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ON TWO-POINT BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER SINGULAR ORDINARY DIFFERENTIAL EQUATIONS

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Consider the differential equation

$$u^{(2n)} = f(t, u, \dots, u^{(n-1)}) \tag{1}$$

with the boundary conditions

$$u^{(i-1)}(a) = 0 \ \ (i = 1, \dots, n), \quad u^{(k-1)}(b) = 0 \ \ (k = 1, \dots, n) \tag{2}$$

or

$$u^{(i-1)}(a) = 0 \ \ (i = 1, \dots, n), \quad u^{(k-1)}(b) = 0 \ \ (k = n+1, \dots, 2n). \tag{3}$$

Here $n \geq 1$, $-\infty < a < b < +\infty$ and the function $f :]a, b[\times \mathbb{R}^n \to \mathbb{R}$ satisfies the local Carathéodory conditions, i.e. $f(t, \cdot, \dots, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous for almost all $t \in]a, b[$, $f(\cdot, x_1, \dots, x_n) :]a, b[\to \mathbb{R}$ is measurable for every $(x_1, \dots, x_n) \in \mathbb{R}^n$, and the function

$$f^*(t, \rho) = \max \left\{ \big| f(t, x_1, \dots, x_n) \big| : \ \sum_{i=1}^n |x_i| \leq \rho \right\} \tag{4}$$

is integrable in the first argument on $[a + \varepsilon, b - \varepsilon]$ for arbitrary $\rho \in [0, +\infty[$ and $\varepsilon \in]0, (b-a)/2[$.

Of special interest for us is the case where the function f (and therefore the function f^*) is non-integrable in the first argument on $[a, b]$, having singularities at the ends of this segment. In this sense the problems (1), (2) and (1), (3) are singular ones.

Singular boundary value problems for ordinary differential equations (including the problems (1), (2) and (1), (3)) have been intensively studied from the 60s of the last century up to the present time (see, e.g., [1]–[19] and the references cited therein). In [11], the problems (1), (2) and (1), (3) are studied in the case where the function f admits the one-sided estimate

$$(-1)^n f(t, x_1, \dots, x_n) \operatorname{sgn} x_1 \leq \sum_{i=1}^n h_i(t) |x_i| + h_0(t),$$

where $h_i :]a, b[\to [0, +\infty[$ $(i = 0, \dots, n)$ are measurable functions satisfying either the conditions

$$\begin{gathered} \int_a^b (s-a)^{2n-i} (b-s)^{2n-i} h_i(t) \, dt < +\infty \ \ (i = 1, \dots, n), \\ \int_a^b (t-a)^{n-\frac{1}{2}} (b-t)^{n-\frac{1}{2}} h_0(t) \, dt < +\infty, \end{gathered} \tag{5}$$

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or the conditions

$$\int_a^b (t-a)^{2n-i} h_i(t)\,dt < +\infty \ \ (i=1,\dots,n), \quad \int_a^b (t-a)^{n-\frac{1}{2}} h_0(t)\,dt < +\infty. \tag{6}$$

In the present paper we consider the case where either

$$h_i(t) = \frac{\ell_i}{(t-a)^{2n+1-i}(b-t)^{2n+1-i}} \ \ (i=1,\dots,n),$$
$$\int_a^b (t-a)^{2n}(b-t)^{2n} h_0(t)\,dt < +\infty,$$

or

$$h_i(t) = \frac{\ell_i}{(t-a)^{2n+1-i}} \ \ (i=1,\dots,n), \quad \int_a^b (t-a)^{2n} h_0^2(t)\,dt < +\infty,$$

and, consequently, the conditions (5) and (6) are violated.

Along with (4), the following notation will be used.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ is the n-dimensional real Euclidean space.

If $x \in \mathbb{R}$, then

$$[x]_+ = \frac{x+|x|}{2}\,.$$

$L^2(]a,b[)$ is the space of square integrable on $[a,b]$ functions $h:]a,b[\to \mathbb{R}$ with the norm

$$\|h\|_{L^2} = \Big(\int_a^b |h(t)|^2\,dt\Big)^{1/2}.$$

$L^2_{\alpha,\beta}(]a,b[)$ is the space of square integrable on $[a,b]$ with the weight $(t-a)^\alpha(b-t)^\beta$ functions $h:]a,b[\to \mathbb{R}$ with the norm

$$\|h\|_{L^2_{\alpha,\beta}} \overset{def}{=} \Big(\int_a^b (t-a)^\alpha(b-t)^\beta h^2(t)\,dt\Big)^{1/2}.$$

$L_{loc}(]a,b[)$ ($L_{loc}(]a,b])$) is the space of functions $h:]a,b[\to \mathbb{R}$ which are integrable on $[a+\varepsilon, b-\varepsilon]$ (on $[a+\varepsilon, b]$) for any $\varepsilon \in]0, \frac{b-a}{2}[$.

$\widetilde{C}^m_{loc}(]a,b[)$ ($\widetilde{C}^m_{loc}(]a,b])$) is the space of functions $u:]a,b[\to \mathbb{R}$ (of functions $u:]a,b] \to \mathbb{R}$) which are absolutely continuous together with $u^{(i)}$ $(i=1,\dots,m)$ on $[a+\varepsilon, b-\varepsilon]$ (on $[a+\varepsilon, b]$) for any $\varepsilon \in]0, \frac{b-a}{2}[$.

$\widetilde{C}^m_2(I)$ is the space of functions $u \in \widetilde{C}^m_{loc}(I)$ satisfying the condition

$$\int_I |u^{(m)}(t)|^2\,dt < +\infty.$$

If $u \in \widetilde{C}^m_2(]a,b[)$, then the functions $u^{(i)}$ $(i=0,\dots,m-1)$ at the points a and b, respectively, have the right and the left limits which in the sequel will be accepted as $u^{(i)}(a)$ and $u^{(i)}(b)$ $(i=0,\dots,m-1)$.

A solution of the problem (1), (2) will be sought in the space

$$\widetilde{C}^{2n-1}_{loc}(]a,b[) \cap \widetilde{C}^n_2(]a,b[),$$

while a solution of the problem (1), (3) will be sought in the space

$$\widetilde{C}^{2n-1}_{loc}(]a,b]) \cap \widetilde{C}^n_2(]a,b]).$$

Let either $h \in L^2_{2n,2n}(]a,b[)$ or $h \in L^2_{2n,0}(]a,b[)$. Consider the perturbed differential equation

$$u^{(2n)} = f(t,u,\dots,u^{(n-1)}) + h(t) \tag{7}$$

and introduce the following definitions.

Definition 1. The problem (1), (2) is called stable with respect to a small perturbation of the right-hand member of the equation (1) if there exists a positive constant ρ_0 such that for any $h \in L^2_{2n,2n}(]a,b[)$ the problem (7), (2) is uniquely solvable in the space $\widetilde{C}^{2n-1}_{loc}(]a,b[) \cap \widetilde{C}^n_2(]a,b[)$, and

$$\big\| u_h^{(n)} - u_0^{(n)} \big\|_{L^2} \le \rho_0 \|h\|_{L^2_{2n,2n}}, \tag{8}$$

where u_0 and $u_h \in \widetilde{C}^{2n-1}_{loc}(]a,b[) \cap \widetilde{C}^n_2(]a,b[)$ are solutions of the problems (1), (2) and (7), (2).

Definition 2. The problem (1), (3) is called stable with respect to a small perturbation of the right-hand member of the equation (1) if there exists a positive constant ρ_0 such that for any $h \in L^2_{2n,0}(]a,b[)$ the problem (1), (3) is uniquely solvable in the space $\widetilde{C}^{2n-1}_{loc}(]a,b]) \cap \widetilde{C}^n_2(]a,b])$, and

$$\big\| u_h^{(n)} - u_0^{(n)} \big\|_{L^2} \le \rho_0 \|h\|_{L^2_{2n,0}}, \tag{9}$$

where u_0 and $u_h \in \widetilde{C}^{2n-1}_{loc}(]a,b]) \cap \widetilde{C}^n_2(]a,b])$ are solutions of the problems (1), (3) and (7), (3).

It should be noted that in view of (2) (in view of (3)) from the inequality (8) (from the inequality (9)) we obtain the following inequalities:

$$\big| u_h^{(i-1)}(t) - u_0^{(i-1)}(t) \big| \le \rho_i(h)(t-a)^{n-i+\frac{1}{2}}(b-t)^{n-i+\frac{1}{2}} \ \text{ for } \ a<t<b \ \ (i=1,\dots,n)$$
$$\Big(\big| u_h^{(i-1)}(t) - u_0^{(i-1)}(t) \big| \le \rho_i(h)(t-a)^{n-i+\frac{1}{2}} \ \text{ for } \ a<t<b \ \ (i=1,\dots,n) \Big),$$

where

$$\rho_i(h) = \frac{1}{(n-i)!\sqrt{2n-2i+1}} \Big(\frac{2}{b-a}\Big)^{n-i+\frac{1}{2}} \rho_0 \|h\|_{L^2_{2n,2n}}$$
$$\Big(\rho_i(h) = \frac{1}{(n-i)!\sqrt{2n-2i+1}} \rho_0 \|h\|_{L^2_{2n,0}} \Big).$$

Theorem 1. *Let there exist non-negative constants ℓ_i $(i=1,\dots,n)$ and a function $\ell \in L^2_{2n,2n}(]a,b[)$ such that on the domain $]a,b[\times\mathbb{R}^n$ the condition*

$$(-1)^n f(t,x_1,\dots,x_n)\operatorname{sgn} x_1 \le \sum_{i=1}^n \frac{\ell_i|x_i|}{(t-a)^{2n+1-i}(b-t)^{2n+1-i}} + \ell(t)$$

holds and

$$\sum_{i=1}^n \frac{\mu_{in}\ell_i}{(b-a)^{2n+1-i}} < 1, \tag{10}$$

where

$$\mu_{in} = 2^{2n+1-i}\Big(\prod_{k=1}^n (4k-3)\Big)^{-\frac{1}{2}}\Big(\prod_{k=1}^{n-i+1}(4k-3)\Big)^{-\frac{1}{2}}.$$

Then the problem (1), (2) has at least one solution in the space $\widetilde{C}^{2n-1}_{loc}(]a,b[) \cap \widetilde{C}^n_2(]a,b[)$.

Theorem 1′. *Let there exist non-negative constants ℓ_i $(i = 1, \dots, n)$ satisfying the inequality* (10) *such that on the domain* $]a, b[\,\times\mathbb{R}^n$ *the condition*

$$(-1)^n \big[f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)\big] \operatorname{sgn}(x_1 - y_1) \leq$$
$$\leq \sum_{i=1}^{n} \frac{\ell_i |x_i - y_i|}{(t-a)^{2n+1-i}(b-t)^{2n+1-i}}$$

holds. Let, moreover,

$$\int_a^b (t-a)^{2n}(b-t)^{2n}\big|f(t, 0, \dots, 0)\big|^2\,dt < +\infty.$$

Then the problem (1), (2) *is uniquely solvable in the space* $\widetilde{C}^{2n-1}_{loc}(]a, b[) \cap \widetilde{C}^n_2(]a, b[)$ *and stable with respect to a small perturbation of the right-hand member of the equation* (1).

Theorem 2. *Let there exist non-negative constants ℓ_i $(i = 1, \dots, n)$ and a function* $\ell \in L^2_{2n,0}(]a, b[)$ *such that on the domain* $]a, b[\,\times\mathbb{R}^n$ *the condition*

$$(-1)^n f(t, x_1, \dots, x_n) \operatorname{sgn} x_1 \leq \sum_{i=1}^{n} \frac{\ell_i |x_i|}{(t-a)^{2n+1-i}} + \ell(t)$$

holds and

$$\sum_{i=1}^{n} \nu_{in}\ell_i < 1, \tag{11}$$

where

$$\nu_{in} = 2^{2n+1-i}\Big(\prod_{k=1}^{n}(2k-1)\Big)^{-1}\Big(\prod_{k=1}^{n-i+1}(2k-1)\Big)^{-1}.$$

Let, moreover,

$$f^*(\cdot, \rho) \in L_{loc}(]a, b]) \ \ \textit{for} \ \ 0 < \rho < +\infty. \tag{12}$$

Then the problem (1), (3) *has at least one solution in the space* $\widetilde{C}^{2n-1}_{loc}(]a, b]) \cap \widetilde{C}^n_2(]a, b])$.

Theorem 2′. *Let there exist non-negative constants ℓ_i $(i = 1, \dots, n)$ satisfying the inequality* (11) *such that on the domain* $]a, b[\,\times\mathbb{R}^n$ *the condition*

$$(-1)^n \big[f(t, x_1, \dots, x_n) - f(t, y_1, \dots, y_n)\big] \operatorname{sgn}(x_1 - y_1) \leq \sum_{i=1}^{n} \frac{\ell_i |x_i - y_i|}{(t-a)^{2n+1-i}}$$

holds. Let, moreover, along with (12) *the condition*

$$\int_a^b (t-a)^{2n}\big|f(t, 0, \dots, 0)\big|^2\,dt < +\infty$$

be fulfilled. Then the problem (1), (3) *is uniquely solvable in the space* $\widetilde{C}^{2n-1}_{loc}(]a, b]) \cap \widetilde{C}^n_2(]a, b])$ *and stable with respect to a small perturbation of the right-hand member of the equation* (1).

Let us consider the differential equations

$$u^{(2n)} = \sum_{i=1}^{n} p_i(t)u^{(i-1)} + q(t, u) \tag{13}$$

and

$$u^{(2n)} = \sum_{i=1}^{n} p_i(t)|u^{(i-1)}|^{\lambda_i} \operatorname{sgn} u^{(i-1)} + q(t, u), \tag{14}$$

where $p_i : L_{loc}(]a, b[)$, $0 < \lambda_i < 1$ $(i = 1, \dots, n)$, and $q :]a, b[\,\times\mathbb{R} \to \mathbb{R}$ is the function satisfying the local Carathéodory conditions.

Suppose

$$q^*(t,\rho)=\max\big\{|q(t,x)|:\ \ |x|\le\rho\big\}\ \text{ for }\ a<t<b,\ \ 0<\rho<+\infty.$$

From the above theorems we have the following corollaries.

Corollary 1. *Let there exist non-negative constants ℓ_i $(i=1,\dots,n)$, satisfying the inequality* (10), *and a function $\ell\in L^2_{2n,2n}(]a,b[)$ such that*

$$\begin{gathered}(-1)^n p_1(t)\le\frac{\ell_1}{(t-a)^{2n}(b-t)^{2n}}\,,\\ |p_i(t)|\le\frac{\ell_i}{(t-a)^{2n+1-i}(b-t)^{2n+1-i}}\ \ (i=2,\dots,n)\ \ \textit{for}\ \ a<t<b\end{gathered}\tag{15}$$

and

$$(-1)^n q(t,x)\operatorname{sgn}x\le\ell(t)\ \ \textit{for}\ \ a<t<b,\ \ x\in\mathbb{R}.\tag{16}$$

Then the problem (13), (2) *has at least one solution in the space* $\widetilde C^{2n-1}_{loc}(]a,b[)\cap\widetilde C^n_2(]a,b[)$.

Corollary 1′. *Let there exist non-negative constants ℓ_i $(i=1,\dots,n)$ satisfying the inequality* (10) *such that the functions p_i $(i=1,\dots,n)$ satisfy the conditions* (15). *Let, moreover,*

$$(-1)^n\big[q(t,x)-q(t,y)\big]\operatorname{sgn}(x-y)\le 0\ \ \textit{for}\ \ a<t<b,\ \ x\in\mathbb{R},\ \ y\in\mathbb{R}\tag{17}$$

and

$$\int\limits_a^b (t-a)^{2n}(b-t)^{2n}|q(t,0)|^2\,dt<+\infty.\tag{18}$$

Then the problem (13), (2) *is uniquely solvable in the space* $\widetilde C^{2n-1}_{loc}(]a,b[)\cap\widetilde C^n_2(]a,b[)$ *and stable with respect to a small perturbation of the right-hand member of the equation* (13).

Corollary 2. *Let there exist non-negative constants ℓ_i $(i=1,\dots,n)$, satisfying the inequality* (11), *and a function $\ell\in L^2_{2n,0}(]a,b[)$ such that*

$$(-1)^n p_1(t)\le\frac{\ell_1}{(t-a)^{2n}}\,,\quad |p_i(t)|\le\frac{\ell_i}{(t-a)^{2n+1-i}}\ \ (i=2,\dots,n)\ \ \textit{for}\ \ a<t<b\tag{19}$$

and the condition (16) *holds. Let, moreover,*

$$p_1\in L_{loc}(]a,b]),\ \ q^*\in(\cdot,\rho)\in L_{loc}(]a,b])\quad\textit{for}\ \ 0<\rho<+\infty.\tag{20}$$

Then the problem (13), (3) *has at least one solution in the space* $\widetilde C^{2n-1}_{loc}(]a,b])\cap\widetilde C^n_2(]a,b])$.

Corollary 2′. *Let there exist non-negative constants ℓ_i $(i=1,\dots,n)$ satisfying the inequality* (11) *such that the functions p_i $(i=1,\dots,n)$ satisfy the conditions* (19). *Let, moreover, along with* (17) *and* (20) *the condition*

$$\int\limits_a^b (t-a)^{2n}|q(t,0)|^2\,dt<+\infty$$

be fulfilled. Then the problem (13), (3) *is uniquely solvable in the space* $\widetilde C^{2n-1}_{loc}(]a,b])\cap\widetilde C^n_2(]a,b])$ *and stable with respect to a small perturbation of the right-hand member of the equation* (13).

Corollary 3. *Let*

$$\int\limits_a^b\Big((t-a)^{2n\lambda_1}(b-t)^{2n\lambda_1}\big[(-1)^n p_1(t)\big]_+\Big)^{\frac{1}{1-\lambda_1}}\,dt<+\infty,$$

$$\int\limits_a^b\Big((t-a)^{(2n+1-i)\lambda_i}(b-t)^{(2n+1-i)\lambda_i}|p_i(t)|\Big)^{\frac{1}{1-\lambda_i}}\,dt<+\infty\ \ (i=2,\dots,n),$$

and there exist a function $\ell \in L^2_{2n,2n}(]a,b[)$ *such that the condition* (16) *holds. Then the problem* (14), (2) *has at least one solution in the space* $\widetilde{C}^{2n-1}_{loc}(]a,b[) \cap \widetilde{C}^n_2(]a,b[)$.

Corollary 4. *Let*

$$\int_a^b \Big((t-a)^{2n\lambda_1}\big[(-1)^n p_1(t)\big]_+\Big)^{\frac{1}{1-\lambda_1}}\,dt < +\infty,$$

$$\int_a^b \Big((t-a)^{(2n+1-i)\lambda_i}|p_i(t)|\Big)^{\frac{1}{1-\lambda_i}}\,dt < +\infty \quad (i=2,\dots,n),$$

and there exist a function $\ell \in L^2_{2n,0}(]a,b[)$ *such that the condition* (16) *holds. Let, moreover, the conditions* (20) *be fulfilled. Then the problem* (14), (3) *has at least one solution in the space* $\widetilde{C}^{2n-1}_{loc}(]a,b]) \cap \widetilde{C}^n_2(]a,b])$.

Finally, we give two examples

$$u^{(n)} = \alpha_0(t)u + \sum_{i=1}^n \frac{\alpha_i u^{(i-1)}}{(t-a)^{2n+1-i}(b-t)^{2n+1-i}} + \\ + \frac{\gamma(t)}{(t-a)^{n+\frac12}(b-t)^{n+\frac12}(1+|\ln(t-a)(b-t)|)} \tag{21}$$

and

$$u^{(n)} = \beta_0(t)u + \sum_{i=1}^n \frac{\beta_i u^{(i-1)}}{(t-a)^{2n+1-i}} + \frac{\gamma(t)}{(t-a)^{n+\frac12}(1+|\ln(t-a)|)}\,, \tag{22}$$

where $\gamma :]a,b[\to \mathbb{R}$ is a bounded measurable function,

$$\alpha_0 \in L_{loc}(]a,b[), \quad \beta_0 \in L_{loc}(]a,b]), \quad (-1)^n\alpha_0(t) \le 0, \quad (-1)^n\beta_0(t) \le 0 \ \text{ for } \ a<t<b,$$

and α_i, β_i $(i=1,\dots,n)$ are real constants satisfying the inequalities

$$\sum_{i=1}^n \frac{\mu_{in}|\alpha_i|}{(b-a)^{2n+1-i}} < 1, \quad \sum_{i=1}^n \nu_{in}|\beta_i| < 1.$$

According to Corollary 1′ (Corollary 2′), the problem (21), (2) (the problem (22), (3)) is uniquely solvable in the space $\widetilde{C}^{2n-1}_{loc}(]a,b[) \cap \widetilde{C}^n_2(]a,b[)$ (in the space $\widetilde{C}^{2n-1}_{loc}(]a,b]) \cap \widetilde{C}^n_2(]a,b])$) and stable with respect to a small perturbation of the right-hand member of the equation (21) (of the equation (22)).

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