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ON LOWER AND UPPER SOLUTIONS OF THE KNESER PROBLEM

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Suppose

$$\mathbb{R}_+ = [0, +\infty[\,, \quad \mathbb{R}_- =]-\infty, 0],$$

and $f :]0, +\infty[\, \times \mathbb{R}_+ \times \mathbb{R}_- \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions, i.e. the function $f(\cdot, x, y) :]0, +\infty[\, \to \mathbb{R}$ is measurable for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_-$, the function $f(t, \cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}_- \to \mathbb{R}$ is continuous for almost all $t \in]0, +\infty[\,$, and the function

$$f^*_\rho(\cdot) = \max\{|f(\cdot, x, y)| : \ 0 \le x \le \rho, \ 0 \le y \le \rho\}$$

is integrable on every compact interval contained in $]0, +\infty[\,$.

For the differential equation

$$u'' = f(t, u, u') \tag{1}$$

we consider the Kneser problem

$$u(0+) = c, \quad u(t) \ge 0, \quad u'(t) \le 0 \quad \text{for} \quad t > 0, \tag{2}$$

where

$$c > 0, \quad u(0+) = \lim_{t \to 0} u(t).$$

A non-increasing function $u :]t_0, +\infty[\, \to \mathbb{R}_+$, where $t_0 \in \mathbb{R}_+$, is said to be **a Kneser type solution of Eq.** (1) defined on $]t_0, +\infty[$ if it is absolutely continuous together with u' on every compact interval contained in $]t_0, +\infty[$ and satisfies Eq. (1) almost everywhere on $]t_0, +\infty[\,$. A Kneser type solution of Eq. (1), defined on $]0, +\infty[$ and satisfying the initial condition

$$u(0+) = c,$$

is said to be **a solution of problem** (1), (2).

A solution $\underline{u}$ (a solution $\overline{u}$) of problem (1), (2) is said to be **a lower solution** (**an upper solution**) of this problem if an arbitrary solution u of problem (1), (2) satisfies the inequality

$$u(t) \ge \underline{u}(t) \quad \big(\, u(t) \le \overline{u}(t) \,\big)$$

on $]0, +\infty[\,$.

Problems of solvability and unique solvability of (1), (2) are studied thoroughly enough (see, e.g., [1]–[5] and the references therein). However in the case where the uniqueness is violated, the problem on the existence of a lower and an upper solution of (1), (2) has remained open. The present paper is concerned with the filling up this gap.

Before formulating the main results, we introduce the following definition.

Definition. Suppose there exist numbers $r > 0$, $a > 0$, $a_0 \in]0, a[\,$, and a continuous function $\rho :]0, a] \to \mathbb{R}_+$ such that

$$\int_0^a \rho(s)ds < +\infty$$

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and for any $t_0 \in]0, a_0]$ and $\lambda \in]0, 1]$, every solution u of the equation

$$u'' = \lambda f(t, u, u'),$$

defined on $[t_0, a]$ and satisfying the inequalities

$$u(t_0) \le r, \quad u(t) \ge 0, \quad u'(t) \le 0 \quad \text{for} \quad t_0 \le t \le a,$$

admits the estimate

$$|u'(t)| \le \rho(t) \quad \text{for} \quad t_0 \le t \le a.$$

Then we say that the function f belongs to the set $\mathcal{B}_r$.

Theorem 1. *Let*

$$f(t, 0, 0) = 0, \quad f(t, x, y) \ge 0 \quad \text{for} \quad t \in]0, +\infty[\,, \quad x \in \mathbb{R}_+, \quad y \in \mathbb{R}_-, \tag{3}$$

and let for some $r > 0$ the condition

$$f \in \mathcal{B}_r \tag{4}$$

hold. Then for any $c \in [0, r]$, problem (1), (2) *has a lower and an upper solution.*

Theorem 2. *Let conditions* (3) *and* (4) *be fulfilled, and $v :]0, +\infty[\to \mathbb{R}_+$ be a non-increasing function, absolutely continuous together with v' on every finite interval and satisfying the differential inequality*

$$f(t, v(t), v'(t)) \ge v''(t) \quad \big(f(t, v(t), v'(t)) \le v''(t) \big)$$

almost everywhere on $]0, +\infty[$. Let, moreover,

$$c \le v(0+) \le r \quad \big(v(0+) \le c \le r \big).$$

Then

$$v(t) \ge \underline{u}(t) \quad \big(v(t) \le \overline{u}(t) \big) \quad \text{for} \quad t \in]0, +\infty[\,,$$

where $\underline{u}$ and $\overline{u}$ are, respectively, the lower and the upper solution of problem (1), (2).

Theorem 3. *Let there exist positive numbers a, r, r_0, and a function $\omega :]0, a] \times \mathbb{R}_- \to \mathbb{R}_+$, satisfying the local Carathéodory conditions, such that along with* (3) *the condition*

$$f(t, x, y) \le \omega(t, y) \quad \text{for} \quad t \in]0, a], \quad x \in [0, r], \quad y \in \mathbb{R}_-$$

holds and the Cauchy problem

$$\frac{dy}{dt} = -\omega(t, y), \quad y(a) = r_0$$

has an upper solution $\overline{y}$, defined on $]0, a]$, such that

$$r < \int_0^a \overline{y}(s)ds < +\infty.$$

Then the conclusions of Theorems 1 *and* 2 *are valid.*

Corollary 1. *Let there exist numbers $\lambda \in \mathbb{R}$, $a > 0$, $r > 0$, and a measurable function $\ell :]0, a] \to \mathbb{R}_+$ such that along with* (3) *the inequality*

$$f(t, x, y) \le \ell(t)(1 + |y|)^\lambda \quad \text{for} \quad t \in]0, a], \quad x \in [0, r], \quad y \in \mathbb{R}_-$$

holds. Let, moreover, λ and ℓ satisfy one of the following three conditions:

(i) $\lambda < 1$, $\int_t^a \ell(s)ds < +\infty$ *for* $0 < t < a$ *and*

$$\int_0^a \Big(\int_t^a \ell(s)ds \Big)^{\frac{1}{1-\lambda}} dt < +\infty;$$

(ii) $\lambda = 1$, $\int\limits_t^a \ell(s)ds < +\infty$ *for* $0 < t < a$ *and*

$$\int\limits_0^a \exp\Big(\int\limits_t^a \ell(s)ds\Big)dt < +\infty;$$

(iii) $\lambda > 1$, $0 < \int\limits_0^t \ell(s)ds < +\infty$ *for* $0 < t \le a$ *and*

$$\int\limits_0^a \Big(\int\limits_0^t \ell(s)ds\Big)^{\frac{1}{1-\lambda}} dt = +\infty.$$

Then the conclusions of Theorems 1 *and* 2 *are valid.*

Corollary 2. *Let there exist numbers* $\lambda \in \mathbb{R}$, $\ell_0 > 0$, $a > 0$ *and* $r > 0$ *such that along with* (3) *the inequality*

$$f(t,x,y) \le \ell_0 t^{\lambda-2+\varepsilon}(1+|y|)^{\lambda} \quad \textit{for} \quad t \in]0,a], \quad x \in [0,r], \quad y \in \mathbb{R}_-$$

holds, where

$$\varepsilon > 0 \quad \textit{for} \quad \lambda \le 1 \quad \textit{and} \quad \varepsilon = 0 \quad \textit{for} \quad \lambda > 1. \tag{5}$$

Then the conclusions of Theorems 1 *and* 2 *are valid.*

As an example, we consider the differential equation

$$u'' = \gamma t^{\lambda-2+\varepsilon}\Big(u^n \cos^2 \frac{1}{u} + \sin^2 u\Big)\big(1+|u'|\big)^{\lambda}, \tag{6}$$

where $n > 0$, $\gamma > 0$, $\lambda \in \mathbb{R}$, and ε satisfies condition (5).

According to Corollary 2, for any $c > 0$ problem (6), (2) has a lower and an upper solution.

From this example it is obvious that Theorem 3 and its corollaries cover the case where the function $f(t,x,y)$ has singularity of arbitrary order for $t = 0$.

Finally, we consider the case where f does not have singularity in the first argument for $t = 0$, i.e.

$$\int\limits_0^a f_\rho^*(s)ds < +\infty \quad \text{for} \quad 0 < \rho < +\infty. \tag{7}$$

Then $f \in \mathcal{B}_r$ for sufficiently small r. Thus the following corollary is true.

Corollary 3. *Let along with* (3) *condition* (7) *hold. Then for a sufficiently small* $r > 0$*, the conclusions of Theorems* 1 *and* 2 *are valid.*

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