i=1,\dots,n), \tag{5.31}$$
$$\Big|\big[(G_1(k)+G_2(k))(I_n+G_1(k))^{-1}\big]_{il}\Big| \leq h_{il}(k)$$
$$\text{for } \ k \in \{0,\dots,m_0\} \ \ (i \neq l; \ \ i,l=1,\dots,n) \tag{5.32}$$

and

$$\bigg|\ell_{1i}(y_1,\dots,y_n)+\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\sum_{l=1}^n \big[I_n+G_1(k_i)\big]_{il}y_l(k_i)-y_i(k_i)\bigg| \leq$$
$$\leq \sum_{l=1}^n \gamma_{il}\bigg\|\sum_{j=1}^n \big[(I_n+G_1(\cdot)\big]_{lj}y_j(\cdot)\bigg\|_\nu \ \ \text{for } \ (y_l)_{l=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)$$
$$(i=1,\dots,n)$$

hold, where $h_{ii} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R})$, $h_{il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+)$ $(i \neq l;\ i,l=1,\dots,n)$, *and* $\gamma_{il} \in \mathbb{R}_+$ $(i,l=1,\dots,n)$, $\nu \geq 2$,

$$\mathcal{H}^* = \Big(\xi_0\big(\big[(I_n+G_1(k_i))^{-1}\big]_{ii}\big)^{-1}\gamma_{il}+\lambda_0\|h_{il}\|_\mu\Big)_{i,l=1}^n,$$

$\frac{1}{\mu}+\frac{2}{\nu}=1$, $\xi_0=m_0^{\frac{1}{\nu}}$, $\lambda_0=\big(\frac{1}{2}\,\sin^{-1}\frac{\pi}{4m_0+2}\big)^{\frac{2}{\nu}}$. *Then the problem* (1.13), (1.16) *has one and only one solution.*

Corollary 5.4. *Let* $G_1(k) \equiv \operatorname{diag}(g_{11}(k),\dots,g_{1n}(k))$ *be a diagonal matrix-function such that the conditions* (4.44), (4.45),

$$(-1)^{j+1}\big((1+g_{1i}(k))^{-1}|1-g_{2ii}(k)|-1\big) \leq h_{ii}(k) \ \ \text{for } \ (-1)^j(k-k_i) \geq 0$$
$$(j=1,2; \ \ i=1,\dots,n),$$
$$(1+g_{1l}(k))^{-1}\big|g_{1il}(k)+g_{2il}(k)\big| \leq h_{il}(k) \ \ \text{for } \ k \in \{0,\dots,m_0\}$$
$$(i \neq l; \ \ i,l=1,\dots,n)$$

and

$$\big|\ell_{1i}(y_1,\dots,y_n)\big| \leq \sum_{l=1}^n \gamma_{il}\big\|(1+g_{1l}(\cdot))y_l(\cdot)\big\|_\nu$$
$$\text{for } \ (y_l)_{l=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n) \ \ (i=1,\dots,n)$$

hold, where $h_{ii} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R})$, $h_{il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}_+)$ $(i \neq l;\ i,l = 1,\dots,n)$, $\gamma_{il} \in \mathbb{R}_+$ $(i,l = 1,\dots,n)$, $\nu \geq 2$, *the matrix* $\mathcal{H}^*$ *is defined as in Theorem* 5.5. *Then the conclusion of Theorem* 5.5 *is true.*

Corollary 5.5. *Let the conditions* (4.33), (4.46), (4.47), (5.30) *and* (5.31) *hold, where* $h_{ii} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R})$, $h_{il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}_+)$ $(i \neq l;\ i,l = 1,\dots,n)$, $\nu \geq 2$, $\mathcal{H}_0 = (\|h_{il}\|_\mu)_{i,l=1}^n$, $\frac{1}{\mu} + \frac{2}{\nu} = 1$. *Then the problem* (1.13), (1.15) *has one and only one solution.*

Corollary 5.6. *Let the conditions* (4.33), (4.46), (4.48),

$$(-1)^{j+1}\Big(\Big|\big[(I_n - G_2(k))(I_n + g_1(k))^{-1}\big]_{ii}\Big| - 1\Big) \leq h_{ii}$$
$$\text{for } (-1)^j(k - k_i) \geq 0 \ \ (j = 1,2;\ \ i = 1,\dots,n)$$

and

$$\Big|\big[(G_1(k) + G_2(k))(I_n + G_1(k))^{-1}\big]_{il}\Big| \leq h_{il} \ \text{ for } k \in \{0,\dots,m_0\}$$
$$(i \neq l;\ \ i,l = 1,\dots,n)$$

hold, where $h_{ii} \in \mathbb{R}$, $h_{il} \in \mathbb{R}_+$ $(i \neq l;\ i,l = 1,\dots,n)$, $\mu \geq 1$, $\nu \geq 2$, $\frac{1}{\mu} + \frac{2}{\nu} = 1$, *and* $\mathcal{H}_0 = (h_{il})_{i,l=1}^n$. *Then the problem* (1.13), (1.15) *has one and only one solution.*

Theorem 5.6. *Let the conditions*

$$(-1)^{j+1}\Big(\Big|\big[(I_n - G_2(k))(I_n + G_1(k))^{-1}\big]_{ii}\Big| - 1\Big) \leq h_{ii}\beta_i(k) + \beta_{1ii}(k)$$
$$\text{for } (-1)^j(k - k_i) \geq 0 \ \ (j = 1,2;\ \ i = 1,\dots,n),$$
$$\Big|\big[(G_1(k) + G_2(k))(I_n + G_1(k))^{-1}\big]_{il}\Big| \leq h_{il}\beta_i(k) + \beta_{1il}(k)$$
$$\text{for } k \in \{0,\dots,m_0\} \ \ (i \neq l;\ \ i,l = 1,\dots,n),$$
$$\big|\big[(I_n + G_1(k))^{-1}\big]_{il}\big| \leq \beta_{2il}(k) \ \text{ for } k \in \{0,\dots,m_0\} \ \ (i,l = 1,\dots,n),$$
$$\Big|\ell_{1i}(y_1,\dots,y_n) + \big[(I_n + G_1(k_i))^{-1}\big]_{ii}\sum_{l=1}^n \big[I_n + G_1(k_i)\big]_{il} y_l(k_i) - y_i(k_i)\Big| \leq$$
$$\leq |\mu_i|\Big|\sum_{l=1}^n \big[I_n + G_1(m_i)\big]_{il} y_l(m_i)\Big| \ \text{ for } (y_l)_{l=1}^n \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}^n) \ \ (i=1,\dots,n),$$
$$\beta_i(k_i) \leq 0, \ \ 0 \leq \beta_i(k) < |\eta_i|^{-1} \ \text{ for } k \in \{k_i+1,\dots,m_0\} \ \ (i = 1,\dots,n)$$

and

$$|\mu_i|\gamma_i(m_i,k) < 1 \ \ (i = 1,\dots,n)$$

hold, where $h_{ii} < 0$, $h_{il} \geq 0$ $(i \neq l;\ i,l = 1,\dots,n)$; $\mu_i \in \mathbb{R}$, $\eta_i < 0$, $m_i \in \widetilde{\mathbb{N}}_{m_0}$, $m_i \neq k_i$ $(i = 1,\dots,n)$; $\beta_{1ii}, \beta_{2il} \in \mathbb{E}(\widetilde{\mathbb{N}}_{m_0}, \mathbb{R}_+)$ $(i,l = 1,\dots,n)$; $\beta_{1il}(k) \geq 0$ $(i \neq l)$ *and* $\beta_i(k) \geq 0$ *for* $k \in \widetilde{\mathbb{N}}_{m_0} \setminus \{k_i\}$ $(i,l = 1,\dots,n)$; $\gamma_i(m,k) \equiv \gamma_{a_i}(m,k)$, *the function* γ_{a_i} *is defined by* (4.49), *and* $a_i(k) \equiv \eta_i(\alpha_i(k) - \alpha_i(k_i))\operatorname{sgn}(k - k_i)$, $\alpha_i(k) \equiv \sum\limits_{l=0}^{k} \beta_i(l)$ $(i = 1,\dots,n)$. *Let, moreover,* *the condition* (2.39) *hold, and let the real part of every characteristic value of the* $n\times n$*-matrix* $(\xi_{il})_{i,l=1}^n$ *be negative, where*

$$\xi_{ii}=h_{ii}(1-g_{ii}),\quad \xi_{il}=h_{il}h_i-h_{ii}g_{il}\ \ (i\neq l;\ \ i,l=1,\dots,n);$$

$$g_{il}=|\mu_i|\big(1-|\mu_i|\gamma_i(m_i,k_i)\big)^{-1}\gamma_{il}(m_i)+\max\big\{\gamma_{il}(0),\gamma_{il}(m_0)\big\}\ \ (i,l=1,\dots,n);$$

$$\gamma_{il}(k_i)=0\ \ (i,l=1,\dots,n),$$

$$\gamma_{il}(k)=\big|\alpha_{il}(k)-\alpha_{il}(k_i)\big|-(1-\delta_{il})\chi_{[0,k_i]}(k)\beta_{1il}(k_i)$$

$$\text{for}\ \ k\in\widetilde{\mathbb{N}}_{m_0}\setminus\{k_i\}\ \ (i,l=1,\dots,n);$$

$$\alpha_{il}(k)\equiv\sum_{m=0}^{k}\beta_{1il}(m)\ \ (i,l=1,\dots,n);$$

$$h_i=1\ \ \text{if}\ \ |\mu_i|\le 1,\quad \text{and}$$

$$h_i=1+\big(|\mu_i|-1\big)\big(1-|\mu_i|\gamma_i(m_i,k_i)\big)^{-1}\ \ \text{if}\ \ |\mu_i|>1\ \ (i=1,\dots,n).$$

Then the problem (1.13), (1.16) *has one and only one solution.*

Theorem 5.7. *Let the condition* (4.33) *hold, and let* $\ell_{0i}:\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}_+^n)\to\mathbb{R}_+$ $(i=1,\dots,n)$ *be linear continuous functionals, a matrix-function* $C=(c_{il})_{i,l=1}^n\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$ *be such that* $c_{il}(k)\ge 0$ *for* $k\in\widetilde{\mathbb{N}}_{m_0}$ $(i\neq l;$ $i,l=1,\dots,n)$, *and the problem* (5.22), (5.23) *have a nontrivial nonnegative solution* $y=(y_i)_{i=1}^n$, *i.e. the condition* (5.27) *be violated. Then there exist a matrix-function* $G_2\in\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^{n\times n})$, *linear continuous functionals* $\ell_i:\mathbb{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n)\to\mathbb{R}$ $(i=1,\dots,n)$ *and numbers* $c_{0i}\in\mathbb{R}$ $(i=1,\dots,n)$ *such that the conditions* (5.28)–(5.30) *are fulfilled, but the problem* (1.13_0), (1.16) *is not solvable. In addition, if the matrix-function* $C=(c_{il})_{i,l=1}^n$ *is such that*

$$\det\big(\big(I_n-\operatorname{diag}(\operatorname{sgn}(k-k_1),\dots,\operatorname{sgn}(k-k_n)\big)C(k)\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_n)\big)\neq 0$$
$$\text{for}\ \ k\in\{0,\dots,m_0\},\tag{5.33}$$

where $\varepsilon_i\in[0,1]$, *then the matrix-function* G_2 *satisfies the condition (5.1).*

Remark 5.1. The condition (5.33) holds, for example, if either

$$\sum_{l=1}^{n}|c_{il}(t)|<1\quad\text{for}\quad k\in\widetilde{\mathbb{N}}_{m_0}\ \ (i=1,\dots,n)$$

or

$$c_{ii}(k)\le 1\quad\text{for}\quad k>k_i\ \ (i=1,\dots,n)$$

and

$$\sum_{l=1,\,l\neq i}^{n}|c_{il}(k)|<|1-\operatorname{sgn}(k-k_i)c_{ii}(k)|\ \ \text{for}\ \ k\in\widetilde{\mathbb{N}}_{m_0}\ \ (i=1,\dots,n)$$

$$\bigg(\sum_{l=1,\,l\neq i}^{n}|c_{li}(k)|<|1-\operatorname{sgn}(k-k_i)c_{ii}(k)|\ \ \text{for}\ \ k\in\widetilde{\mathbb{N}}_{m_0}\ \ (i=1,\dots,n)\bigg).$$

6. Auxiliary Propositions

6.1. General Lemmas.

Lemma 6.1. *Let $g = (g_i)_{i=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^n)$, and let $(c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ be such that c_{il} $(i \neq l;\ i,l = 1,\dots,n)$ are nondecreasing functions. Let, moreover, $B = (b_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ be a matrix-function satisfying the conditions*

$$S_0(b_{ii})(t) - S_0(b_{ii})(s) \leq \big[S_0(c_{ii})(t) - S_0(c_{ii})(s)\big]\operatorname{sgn}(t-s)$$
$$\text{for } (t-s)(s-t)_i > 0 \ \ (i = 1,\dots,n), \tag{6.1}$$

$$(-1)^{j+m}\Big(\big|1 + (-1)^m d_m b_{ii}(t)\big| - 1\Big) \leq d_m c_{ii}(t)$$
$$\text{for } (-1)^j(t-t_i) \geq 0 \ \ (j,m = 1,2;\ \ i = 1,\dots,n), \tag{6.2}$$

$$\big|S_0(b_{il})(t) - S_0(b_{il})(s)\big| \leq S_0(c_{il})(t) - S_0(c_{il})(s)$$
$$\text{for } a \leq s < t \leq b \ \ (i \neq l;\ \ i,l = 1,\dots,n) \tag{6.3}$$

and

$$|d_j b_{il}(t)| \leq d_j c_{il}(t) \ \text{ for } t \in [a,b] \ \ (i \neq l;\ \ i,l = 1,\dots,n). \tag{6.4}$$

Then every solution $x = (x_i)_{i=1}^n$ of the system

$$dx(t) = dB(t)\cdot x(t) + dg(t)$$

will be a solution of the system

$$\Big[d|x_i(t)| - \operatorname{sgn}(t-t_i)\sum_{l=1}^n |x_l(t)|dc_{il}(t) - \operatorname{sgn} x_i(t)\cdot dg_i(t)\Big]\operatorname{sgn}(t-t_i) \leq 0$$
$$(i = 1,\dots,n),$$

$$(-1)^j d_j|x_i(t_i)| \leq \sum_{l=1}^n |x_l(t_i)| d_j c_{il}(t_i) +$$
$$+ (-1)^j \operatorname{sgn} x_i(t_i)\cdot d_j g_i(t_i) \ \ (j = 1,2;\ \ i = 1,\dots,n).$$

Proof. This lemma with the supplementary condition

$$d_j c_{ii}(t_i) \geq 0, \quad d_j c_{ii}(t) > -1 \ \text{ for } (-1)^j(t-t_i) > 0 \ \ (j = 1,2;\ \ i = 1,\dots,n)$$

is proved in [6, Lemma 2.2]. We only note that the last condition follows immediately from the condition (6.2). □

Lemma 6.2. *Let $t_0 \in [a,b]$, $c_0 \in \mathbb{R}^n$, $g \in \mathrm{BV}([a,b],\mathbb{R}^n)$, and let the matrix-function $B = (b_{ik})_{i,k=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$, where b_{ik} $(i \neq k;\ i,k = 1,\dots,n)$ are nondecreasing functions on $[a,b]$, be such that*

$$\det(I_n + d_j B(t)) \neq 0 \ \text{ for } t \in [a,b]\setminus\{t_0\} \ \ (j = 1,2),$$
$$1 + d_j b_{ii}(t) > 0 \ \text{ for } (-1)^j(t-t_0) \geq 0 \ \ (j = 1,2)$$

and

$$\sum_{i=1}^n d_j b_{ik}(t) < 1 \ \text{ for } (-1)^j(t-t_0) < 0 \ \ (j = 1,2;\ \ k = 1,\dots,n).$$

Let, moreover, the vector-function $z : [a,b] \to \mathbb{R}^n$ ($z \in \mathrm{BV}([c,d],\mathbb{R}^n)$ *for every* $[c,d] \subset [a,t_0[\cup]t_0,b]$) *be a solution of the system of linear differential inequalities*

$$\operatorname{sgn}(t-t_0)dz(t) \le dB(t)\cdot z(t) + dg(t)$$

on the intervals $[a,t_0[$ *and* $]t_0,b]$, *satisfying the condition*

$$z(t_0) + (-1)^j d_j z(t_0) \le c_0 + d_j B(t_0)\cdot c_0 + d_j g(t) \ \ (j=1,2).$$

Then the estimate

$$z(t) \le u(t)$$

holds, where $u \in \mathrm{BV}([a,b],\mathbb{R}^n)$ *is the unique solution of the system*

$$\operatorname{sgn}(t-t_0)du(t) = dB(t)\cdot u(t) + dg(t)$$

on the intervals $[a,t_0[$ *and* $]t_0,b]$, *satisfying the conditions*

$$(-1)^j d_j u(t_0) = d_j B(t_0)\cdot u(t_0) + d_j b(t_0) \ \ (j=1,2)$$

and

$$u(t_0) = c_0.$$

Proof. This lemma is proved in [8, Lemma 2.7]. □

6.2. **On the Set** $U(t_1,\dots,t_n)$**.**

The following lemmas make more precise the ones given in [3].

Lemma 6.3. *Let the conditions* (2.26), (2.27) *and*

$$\big|c_{il}(t) - c_{il}(s)\big| \le \int\limits_s^t h_{il}(\tau)\,d\alpha_l(\tau) \ \ \text{for } a \le s < t \le b \ \ (i,l=1,\dots,n)$$

hold, where $c_{il} \in \mathrm{BV}([a,b],\mathbb{R})$ $(i,l=1,\dots,n)$, α_l $(l=1,\dots,n)$ *are functions, nondecreasing on* $[a,b]$ *and having not more than a finite number of points of discontinuity;* $h_{il} \in L^\mu([a,b],\mathbb{R}_+;\alpha_l)$ $(i \ne l)$, $h_{ii} \in L^\mu([a,b],\mathbb{R};\alpha_l)$ $(i,l=1,\dots,n)$, $1 \le \mu \le +\infty$; $\ell_{mik} \in \mathbb{R}_+$ $(m=0,1,2;\ i,k=1,\dots,n)$, $\frac{1}{\mu}+\frac{2}{\nu}=1$, *and* $\mathcal{H} = (\mathcal{H}_{j+1,m+1})_{j,m=0}^2$ *is the* $3n\times 3n$*-matrix defined as in Theorem* 2.6. *Then the problem* (1.23), (1.24) *has no nontrivial nonnegative solution. In addition, if* c_{il} $(i \ne l;\ i,l=1,\dots,n)$ *are functions nondecreasing on* $[a,b]$, *then the condition* (2.16) *holds for* $C=(c_{il})_{i,l=1}^n$ *and* $\ell_0 = (\ell_{0i})_{i=1}^n$,

$$\ell_{0i}(x_1,\dots,x_n) = \sum_{m=0}^{2}\sum_{k=1}^{n} \ell_{mik}\|x\|_{\nu,s_m(\alpha_k)}$$
$$\text{for } (x_i)_{i=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^n) \ \ (i=1,\dots,n).$$

Lemma 6.4. *Let the conditions* (2.36)–(2.38),

$$S_j(c_{il})(t) - S_j(c_{il})(s) \le h_{il}\big[S_j(\alpha_i)(t) - S_j(\alpha_i)(s)\big] + S_j(\beta_{il})(t) - S_j(\beta_{il})(s)$$
$$\text{for } a \le s < t < t_i \ \text{ and } \ t_i < s \le t \le b \ \ (j=0,1,2;\ \ i,l=1,\dots,n)$$

and

$$d_j c_{ii}(t_i) \le h_{ii} d_j \alpha_i(t_i) + d_j \beta_{ii}(t_i) \ \ (j=1,2;\ \ i=1,\dots,n)$$

hold, where $h_{il} \ge 0$ $(i \ne l)$, $h_{ii} < 0$ $(i,l=1,\dots,n)$*;* $\mu_i \ge 0$, $\eta_i < 0$, $s_i \in [a,b]$, $s_i \ne t_i$ $(i=1,\dots,n)$*;* β_{ii} $(i=1,\dots,n)$ *are functions nondecreasing on* $[a,b]$*;* β_{il} $(i \ne l)$ *and* α_i $(i,l=1,\dots,n)$ *are functions on* $[a,b]$ *nondecreasing on every interval* $[a,t_i[$ *and* $]t_i,b]$*;* $\lambda_i(t) \equiv \gamma_{a_i}(t,t_i)$*, the function* $\gamma_{a_i}(t,t_i)$ *is defined according to* (1.22)*, and* $a_i(t) \equiv \eta_i(\alpha_i(t)-\alpha_i(t_i))\operatorname{sgn}(t-t_i)$ $(i \ne l;$ $i=1,\dots,n)$*. Let, moreover, the condition* (2.39) *hold, and let the real part of every characteristic value of the matrix* $(\xi_{il})_{i,l=1}^n$ *be negative, where* g_{ii} *and* ξ_{il} $(i,l=1,\dots,n)$ *are defined as in Theorem* 2.7*. Then the problem* (1.23)*,* (1.24) *has no nontrivial nonnegative solution. Moreover, if* c_{il} $(i \ne l;$ $i,l=1,\dots,n)$ *are nondecreasing functions, then the condition* (2.16) *holds for* $C=(c_{il})_{i,l=1}^n$ *and* $\ell_0=(\ell_{0i})_{i=1}^n$*,*

$$\ell_{0i}(x_1,\dots,x_n)=\mu_i x_i(s_i) \ \ \text{for} \ \ (x_l)_{l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}_+^n) \ \ (i=1,\dots,n).$$

Proofs of Lemmas 6.3 *and* 6.4 are analogous to those of Lemmas 2.6 and 2.7, respectively, given in [3].

Lemma 6.5. *Let* $t_{k1},\dots,t_{kn} \in [a,b]$ $(k=1,2)$*,* $\ell_{0ki} : \mathrm{BV}([a,b],\mathbb{R}_+^n) \to \mathbb{R}_+$ $(k=1,2;$ $i=1,\dots,n)$ *be linear continuous functionals, and* $C_{kj} = (c_{kjil})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ $(k,j=1,2)$ *be such that the system*

$$\begin{gathered}
\operatorname{sgn}(t-t_{1i})\cdot dx_{1i}(t) \le \sum_{l=1}^n x_{1l}(t)\,dc_{11il}(t) + \sum_{l=1}^n x_{2l}(t)\,dc_{12il}(t) \\
\text{for} \ \ t\in[a,b], \ \ t \ne t_{1i} \ \ (i=1,\dots,n), \\
(-1)^j d_j x_{1i}(t_{1i}) \le \sum_{l=1}^n x_{1l}(t_{1i})\,d_j c_{11il}(t_{1i}) + \sum_{l=1}^n x_{2l}(t_{1i})\,d_j c_{12il}(t_{1i}) \\
(j=1,2;\ \ i=1,\dots,n), \\
dx_{2i}(t) = \sum_{l=1}^n x_{1l}(t)\,dc_{21il}(t) + \sum_{l=1}^n x_{2l}(t)\,dc_{22il}(t) \\
\text{for} \ \ t\in[a,b] \ \ (i=1,\dots,n)
\end{gathered} \tag{6.5}$$

has a nontrivial nonnegative solution under the condition

$$x_{ki}(t_{1i}) \le \ell_{0ki}(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n}) \ \ (k=1,2;\ \ i=1,\dots,n). \tag{6.6}$$

Then there exist a matrix-function $A \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$*, linear continuous functionals* $\ell_i : \mathrm{BV}([a,b],\mathbb{R}^{2n}) \to \mathbb{R}$ $(i=1,\dots,2n)$ *and numbers* $c_{0i} \in \mathbb{R}$ $(i=1,\dots,2n)$ *such that the* $2n$*-system*

$$dx(t) = d\widetilde{A}(t)\cdot x(t) \tag{6.7}$$

under the $2n$*-condition* (1.4) *is unsolvable, where* $t_i = t_{1i}$ $(i=1,\dots,n)$*,* $t_{n+i} = t_{2i}$ $(i=1,\dots,n)$*, and*

$$\widetilde{A}(t) \equiv \begin{pmatrix} A(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{pmatrix}. \tag{6.8}$$

Proof. Let $x = (x_k)_{k=1}^2$, $x_k = (x_{ki})_{i=1}^n$ $(k = 1, 2)$ be the nonnegative solution of the problem (6.5), (6.6).

Let $b_i, \varphi_i \in \mathrm{BV}([a, b], \mathbb{R})$ $(i = 1, \dots, n)$ be the functions defined by

$$b_i(t) \equiv \big(S_0(c_{11\,ii})(t) - S_0(c_{11\,ii})(s)\big)\,\mathrm{sgn}(t - t_{1i}) \quad (i = 1, \dots, n)$$

and

$$\varphi_i(t) \equiv \bigg(\sum_{l=1}^n \Big(\int_{t_{1i}}^t x_{1l}(\tau)\,dc_{11\,il}(\tau) + \int_{t_{1i}}^t x_{2l}(\tau)\,dc_{12\,il}(\tau)\Big) - \\ - \int_{t_{1i}}^t x_{1i}(\tau) dS_0(c_{11\,ii})(\tau)\bigg)\,\mathrm{sgn}(t - t_{1i}) \quad (i = 1, \dots, n).$$

It is evident that the Cauchy problem

$$dy(t) = y(t)\,db_i(t) + d\varphi_i(t), \tag{6.9}$$

$$y(t_{1i}) = x_{1i}(t_{1i}) \tag{6.10}$$

has a unique solution y_{1i} for every $i \in \{1, \dots, n\}$.

Moreover, it is easy to verify that the function $z(t) = z_i(t)$,

$$z_i(t) \equiv x_{1i}(t) - y_{1i}(t)$$

satisfies the conditions of Lemma 6.2 and the problem

$$du(t) = u(t)\,db_i(t), \quad u(t_{1i}) = 0$$

has only the trivial solution for every $i \in \{1, \dots, n\}$.

According to this lemma we have

$$x_{1i}(t) \le y_{1i}(t) \quad \text{for} \quad t \in [a, b] \ \ (i = 1, \dots, n)$$

and therefore

$$x_{1i}(t) = \eta_i(t) y_{1i}(t) \quad \text{for} \quad t \in [a, b] \quad (i = 1, \dots, n),$$

where, by Theorem I.4.25 of [28], $\eta_i : [a, b] \to [0, 1]$ $(i = 1, \dots, n)$ are functions such that the integrals $\int_{t_i}^t \eta_i(\tau)\,dc_{11\,il}(\tau)$ $(i, l = 1, \dots, n)$ exist for every $t \in [a, b]$.

Let us introduce the notation

$$a_{ii}(t) \equiv b_i(t) + \mathrm{sgn}(t - t_{1i}) \int_{t_{1i}}^t \eta_i(\tau)\,d(c_{11\,ii}(\tau) - S_0(c_{11\,ii})(\tau)) \\ (i = 1, \dots, n), \tag{6.11}$$

$$a_{il}(t) \equiv \mathrm{sgn}(t - t_{1i}) \int_{t_{1i}}^t \eta_l(\tau)\,dc_{11\,il}(\tau) \quad (i \ne l, \ \ i, l = 1, \dots, n).$$

Due to (6.6), (6.9) and (6.10), the vector-function $y = (y_i)_{i=1}^{2n}$, $y_i(t) = x_{1i}(t)$ $(i = 1, \dots, n)$, $y_{n+i}(t) = x_{2i}(t)$ $(i = 1, \dots, n)$, is a nontrivial nonnegative solution of the $2n$-problem

$$dy(t) = d\widetilde{A}(t) \cdot y(t), \tag{6.12}$$

$$y_i(t_i) = \delta_i \ell_{0i}(y_1, \dots, y_{2n}) \quad (i = 1, \dots, 2n), \tag{6.13}$$

where $\delta_i \in [0,1]$ $(i = 1, \dots, n)$, $\delta_{n+i} = 1$ $(i = 1, \dots, n)$, $A(t) = (a_{il}(t))_{i,l=1}^{n}$,

$$\ell_{0i}(y_1, \dots, y_{2n}) = \ell_{0ki}(y_1, \dots, y_{2n})$$

$$\text{for } \ (y_l)_{l=1}^{2n} \in \mathrm{BV}([a,b], \mathbb{R}^{2n}) \ \ (k = 1,2; \ \ i = (k-1)n + 1, \dots, kn). \tag{6.14}$$

Let $\ell_i : \mathrm{BV}([a,b], \mathbb{R}^{2n}) \to \mathbb{R}$ $(i = 1, \dots, 2n)$ be linear functionals defined by

$$\ell_i(x_1, \dots, x_{2n}) = \delta_i \left(\ell_{0i}([x_1]_+, \dots, [x_{2n}]_+) - \ell_{0i}([x_1]_-, \dots, [x_{2n}]_-) \right)$$
$$\text{for} \quad (x_l)_{l=1}^{2n} \in \mathrm{BV}([a,b], \mathbb{R}^{2n}) \quad (i = 1, \dots, 2n), \tag{6.15}$$

where $[x_i]_+(t) = \frac{1}{2}(|x_i(t)| + x_i(t))$ and $[x_i]_-(t) = \frac{1}{2}(|x_i(t)| - x_i(t))$ $(i = 1, \dots, 2n)$ are the positive and negative parts of the function x_i, respectively.

By (6.11)–(6.13), $y = (y_i)_{i=1}^{2n}$ is a nontrivial, nonnegative solution of the system (6.7) under the boundary condition (1.4).

On the other hand, by Remark 1.2, there exist numbers $c_{0i} \in \mathbb{R}$ $(i = 1, \dots, 2n)$ such that the problem (6.7), (1.4) is not solvable, where the matrix-function $\widetilde{A}(t)$ is defined by (6.8), (6.11), and the linear functionals ℓ_i $(i = 1, \dots, 2n)$ are defined by (6.15). The lemma is proved. □

7. Proof of the Main Results

7.1. Proof of the Results of Section 2.

Proof of Theorem 2.1. Let $l = 1$.

We introduce the following sequence of operators: $p_i : \mathrm{BV}([a,b], \mathbb{R}^{n \times l}) \to \mathrm{BV}([a,b], \mathbb{R}^{n \times l})$ $(i = 0, 1, \dots)$:

$$p_0(X)(t) \equiv X(t),$$
$$p_i(X)(t) \equiv (I_n - d_1 A(t))^{-1} \int_a^t dA(\tau-) \cdot p_{i-1}(X)(\tau) \quad (i = 1, 2, \dots). \tag{7.1}$$

To prove the theorem, we have to show that the conditions of the theorem are necessary and sufficient for the absence of nontrivial solutions to the homogeneous problem (1.1_0), (1.5_0).

Let us show the sufficiency. Let $x = (x_i)_{i=1}^{n}$ be an arbitrary solution of the homogeneous problem (1.1_0), (1.5_0). Then

$$x(t) = c + \int_a^t dA(\tau) \cdot x(\tau) \quad \text{for} \quad t \in [a,b], \tag{7.2}$$

where $c = x(a)$. This, by (1.21), (1.31) and (7.1), yields

$$x(t) = c + \int_a^t dA(\tau-)\cdot x(\tau) + d_1A(t)\cdot x(t)$$

and

$$\begin{aligned}
x(t) &= (I_n - d_1A(t))^{-1}c + (I_n - d_1A(t))^{-1}\int_a^t dA(\tau-)\cdot x(\tau) = \\
&= [A(t)]_0\cdot c + p_1(x)(t) = [A(t)]_0\cdot c + p_1([A(\cdot)]_0\cdot c + p_1(x))(t) = \\
&= [A(t)]_0\cdot c + p_1([A(\cdot)]_0\cdot c)(t) + p_1(p_1(x))(t) = \\
&= ([A(t)]_0 + [A(t)]_1)\cdot c + p_2(x)(t) = ([A(t)]_0 + [A(t)]_1)\cdot c + \\
&\quad + p_2([A(\cdot)]_0\cdot c + p_1(x))(t) = \\
&= ([A(t)]_0 + [A(t)]_1)\cdot c + p_2([A(\cdot)]_0\cdot c)(t) + p_2(p_1(x))(t) = \\
&= ([A(t)]_0 + [A(t)]_1 + [A(t)]_2)\cdot c + p_3(x)(t) \quad \text{for} \quad t\in[a,b],
\end{aligned}$$

etc. Continuing this process infinitely, we obtain

$$x(t) = \left(\sum_{i=0}^{j-1}[A(t)]_i\right)c + p_j(x)(t) \quad \text{for} \quad t\in[a,b] \tag{7.3$_j$}$$

for every natural number j.

According to (1.17_1), (1.19_1) and (7.1), from (1.5_0) and (7.3_k) we find that

$$M_k c - \int_a^b dL(t)\cdot p_k(x)(t) = 0.$$

Therefore in view of the fact that M_k is a nonsingular matrix, we have

$$c = M_k^{-1}\int_a^b dL(t)\cdot p_k(x)(t).$$

Substituting this value of c into (7.3_m), we get

$$x(t) = p_m(x)(t) + \left(\sum_{i=0}^{m-1}[A(t)]_i\right)\int_a^b d(M_k^{-1}L(t))\cdot p_k(x)(t). \tag{7.4}$$

On the other hand, by (1.19_1) and (7.1), we have

$$|p_j(x)(t)| \le V_j(A)(t)\cdot |x|_s \quad \text{for} \quad t\in[a,b] \quad (j=1,2,\dots).$$

From this and (2.3), owing to (7.4), it follows that

$$|x|_s \le M_{k,m}|x|_s$$

and

$$(I_n - M_{k,m})|x|_s \le 0.$$

Hence according to (2.2), we obtain

$$|x|_s \leq 0.$$

Consequently, $x(t) \equiv 0$. Thus the sufficiency of the conditions of the theorem is proved for the absence of nontrivial solutions to the problem (1.1_0), (1.5_0).

Let us now prove the necessity. Let the problem (1.1_0), (1.5_0) have no nontrivial solutions. Then the inequality (1.28) holds, where the matrix D is defined by (1.25), and Y is an arbitrary fundamental matrix of the system (1.1_0). For definiteness we mean that

$$Y(a) = I_n.$$

Assume

$$Y_k(t) = \sum_{i=0}^{k-1} [A(t)]_i \quad \text{for} \quad t \in [a,b] \quad (k = 1, 2, \dots). \tag{7.5}$$

Analogously to (7.3_j) we show that

$$Y(t) = \sum_{i=0}^{k-1} [A(t)]_i + p_k(Y)(t) \quad \text{for} \quad t \in [a,b] \quad (k = 1, 2, \dots). \tag{7.6}$$

We now estimate $\|p_k(Y)\|_s$. Let $r_0 = \|Y\|_s$. It is clear that $(I_n - d_1 A(t))^{-1}$ is a bounded matrix-function on $[a,b]$. Therefore

$$r = \sup \big\{ \|(I_n - d_1 A(t))^{-1}\| : t \in [a,b] \big\} < \infty.$$

Taking into account the fact that A_- is a continuous from the left matrix-function and $V(A_-)$ is nondecreasing, by (1.34) we estimate

$$\|p_1(Y)(t)\| \leq \|(I_n - d_1 A(t))^{-1}\| \int_a^t \|Y(\tau)\| d\|V(A_-)(\tau)\| \leq r r_0 \|V(A_-)(t)\|,$$

$$\|p_2(Y)(t)\| \leq \|(I_n - d_1 A(t))^{-1}\| \int_a^t \|p_1(Y)(\tau)\| d\|V(A_-)(\tau)\| \leq$$

$$\leq r^2 r_0^2 \int_a^t \|V(A_-)(\tau)\| d\|V(A_-)(\tau)\| \leq \frac{r_0 r^2}{2!} \|V(A_-)(t)\|^2.$$

Using the method of induction, we obtain

$$\|p_k(Y)(t)\| \leq \frac{r_0 (r\|V(A_-)(t)\|)^k}{k!} \leq \frac{r_0 (r\|V(A_-)(b)\|)^k}{k!} \quad \text{for} \quad t \in [a,b] \quad (k = 1, 2, \dots). \tag{7.7}$$

According to (7.7), from (7.5) and (7.6) it follows that

$$\lim_{k \to \infty} \|Y_k - Y\|_s = 0. \tag{7.8}$$

Moreover,

$$\|\ell(Y_k)-\ell(Y)\| \le \int\limits_a^b \|Y_k(t)-Y(t)\| d\|V(L)(t)\| \le \|V(L)(b)\| \cdot \|Y_k - Y\|_s.$$

Therefore by (7.8) we have

$$\lim_{k\to\infty} \ell(Y_k) = \ell(Y).$$

But in view of (2.1) and (7.5), we have

$$\ell(Y_k) = -M_k,$$

and hence

$$\lim_{k\to\infty} M_k = -\ell(Y).$$

From the above arguments and (1.28), there exist a natural number k_0 and a positive number α such that

$$\det(M_k) \ne 0, \quad \|M_k^{-1}\| < \alpha \quad (k = k_0, k_0+1, \dots). \tag{7.9}$$

Moreover, as above, it is easy to verify that

$$\|V_1(A)(t)\| \le r\|V(A_-)(t)\| \quad \text{for} \quad t \in [a,b],$$

$$\|V_2(A)(t)\| \le \int\limits_a^t \|V_1(A)(\tau)\| d\|V(A_-)(\tau)\| \le$$

$$\le r^2 \int\limits_a^t \|V(A_-)(\tau)\| d\|V(A_-)(\tau)\| \le$$

$$\le \frac{r^2}{2!}\, \|V(A_-)(t)\|^2 \quad \text{for} \quad t \in [a,b],$$

and so on. Thus

$$\|V_k(A)(t)\| \le \frac{1}{k!}\,(r\|V(A_-)(t)\|)^k \le \frac{1}{k!}\,(r\|V(A_-)(b)\|)^k$$
$$\text{for} \quad t \in [a,b] \quad (k = 1,2,\dots).$$

Taking into account these estimates and (7.9), from (2.3) we get

$$\lim_{k,m\to\infty} M_{k,m} = O_{n\times n}.$$

Thus the inequality (2.2) holds for some sufficiently large k and m. The theorem has been proved for $l = 1$.

Let now $l = 2$. For this case we define the operators p_i $(i = 0,1,\dots)$ by

$$p_0(X)(t) = X(t),$$

$$p_i(X)(t) \equiv (I_n + d_2A(t))^{-1} \int\limits_b^t dA(\tau+)\cdot p_{i-1}(X)(\tau) \quad (i = 1,2,\dots)$$

instead of (7.1).

We use the equality

$$x(t) = c + \int_b^t dA(\tau) \cdot x(\tau) \quad \text{for} \quad t \in [a,b]$$

instead of (7.2).

Acting analogously as in proving the case $l = 1$, we can easily show that the theorem is true in this case, as well. □

Proof of Theorem 2.1′. The proof is analogous to that of Theorem 2.1.

Let $l = 1$, and let $p_i : \mathrm{BV}([a,b], \mathbb{R}^{n\times l}) \to \mathrm{BV}([a,b], \mathbb{R}^{n\times l})$ $(i = 0, 1, \dots)$ be the operators the defined by

$$p_0(X)(t) \equiv X(t), \quad p_i(X)(t) \equiv \int_a^t dA(\tau) \cdot p_{i-1}(X)(\tau) \quad (i = 1, 2, \dots).$$

Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of the problem (1.1_0), (1.5_0). Then by virtue of (7.2),

$$\begin{aligned} x(t) = c + p_1(x)(t) = c + \int_a^t dA(\tau) \cdot (c + p_1(x)(\tau) = \\ = [I_n + (A(t))_1]c + \int_a^t dA(\tau) \cdot p_1(x)(\tau) = [I_n + (A(t))_1]c + p_2(x)(t) = \\ = [I_n + A(t))_1]c + \int_a^t dA(\tau) \cdot \int_a^\tau dA(s)(c + p_1(x)(s)) = \\ = [I_n + (A(t))_1 + (A(t))_2]c + p_3(x)(t) \quad \text{for} \ t \in [a,b], \end{aligned}$$

and so on. Continuing this process infinitely, we obtain

$$x(t) = \left[I_n + \sum_{i=0}^{j-1} (A(t))_i \right] c + p_j(x)(t) \quad \text{for} \quad t \in [a,b] \ \ (j = 1, 2, \dots). \tag{7.10}$$

According to (2.4) and (2.5), from (1.5_0) and (7.10) we can find c as above. Substituting the value of c in (7.10) and acting as above, we find that $x(t) \equiv 0$. The theorem has been proved for $l = 1$.

The proof of the theorem is analogous for the case $l = 2$. We only note that the operators p_i $(i = 0, 1, \dots)$ are defined by

$$p_0(X)(t) \equiv X(t), \quad p_i(X)(t) \equiv \int_b^t dA(\tau) \cdot p_{i-1}(X)(\tau) \quad (i = 1, 2, \dots).$$

The theorem is proved. □

Proof of Corollary 2.1. Let $A_\varepsilon(t) \equiv \varepsilon A(t)$. It is evident that

$$\lim_{\varepsilon\to 0}(I_n + (-1)^j \varepsilon d_j A(t)) = I_n \quad \text{uniformly on } [a,b] \ (j=1,2).$$

Therefore there exists $\varepsilon_1 > 0$ such that

$$\det(I_n + (-1)^j d_j A_\varepsilon(t)) \neq 0 \quad (t \in [a,b], \quad j=1,2)$$

for every $\varepsilon \in]0, \varepsilon_1]$.

If the condition (2.6) holds, then we assume $k = 1$, while if the conditions (2.7)–(2.9) hold, we assume $k = l + 1$. Moreover, we put

$$M_k(\varepsilon) = L(a) - \sum_{i=0}^{k-1} \int_a^b dL(t) \cdot (\varepsilon A(t))_i$$

and

$$M_{k,1}(\varepsilon) = (V(\varepsilon A)(b))_1 + |M_k^{-1}(\varepsilon)| \int_a^b dV(L)(t) \cdot (V(\varepsilon A)(t))_k.$$

In view of the condition (2.6) (of the conditions (2.7)–(2.9)), we can easily verify that

$$M_k(\varepsilon) = \varepsilon^{k-1} M_k, \quad \det(M_k) \neq 0, \quad M_{k,1}(\varepsilon) = \varepsilon M_{k,1},$$

where M_k and M_{k1} are the matrices defined by (2.4) and (2.5), respectively. Let

$$\varepsilon_0 = \min\left\{ \frac{1}{r(M_{k,1})}, \varepsilon_1 \right\}.$$

Then we have

$$r(M_{k,1}(\varepsilon)) < 1$$

for every $\varepsilon \in]0, \varepsilon_0[$. Therefore, according to Theorem 2.1$'$, the problem (1.6), (1.5) has one and only one solution for every $\varepsilon \in]0, \varepsilon_0[$. Thus the corollary is proved. $\square$

Proof of Theorem 2.2. Let $x = (x_i)_{i=1}^n$ be an arbitrary solution of the problem (1.1_0), (1.5_0). Since the problem (2.10), (1.5_0) has only the trivial solution, by (1.27) and the equality

$$dx(t) = dA_0(t) \cdot x(t) + d\bigg(\int_a^t d(A(\tau) - A_0(\tau)) \cdot x(\tau) \bigg)$$

we have the representation

$$x(t) = \int_a^t d_\tau \mathcal{G}_0(t,\tau) \cdot \int_a^\tau d(A(s) - A_0(s)) \cdot x(s).$$

Therefore using the integration by part formula (1.32) and (1.33), we have

$$\begin{aligned}
x(t) = -\int_a^t & \mathcal{G}_0(t,\tau)d(A(\tau)-A_0(\tau))\cdot x(\tau)+ \\
&+\sum_{a<\tau\le t}(\mathcal{G}_0(t,\tau)-\mathcal{G}_0(t,\tau-))\cdot d_1\,(A(\tau)-A_0(\tau))\cdot x(\tau)- \\
&-\sum_{a\le\tau<t}(\mathcal{G}_0(t,\tau+)-\mathcal{G}_0(t,\tau))\cdot d_2\,(A(\tau)-A_0(\tau))\cdot x(\tau)= \\
=-\int_a^t & \mathcal{G}_0(t,\tau)dS_0(A-A_0)(\tau)\cdot x(\tau)- \\
&-\sum_{a<\tau\le t}\mathcal{G}_0(t,\tau-)d_1\,(A(\tau)-A_0(\tau))\cdot x(\tau)- \\
&-\sum_{a\le\tau<t}\mathcal{G}_0(t,\tau+)d_2\,(A(\tau)-A_0(\tau))\cdot x(\tau)
\end{aligned}$$

and

$$\begin{aligned}
|x(t)|\le\int_a^t & |\mathcal{G}_0(t,\tau)|dV(S_0(A-A_0))(\tau)\cdot|x(\tau)|+ \\
&+\sum_{a<\tau\le t}|\mathcal{G}_0(t,\tau-)\,d_1(A(\tau)-A_0(\tau))|\cdot|x(\tau)|+ \\
&+\sum_{a\le\tau<t}|\mathcal{G}_0(t,\tau+)\,d_2(A(\tau)-A_0(\tau))|\cdot|x(\tau)|\le \\
&\le M|x|_s \quad\text{for}\quad t\in[a,b].
\end{aligned}$$

Hence

$$(I_n-M)|x|_s\le 0.$$

From the above, owing to (2.12), it follows that $x(t)\equiv 0$. Consequently, the problem (1.1), (1.5) has one and only one solution. Thus the theorem is proved. □

Theorems 2.3 and 2.4 and Corollaries 2.2 and 2.4 follow immediately from Theorems 1.1 and 2.1 and Corollaries 1.1 and 2.1 if we assume that

$$L(t)=-\sum_{j=1}^{n_0}\chi_{[a,t_j]}(t)L_j,$$

where $\chi_{[a,t_j]}$ is the characteristic function of the set $[a,t_j]$ $(j=1,\dots,n_0)$.

Corollary 2.3 follows from Theorem 2.4$'$ if we assume $k=1$ and $m=1$.

Proof of Theorem 2.5. According to Theorem 1.1, to prove the theorem it is sufficient to verify that the homogeneous problem (1.1_0), (1.4_0) has only the trivial solution.

Let $(x_i)_{i=1}^n$ be an arbitrary solution of the problem (1.1_0), (1.4_0). We assume

$$\overline{x}_i(t) = |x_i(t)| \quad (i = 1, \dots, n).$$

Then by (2.17)–(2.21) and Lemma 6.1, we have

$$\operatorname{sgn}(t - t_i) d\overline{x}_i(t) \le \sum_{l=1}^n \overline{x}_l(t) dc_{il}(t) \quad \text{for} \quad t \in [a, b], \ \ t \ne i_i \ \ (i = 1, \dots, n),$$

$$(-1)^j d_j \overline{x}_i(t_i) \le \sum_{l=1}^n \overline{x}_l(t_i) d_j c_{il}(t_i) \quad (j = 1, 2; \ \ i = 1, \dots, n)$$

and

$$\overline{x}_i(t_i) \le \ell_{0i}(\overline{x}_1, \dots, \overline{x}_n) \quad (i = 1, \dots, n).$$

Hence $(\overline{x}_i)_{i=1}^n$ is a nonnegative solution of the problem (1.23), (1.24). Therefore by (2.16), $\overline{x}_i(t) \equiv 0$ $(i = 1, \dots, n)$ and

$$x_i(t) \equiv 0 \quad (i = 1, \dots, n). \qquad \square$$

Proof of Theorem 2.6. By Lemma 6.3, the condition (2.16) holds for $C = (c_{il})_{i,l=1}^n$ and $\ell_0 = (\ell_{0i})_{i=1}^n$, where

$$c_{il}(t) = \int_a^t h_{il}(\tau) d\alpha_l(\tau) \quad \text{for} \quad t \in [a, b] \quad (i, l = 1, \dots, n)$$

and

$$\ell_{0i}(x_1, \dots, x_n) = \sum_{m=0}^2 \sum_{k=1}^n \ell_{mik} \|x_k\|_{\nu, S_m(\alpha_k)}$$
$$\text{for} \quad (x_l)_{l=1}^n \in \mathrm{BV}([a, b], \mathbb{R}^n) \quad (i = 1, \dots, n).$$

Therefore the theorem follows from Theorem 2.5. $\square$

Remark 2.1 follows from the fact that Lemma 6.3 is also true for the $n \times n$-matrix described in this remark.

Corollary 2.5 is a particular case of Theorem 2.6, when $\ell_{mki} = 0$ $(m = 0, 1, 2;\ i, k = 1, \dots, n)$.

Proof of Theorem 2.7. By Lemma 6.4, the condition (2.16) holds for $C = (c_{il})_{i,l=1}^n$ and $\ell = (\ell_{0i})_{i=1}^n$, where

$$c_{il}(t) = h_{il} \alpha_i(t) + \alpha_{il}(t) \quad \text{for} \quad t \in [a, b] \quad (i, l = 1, \dots, n)$$

and

$$\ell_{0i}(x_1, \dots, x_n) = |\mu_i| x_i(s_i) \ \ \text{for} \ \ (x_l)_{l=1}^n \in \mathrm{BV}([a, b], \mathbb{R}_+^n) \ \ (i = 1, \dots, n).$$

Therefore the theorem follows from Theorem 2.5. $\square$

Proof of Theorem 2.8. Note that the problem (1.23), (1.24) is a particular case of the problem (6.5), (6.6) if we assume in it $C_{11}(t) \equiv C(t)$, $C_{12}(t) = C_{21}(t) = C_{22}(t) \equiv O_{n \times n}$ and $\ell_{01i}(x_1, \dots x_{2n}) \equiv \ell_{0i}(x_1, \dots, x_n)$ $(i = 1, \dots, n)$, $\ell_{02i}(x_1, \dots, x_{2n}) \equiv 0$ $(i = 1, \dots, n)$.

By Lemma 6.5 there exist a matrix-function $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and linear continuous functionals ℓ_i $(i=1,\dots,2n)$ defined by (6.11) and (6.15), respectively, and numbers c_{0i} $(i=1,\dots,2n)$ such that the $2n$-system (6.7) is unsolvable under the $2n$-condition (1.4), where $\widetilde{A}(t)$ is defined by (6.8). Moreover, it is evident that the system (6.7) is equivalent to the system (1.1_0). Therefore the problem (1.1_0), (1.4) is unsolvable for the matrix-function A and linear functionals ℓ_i $(i=1,\dots,n)$.

Due to (2.40), (6.11) and (6.15), it is not difficult to verify that the conditions (2.17)–(2.21) are fulfilled.

Let now the condition (2.41) hold. By (6.11), we get

$$d_jA(t) = \operatorname{diag}\left(\operatorname{sgn}(t-t_1),\dots,\operatorname{sgn}(t-t_n)\right) d_jC(t) \operatorname{diag}(\eta_1(t),\dots,\eta_n(t)) \quad \text{for} \quad t\in[a,b] \quad (j=1,2).$$

Therefore, in view of (2.41), the condition (1.21) holds. Thus the theorem is proved. □

Consider *Remark* 2.4. The first case is evident. Indeed, by (6.11),

$$d_ja_{il}(t) = \operatorname{sgn}(t-t_i)\eta_l(t)d_jc_{il}(t) \quad \text{for} \quad t\in[a,b] \quad (j=1,2;\ \ i,l=1,\dots,n)$$

and

$$|d_ja_{il}(t)| \le |d_jc_{il}(t)| \quad \text{for} \quad t\in[a,b] \quad (j=1,2;\ \ i,l=1,\dots,n).$$

Taking this into account, by (2.42) we have

$$\sum_{l=1}^n |d_ja_{il}(t)| < 1 \quad \text{for} \quad t\in[a,b] \quad (j=1,2;\ \ i=1,\dots,n).$$

Hence the condition (1.21) holds.

Let now the conditions (2.43) and (2.44) be valid. Then from (2.44) we have

$$\sum_{l=1,\ l\neq i}^n |\operatorname{sgn}(t-t_l)\cdot\varepsilon_i d_jc_{il}(t)| \le |\varepsilon_i + (-1)^j\operatorname{sgn}(t-t_i)\cdot\varepsilon_i d_jc_{ii}(t)| \quad \text{for} \quad t\in[a,b] \quad (j=1,2;\ \ i=1,\dots,n). \tag{7.11}$$

Using (2.43), we obtain

$$|\varepsilon_i + (-1)^j\operatorname{sgn}(t-t_i)\cdot\varepsilon_i d_jc_{ii}(t)| \le 1 + (-1)^j\operatorname{sgn}(t-t_i)\varepsilon_i d_jc_{ii}(t) \quad \text{for} \quad t\in[a,b] \quad (j=1,2;\ \ i=1,\dots,n).$$

This and (7.11) yield

$$\sum_{l=1,\ l\neq i}^n |\operatorname{sgn}(t-t_l)\cdot\varepsilon_i d_jc_{il}(t)| < 1 + (-1)^j\operatorname{sgn}(t-t_i)\varepsilon_i d_jc_{ii}(t) \quad \text{for} \quad t\in[a,b] \quad (j=1,2;\ \ i=1,\dots,n).$$

Therefore by Hadamard's theorem (see [12, p. 382]), the condition (1.21) holds. Remark 2.4 is proved analogously for the second case of (2.44). □

7.2. **Proof of Results of Section 3.** Below, for the theorems and corollaries of Section 3 we assume that the matrix-function $A(t)$ and the vector-function $f(t)$ are defined by

$$\begin{gathered}A(a)=O_{n\times n},\quad f(a)=0,\\ A(t)=\int\limits_a^t P(\tau)\,d\tau+\sum_{a\le\tau_k<t}G_k,\\ f(t)=\int\limits_a^t q(\tau)\,d\tau+\sum_{a\le\tau_k<t}g_k\ \text{ for }\ a<t\le b.\end{gathered}\tag{7.12}$$

Note that by virtue of (1.8) we have $A\in\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and $f\in\mathrm{BV}([a,b],\mathbb{R}^n)$. In addition, A and f are continuous from the left, i.e.

$$d_1A(t)=O_{n\times n}\quad\text{and}\quad d_1f(t)=0.\tag{7.13}$$

Moreover,

$$\begin{gathered}d_2A(t)=O_{n\times n}\quad\text{and}\quad d_2f(t)=0\quad\text{if}\ \ t\notin\{\tau_0,\tau_1,\dots\},\\ d_2A(\tau_k)=G_k\quad\text{and}\quad d_2f(\tau_k)=g_k\quad(k=1,2,\dots),\end{gathered}\tag{7.14}$$

$$S_0(A)(t)\equiv\int\limits_a^t P(\tau)\,d\tau,\quad S_0(f)(t)\equiv\int\limits_a^t q(\tau)\,d\tau,\tag{7.15}$$

$$\begin{gathered}S_1(A)(t)\equiv O_{n\times n},\quad S_1(f)(t)\equiv 0,\\ S_2(A)(t)\equiv\sum_{a\le\tau_k<t}G_k,\quad S_2(f)(t)\equiv\sum_{a\le\tau_k<t}g_k.\end{gathered}\tag{7.16}$$

Theorems 3.1–3.5 and *Corollaries* 3.1–3.3 follow from Theorems 1.1, 2.1–2.4 and Corollaries 2.1–2.4, respectively, taking into account (7.12)–(7.16). □

Proof of Theorem 3.6. The theorem follows from Theorem 2.5 if we assume in it that

$$c_{il}(t)\equiv\int\limits_a^t q_{il}(\tau)\,d\tau+\sum_{a\le\tau_k<t}h_{kil}\quad(i,l=1,\dots,n)\tag{7.17}$$

and take into account that owing to the equalities (7.13)–(7.16) the condition (2.16) has the form (3.7), the inequalities (2.17) and (2.19) are equivalent to the inequalities (3.8) and (3.10), respectively, and the inequalities (2.18) and (2.20) coincide with the inequalities (3.9) and (3.11), respectively. □

Proof of Theorem 3.7. In Theorem 2.6 we assume

$$\alpha_i(t)=t-a+\sum_{a_k\le\tau_l<t}\alpha_{li}\quad(i=1,\dots,n)$$

and

$$\ell_{1ik} = \gamma_{1ik}, \quad \ell_{2ik} = 0, \quad \ell_{3ik} = \gamma_{2ik} \quad (i,k = 1,\dots,n).$$

Then by virtue of (7.13)–(7.15) the conditions (2.22)–(2.25) are transformed into the conditions (3.12)–(3.15), respectively, and for the $3n \times 3n$-matrix $\mathcal{H} = (\mathcal{H}_{j+1,m+1})_{j,m=0}^{2}$ we have $\mathcal{H}_{2j} = O_{n\times n}$ $(j = 1,2,3)$, $\mathcal{H}_{j2} = O_{n\times n}$ $(j = 1,2,3)$, $\mathcal{H}_{11} = \mathcal{H}_{011}$, $\mathcal{H}_{13} = \mathcal{H}_{012}$, $\mathcal{H}_{31} = \mathcal{H}_{021}$, $\mathcal{H}_{33} = \mathcal{H}_{022}$ and $r(\mathcal{H}) = r(\mathcal{H}_0)$, since in this case

$$\xi_{i0} = (b-a)^{\frac{1}{\nu}}, \quad \xi_{i1} = 0, \quad \xi_{i2} = \left(\sum_{k=0}^{m_0} \alpha_{ki}\right)^{\frac{1}{\nu}} \quad (i = 1,\dots,n),$$

$$\lambda_{k0i0} = \left[\frac{2}{\pi}\,(b-a)\right]^{\frac{2}{\nu}} \quad (i,k = 1,\dots,n),$$

$$\lambda_{k1ij} = 0 \quad (j = 0,1,2;\ \ i,k = 1,\dots,n),$$

$$\lambda_{kji1} = 0 \quad (j = 0,2;\ \ i,k = 1,\dots,n),$$

$$\lambda_{k2i2} = \left(\frac{1}{4}\,\mu_i\mu_k \sin^{-2}\frac{\pi}{4n_k+2}\right)^{\frac{1}{\nu}} \quad (i,k = 1,\dots,n).$$

Thus Theorem 3.7 follows from Theorem 2.6. □

Corollary 3.4 is a particular case of Theorem 3.6, when $\gamma_{mik} = 0$ $(m = 1,2;\ i,k = 1,\dots,n)$.

Corollary 3.5 is a particular case of Corollary 3.4 if we assume $h_{il}(t) \equiv h_{il} = const$ $(i,l = 1,\dots,n)$ and $\mu = +\infty$.

Theorem 3.8 follows from Theorem 2.7 if we assume

$$\alpha_i(t) \equiv \int_a^t \beta_i(\tau)\,d\tau + \sum_{a\le\tau_k<t} \beta_{ki} \quad (i = 1,\dots,n),$$

$$\alpha_{il}(t) \equiv \int_a^t \beta_{il}(\tau)\,d\tau + \sum_{a\le\tau_k<t} \beta_{kil} \quad (i,l = 1,\dots,n)$$

and apply the equalities (7.13) and (7.14).

Corollary 3.6 is a particular case of Theorem 3.7, when

$$\beta_i(t) \equiv 1, \quad \beta_{il}(t) \equiv 0 \quad (i,l = 1,\dots,n),$$

$$\beta_{ki} = 0 \quad (i = 1,\dots,n; \quad k = 1,\dots,m_0).$$

Corollary 3.7 is a particular case of Corollary 3.6, when $g_{kii} = 0$ $(i = 1,\dots,n;\ k = 1,\dots,m_0)$.

Theorem 3.9 follows from Theorem 2.8 if we define the functions c_{il} $(i,l = 1,\dots,n)$ by (7.17) and take into account (7.13) and (7.14).

7.3. **Proof of the Results of Section 4.**

Proof of Theorem 4.1. The problem (1.9), (1.10) is equivalent to the $2n$-problem (1.1), (1.2), where $A(t)$ and $f(t)$ are defined by (4.7), $n_0 = n_{01}+n_{02}$,

t_j ($j = 1, \dots, n_0$) are defined by (4.8), and L_j are defined by (4.9). The above-said shows that

$$d_1 A(t) = O_{2n\times 2n} \quad\text{and}\quad d_1 f(t) = 0 \qquad \text{for } \ t \in [0, m_0] \setminus \left\{\frac{1}{2}, \frac{3}{2}, \dots, m_0 - \frac{1}{2}\right\}, \tag{7.18}$$

$$d_1 A\left(k - \frac{1}{2}\right) = G(k) \quad\text{and}\quad d_1 f\left(k - \frac{1}{2}\right) = g(k) \ \text{ for } \ k \in \mathbb{N}_{m_0}, \tag{7.19}$$

$$d_2 A(t) \equiv O_{2n\times 2n}, \quad d_2 f(t) \equiv 0, \tag{7.20}$$

$$S_0(A)(t) \equiv O_{2n\times 2n}, \quad S_0(f)(t) \equiv 0, \tag{7.21}$$

$$\begin{aligned} &S_1(A)(t) \equiv \sum_{i=1}^{[t+\frac{1}{2}]} G(i), \quad S_1(f)(t) \equiv \sum_{i=1}^{[t+\frac{1}{2}]} f(i), \\ &S_2(A)(t) \equiv O_{2n\times 2n}, \quad S_2(f)(t) \equiv 0, \end{aligned} \tag{7.22}$$

where $G(k) = (G_{lm}(k))_{l,m=1}^2$ and $g(k) = (g_l(k))_{l=1}^2$ are defined, respectively, by (4.4) and (4.5).

Using Theorem 1.1, we can see that the first part of Theorem 4.1 is valid.

Let us show the representation (4.28). By (1.32), (1.33), (2.14) and (7.18)–(7.22), we have

$$\begin{aligned} &\int_0^{m_0} d_s \mathcal{G}(t,s) \cdot f(s) = \\ &\quad = \mathcal{G}(t, m_0) \cdot f(m_0) - \int_0^{m_0} \mathcal{G}(t,s)\, df(s) + \sum_{0<s\le m_0} d_1 \mathcal{G}(t,s) \cdot d_1 f(s) = \\ &\quad = -\sum_{0<s\le m_0} \mathcal{G}(t,s) \cdot d_1 f(s) + \sum_{0<s\le m_0} d_1 \mathcal{G}(t,s) \cdot d_1 f(s) = \\ &\quad = -\sum_{0<s\le m_0} \mathcal{G}(t,s-)\, d_1 f(s) = -\sum_{i=1}^{m_0} \mathcal{G}(t,i-) \cdot g(i) \quad \text{for} \quad t \in [0, m_0]. \end{aligned}$$

Therefore, by (1.27),

$$x(t) = x_0(t) - \sum_{i=1}^{m_0} \mathcal{G}(t, i-) \cdot g(i) \quad \text{for} \quad t \in [0, m_0], \tag{7.23}$$

where x_0 is a solution of the problem (1.1_0), (1.2) corresponding to the problem (1.9_0), (1.10).

If we take into account the fact that the functions x, x_0, $\mathcal{G}(\cdot, i)$ and $\mathcal{G}(k, \cdot)$ are jump functions continuous from the right, then from (7.23) we get

$$x(k) = x_0(k) - \sum_{i=1}^{m_0} \mathcal{G}(k, i-) \cdot g(i) \quad \text{for} \quad k = 1, \dots, m_0,$$

whence by virtue of (4.2), (4.5) and (4.6) we obtain

$$(I_n + G_1(k))y(k) = (I_n + G_1(k))y_0(k) - \sum_{i=1}^{m_0} \mathcal{G}_{11}(k, i-1) \cdot g(i) \quad \text{for} \quad k = 1, \dots, m_0.$$

Consequently, the equalities (4.28) hold. The theorem is proved. □

Remark 4.1 follows from Remark 1.2.

Proof of Corollary 4.1. By (4.10), (7.18)–(7.22) and Corollary 2.2, the problem (1.9), (1.10) is uniquely solvable iff

$$\det\left(\sum_{j=1}^{n_0} D_j\right) \neq 0, \tag{7.24}$$

where $n_0 = n_{01} + n_{02}$,

$$\begin{aligned} D_j &= L_j \prod_{i=k_j}^{1} (I_{2n} - G(i))^{-1} \quad (j = 1, \dots, n_{01}), \\ D_j &= L_j \prod_{i=k_j+1}^{1} (I_{2n} - G(i))^{-1} \quad (j = n_{01}+1, \dots, n_0), \end{aligned} \tag{7.25}$$

$G(i) = (G_{lm}(i))_{l,m=1}^2$ is defined by (4.4), and L_j $(j = 1, \dots, n_0)$ are defined by (4.9).

It is not difficult to verify that

$$(I_{2n} - G(i))^{-1} \equiv \begin{pmatrix} O_{n\times n} & I_n + G_1(i) \\ -G_3^{-1}(i) & G_3^{-1}(i)(I_n - G_2(i)) \end{pmatrix}. \tag{7.26}$$

From this, in view of (7.25), we have

$$\begin{aligned} D_j &= L_j Y(k_j) \ \ (j = 1, \dots, n_{01}), \\ D_j &= L_j Y(k_j + 1) \ \ (j = n_{01}+1, \dots, n_0), \end{aligned} \tag{7.27}$$

where $Y(k) \equiv (Y_{lm}(k))_{l,m=1}^2$ is defined by (4.19).

According to (4.9), we conclude that

$$D_j = \begin{pmatrix} L_{1j}(I_n + G_1(k_j))^{-1} Y_{11}(k_j) & L_{1j}(I_n + G_1(k_j))^{-1} Y_{12}(k_j) \\ O_{n\times n} & O_{n\times n} \end{pmatrix} \quad (j = 1, \dots, n_{01})$$

and

$$D_j = \begin{pmatrix} O_{n\times n} & O_{n\times n} \\ L_{2j} Y_{21}(k_j + 1) & L_{2j} Y_{22}(k_j + 1) \end{pmatrix} \ \ (j = n_{01}+1, \dots, n_0).$$

Therefore, by (7.24), the condition (4.29) holds. □

Theorems 4.2 and 4.3 and *Corollaries* 4.2 and 4.3 follow from the corresponding results of Section 2 if we apply them to the $2n$-problem (1.1), (1.2) corresponding to the problem (1.9), (1.10), and take into account the equalities (4.6)–(4.9) and (4.20)–(4.24).

Proof of Theorem 4.4. To prove the theorem, we use Theorem 2.5. Consider the $2n$-problem (1.1), (1.4), where the matrix- and the vector-functions $A(t)$ and $f(t)$ are defined by (4.4), (4.5) and (4.7), t_i $(i=1,\dots,2n)$ and $c_{0i}(i=1,\dots,2n)$ are defined by (4.13), and ℓ_i $(i=1,\dots,2n)$ are defined by (4.14) and (4.15).

Consider the matrix-function $C=(c_{il})_{i,l=1}^{2n}$ and the operator $\ell_0=(\ell_{0i})_{i=1}^{n}$ appearing in the condition (2.16).

Let

$$C(t)=(C_{jm}(t))_{j,m=1}^{2} \quad \text{for} \quad t\in[0,m_0],$$

where the matrix-functions $C_{jm}(t)$ $(j,m=1,2)$ are defined by (4.27).

For $x=(x_l)_{l=1}^{2}\in \mathrm{BV}([0,m_0],\mathbb{R}_+^{2n})$, $x_l=(x_{li})_{i=1}^{n}$ $(l=1,2)$, we define the nonnegative operators

$$\begin{aligned}\ell_{0i}(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n}) = \\ =\big([(I_n+G_1(k_i))^{-1}]_{ii}\big)^{-1}\,\ell_{01i}(y_1,\dots,y_n)+ \\ +\big((1+g_{1ii}(k_i))[(I_n+G_1(k_i))^{-1}]_{ii}\big)^{-1}\times \\ \times \sum_{l,j=1;\,l\neq i,\,j\neq i}^{n} g_{1il}(k_i)\,[(I_n+G_1(k_i))^{-1}]_{lj}x_{1l}(k_i) \quad (i=1,\dots,n)\end{aligned} \tag{7.28}$$

and

$$\ell_{0\,n+i}(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})=\ell_{02i}(y_1,\dots,y_n) \ \ (i=1,\dots,n), \tag{7.29}$$

where

$$(y_l(k))_{l=1}^{n}=(I_n+G_1(k))^{-1}x_1(k) \quad \text{for} \quad k\in\{0,\dots,m_0\}. \tag{7.30}$$

Let us now verify the conditions of Theorem 2.5.

The conditions (2.17) and (2.19) are evident owing to $S_0(A)(t)=S_0(C)(t)\equiv O_{n\times n}$. The conditions (2.18) and (2.20) are of the form (4.35)–(4.37). As to the estimate (2.21), it follows from (4.38) due to the condition (4.33).

We now show that the conclusion (2.16) is true.

Let $x=(x_l)_{l=1}^{2}\in \mathrm{BV}([0,m_0],\mathbb{R}_+^{2n})$, $x_l=(x_{li})_{i=1}^{n}$ $(l=1,2)$ be a nonnegative solution of the problem (1.23), (1.24). Then by the definition of the matrix-function $C=(C_{jm})_{j,m=1}^{2}$ and due to (4.33), the vector-function $y(k)=(y_l(k))_{l=1}^{n}$ defined by (4.10) will be a nonnegative solution of the system of difference inequalities (4.25).

Moreover, by (4.16) and (7.28), we have

$$\begin{aligned}\ell_{0i}(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})-x_{1i}(k_i)= \\ =\big([(I_n+G_1(k_i))^{-1}]_{ii}\big)^{-1}\,(\ell_{01i}(y_1,\dots,y_n)-y_i(k_i)) \quad (i=1,\dots,n).\end{aligned} \tag{7.31}$$

With regard for (1.24), (4.15), (4.38), (7.30) and (7.31), the vector-function $y(k) = (y_l(k))_{l=1}^2$ satisfies the inequalities (4.26). Therefore, due to the condition (4.34), we have

$$y(k) = 0 \quad \text{for} \quad k \in \{0, \dots, m_0\}.$$

As a result, from the above reasoning and (4.1) we obtain

$$x_1(k) = 0 \quad \text{and} \quad x_2(k) = 0 \quad (k = 0, \dots, m_0).$$

Thus the conditions of Theorem 2.5 are fulfilled. On the basis of this theorem, the generalized boundary value problem corresponding to the problem (1.9), (1.12) has one and only one solution. Consequently, the problem (1.9), (1.12) has one and only one solution, as well. Thus the theorem is proved. □

Proof of Theorem 4.5. Consider the $2n$-problem (1.1), (1.4) corresponding to the problem (1.9), (1.12), where the matrix– and the vector-functions $A(t)$ and $f(t)$ are defined by (4.4), (4.5) and (4.7), t_i ($i = 1, \dots, 2n$) and c_{0i} ($i = 1, \dots, 2n$) are defined by (4.13), and ℓ_i ($i = 1, \dots, 2n$) are defined by (4.14) and (4.15).

Let

$$\alpha_l(t) = [t] \quad \text{for} \quad 0 \le t \le m_0 \quad (l = 1, \dots, n).$$

We verify the conditions of Theorem 2.6.

The conditions (2.22) and (2.24) are trivial. By the definition of the matrix-function $A(t)$, every condition from (2.23) and (2.25) contains $2n$ inequalities which are equivalent to the conditions (4.39)–(4.41) if we assume

$$\begin{aligned}
&h_{il}(t) \equiv h_{1il}([t]) \quad (i, l = 1, \dots, n),\\
&h_{il}(t) \equiv h_{2il}([t]) \quad (i = 1, \dots, n; \ l = n+1, \dots, 2n),\\
&h_{il}(t) \equiv [(I_n + G_1([t]))^{-1}]_{il} \quad (i = n+1, \dots, 2n; \ l = 1, \dots, n),\\
&h_{il}(t) \equiv \delta_{il} \quad (i, l = n+1, \dots, 2n).
\end{aligned}$$

Consider now the condition (2.26). Let $x = (x_l)_{l=1}^2$, $x_l = (x_{li})_{i=1}^n \in \mathrm{BV}([0, m_0], \mathbb{R}^n)$ $(l=1,2)$, and let $\ell_i(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n})$ ($i = 1, \dots, n$) and $\ell_{n+i}(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n})$ be defined by (4.14) and (4.15), respectively, where $y(k) = (y_i(k))_{i=1}^n$ is defined by (4.10). Then taking into account (4.33), from (4.17) and (4.42) we obtain

$$\begin{aligned}
&|\ell_i(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n})| \le \big([(I_n + G_1(k_i))^{-1}]_{ii}\big)^{-1} \times\\
&\times \left(\left|\ell_{1i}(y_1, \dots, y_n) + [(I_n + G_1(k_i))^{-1}]_{ii} \sum_{l=1}^n [I_n + G_1(k_i)]_{il} y_l(k_i) - y_i(k_i)\right|\right) \le\\
&\le \big([(I_n + G_1(k_i))^{-1}]_{ii}\big)^{-1} \sum_{l=1}^n \gamma_{1il} \|x_{1l}\|_{\nu, \alpha_l}
\end{aligned}$$

$$\text{for} \quad x_1 = (x_{1l})_{l=1}^n, \ x_2 = (x_{2l})_{l=1}^n \in \mathrm{BV}([0, m_0], \mathbb{R}^n) \quad (i = 1, \dots, n),$$

and from (4.15) and (4.43) we have

$$|\ell_{n+i}(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})| \le \sum_{l=1}^{n} \gamma_{2il}\|x_{2il}\|_{\nu,\alpha_l}$$
$$\text{for} \quad x_1=(x_{1l})_{l=1}^n\,,\ x_2=(x_{2l})_{l=1}^n \in \mathrm{BV}([0,m_0],\mathbb{R}^n) \quad (i=1,\dots,n).$$

Therefore

$$|\ell_i(x_{11},\dots,x_{1n},x_{21},\dots,x_{2n})| \le \sum_{k=1}^{n} \ell_{ik}\|x_{1k}\|_{\nu,\alpha_k} + \sum_{k=n+1}^{2n} \ell_{2k}\|x_{2k}\|_{\nu,\alpha_k}$$
$$\text{for} \quad x_1=(x_{1l})_{l=1}^n\,,\ x_2=(x_{2l})_{l=1}^n \in \mathrm{BV}([0,m_0],\mathbb{R}^n) \quad (i=1,\dots,2n),$$

where

$$\begin{aligned}
&\ell_{ik} = \big([(I_n+G_1(k_i))^{-1}]_{ii}\big)^{-1}\gamma_{1ik} \quad (i,k=1,\dots,n),\\
&\ell_{ik}=0 \quad (i=1,\dots,n;\ \ k=n+1,\dots,2n),\\
&\ell_{ik}=0 \quad (i=n+1,\dots,2n;\ \ k=1,\dots,n),\\
&\ell_{ik}=\gamma_{2ik} \quad (i,k=n+1,\dots,2n).
\end{aligned}$$

Thus the condition (2.26) holds for $\ell_{1ik}=\ell_{ik}$ $(i,k=1,\dots,2n)$, because $S_0(\alpha_l)(t)\equiv 0$ and $S_2(\alpha_l)(t)\equiv 0$.

Let $\mathcal{H}=(\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$, $\mathcal{H}_{j+1\,m+1}\in\mathbb{R}^{2n\times 2n}$ $(j,m=0,1,2)$ be the $6n\times 6n$-matrix appearing in Theorem 2.6. Then by the definition of the functions $\alpha_l(t)$ $(l=1,\dots,n)$, it is not difficult to verify that

$$\begin{aligned}
&\mathcal{H}_{1m}=O_{2n\times 2n}, \quad \mathcal{H}_{3m}=O_{2n\times 2n} \quad (m=1,2,3),\\
&\mathcal{H}_{21}=O_{2n\times 2n}, \quad \mathcal{H}_{23}=O_{2n\times 2n}
\end{aligned}$$

and

$$\mathcal{H}_{22}=\mathcal{H}^*.$$

Therefore the condition (2.27) is equivalent to the condition (4.44). Consequently, using Theorem 2.6, we have proved Theorem 4.5. □

Corollary 4.4 is a particular case of Theorem 4.5, when $G_1(k)=\operatorname{diag}(g_{11}(k),\dots,g_{1n}(k))$ is a diagonal matrix.

Proof of Corollary 4.5. In Theorem 4.5 we take $\gamma_{1il}=\gamma_{2il}=0$ $(i,l=1,\dots,n)$. Then by virtue of (4.42) and (4.43), we have

$$\ell_{1i}(y_1,\dots,y_n)=y_i(k_i)-[(I_n+G_1(k_i))^{-1}]_{ii}[I_n+G_1(k_i)]_{ii}\,y_i(k_i)$$
$$\text{for} \quad (y_l)_{l=1}^n\in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n) \quad (i=1,\dots,n)$$

and

$$\ell_{2i}(y_1,\dots,y_n)=0 \quad \text{for} \quad (y_l)_{l=1}^n\in \mathrm{E}(\widetilde{\mathbb{N}}_{m_0},\mathbb{R}^n) \quad (i=1,\dots,n).$$

On the other hand, the condition (4.44) is equivalent to the condition (4.47), because in this case

$$\mathcal{H}^* = \left(\frac{1}{2}\sin^{-1}\frac{\pi}{4m_0+2}\right)^{\frac{2}{\nu}}\mathcal{H}_0.$$

Therefore the system (1.9_0) has a unique solution $y = (y_l)_{l=1}^n$ satisfying

$$[(I_n + G_1(k_i))^{-1}]_{ii}[I_n + G_1(k_i)]_{ii}\, y_i(k_i) = 0 \quad (i = 1, \dots, n),$$
$$y_i(k_i + 1) = 0 \quad (i = 1, \dots, n).$$

But owing to (4.33) and (4.45), the above equalities are equivalent to the condition (1.11_0). Thus the corollary is proved. □

Corollary 4.6 is a particular case of Corollary 4.5.

Proof of Theorem 4.6. Consider the $2n$-problem (1.1), (1.4) corresponding to the problem (1.9), (1.12), where $A(t)$ and $f(t)$ are defined by (4.4), (4.5) and (4.7), t_i $(i = 1, \dots, 2n)$ and c_{0i} $(i = 1, \dots, 2n)$ are defined by (4.13).

To prove this theorem we use Theorem 2.7. We construct the functions and numbers appearing in Theorem 2.7, which are based on the functions given in the conditions of Theorem 4.6.

Let

$$h_{il} = h_{1+j\,il} \quad (j = 0, 1;\ \ i = 1, \dots, n;\ \ l = nj + 1, \dots, nj + n),$$

$$h_{il} = 0 \ \ (i = n + 1, \dots, 2n;\ \ l = 1, \dots, n), \ \ h_{il} = -\delta_{il} \ \ (i, l = n + 1, \dots, 2n),$$

$$\alpha_i(t) = \sum_{k=0}^{[t]} \beta_i(k) \quad \text{for} \quad t \in [0, m_0] \quad (i = 1, \dots, n),$$

$$\alpha_i(t) = 0 \quad \text{for} \quad t \in [0, m_0] \quad (i = n + 1, \dots, 2n),$$

$$\alpha_{il}(t) = \sum_{k=0}^{[t]} \beta_{1+\mu\,1+\nu\,il}(k) \quad \text{for} \quad t \in [0, m_0] \ \ (\mu, \nu = 0, 1;$$
$$\mu + \nu \le 1;\ \ i = n\mu + 1, \dots, n\mu + n;\ \ l = n\nu + 1, \dots, n\nu + n),$$

$$\alpha_{il}(t) = \delta_{il} \sum_{k=0}^{[t]} \chi_{[k_i+1, m_0]}(k) \quad \text{for} \quad t \in [0, m_0] \quad (i, l = n + 1, \dots, n),$$

$$h_i = 1 \quad (i = n + 1, \dots, 2n).$$

It is not difficult to verify that the conditions of Theorem 2.7 coincide with those of Theorem 4.6. In addition, just as in the proof of Theorem 4.5, we can easily verify that the estimates (4.50) and (4.51) guarantee the estimate (2.36).

Therefore, by Theorem 2.7, the generalized $2n$-problem (1.1), (1.4), corresponding to the problem (1.9), (1.12), is uniquely solvable and hence the problem (1.9), (1.12) is uniquely solvable, too. Thus the theorem is proved. □

Proof of Theorem 4.7. Let $y = (y_i)_{i=1}^n$ be a nontrivial nonnegative solution of the problem (4.25), (4.26). Then due (4.33), the vector-function $x(t) = (x_i(t))_{i=1}^n$ defined by (4.2) and (4.6) is a nontrivial, nonnegative solution of the problem (6.5), (6.6) on $[0, m_0]$, where the matrix-function $C(t) = (C_{jm}(t))_{j,m=1}^n$ is defined by

$$\begin{aligned} &C_{11}(t) = \sum_{k=0}^{[t+\frac{1}{2}]} C_1(k), \quad C_{12}(t) = \sum_{k=0}^{[t+\frac{1}{2}]} C_2(k), \\ &C_{21}(t) = -\sum_{k=0}^{[t+\frac{1}{2}]} (I_n + G_1(k))^{-1}, \quad C_{22}(t) = \left(\left[t + \frac{1}{2}\right] + 1\right) I_n, \end{aligned} \tag{7.32}$$

the functionals ℓ_{0i}, $\ell_{0\,n+i}$ $(i = 1, \dots, n)$ and the points t_i, t_{n+i} $(i = 1, \dots, n)$ are defined by (7.28), (7.29) and (4.13), respectively.

According to Lemma 6.5, there exist a matrix-function $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}([0, m_0], \mathbb{R}^{n\times n})$, linear continuous functionals $\ell_i : \mathrm{BV}([0, m_0], \mathbb{R}^{2n}) \to \mathbb{R}$ $(i = 1, \dots, n)$ defined by (6.11) and (6.15), respectively, and numbers $c_{0i} \in \mathbb{R}$ $(i = 1, \dots, 2n)$ such that the $2n$-system (6.7) is unsolvable under the $2n$-condition (1.4), where $\widetilde{A}(t)$ is defined by (6.8).

By (6.8), (6.11) and (7.32) we have

$$\begin{aligned} d_2\widetilde{A}(t) &= O_{2n\times 2n} \quad \text{for} \quad t \in [0, m_0], \\ d_1 A(k) &= C(k), \quad d_1 C_{12}(k) = C_2(k) \quad \text{for} \quad k \in \widetilde{N}_{m_0}, \\ d_1 C_{21}(k) &= -(I_n + G_1(k))^{-1}, \quad d_1 C_{22}(k) = I_n \ \text{ for } \ k \in \widetilde{\mathbb{N}}_{m_0}, \end{aligned}$$

where

$$C(k) = \big(\operatorname{sgn}(k - k_i)\eta_l(k) c_{1il}(k)\big)_{i,l=1}^n.$$

Basing on the above-said, by virtue of (4.4) and (4.7), we define

$$G_2(k) = C(k)(I_n + G_1(k)) - G_1(k) \quad \text{for} \quad k \in \widetilde{\mathbb{N}}_{m_0}$$

and

$$G_3(k) = C_2(k) \quad \text{for} \quad k \in \widetilde{\mathbb{N}}_{m_0}.$$

Let, moreover, the functionals $\ell_{1i}(y_1, \dots, y_n)$ $(i = 1, \dots, n)$ and $\ell_{2i}(y_1, \dots, y_n)$ $(i = 1, \dots, n)$ be defined by means of (4.14) and (4.15), respectively, where

$$x_{1i}(k) = \sum_{l=1}^n (\delta_{il} + g_{1il}(k)) y_l(k) \quad \text{for} \ \ k \in \widetilde{\mathbb{N}}_{m_0} \ \ (i = 1, \dots, n)$$

and

$$x_{2i}(k) = y_i(k+1) \quad \text{for} \ \ k \in \widetilde{\mathbb{N}}_{m_0} \ \ (i = 1, \dots, n),$$

and

$$c_{01i} = (1 + g_{1ii}(k_i))^{-1} c_{0i}, \quad c_{02i} = c_{0\,n+i} \quad (i = 1, \dots, n).$$

Then, by (4.53) and (4.54), it is not difficult to verify that the conditions (4.1), (4.35)–(4.38) are fulfilled, but the problem (1.9_0), (1.12) is not solvable. Thus the theorem is proved. □

The results of Section 5 follow immediately from the corresponding results of Section 2, because the system (1.13) is a particular case of the system (1.1), where the matrix- and the vector-functions $A(t)$ and $f(t)$ are defined by (5.6). We only note that in Theorem 5.7 the matrix-function $G_2(k)$ is defined as in Theorem 4.7.

References

1. R. P. Agarwal, Difference equations and inequalities. Theory, methods, and applications. *Monographs and Textbooks in Pure and Applied Mathematics,* 155. *Marcel Dekker, Inc., New York,* 1992.
2. M. Ashordia, On the stability of solutions of nonlinear boundary value problems for systems of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **5** (1995), 119–121.
3. M. Ashordia, On the stability of solutions of the multipoint boundary value problem for the system of generalized ordinary differential equations. *Mem. Differential Equations Math. Phys.* **6** (1995), 1–57.
4. M. T. Ashordia, A criterion for the solvability of a multipoint boundary value problem for a system of generalized ordinary differential equations. (Russian) *Differ. Uravn.* **32** (1996), No. 10, 1303–1311, 1437; English transl.: *Differential Equations* **32** (1996), No. 10, 1300–1308 (1997)
5. M. Ashordia, On the correctness of nonlinear boundary value problems for systems of generalized ordinary differential equations. *Georgian Math. J.* **3** (1996), No. 6, 501–524.
6. M. Ashordia, Conditions of existence and uniqueness of solutions of the multipoint boundary value problem for a system of generalized ordinary differential equations. *Georgian Math. J.* **5** (1998), No. 1, 1–24.
7. M. Ashordia, On the solvability of linear boundary value problems for systems of generalized ordinary differential equations. *Funct. Differ. Equ.* **7** (2000), No. 1-2, 39–64 (2001).
8. M. Ashordia, Lyapunov stability of systems of generalized ordinary differential equations. *Comput. Math. Appl.* **50** (2005), 957–982.
9. F. M. Atici and A. Cabada, Existence and uniqueness results for discrete second-order periodic boundary value problems. *Advances in difference equations, IV. Comput. Math. Appl.* **45** (2003), No. 6-9, 1417–1427.
10. F. V. Atkinson, Discrete and continuous boundary problems. *Mathematics in Science and Engineering,* Vol. 8. *Academic Press, New York–London,* 1964.
11. S. N. Elaydi, An introduction to difference equations. Second edition. *Undergraduate Texts in Mathematics. Springer-Verlag, New York,* 1999.
12. F. R. Gantmakher, Theory of matrices. Fourth edition. (Russian) *Nauka, Moscow,* 1988.
13. Sh. Gelashvili and I. Kiguradze, On multi-point boundary value problems for systems of functional-differential and difference equations. *Mem. Differential Equations Math. Phys.* **5** (1995), 1–113.
14. J. Groh, A nonlinear Volterra-Stieltjes integral equation and a Gronwall inequality in one dimension. *Illinois J. Math.* **24** (1980), No. 2, 244–263.
15. T. H. Hildebrandt, On systems of linear differentio-Stieltjes-integral equations. *Illinois J. Math.* **3** (1959), 352–373.

16. I. T. Kiguradze, On the modified problem of Cauchy–Nicoletti. *Ann. Mat. Pura Appl. (4)* **104** (1975), 177–186.
17. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh.* **30** (1987), 3–103; English transl.: *J. Sov. Math.* **43** (1988), No. 2, 2259–2339.
18. I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I. Linear theory. (Russian) *Metsniereba, Tbilisi,* 1997.
19. I. Kiguradze and B. Půža, Certain boundary value problems for a system of ordinary differential equations. (Russian) *Differencial'nye Uravnenija* **12** (1976), No. 12, 2139–2148.
20. T. Kiguradze, Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Phys.* **1** (1994), 144 pp.
21. V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, Theory of impulsive differential equations. *Series in Modern Applied Mathematics,* 6. *World Scientific Publishing Co., Inc., Teaneck, NJ,* 1989.
22. A. Lasota, A discrete boundary value problem. *Ann. Polon. Math.* **20** (1968), 183–190.
23. A. Lasota and C. Olech, An optimal solution of Nicoletti's boundary value problem. *Ann. Polon. Math.* **18** (1966), No. 2, 131–139.
24. O. Nicoletti, Sulle condizioni iniziale che determiniano gli integrali della equazioni differenziali ordinaire. (Italian) *Atti della R. Acc. Sc. Torino* **33** (1897–1898), 746–759.
25. A. Samoilenko, S. Borysenko, C. Cattani, G. Matarazzo, and V. Yasinsky, Differential models. Stability, inequalities & estimates. *Naukova Dumka, Kiev,* 2001.
26. A. M. Samoĭlenko and N. A. Perestyuk, Impulsive differential equations. (Translated from the Russian) *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises,* 14. *World Scientific Publishing Co., Inc., River Edge, NJ,* 1995.
27. Š. Schwabik and M. Tvrdý, Boundary value problems for generalized linear differential equations. *Czechoslovak Math. J.* **29(104)** (1979), No. 3, 451–477.
28. Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht–Boston, Mass.–London,* 1979.
29. S. T. Zavalishchin and A. N. Sesekin, Impulse processes: models and applications. (Russian) *Nauka, Moscow,* 1991.
30. F. Zhang, Zh. Ma, and J. Yan, Functional boundary value problem for first order impulsive differential equations at variable times. *Indian J. Pure Appl. Math.* **34** (2003), No. 5, 733–742.

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