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THE CONSTRUCTION OF THE LINEAR PFAFF SYSTEM WITH THE ARBITRARY GIVEN DEGREE SETS AND TRIVIAL CHARACTERISTIC SETS

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Consider the linear Pfaff system

$$\partial x/\partial t_i = A_i(t)x, \quad x \in R^n, \quad t = (t_1, t_2) \in R^2_{>1}, \quad i = 1, 2, \tag{1}$$

with bounded continuously differentiable matrices $A_i(t)$ satisfying the complete integrability condition [1, pp. 43–44; 2. pp. 21–24]

$$\partial A_1(t)/\partial t_2 + A_1(t)A_2(t) = \partial A_2(t)/\partial t_1 + A_2(t)A_1(t), \quad t \in R^2_{>1}.$$

Suppose that the lower characteristic [3] P_x and the characteristic [4] Λ_x sets of a nontrivial solution $x : R^2_{>1} \to R^n \setminus \{0\}$ of the system (1) are trivial, i.e., $P_x = \{p^0\}$ and $\Lambda_x = \{\lambda^0\}$.

On the basis of Demidovich's definition of the characteristic degree [5] of a solution of the ordinary differential system, the lower [6] $\underline{d} = \underline{d}_x(p^0) \in R^2$ and upper [7] $\overline{d} = \overline{d}_x(\lambda^0) \in R^2$ characteristic degrees of the solution $x \neq 0$ of the system (1) are defined by the conditions

$$\underline{ln}_x(p^0, \underline{d}) \equiv \varliminf_{t\to\infty} \frac{\ln\|x(t)\| - (p^0, t) - (\underline{d}, \ln t)}{\|\ln t\|} = 0, \;\; \underline{ln}_x(p^0, \underline{d} + \varepsilon e_i) < 0, \;\; \forall \varepsilon > 0,$$

$$\overline{ln}_x(\lambda^0, \overline{d}) \equiv \varlimsup_{t\to\infty} \frac{\ln\|x(t)\| - (\lambda^0, t) - (\overline{d}, \ln t)}{\|\ln t\|} = 0, \;\; \overline{ln}_x(\lambda^0, \overline{d} - \varepsilon e_i) > 0, \;\; \forall \varepsilon > 0,$$

$i = 1, 2$. The sets $\underline{D}_x \equiv \{\underline{d}_x(p^0)\}$ and $\overline{D}_x \equiv \{\overline{d}_x(\lambda^0)\}$ are called the lower and upper degree sets.

Necessary properties of the lower degree set $\underline{D}_x$ and the upper degree set $\overline{D}_x$ of a solution x of the system (1) were obtained in paper [8]. More precisely, it was shown that the nonempty lower $\underline{D}_x$ (upper $\overline{D}_x$) degree set of x is a continuous closed decreasing concave (convex) curve on the two-dimensional plane. In the present paper we prove the sufficiency of these properties.

Theorem 1. *Let n be a positive integer, $D \in R^2$ be a continuous closed decreasing concave curve on the two-dimensional plane and $p^0 \in R^2$ be a point. Then there exists a completely integrable Pfaff system* (1) *with infinitely differentiable bounded coefficients such that its arbitrary nontrivial solution $x : R^2_{>1} \to R^n \setminus \{0\}$ has the trivial lower characteristic set $P_x = \{p^0\}$ and the lower degree set $\underline{D}_x = D$.*

Construction of the Pfaff system. First, we note that it suffices to construct a completely integrable linear Pfaff equation

$$\partial x/\partial t_i = a_i(t)x, \quad x \in R, \quad t \in R^2_{>1}, \quad i = 1, 2, \tag{1_1}$$

with infinitely differentiable bounded coefficients and with the desired lower characteristic set and the desired lower degree set.

We construct the desired equation (1_1) by constructing a nontrivial solution.

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We define the solution x of equation (1_1) by $x = \phi\psi$, where $\ln \phi(t) = (p^0, t)$, $t \in R^2_{>1}$.

We define the function ψ so as to ensure that the lower characteristic set of the resulting solution x coincides with the corresponding set $P_\phi = \{p^0\}$ of the function ϕ and the solution x has the lower degree set $\underline{D}_x = D$.

It follows from the properties of the curve D that it necessarily has one of the following ten forms: 1) unbounded from below, left and right, bounded from above; 2) unbounded from below, above, left and right; 3) bounded from above and left, unbounded from below and right; 4) unbounded from below, above and left, bounded from right; 5) bounded from below and right, unbounded from above and left; 6) bounded from below, above and right, unbounded from left; 7) bounded from above, left and right, unbounded from below; 8) bounded from above and right, unbounded from below and left; 9) bounded from above, below, left and right; 10) coinciding with a point.

1. We first suppose that **the curve D has one of the forms 1) − 9).** Then to construct a function ψ realizing the lower degree set $\underline{D}_x = D$, we perform the following partition of the curve D.

1.1. Partition of the curve D. If the curve D has the form 1) or 2), then its lth partition D_l, $l \in N$, consists of the points $\Delta(i,l) \in D$ with first components $\Delta_1(i,l) = (i \cdot 2^{1-l} - l)\gamma$, $i \in \{1, 2, \dots, l \cdot 2^l\} \equiv I_l$; in case 3) of the curve D with the left boundary point $\Delta' \in D$, we construct its lth partition D_l using the points $\Delta(i,l) \in D$ with first components $\Delta_1(i,l) = \Delta'_1 + i\gamma \cdot 2^{-l}$, $i \in I_l$.

If the curve D has the form 4), then its lth partition $D_l = \cup_{i \in I_l} \{\Delta(i,l)\} \subset D$ consists of the points $\Delta(i,l) \in D$ with second components $\Delta_2(i,l) = (i \cdot 2^{1-l} - l)\gamma$, $i \in I_l$.

In cases 5) and 6) of the curve D with the right boundary point $\Delta'' \in D$, and also in the case 8) of the curve D with the vertical asymptote $d_1 = \Delta''_1$, we construct its lth partition D_l using the points $\Delta(i,l) \in D$ with first components $\Delta_1(i,l) = \Delta''_1 - i\gamma \cdot 2^{-l}$, $i \in I_l$.

In case 7) of the curve D with the left boundary point $\Delta' \in D$, its lth partition D_l consists of the points $\Delta(i,l) \in D$ with second components $\Delta_2(i,l) = \Delta'_2 - i\gamma \cdot 2^{-l}$, $i \in I_l$.

If the curve D has one of the forms 1) − 8), then we denote by $i_l \equiv l \cdot 2^l$ the last element of the set I_l.

In case 9) of the curve D with the left $\Delta' \in D$ and right $\Delta'' \in D$ boundary points, we construct its lth partition D_l using the points $\Delta(i,l) \in D$ with first components $\Delta_1(i,l) = \Delta'_1 + (\Delta''_1 - \Delta'_1)i \cdot 2^{-(l+1)}$, $i = 1, 2, \dots, 2^{l+1} - 1$. We denote by I_l the set $\{1, 2, \dots, 2^{l+1} - 1\}$ in this case and set $i_l \equiv 2^{l+1} - 1$.

By continuing the partition of the curve D infinitely, we obtain a countable set $D_\infty = \cup_{l \in N} \cup_{i \in I_l} \{\Delta(i,l)\} \subset D$, which is everywhere dense in D.

We note that $D_l \subset D_{l+1}$, $l \in N$.

1.2. Construction of a solution. At the ith point $\Delta(i,l) \in D$, $i \in I_l$, of the lth partition, $l \in N$, we draw some straight line of support

$$d_2 - \Delta_2(i,l) = k(i,l)(d_1 - \Delta_1(i,l)), \quad k(i,l) \in (-\infty, 0), \quad (d_1, d_2) \in R^2$$

to D, which does not lie below this curve. The existence of such a straight line of support follows from the concavity of D, its decreasing character and from the fact that by construction all points $\Delta(i,l)$ of each lth partition D_l are interior points of D. Moreover, if a point has been used in the partition, then for all subsequent partitions, we draw the same straight line of support at this point. This will ensure the existence of a sequence realizing the limit $\underline{ln}_x(p^0, d)$ in the definition of lower characteristic degree.

We set

$$\Theta_{i,l} \equiv 1/|k(i,l)|, \quad i \in I_l, \quad \Theta_l \equiv \max_{i \in I_l}\{\Theta_{i,l}\}, \quad \Omega_l \equiv \min_{i \in I_l}\{\Theta_{i,l}\},$$

$$\Delta_1(l) \equiv \max_{i \in I_l}\{\|\Delta(i,l)\|\}, \quad \Delta_2(l) \equiv 2^{-l}\|\Delta(i_l, l) - \Delta(1,l)\|^{-1}, \quad l \in N.$$

To sew the different infinitely differentiable functions together into a single infinitely differentiable function, we introduce the infinitely differentiable functions

$$e_{101}(\tau;\alpha_1,\alpha_2,\alpha_3)=e_{01}(\tau;\alpha_2,\alpha_3)+[1-e_{01}(\tau;\alpha_1,\alpha_2)],$$

$e_{0110}(\tau;\alpha_1,\alpha_2,\alpha_3,\alpha_4)=e_{01}(\tau;\alpha_1,\alpha_2)\cdot(1-e_{01}(\tau;\alpha_3,\alpha_4))$, $\alpha_1<\alpha_2<\alpha_3<\alpha_4$, $\tau\in R$, defined on the basis of the infinitely differentiable function [9]

$$e_{01}(\tau;\tau_1,\tau_2)=\begin{cases}\exp\{-(\tau-\tau_1)^{-2}\exp[-(\tau-\tau_2)^{-2}]\}, & \tau\in(\tau_1,\tau_2),\\ (1+\operatorname{sgn}(\tau-2^{-1}(\tau_1+\tau_2)))/2, & \tau\notin(\tau_1,\tau_2),\end{cases}$$

$-\infty<\tau_1<\tau_2<+\infty$.

We define the functions $\psi_{i,l}$ by

$$\ln\psi_{i,l}(t)\equiv(\Delta(i,l),\ln t)e_{0110}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\frac{5\tau_l}{4},\Theta_{i,l}-\tau_l,\Theta_{i,l}+\tau_l,\Theta_{i,l}+\frac{5\tau_l}{4}\right)+$$
$$+\|\ln t\|^2e_{101}\left(\frac{\ln t_2}{\ln t_1};\Theta_{i,l}-\tau_l,\Theta_{i,l},\Theta_{i,l}+\tau_l\right),\quad t\in R^2_{>1},\quad i\in I_l,\quad l\in N,$$

$\tau_l\equiv\min\{1/2;\Omega_l/2,\Delta_2(l)\}$.

It follows from the definition of the function $\psi_{i,l}$ that there exists a number $T_l\geq 1$ such that

$$\ln\psi_{i,l}(t)-(d,\ln t)\geq 0,\quad t\in R^2_{>1}\setminus S(i,l),\ S(i,l)\equiv\left\{t\in R^2_{>1}:\left|\frac{\ln t_2}{\ln t_1}-\Theta_{i,l}\right|\leq\tau_l\right\},$$
$$\|t\|\geq T_l,\ d\in D_l,\ i\in I_l.$$

We split the domain of the solution $x:R^2_{>1}\to R\setminus\{0\}$ by lines of the forms $\zeta(t)\equiv t_1+t_2=\text{const}$ into disjoint strips. By using some values $\eta_1\geq T_1$ and $c\geq\exp(100)$, we introduce the numbers

$$\nu_l=c(\Theta_l^6+\Omega_l^{-2})(\Delta_1^2(l)+1)\exp(c\tau_l^{-2}),\quad \alpha_{i,l}=\left(\eta_l+\nu_l^{4(\Theta_l+\Omega_l^{-1})}\right)\exp(\exp i),$$
$$\beta_{i,l}=e^2\alpha_{i,l},\ i\in I_l,\quad \eta_{l+1}=\beta_{i_l,l}+T_{l+1}+2^l,\ l\in N.$$

We introduce the “basic” strips

$$\Pi(i,l)=\{t\in R^2_{>1}:\beta_{i,l}\leq\zeta(t)\equiv t_1+t_2\leq\alpha_{i+1,l}\},\ i\in I_l\setminus\{i_l\}\equiv I_l^1,$$
$$\Pi(i_l,l)=\{t\in R^2_{>1}:\beta_{i_l,l}\leq\zeta(t)\leq\alpha_{1,l+1}\},\ l\in N,$$

the “transition” strips

$$P(i,l)=\{t\in R^2_{>1}:\alpha_{i,l}<\zeta(t)<\beta_{i,l}\},\ i\in I_l,\ l\in N,$$

and the triangle $T=\{t\in R^2_{>1}:\zeta(t)\leq\alpha_{1,1}\}$.

Let us proceed to the construction of the function ψ used in the realization of the desired lower degree set $\underline{D}_x=D$ of x. First we introduce the auxiliary function $\tilde\psi$ by

$$\ln\tilde\psi(t)=\ln\psi_{i,l}(t)+[\ln\psi_{i+1,l}(t)-\ln\psi_{i,l}(t)]e_{01}(\ln\zeta(t);\ln\alpha_{i+1,l},\ln\beta_{i+1,l}),$$
$$t\in\Pi(i,l)\cup P(i+1,l)\cup\Pi(i+1,l),\quad i\in I_l^1,\quad l\in N,$$
$$\ln\tilde\psi(t)=\ln\psi_{i_l,l}(t)+[\ln\psi_{1,l+1}(t)-\ln\psi_{i_l,l}(t)]e_{01}(\ln\zeta(t);\ln\alpha_{1,l+1},\ln\beta_{1,l+1}),$$
$$t\in P(1,l+1),\quad l\in N,$$
$$\ln\tilde\psi(t)=\ln\psi_{1,1}(t)e_{01}(\ln\zeta(t);\ln\alpha_{1,1},\ln\beta_{1,1}),\quad t\in T\cup P(1,1).$$

We set $\psi(t)=\tilde\psi(t)$, $t\in R^2_{>1}$ in case of the curve D of one of the forms 1), 2), 4) or 8).

We define the function ψ by

$$\ln\psi(t)=\ln\tilde\psi(t)+[(\Delta',\ln t)-\ln\tilde\psi(t)]e_{01}\left(\frac{\ln t_2}{\sqrt[3]{t_1}\ln t_1};1,3\right),\quad t\in R^2_{>1},$$

in case of the curve D of the forms 3) or 7) with the left boundary point $\Delta'\in D$.

We set

$$\ln\psi(t)=\ln\tilde\psi(t)+[(\Delta'',\ln t)-\ln\tilde\psi(t)]e_{01}\left(\frac{\ln t_1}{\sqrt[3]{t_2}\ln t_2};1,3\right),\quad t\in R^2_{>1},$$

in case of the curve D of the form 5) or 6) with the right boundary point $\Delta'' \in D$.

Finally, we define the function ψ by

$$\ln\psi(t) = \ln\tilde{\psi}(t) + [(\Delta', \ln t) - \ln\tilde{\psi}(t)]e_{01}\left(\frac{\ln t_2}{\sqrt[3]{t_1}\ln t_1}; 1, 3\right) +$$
$$+[(\Delta'', \ln t) - \ln\tilde{\psi}(t)]e_{01}\left(\frac{\ln t_1}{\sqrt[3]{t_2}\ln t_2}; 1, 3\right), \quad t \in R^2_{>1},$$

in case of the curve D of the form 9) with the left $\Delta' \in D$ and right $\Delta'' \in D$ boundary points.

2. In case 10) of the curve D consisting of one point $\Delta \in R^2$, we set $\ln\psi(t) = (\Delta, \ln t), \quad t \in R^2_{>1}$.

Construction of the equation. The above-constructed function $x > 0$ is a solution of the Pfaff equation (1_1) with infinitely differentiable bounded coefficients $a_k(t) = x^{-1}(t)\partial x(t)/\partial t_k = \partial \ln x(t)/\partial t_k$, $t \in R^2_{>1}$, $k = 1, 2$, satisfying the complete integrability condition in $R^2_{>1}$ and this solution has the trivial characteristic set $P_x = \{p^0\}$ and the lower degree set $\underline{D}_x = D$.

Theorem 2. *Let n be a positive integer, $D \in R^2$ be a continuous closed decreasing convex curve on a two-dimensional plane and $\lambda^0 \in R^2$ be a point. Then there exists a completely integrable Pfaff system* (1) *with infinitely differentiable bounded coefficients such that its arbitrary nontrivial solution $x : R^2_{>1} \to R^n \backslash \{0\}$ has the trivial characteristic set $\Lambda_x = \{\lambda^0\}$ and the upper degree set $\overline{D}_x = D$.*

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