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**NEW TYPE ENERGY ESTIMATES FOR  
MULTIDIMENSIONAL OBSTACLE PROBLEMS**

**Abstract.** Using the methods of stochastic analysis, new type energy estimates are obtained in the theory of variational inequalities, in particular, for obstacle problems. The derivation of these estimates is essentially based on our previously obtained stochastic a priori estimates for Snell envelopes and on the connection between the optimal stopping problem and a variational inequality. Using these results, energy estimates are obtained for the solution of an obstacle problem when only the continuity is required of the obstacle function  $g = g(x)$ .

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**Key words and phrases:** Variational inequality, Snell envelope, semimartingale, multidimensional diffusion process, optimal stopping problem.

**რეზიუმე.** ნაშრომში მიღებულია ახალი სახის ენერგეტიკული შეფასებები ვარიაციულ უტოლობათა თეორიაში (კერძოდ, დაბრკოლების ამოცანებში) სტოქასტური ანალიზის ტექნიკით. ამ შეფასებების გამოყვანისას ჩვენ არსებითად ვყვრდებით ჩვენს მიერ მიღებულ სტოქასტურ აპრიორულ შეფასებებს სნელის მომვლეთათვის და კავშირს ოპტიმალური გაჩერების ამოცანასა და ვარიაციულ უტოლობას შორის. აღნიშნულ შედეგებზე დაყრდნობით ვღებულობთ ენერგეტიკულ შეფასებებს დაბრკოლების ამოცანის ამონახსნისათვის, როდესაც დაბრკოლების  $g = g(x)$  ფუნქციას მოეთხოვება მხოლოდ უწყვეტობა.

## 1. INTRODUCTION

We consider a probability space  $(\Omega, \mathcal{F}, P)$  and an  $n$ -dimensional Wiener process  $w_t = (w_t^1, \dots, w_t^n)$  on it. Denote by  $F^w = (\mathcal{F}_t^w)_{t \geq 0}$  the completed filtration of the Wiener process.

Let  $D$  be a bounded domain in  $\mathbb{R}^n$  with a smooth boundary ( $\partial D \in C^2$ ). Denote by  $\sigma(D) = \inf\{t \geq 0 : w_t \in \bar{D}\}$  the first time at which the Wiener process  $w_t$  leaves the domain  $D$ .

It is assumed that  $g = g(x)$  and  $c = c(x)$  are continuous functions:  $g(x), c(x) \in C(\bar{D})$ , and  $g(x) \leq 0$  when  $x \in \partial D$ .

Let us pose the following optimal stopping problem of the Wiener process  $w_t$  in the domain  $D$ :

$$S(x) = \sup_{\tau \in \mathfrak{m}} E_x \left( g(w_\tau) I_{(\tau < \sigma(D))} + \int_0^{\tau \wedge \sigma(D)} c(w_t) dt \right), \quad (1.1)$$

where  $P_x$  is the probability measure corresponding to the initial condition  $w_0(\omega) = x$  and  $\mathfrak{m}$  is the class of all stopping times with respect to the filtration  $F^w = (\mathcal{F}_t^w)_{t \geq 0}$ . We call the function  $g = g(x)$  the payoff function;  $-c = -c(x)$  has the meaning of instantaneous cost of observation, and  $S(x)$  is the value function of the optimal stopping problem.

The optimal stopping problem consists in finding a value function  $S(x)$  and in defining the optimal stopping time  $\tau^*$  at which the supremum of this problem (1.1) is achieved.

In [1, Ch. VII, §4], A. Bensoussan established a connection between the optimal stopping problem and the corresponding variational inequality. Below we give a brief account of A. Bensoussan's main results.

Denote by  $H^1(D)$  the first order Sobolev space of functions  $v = v(x)$  defined on  $D$ , i.e.,

$$v \in L^2(D), \quad \frac{\partial v}{\partial x_i} \in L^2(D), \quad i = 1, \dots, n,$$

where  $\frac{\partial v}{\partial x_i}$ ,  $i = 1, \dots, n$ , are first order generalized derivatives of the function  $v$ . It is well known that if we introduce the scalar product

$$(u, v)_{H^1(D)} = \int_D u(x)v(x)dx + \sum_{i=1}^n \int_D \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx,$$

then the space  $H^1(D)$  becomes a Hilbert space.

Denote by  $H_0^1(D)$  the subspace of the space  $H^1(D)$  consisting of the functions  $v = v(x)$  that are "equal to zero" on the boundary  $\partial D$  of the domain  $D$  (in the sense of  $H^1(D)$ ).

On the product  $H_0^1(D) \times H_0^1(D)$ , consider the symmetric bilinear form (i.e., the scalar product in  $H_0^1(D)$ )

$$a(u, v) = \sum_{i=1}^n \int_D \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx.$$

Let us introduce the closed convex subset of the space  $H_0^1(D)$ :

$$K = \{v : v \in H_0^1(D), v(x) \geq g(x)\}. \quad (1.2)$$

In the sequel it will always be assumed that  $K \neq \emptyset$ .

The variational inequality is formulated as follows:

Find a function  $u(x) \in K$  such that the inequality

$$a(u, v - u) \geq \int_D c(x)(v(x) - u(x))dx \quad (1.3)$$

is fulfilled for any function  $v(x) \in K$ .

In [1, Ch. VII, Theorem 3.2], the fundamental result obtained by A. Bensoussan asserts the existence and uniqueness of the solution of the variational inequality (1.3). Moreover, he established the fundamental connection between the optimal stopping problem and the corresponding variational inequality (see [1, Ch. VII, Theorem 4.1]). In particular, he showed that

$$u(x) = S(x), \quad x \in \overline{D}. \quad (1.4)$$

In [1, Ch. VII, Lemma 3.4], A. Bensoussan also showed that

$$\sup_{x \in \overline{D}} |u^2(x) - u^1(x)| \leq \sup_{x \in \overline{D}} |g^2(x) - g^1(x)| \quad (1.5)$$

holds, where the functions  $u^i(x)$ ,  $i = 1, 2$ , represent the solution of the variational inequality (1.3) for the obstacle function  $g^i(x)$ ,  $i = 1, 2$ .

The aim of our present work is to give an answer to the following question: does the uniform closeness of the obstacle functions  $g^1(x)$  and  $g^2(x)$  imply in a certain sense the closeness of the partial derivatives  $\frac{\partial u^1(x)}{\partial x_i}$ ,  $\frac{\partial u^2(x)}{\partial x_i}$ ,  $i = 1, \dots, n$ , of the respective solutions  $u^1(x)$  and  $u^2(x)$  of the variational problem (1.3)?

We use the estimates from [3] and obtain new type energy estimates formulated as follows.

**Theorem I.** *Let  $g^i(x)$ ,  $c^i(x)$ ,  $i = 1, 2$ , be two initial pairs of the variational inequality (1.3). Then for the solution  $u^i(x)$ ,  $i = 1, 2$ , of the problem (1.3) the global estimate*

$$\begin{aligned} & \int_D d^2(x, \partial D) |\text{grad}(u^2 - u^1)(x)|^2 dx + \int_D (u^2(x) - u^1(x))^2 dx \leq \\ & \leq C \left[ \left( \sup_{x \in \overline{D}} |g^2(x) - g^1(x)| + \sup_{x \in \overline{D}} |c^2(x) - c^1(x)| \right) \left( \sup_{x \in \overline{D}} |g^1(x)| + \right. \right. \end{aligned}$$

$$+ \sup_{x \in D} |c^1(x)| + \sup_{x \in D} |g^2(x)| + \sup_{x \in D} |c^2(x)| \Big) + \left( \sup_{x \in D} |c^2(x) - c^1(x)| \right)^2 \Big]$$

is valid, where  $d(x, \partial D)$  is the distance from the point  $x$  to the boundary  $\partial D$ ,  $C$  is a constant depending on the dimension of the space  $\mathbb{R}^n$  and on the Lebesgue measure of  $D$ , i.e.  $C = C(n, \text{mes}(D))$ .

**Theorem II.** Let  $B \subset D$  be some smooth ( $\partial B \in C^2$ ) domain. If  $g^i(x)$ ,  $c^i(x)$ ,  $i = 1, 2$ , are two initial pairs of the variational inequality (1.3), then for the solution  $u^i(x)$ ,  $i = 1, 2$ , of the problem (1.3) the local energy estimate

$$\begin{aligned} & \int_B d^2(x, \partial B) |\text{grad}(u^2 - u^1)(x)|^2 dx + \int_B (u^2(x) - u^1(x))^2 dx \leq \\ & \leq C \left[ \left( \sup_{x \in B} |g^2(x) - g^1(x)| + \sup_{x \in B} |c^2(x) - c^1(x)| \right) \times \right. \\ & \times \left( \sup_{\substack{x \in B \\ y \in \partial B}} |g^1(x) - g^1(y)| + \sup_{\substack{x \in B \\ y \in \partial B}} |c^1(x) - c^1(y)| + \sup_{\substack{x \in B \\ y \in \partial B}} |g^2(x) - g^2(y)| + \right. \\ & \left. \left. + \sup_{\substack{x \in B \\ y \in \partial B}} |c^2(x) - c^2(y)| \right) + \left( \sup_{x \in B} |c^2(x) - c^1(x)| \right)^2 + \sup_{y \in \partial B} (u^2(y) - u^1(y))^2 \right] \end{aligned}$$

is valid, where  $d(x, \partial B)$  is the distance from the point  $x$  to the boundary  $\partial B$ ,  $C$  is the constant depending on the dimension of the space  $\mathbb{R}^n$  and on the Lebesgue measure of  $B$ , i.e.  $C = C(n, \text{mes}(B))$ .

## 2. AUXILIARY PROPOSITIONS

Let  $[0, T]$  be a finite or an infinite time interval. Consider the space  $S^2$  of the processes  $X = (X_t, \mathcal{F}_t^w)_{0 \leq t \leq T}$  which are right continuous with left-hand limits (cadlag in French terminology). The norm of this space is defined by

$$\|X\|_{S^2} = \left\| \sup_{0 \leq t \leq T} |X_t| \right\|_{L^2}.$$

Consider the space  $H^2$  of the semimartingales  $X = (X_t, \mathcal{F}_t^w)_{0 \leq t \leq T}$  which are right continuous with left-hand limits. The norm of this space is defined by

$$\|X\|_{H^2} = \left\| \left[ m_T^{1/2} + \int_0^T |dA_s| \right] \right\|_{L^2},$$

where  $m_t$  and  $A_t$  are the processes from the Doob-Meyer decomposition of the semimartingale  $X_t = m_t + A_t$ . For a fixed process  $X = (X_t, \mathcal{F}_t^w)_{0 \leq t \leq T}$  from the space  $S^2$ , we consider the closed convex subset from  $S^2$

$$Q = \{V_t : V_t \in S^2, \quad V_t \geq X_t, \quad 0 \leq t \leq T, \quad V_T = X_T\}. \quad (2.1)$$

The stochastic variational inequality is introduced as follows:

Find an element  $U_t \in Q \cap H^2$  such that for any element  $V_t \in K$  the following inequality

$$E\left(\int_{\tau_1}^{\tau_2} (U_t - V_t) dU_t \mid \mathcal{F}_{\tau_1}^w\right) \geq 0 \quad (2.2)$$

holds for each pair  $\tau_1, \tau_2, 0 \leq \tau_1 \leq \tau_2 \leq T$  of stopping times (the stochastic integral is understood in the Itô sense). The question of the existence and uniqueness of a solution of a stochastic variational inequality is studied in [3], where in particular the following theorem is formulated.

**Theorem 2.1** ([3, Theorem 2.1]). *The stochastic variational inequality has a unique solution  $U_t$  and this solution is “the Snell envelope” of the process  $X_t$  (the minimal supermartingale that majorizes the random process  $X_t$ ). The solution  $U_t$  satisfies the relation*

$$U_t = \operatorname{ess\,sup}_{\tau \geq t} E(X_\tau \mid \mathcal{F}_t^w), \quad (2.3)$$

where the supremum is taken with respect to the class of all stopping times with values from the set  $[t, T]$ .

Consider the processes  $X_t^1$  and  $X_t^2$  from the space  $S^2$  and the corresponding closed convex sets

$$\begin{aligned} Q_1 &= \{V_t^1 : V_t^1 \in S^2, \quad V_t^1 \geq X_t^1, \quad 0 \leq t \leq T \quad V_T^1 = X_T^1\} \\ Q_2 &= \{V_t^2 : V_t^2 \in S^2, \quad V_t^2 \geq X_t^2, \quad 0 \leq t \leq T \quad V_T^2 = X_T^2\}. \end{aligned}$$

The following a priori stochastic estimate is valid.

**Theorem 2.2** ([3, Theorem 2.3]). *If the processes  $U_t^1$  and  $U_t^2$  are solutions of the stochastic variational inequality (1.3) for the processes  $X_t^1$  and  $X_t^2$ , respectively, then the following stochastic a priori estimate is valid:*

$$\begin{aligned} &E([U^2 - U^1]_{\tau_2} - [U^2 - U^1]_{\tau_1}) + E(U_{\tau_1}^2 - U_{\tau_1}^1)^2 \leq \\ &\leq 4 \left\| \sup_{\tau_1 < t < \tau_2} |X_t^1 - X_t^2| \right\|_{L^2} \left[ \left\| \sup_{\tau_1 \leq t \leq \tau_2} |X_t^1 - X_{\tau_2}^1| \right\|_{L^2} + \right. \\ &\quad \left. + \left\| \sup_{\tau_1 \leq t \leq \tau_2} |X_t^2 - X_{\tau_2}^2| \right\|_{L^2} \right] + E(U_{\tau_2}^2 - U_{\tau_2}^1)^2. \end{aligned} \quad (2.4)$$

In the sequel we will frequently make use of the following lemma proved in [2, Ch. VI, Lemma 1.2].

**Lemma 2.1.** *Let  $f(x) \in L^{p/2}(D)$ ,  $p > n$ . Then the following estimate is valid:*

$$\sup_{x \in \bar{D}} E_x \int_0^{\sigma(D)} |f(w_s)| ds \leq C \|f(x)\|_{L^{p/2}(D)}, \quad (2.5)$$

where the constant  $C$  does not depend on the choice of the function  $f(x)$ .

*Proof.* Note that if  $f(x) = 1$ , then from the estimate (2.5) we obtain

$$\sup_{x \in \overline{D}} E_x \sigma(D) < \infty. \quad (2.6)$$

Next, we introduce the notation  $w_t^{\sigma(D)} \equiv w_{t \wedge \sigma(D)}$ ,  $t \geq 0$ . Note that there holds the following relation:

$$I_{(s < \sigma(D))} = \chi_D(w_s^{\sigma(D)}), \quad (2.7)$$

where  $\chi_D(x)$  is the characteristic function of the domain  $D$ .

Taking into account (2.7), we have

$$\int_0^{t \wedge \sigma(D)} c(w_s) ds = \int_0^t I_{(s < \sigma(D))} c(w_s) ds = \int_0^t \bar{c}(w_s^{\sigma(D)}) ds,$$

where  $\bar{c}(x) = c(x) \cdot \chi_D(x)$ .

It is well-known [4, Ch. X, Theorem 10.3] that the process  $(w_t^{\sigma(D)}, \mathcal{F}_t, P_x)$ ,  $t \geq 0$ ,  $x \in \overline{D}$ , is the standard Markov process.

Rewrite the optimal stopping problem (1.1) in terms of a standard Markov process  $(w_t^{\sigma(D)}, \mathcal{F}_t, P_x)$ ,  $t \geq 0$ :

$$S(x) = \sup_{\tau \in \mathfrak{m}} E_x \left[ \bar{g}(w_\tau^{\sigma(D)}) + \int_0^\tau \bar{c}(w_t^{\sigma(D)}) dt \right], \quad x \in \overline{D}, \quad (2.8)$$

where  $\bar{g}(x) = g(x) \cdot \chi_D(x)$ .

Note that

$$|\bar{g}(w_\tau^{\sigma(D)})| \leq \sup_{x \in D} |g(x)| < \infty$$

and

$$\int_0^\infty |\bar{c}(w_t^{\sigma(D)})| dt = \int_0^{\sigma(D)} |c(w_t)| dt.$$

By virtue of Lemma 2.1,

$$\sup_{x \in \overline{D}} E_x \int_0^{\sigma(D)} |c(w_s)| ds \leq C \|c(x)\|_{L^\infty(D)} < \infty.$$

Therefore the optimal stopping problem is defined correctly and  $S(x)$  is a bounded function of the variable  $x$ :

$$\sup_{x \in \overline{D}} |S(x)| < \infty.$$

Denote by  $f(x)$  the expression

$$f(x) = E_x \int_0^\infty \bar{c}(w_t^{\sigma(D)}) dt = E_x \int_0^{\sigma(D)} c(w_t) dt. \quad (2.9)$$

By the strong Markovian property we have

$$E_x \left( \int_{\tau}^{\infty} \bar{c}(w_t^{\sigma(D)}) dt | \mathcal{F}_{\tau}^w \right) = f(w_{\tau}^{\sigma(D)}),$$

$$\int_0^{\tau} \bar{c}(w_t^{\sigma(D)}) dt = E_x \left( \int_0^{\infty} \bar{c}(w_t^{\sigma(D)}) dt | \mathcal{F}_{\tau}^w \right) - f(w_{\tau}^{\sigma(D)}).$$

This allows us to rewrite the optimal stopping problem (2.8) as

$$S(x) = \sup_{\tau \in \mathfrak{m}} E_x(\bar{g}(w_{\tau}^{\sigma(D)}) - f(w_{\tau}^{\sigma(D)}) + f(x)).$$

Therefore

$$S(x) - f(x) = \sup_{\tau \in \mathfrak{m}} E_x[\bar{g}(w_{\tau}^{\sigma(D)}) - f(w_{\tau}^{\sigma(D)})]. \quad (2.10)$$

The equality (2.10) and the general theory of optimal stopping of a standard Markov process (see [5, Ch. III, §3]) imply that the stochastic process  $S(w_t^{\sigma(D)}) - f(w_t^{\sigma(D)})$  is a minimal supermartingale (on the time interval  $[0, \infty)$ ) that majorizes the process  $\bar{g}(w_t^{\sigma(D)}) - f(w_t^{\sigma(D)})$ .  $\square$

**Lemma 2.2** (see [10, Lemma 2]). *Let  $v(x) \in W^{2,p}(D)$ ,  $p > n$ ,  $W^{2,p}(D)$  be a Sobolev space. Then for the process  $v(w_t^{\sigma(D)})$ , the Itô formula*

$$v(w_{t \wedge \sigma(D)}) = v(x) + \int_0^{t \wedge \sigma(D)} \text{grad } v(w_s) dw_s + \int_0^{t \wedge \sigma(D)} \Delta v(w_s) ds, \quad t \geq 0, \quad (2.11)$$

holds true, where  $\Delta$  denotes the  $\frac{1}{2}$  multiplied by the Laplace operator.

*Proof.* Consider a sequence  $v_n(x)$  from the space  $C^2(D)$  such that

$$\|v_n(x) - v(x)\|_{W^{2,p}(D)} \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

Write the Itô formula for the process  $v_n(w_{t \wedge \sigma(D)})$  in the form

$$v_n(w_{t \wedge \sigma(D)}) = v_n(x) + \int_0^{t \wedge \sigma(D)} \Delta v_n(w_s) ds + \int_0^{t \wedge \sigma(D)} \text{grad } v_n(w_s) dw_s. \quad (2.12)$$

Consider the expressions

$$E_x \int_0^{\sigma(D)} |\Delta v(w_s)| ds, \quad E_x \int_0^{\sigma(D)} |\text{grad } v(w_s)|^2 ds.$$

By virtue of Lemma 2.1, we have

$$E_x \int_0^{\sigma(D)} |\Delta v(w_s)| ds \leq C \|\Delta v(x)\|_{L^{p/2}(D)} \leq C_1 \|v(x)\|_{W^{2,p}(D)} < \infty,$$



$$\begin{aligned} E_x \int_0^{\sigma(D)} |\text{grad } v(w_s)|^2 ds &\leq C \|\text{grad } v(x)\|_{L^{p/2}(D)}^2 = \\ &= C \|\text{grad } v(x)\|_{L^p(D)}^2 \leq C_1 \|v(x)\|_{W^{2,p}(D)}^2 < \infty. \end{aligned}$$

Hence it follows that the process  $\int_0^{t \wedge \sigma(D)} \text{grad } v(w_s) dw_s$  is a square integrable martingale, while the process  $\int_0^{t \wedge \sigma(D)} \Delta v(w_s) ds$  is a process with integrable variation.

Analogously to the estimates obtained above, we have

$$\begin{aligned} E_x \int_0^{t \wedge \sigma(D)} |\Delta v_n(w_s) - \Delta v(w_s)| ds &\leq C \|\Delta v_n(x) - \Delta v(x)\|_{L^{p/2}(D)} \leq \\ &\leq C_1 \|v_n(x) - v(x)\|_{W^{2,p/2}(D)}. \\ E_x \left| \int_0^{t \wedge \sigma(D)} (\text{grad } v_n(w_s) - \text{grad } v(w_s)) dw_s \right|^2 &\leq \\ &\leq E_x \int_0^{t \wedge \sigma(D)} |\text{grad } v_n(w_s) - \text{grad } v(w_s)|^2 ds \leq \\ &\leq C \|\text{grad}(v_n(x) - v(x))\|_{L^{p/2}(D)}^2 = C \|\text{grad}(v_n(x) - v(x))\|_{L^p(D)}^2 \leq \\ &\leq C_1 \|v_n(x) - v(x)\|_{W^{2,p}(D)}^2. \end{aligned}$$

Thus, passing to the limit in the equality (2.12) as  $n \rightarrow \infty$  and taking into account that by virtue of the well-known Sobolev lemma (see [6, §7.7])

$$\sup_{x \in \bar{D}} |v_n(x) - v(x)| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

we obtain the validity of the equality given in Lemma 2.2.  $\square$

**Lemma 2.3** (see [10, Lemma 3]). *The function  $f(x) = E_x \int_0^{\sigma(D)} c(w_s) ds$  belongs to the Sobolev space  $W^{2,p}(D)$ ,  $\forall p > n$ , and the equality*

$$f(w_{t \wedge \sigma(D)}) = f(x) + \int_0^{t \wedge \sigma(D)} \text{grad } f(w_s) dw_s + \int_0^{t \wedge \sigma(D)} (-c(w_s)) ds, \quad t \geq 0,$$

is valid.

*Proof.* We have  $c(x) \in C(\bar{D})$ . Therefore  $c(x) \in L^p(D)$ ,  $\forall p > n$ .

Consider the following problem: find  $v(x) \in W^{2,p}(D)$  such that

$$\begin{aligned} \Delta v(x) &= -c(x), \quad x \in D, \quad \text{a.e.}, \\ v(x) &= 0 \quad x \in \partial D. \end{aligned}$$

In [6, Ch. IX, Theorem 9.15] it is shown that this problem has a unique solution  $v(x) \in W^{2,p}(D)$ . Applying Lemma 2.2 to the latter function, we have

$$v(w_{t \wedge \sigma(D)}) = v(x) + \int_0^{t \wedge \sigma(D)} \text{grad } v(w_s) dw_s + \int_0^{t \wedge \sigma(D)} \Delta v(w_s) ds, \quad t \geq 0.$$

Passing to the limit in both sides of the equality as  $t \rightarrow \infty$ , we obtain

$$0 = v(x) + \int_0^{\sigma(D)} \text{grad } v(w_s) dw_s + \int_0^{\sigma(D)} \Delta v(w_s) ds.$$

The formulation of the problem clearly implies that

$$\int_0^{\sigma(D)} \Delta v(w_s) ds = - \int_0^{\sigma(D)} c(w_s) ds.$$

Taking the mathematical expectation of both sides of this equality, we obtain

$$v(x) = E_x \int_0^{\sigma(D)} c(w_s) ds = f(x).$$

Thus we have shown that  $f(x) = v(x)$ .  $\square$

### 3. PROOF OF A PRIORI ESTIMATES FOR THE OBSTACLE PROBLEM

Recall that  $D$  is a bounded domain with the smooth boundary ( $\partial D \in C^2$ ) in  $\mathbb{R}^n$ . Consider a domain  $\tilde{D}$  in  $\mathbb{R}^n$  such that  $D \subset \tilde{D}$ . Assume that  $\tilde{D}$  is a bounded domain with the smooth boundary ( $\partial \tilde{D} \in C^2$ ).

In §1 we have introduced the continuous function  $g(x)$  on the set  $\overline{D}$ . It is well known (Tietze's theorem) that there exists a continuous function  $\tilde{g}(x)$  on the domain  $\tilde{D}$  such that  $\tilde{g}(x) = g(x)$  when  $x \in \overline{D}$ .

We define the averaged function  $\tilde{g}_h(x)$  of the function  $\tilde{g}(x)$  by the following rule: for the function  $\tilde{g}(x)$  and  $h > 0$

$$\tilde{g}_h(x) = h^{-n} \int_{\tilde{D}} \rho\left(\frac{x-y}{h}\right) \tilde{g}(y) dy, \quad (3.1)$$

where  $\rho(x)$  is a non-negative function from the space  $C^\infty(\mathbb{R}^n)$ , equal to zero outside the unit ball  $B_1(0)$ , and satisfies the condition  $\int \rho(x) dx = 1$ .

It is well known (see [6, Ch. VII, §2]) that the function  $\tilde{g}_h(x)$  belongs to the space  $C_0^\infty(\mathbb{R}^n)$  and, moreover, on the set  $\overline{D}$  there holds the uniform convergence

$$\sup_{x \in \overline{D}} |\tilde{g}_h(x) - g(x)| \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad (3.2)$$

Denote by  $g_h(x)$  the restriction of the function  $\tilde{g}_h(x)$  on  $\overline{D}$ . It is clear that  $g_h(x) \in C^2(\overline{D})$ .

By virtue of (3.2), for every  $m$  there exists  $h_m$  such that  $g_{h_m}(x) - \frac{1}{m} \leq g(x) \leq g_{h_m}(x) + \frac{1}{m}$ . Let us introduce the notation  $g_{h_m}(x) - \frac{1}{m} \equiv g_m(x)$ . Then  $g_m(x) \leq g(x) \leq g_m(x) + \frac{2}{m}$ , i.e.  $\sup_{x \in D} |g_m(x) - g(x)| \leq \frac{2}{m} \rightarrow 0$  as  $m \rightarrow \infty$ .

For the payoff function  $g_m(x)$ , we formulate in the domain  $D$  the following optimal stopping problem of the Wiener process  $w_t$ :

$$S_m(x) = \sup_{\tau \in \mathfrak{m}} E_x \left( g_m(w_\tau) \cdot I_{(\tau < \sigma(D))} + \int_0^{\tau \wedge \sigma(D)} c(w_t) dt \right), \quad (3.3)$$

where  $P_x$  is the probability measure corresponding to the initial condition  $w_0(\omega) = x$ , and  $\mathfrak{m}$  is the class of all stopping times with respect to the filtration  $F^w = (\mathcal{F}_t^w)_{t \geq 0}$ .

The optimal stopping problem (3.3) can be rewritten in terms of the standard Markov process  $(w_t^{\sigma(D)}, \mathcal{F}_t^w, P_x)$  as

$$S_m(x) = \sup_{\tau \in \mathfrak{m}} E_X \left( \bar{g}_m(w_\tau^{\sigma(D)}) + \int_0^\tau \bar{c}(w_\tau^{\sigma(D)}) dt \right), \quad (3.4)$$

where  $\bar{g}_m(x) = g_m(x) \cdot \chi_D(x)$  and  $\bar{c}(x) = c(x) \cdot \chi_D(x)$ .

Let us consider the following obstacle problem: given the initial data  $g_m(x) \in W^{2,p}(D)$ ,  $c(x) \in L^p(D)$ ,  $p > n$ ,  $g_m(x) \leq 0$  as  $x \in \partial D$ , find  $u_m(x) \in W^{2,p}(D)$  such that

$$\begin{cases} \Delta u_m(x) + c(x) \leq 0 & u_m(x) \geq g_m(x), \\ (\Delta u_m(x) + c(x))(u_m(x) - g_m(x)) = 0. \end{cases} \quad (3.5)$$

It is known (see [1, Ch. VII, Theorem 2.2]), that the problem (3.5) has a unique solution  $u_m(x) \in W^{2,p}(D)$ ,  $\forall p > n$ , and this solution coincides with the solution of the corresponding variational inequality (1.3) (see [1, Ch. VII, Remark 3.1]). It is also known (see [1, Ch. VII, Theorem 4.1]) that the solution  $u_m(x)$  of the obstacle problem (3.5) coincides with the value function

$$u_m(x) = S_m(x) \quad (3.6)$$

of the optimal stopping problem (3.3).

Note that by virtue of the equality (2.10) the stochastic process  $u_m(w_t^{\sigma(D)}) - f(w_t^{\sigma(D)})$  is a minimal supermartingale (on the time interval  $[0, \infty)$ ) that majorizes the process  $\bar{g}_m(w_t^{\sigma(D)}) - f(w_t^{\sigma(D)})$ .

Now we proceed to proving the main theorems of this paper.

**Theorem 3.1.** *Let  $g^i(x)$ ,  $c^i(x)$ ,  $i = 1, 2$ , be two initial pairs of the variational inequality (1.3) such that  $g^i(x)$ ,  $c^i(x) \in C(\overline{D})$ ,  $g^i(x) \leq 0$ ,  $x \in$*

$\partial D$  and  $K^i \neq \emptyset$ ,  $i = 1, 2$ . Then for the solution  $u^i(x)$ ,  $i = 1, 2$ , of the problem (1.2), (1.3) there holds the following global energy estimate:

$$\begin{aligned} & \int_D d^2(x, \partial D) |\text{grad}(u^2 - u^1)(x)|^2 dx + \int_D (u^2(x) - u^1(x))^2 dx \leq \\ & \leq C \left[ \left( \sup_{x \in D} |g^2(x) - g^1(x)| + \sup_{x \in D} |c^2(x) - c^1(x)| \right) \times \right. \\ & \quad \times \left( \left( \sup_{x \in D} |g^1(x)| + \sup_{x \in D} |c^1(x)| \right) + \right. \\ & \quad \left. \left. + \left( \sup_{x \in D} |g^2(x)| + \sup_{x \in D} |c^2(x)| \right) \right) + \left. \left( \sup_{x \in D} |c^2(x) - c^1(x)| \right)^2 \right], \end{aligned}$$

where  $d(x, \partial D)$  is the distance from the point  $x$  to the boundary  $\partial D$ ,  $C$  is a constant depending on the dimension of the space  $\mathbb{R}^n$  and on the Lebesgue measure of  $D$ .

*Proof.* The main tool of proving the above estimate is the general semimartingale inequality for the ‘‘Snell envelope’’ used in Theorem 2.2.

Since the processes  $u_m^i(w_t^{\sigma(D)}) - f^i(w_t^{\sigma(D)})$ ,  $i = 1, 2$ , are Snell envelopes for the processes  $\bar{g}_m^i(w_t^{\sigma(D)}) - f^i(w_t^{\sigma(D)})$ ,  $i = 1, 2$ , the processes  $u_m^i(w_t^{\sigma(D)}) - f^i(w_t^{\sigma(D)})$  are solutions of the stochastic variational inequality (2.1), (2.2).

By virtue of Lemmas 2.2 and 2.3, we can apply the Itô formula to the processes  $u_m^i(w_t^{\sigma(D)}) - f^i(w_t^{\sigma(D)})$ . As a result, we obtain

$$\begin{aligned} & [(u_m^2 - u_m^1)(w^{\sigma(D)}) - (f^2 - f^1)(w^{\sigma(D)})]_{\sigma(D)} - \\ & - [(u_m^2 - u_m^1)(w^{\sigma(D)}) - (f^2 - f^1)(w^{\sigma(D)})]_0 = \\ & = \int_0^{\sigma(D)} |\text{grad}[(u_m^2 - u_m^1) - (f^2 - f^1)](w_t)|^2 dt. \end{aligned}$$

Further, we use the result obtained in Theorem 2.2. Since the stopping times  $\tau_1(\omega)$  and  $\tau_2(\omega)$  in the estimate (2.4) are arbitrary, we can write  $\tau_1(\omega) = 0$  and  $\tau_2(\omega) = \sigma(D)$ . Note that  $\bar{g}_m^i(x) = 0$ ,  $x \in \partial D$ , and  $f^i(x) = 0$  when  $x \in \partial D$ .

Therefore, by virtue of the estimate (2.4), we obtain

$$\begin{aligned} & E_x \int_0^{\sigma(D)} |\text{grad}[(u_m^2 - u_m^1) - (f^2 - f^1)](w_t)|^2 dt + \\ & + [(u_m^2(x) - u_m^1(x)) - (f^2(x) - f^1(x))]^2 \leq \\ & \leq 4 \sup_{x \in D} |(\bar{g}_m^2(x) - \bar{g}_m^1(x)) - (f^2(x) - f^1(x))| \times \\ & \quad \times \left[ \sup_{x \in D} |\bar{g}_m^1(x) - f^1(x)| + \sup_{x \in D} |\bar{g}_m^2(x) - f^2(x)| \right]. \quad (3.7) \end{aligned}$$

As is known (see [8, Ch. II, §7]), for every continuous function  $\psi(x)$  we have

$$E_x \int_0^{\sigma(D)} \psi(w_s) ds = \int_D G_D(x, y) \psi(y) dy, \quad (3.8)$$

where  $G_D(x, y)$  is the Green function. As is also known (see [4, Ch. XIV, §1]), the Green function  $G_D(x, y)$  is symmetric with respect to the variables  $x, y$ :

$$G_D(x, y) = G_D(y, x), \quad x, y \in D. \quad (3.9)$$

The equality (3.8) implies that if  $\psi(x) = 1$ , then

$$E_x \sigma(D) = \int_D G_D(x, y) dy = \int_D G_D(y, x) dy. \quad (3.10)$$

Consider the expression

$$\int_D \psi(y) G_D(x, y) dy.$$

Due to the symmetry of the Green function, we have

$$\int_D \psi(y) G_D(x, y) dy = \int_D \psi(y) G_D(y, x) dy. \quad (3.11)$$

Integrating both sides of the equality (3.11) with respect to the initial point  $x$  and applying the Fubini theorem, we obtain

$$\begin{aligned} \int_D \int_D \psi(y) G_D(x, y) dy dx &= \int_D \int_D \psi(y) G_D(y, x) dy dx = \\ &= \int_D \psi(y) \int_D G_D(y, x) dx dy = \int_D \psi(y) E_y \sigma(D) dy. \end{aligned} \quad (3.12)$$

Consider the first summand on the left-hand side of (3.7):

$$I_m \equiv E_x \int_0^{\sigma(D)} |\text{grad}[(u_m^2 - u_m^1) - (f^2 - f^1)](w_t)|^2 dt.$$

By virtue of (3.8), we obtain

$$I_m = \int_D G_D(x, y) |\text{grad}[(u_m^2 - u_m^1) - (f^2 - f^1)](y)|^2 dy.$$

Thus the inequality (3.7) takes the form

$$\begin{aligned} &\int_D G_D(x, y) |\text{grad}[(u_m^2 - u_m^1) - (f^2 - f^1)](y)|^2 dy + \\ &+ [(u_m^2(x) - u_m^1(x)) - (f^2(x) - f^1(x))]^2 \leq \end{aligned}$$

$$\begin{aligned} &\leq 4 \sup_{x \in D} |(\bar{g}_m^2(x) - \bar{g}_m^1(x)) - (f^2(x) - f^1(x))| \times \\ &\times \left[ \sup_{x \in D} |\bar{g}_m^1(x) - f^1(x)| + \sup_{x \in D} |\bar{g}_m^2(x) - f^2(x)| \right]. \end{aligned} \quad (3.13)$$

If we integrate both sides of (3.13) with respect to the initial point  $x$  and take into account (3.12), then we have

$$\begin{aligned} &\int_D E_x \sigma(D) \text{grad}[(u_m^2 - u_m^1) - (f^2 - f^1)](x)^2 dx + \\ &+ \int_D [(u_m^2(x) - u_m^1(x)) - (f^2(x) - f^1(x))]^2 dx \leq \\ &\leq 4 \text{mes}(D) \sup_{x \in D} |(\bar{g}_m^2(x) - \bar{g}_m^1(x)) - (f^2(x) - f^1(x))| \times \\ &\times \left[ \sup_{x \in D} |\bar{g}_m^1(x) - f^1(x)| + \sup_{x \in D} |\bar{g}_m^2(x) - f^2(x)| \right]. \end{aligned} \quad (3.14)$$

Let us consider an arbitrary point  $x$  from the domain  $D$ . We denote by  $B(d(x))$  the ball with center at the point  $x$  and radius  $r = d(x)$ , where  $d(x)$  denotes the distance from the point  $x$  to the boundary  $\partial D$ ,  $d(x) = \text{dist}(x, \partial D)$ . Note that

$$\sigma(D) \geq \sigma(B(d(x))).$$

Hence we have

$$E_x \sigma(D) \geq E_x \sigma(B(d(x))). \quad (3.15)$$

It is known that ([9, Ch. II, §2])

$$E_x \sigma(B(x, r)) = \frac{1}{n} r^2, \quad (3.16)$$

where  $B(x, r)$  is the ball with center at the point  $x$  and radius  $r$ . By virtue of (3.15) and (3.16), we have

$$E_x \sigma(D) \geq \frac{1}{n} r^2 = \frac{1}{n} d^2(x). \quad (3.17)$$

Note that according to Lemma 2.3, the function  $f^2(x) - f^1(x)$  satisfies the differential equation

$$\Delta(f^2 - f^1)(x) = -(c^2 - c^1)(x), \quad x \in D. \quad (3.18)$$

Multiplying both sides of (3.18) by the function  $(f^2 - f^1)(x)$  and integrating on the domain  $D$ , we obtain

$$\int_D (f^2 - f^1)(x) \Delta(f^2 - f^1)(x) dx = - \int_D (c^2 - c^1)(x) (f^2 - f^1)(x) dx. \quad (3.19)$$

By virtue of the Green formula,

$$\int_D (f^2 - f^1)(x) \Delta(f^2 - f^1)(x) dx = - \int_D |\text{grad}(f^2 - f^1(x))|^2 dx. \quad (3.20)$$

Consider the first summand on the left-hand side of the inequality (3.14)

$$I_m^1 \equiv \int_D E_x \sigma(D) |\text{grad}[(u_m^2 - u_m^1) - (f^2 - f^1)](x)|^2 dx.$$

Taking into account the inequality (3.17), we obtain

$$\begin{aligned} I_m^1 &\geq \frac{1}{2n} \int_D d^2(x, \partial D) |\text{grad}(u_m^2 - u_m^1)(x)|^2 dx - \\ &\quad - \frac{1}{n} \int_D d^2(x, \partial D) |\text{grad}(f^2 - f^1)(x)|^2 dx. \end{aligned}$$

Taking into account the inequalities (3.19), (3.20), we obtain

$$\begin{aligned} I_m^1 &\geq \frac{1}{2n} \int_D d^2(x, \partial D) |\text{grad}(u_m^2 - u_m^1)(x)|^2 dx - \\ &\quad - \frac{d_1^2}{n} \int_D (c^2 - c^1)(x)(f^2 - f^1)(x) dx, \end{aligned} \quad (3.21)$$

where  $d_1 = \max_{x, y \in D} \rho_1(x, y)$  ( $\rho_1(x, y)$  denotes the distance in the space  $\mathbb{R}^n$ ).

By virtue of (3.21), the inequality (3.14) implies

$$\begin{aligned} &\int_D d^2(x, \partial D) |\text{grad}(u_m^2 - u_m^1)(x)|^2 dx + \int_D (u_m^2(x) - u_m^1(x))^2 dx \leq \\ &\leq C \left[ \sup_{x \in D} |(\bar{g}_m^2(x) - \bar{g}_m^1(x)) - (f^2(x) - f^1(x))| \times \right. \\ &\quad \times [\sup_{x \in D} |\bar{g}_m^1(x) - f^1(x)| + \sup_{x \in D} |\bar{g}_m^2(x) - f^2(x)|] + \\ &\quad \left. + d_1^2 \int_D (c^2 - c^1)(x)(f^2 - f^1)(x) dx + \int_D (f^2(x) - f^1(x))^2 dx \right]. \end{aligned} \quad (3.22)$$

Note that

$$\sup_{x \in D} |\bar{g}_m^2(x) - \bar{g}_m^1(x)| = \sup_{x \in D} |g_m^2(x) - g_m^1(x)|.$$

By Lemma 2.1,

$$\begin{aligned} \sup_{x \in D} |\bar{g}_m^i(x) - f^i(x)| &\leq \sup_{x \in D} |\bar{g}_m(x)| + \sup_{x \in D} |f^i(x)| \leq \\ &\leq \sup_{x \in D} |\bar{g}_m^i(x)| + C_1 |c^i(x)|, \quad i = 1, 2, \end{aligned}$$

$$\int_D (f^2(x) - f^1(x))^2 dx \leq C_2 \text{mes}(D) (\sup_{x \in D} |c^2(x) - c^1(x)|)^2,$$

$$d_1^2 \int_D (c^2 - c^1)(x) \cdot (f^2 - f^1)(x) dx \leq d_1^2 \text{mes}(D) \cdot C_3 (\sup_{x \in D} |c^2(x) - c^1(x)|)^2.$$

Hence the inequality (3.22) implies

$$\begin{aligned} & \int_D d^2(x, \partial D) |\text{grad}(u_m^2 - u_m^1)(x)|^2 dx + \int_D (u_m^2(x) - u_m^1(x))^2 dx \leq \\ & \leq \bar{C} \left[ \left( \sup_{x \in D} |(\bar{g}_m^2(x) - \bar{g}_m^1(x))| + \sup_{x \in D} |c^2(x) - c^1(x)| \right) \times \right. \\ & \quad \times \left( \sup_{x \in D} |\bar{g}_m^1(x)| + \sup_{x \in D} |c^1(x)| + \sup_{x \in D} |\bar{g}_m^2(x)| + \right. \\ & \quad \left. \left. + \sup_{x \in D} |c^2(x)| \right) + \left( \sup_{x \in D} |c^2(x) - c^1(x)| \right)^2 \right]. \end{aligned} \quad (3.23)$$

Note that

$$\sup_{x \in D} |\bar{g}_m^i(x) - g^i(x)| \rightarrow 0, \quad m \rightarrow \infty, \quad i = 1, 2, \quad (3.24)$$

$$g_m^i(x) \leq g^i(x) \leq g_m^i(x) + \frac{2}{m}, \quad i = 1, 2. \quad (3.25)$$

Let us now show that functions  $u_m^i(x)$  are weakly convergent to a solution  $u^i(x)$ ,  $i = 1, 2$ , of the variational inequality (1.3) for the obstacle functions  $g^i(x)$ .

Since by virtue of (1.2) the defined sets  $K^i$ ,  $i = 1, 2$ , are nonempty, there exist functions  $v_0^i \in K^i$  such that  $v_0^i(x) \in H_0^1(D)$  and  $v_0^i(x) \geq g^i(x)$ ,  $i = 1, 2$ .

Let us consider the closed convex sets (3.26)

$$K_m^i = \{v^i(x) : v^i(x) \in H_0^1(D), \quad v^i(x) \geq g_m^i(x) \text{ a.e.}\} \quad (3.26)$$

for each  $m$ ,  $m = 1, 2, \dots$ , it follows from (3.25) that  $K^i \subseteq K_m^i$ ,  $m = 1, 2, \dots$

Let us consider the following problem that corresponds to the problem (1.3): find  $u_m^i(x) \in K_m^i$  such that for any function  $v^i(x) \in K_m^i$  the inequality

$$a(u_m^i, v^i - u_m^i) \geq \int_D c^i(x)(v^i(x) - u_m^i(x))dx, \quad i = 1, 2, \quad (3.27)$$

is valid.

We know that the problem (3.26), (3.27) has a unique solution. If instead of the functions  $v^i(x)$  we take the functions  $v_0^i(x) \in K_m^i$ ,  $m = 1, 2, \dots$ ,  $i = 1, 2$ , we will have

$$a(u_m^i, v_0^i - u_m^i) \geq \int_D c^i(x)(v_0^i(x) - u_m^i(x))dx, \quad i = 1, 2,$$

$$\|u_m^i\|_{H_0^1(D)}^2 = a(u_m^i, u_m^i) \leq a(u_m^i, v_0^i) + \int_D c^i(x)(v_0^i(x) - u_m^i(x))dx, \quad i = 1, 2.$$

From these formulas it follows that

$$\|u_m^i\|_{H_0^1(D)} \leq \tilde{C}.$$

Let us show that the sequence  $u_m^i(x)$  is weakly convergent to the solution  $u^i(x)$ ,  $i = 1, 2$ , of the problem (1.3). Consider some weakly convergent



subsequences  $u_{m_k}^i(x)$ ,  $i = 1, 2$ , of the sequences  $u_m^i(x)$ ,  $i = 1, 2$ . Denote their limit by  $\widehat{u}^i(x) \in H_0^1(D)$ ,  $i = 1, 2$ , respectively.

Note that  $u_{m_k}^i(x)$  is a solution of the problem (3.26), (3.27) for the respective obstacle functions  $g_{m_k}^i(x)$ .

Let us show that the functions  $\widehat{u}^i(x)$ ,  $i = 1, 2$ , are solutions of the problem (1.3). Indeed for every function  $v^i(x) \in K^i \subseteq \widetilde{K}_{m_k}^i$  we have

$$a(u_{m_k}^i, v^i - u_{m_k}^i) \geq \int_D c^i(x)(v^i(x) - u_{m_k}^i(x))dx, \quad i = 1, 2, \quad (3.28)$$

which implies

$$a(u_{m_k}^i, v^i) \geq a(u_{m_k}^i, u_{m_k}^i) + \int_D c^i(x)(v^i(x) - u_{m_k}^i(x))dx, \quad i = 1, 2.$$

Note that the quadratic form  $a(u, u)$  is weakly lower semicontinuous. Passing to the limit as  $k \rightarrow \infty$ , we obtain

$$a(\widehat{u}^i, v^i - \widehat{u}^i) \geq \int_D c^i(x)(v^i(x) - \widehat{u}^i(x))dx, \quad i = 1, 2.$$

Now let us show that  $\widehat{u}^i(x) \in K^i$ ,  $i = 1, 2$ .

Since  $u_{m_k}^i$ ,  $i = 1, 2$ , are the solutions of the problem (3.26), (3.27) for the obstacle functions  $g_{m_k}^i(x)$ ,  $i = 1, 2$ , respectively, we conclude that  $u_{m_k}^i(x) \geq g_{m_k}^i(x)$  a. e.

From (3.25) we obtain  $u_{m_k}^i(x) + \frac{2}{m_k} \geq g^i(x)$ ,  $i = 1, 2$ . Therefore  $u_{m_k}^i(x) + \frac{2}{m_k} \in \widetilde{K}^i$ ,  $i = 1, 2$ , where

$$\widetilde{K}^i = \{v^i(x) : v^i(x) \in H^1(D), \quad v^i(x) \geq g^i(x) \text{ a.e.}\}.$$

Moreover,  $u_{m_k}^i(x) + \frac{2}{m_k}$  is weakly convergent to the functions  $\widehat{u}^i(x)$ . Since in the Hilbert space a closed convex set is weakly closed, we conclude that  $\widehat{u}^i(x) \in \widetilde{K}^i$ . But  $\widehat{u}^i(x) \in H_0^1(D)$  and therefore  $\widehat{u}^i(x) \in K^i$ ,  $i = 1, 2$ . By virtue of the uniqueness of the solution of the problem (1.3), we have  $\widehat{u}^i(x) = u^i(x)$ ,  $i = 1, 2$ . We have thus shown that the entire sequence  $u_m^i(x)$ ,  $i = 1, 2$ , is weakly convergent to the function  $u^i(x)$ ,  $i = 1, 2$ .

Denote by  $\widetilde{a}(u, v)$  the following bilinear form:

$$\widetilde{a}(u, v) = \int_D d^2(x, \partial D) \operatorname{grad} u(x) \operatorname{grad} v(x) dx + \int_D u(x)v(x) dx. \quad (3.29)$$

It is easy to verify that the quadratic form  $\widetilde{a}(u, u)$  is weakly lower semicontinuous. Since  $u_m^i(x)$  is weakly convergent to the function  $u^i(x)$ ,  $i = 1, 2$ , in  $H_0^1(D)$ , we have

$$\varliminf_{m \rightarrow \infty} \widetilde{a}(u_m^i, u_m^i) \geq \widetilde{a}(u^i, u^i), \quad i = 1, 2.$$

Recall that  $\sup_{x \in D} |\bar{g}_m^i(x) - g^i(x)| \rightarrow 0$  as  $m \rightarrow \infty$ ,  $i = 1, 2$ . Hence if in the inequality (3.23) we pass to limit when  $m \rightarrow \infty$ , we complete the proof of the theorem.  $\square$

Now let us prove the theorem on local energy estimates for the domain  $B \subset D$ . Assume that the boundary  $\partial B$  of the domain  $B$  is smooth ( $\partial B \in C^2$ ).

**Theorem 3.2.** *Let  $B \subset D$  be a smooth ( $\partial B \in C^2$ ) domain. If  $g^i(x)$ ,  $c^i(x)$ ,  $i = 1, 2$ , are two initial pairs of the variational inequality (1.3) such that  $g^i(x)$ ,  $c^i(x) \in C(\bar{D})$ ,  $g^i(x) \leq 0$ ,  $x \in \partial D$ , and  $K^i \neq \emptyset$ ,  $i = 1, 2$ , then for the solution  $u^i(x)$ ,  $i = 1, 2$ , of the problem (1.2), (1.3) the following local energy estimate is valid:*

$$\begin{aligned} & \int_B d^2(x, \partial B) |\text{grad}(u^2 - u^1)(x)|^2 dx + \int_B (u^2(x) - u^1(x))^2 dx \leq \\ & \leq C \left[ \left( \sup_{x \in B} |g^2(x) - g^1(x)| + \sup_{x \in B} |c^2(x) - c^1(x)| \right) \times \right. \\ & \times \left( \sup_{\substack{x \in B \\ y \in \partial B}} |g^1(x) - g^1(y)| + \sup_{\substack{x \in B \\ y \in \partial B}} |c^1(x) - c^1(y)| + \sup_{\substack{x \in B \\ y \in \partial B}} |g^2(x) - g^2(y)| + \right. \\ & \left. \left. + \sup_{\substack{x \in B \\ y \in \partial B}} |c^2(x) - c^2(y)| \right) + \left( \sup_{x \in B} |c^2(x) - c^1(x)| \right)^2 + \sup_{y \in \partial B} (u^2(y) - u^1(y))^2 \right]. \end{aligned}$$

where  $d(x, \partial B)$  is the distance from the point  $x$  to the boundary  $\partial B$  of the domain  $B$ ,  $C$  is a constant depending on the dimension of the space  $\mathbb{R}^n$  and on the measure of the domain  $B$ .

*Proof.* By virtue of the estimate (2.4), analogously to the inequality (3.7), we obtain

$$\begin{aligned} & E_x \int_0^{\sigma(B)} |\text{grad}[(u_m^2 - u_m^1) - (f^2 - f^1)](w_t)|^2 dt + [(u_m^2(x) - u_m^1(x)) - \\ & - (f^2(x) - f^1(x))]^2 \leq 4 \sup_{x \in B} |(g_m^2(x) - g_m^1(x)) - (f^2(x) - f^1(x))| \times \\ & \times \left[ \sup_{\substack{x \in B \\ y \in \partial B}} |(g_m^1(x) - g_m^1(y)) - (f^1(x) - f^1(y))| + \right. \\ & \left. + \sup_{\substack{x \in B \\ y \in \partial B}} |(g_m^2(x) - g_m^2(y)) - (f^2(x) - f^2(y))| \right] + \\ & + \sup_{y \in \partial B} \left[ (u_m^2(y) - u_m^1(y)) - (f^2(y) - f^1(y)) \right]^2. \quad (3.30) \end{aligned}$$

Analogously to the inequality (3.23), the estimate (3.30) implies

$$\begin{aligned}
& \int_B d^2(x, \partial B) |\text{grad}(u_m^2 - u_m^1)(x)|^2 dx + \int_B (u_m^2(x) - u_m^1(x))^2 dx \leq \\
& \leq \overline{C} \left[ \left( \sup_{x \in B} |g_m^2(x) - g_m^1(x)| + \sup_{x \in B} |c^2(x) - c^1(x)| \right) \times \right. \\
& \quad \times \left( \sup_{\substack{x \in B \\ y \in \partial B}} |g_m^1(x) - g_m^1(y)| + \sup_{\substack{x \in B \\ y \in \partial B}} |c^1(x) - c^1(y)| + \right. \\
& \quad \left. \left. + \sup_{\substack{x \in B \\ y \in \partial B}} |g_m^2(x) - g_m^2(y)| + \sup_{\substack{x \in B \\ y \in \partial B}} |c^2(x) - c^2(y)| \right) + \right. \\
& \quad \left. + \left( \sup_{x \in B} |c^2(x) - c^1(x)| \right)^2 + \sup_{y \in \partial B} (u_m^2(y) - u_m^1(y))^2 \right]. \quad (3.31)
\end{aligned}$$

Since the functions  $g_m^i(x)$  are uniformly convergent to the function  $g^i(x)$ ,  $i = 1, 2$ , on arbitrary subset of the domain  $D$  we have

$$\sup_{x \in B} |g_m^i(x) - g^i(x)| \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad i = 1, 2.$$

In Theorem 3.1 it is shown that the functions  $u_m^i(x)$ ,  $i = 1, 2$ , are respectively weakly convergent to the solutions  $u^i(x)$ ,  $i = 1, 2$ , of the problem (1.3) and that

$$\varliminf_{m \rightarrow \infty} \tilde{a}(u_m^i, u_m^i) \geq \tilde{a}(u^i, u^i), \quad i = 1, 2,$$

where the form  $\tilde{a}(u, v)$  is defined by the expression (3.29).

Hence, denoting the constant  $\overline{C}$  in the inequality (3.31) by the symbol  $C$ , we complete the proof of the theorem.  $\square$

*Remark 3.1.* Let  $\mathbb{K}$  be a compact set from the domain  $D$ ,  $\mathbb{K} \subset D$ . If  $g^i(x)$ ,  $c^i(x)$ ,  $i = 1, 2$ , is an initial pair of the variational inequality (1.3) such that  $g^i(x)$ ,  $c^i(x) \in C(\overline{D})$ ,  $g^i(x) \leq 0$ ,  $x \in \partial D$ , and  $K^i \neq \emptyset$ ,  $i = 1, 2$ , then for the respective solutions  $u^i(x)$ ,  $i = 1, 2$ , of the problem (1.3) the following estimate is valid:

$$\begin{aligned}
& \int_{\mathbb{K}} |\text{grad}(u^2 - u^1)(x)|^2 dx + \int_{\mathbb{K}} |(u^2(x) - u^1(x))^2 dx \leq \\
& \leq C \left[ \left( \sup_{x \in D} |g^2(x) - g^1(x)| + \sup_{x \in D} |c^2(x) - c^1(x)| \right) \left( \sup_{x \in D} |g^1(x)| + \right. \right. \\
& \quad \left. \left. + \sup_{x \in D} |c^1(x)| + \sup_{x \in D} |g^2(x)| + \sup_{x \in D} |c^2(x)| \right) + \left( \sup_{x \in D} |c^2(x) - c^1(x)| \right)^2 \right].
\end{aligned}$$

The proof immediately follows from Theorem 3.1. Note that here the constant  $C$  becomes dependant on the distance from the compact set  $\mathbb{K}$  to the boundary of the domain  $D$ .

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