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POISSON PROBLEMS WITH HALF-DIRICHLET BOUNDARY CONDITIONS FOR DIRAC OPERATORS ON NONSMOOTH MANIFOLDS

Abstract. We consider Dirichlet and Poisson type problems for the Maxwell-Dirac operator $\mathbb{D}_k = d + \delta + k\,dt\cdot$ on a Lipschitz subdomain Ω of a smooth, Riemannian manifold $\mathcal{M}$. The emphasis is on solutions of finite L^p energy, i.e. sections u satisfying $\iint_\Omega [|u|^p + |du|^p + |\delta u|^p]\, d\mathrm{Vol} < +\infty$. In this context, we prove well-posedness for p near 2. Our approach relies heavily on the analysis of the spectra of Cauchy type operators naturally associated with the problems at hand.

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Key words and phrases: Dirac operators, Poisson problem, Dirichlet problem, Cauchy operators, Maxwell's equations, Lipschitz domains.

რეზიუმე. ნაშრომში მაქსველ–დირაკის $\mathbb{D}_k = d + \delta + k\,dt\cdot$ ოპერატორისთვის დირიხლესა და პუასონის ტიპის ამოცანები განიხილება გლუვი რიმანის $\mathcal{M}$ მრავალსახეობის ლიპშიცის Ω ქვეარეში. განსაკუთრებული ყურადღება ეთმობა სასრული L^p ენერგიის მქონე ამონახსნებს, ე.ი. ისეთ u ამონახსნებს, რომლებიც აკმაყოფილებს $\iint_\Omega [|u|^p + |du|^p + |\delta u|^p]\, d\mathrm{Vol} < +\infty$ პირობას. მტკიცდება ამოცანის კორექტულობა, როდესაც p ახლოსაა 2-თან. ნაშრომში გამოყენებული მიდგომა არსებითად ემყარება განსახილველ ამოცანებთან ბუნებრივად ასოცირებული კოშის ტიპის ოპერატორების სპექტრალურ ანალიზს.

1. Introduction

In a series of recent papers [19], [21], [18], [22], [17], we have initiated a program aimed at studying boundary value problems for differential forms in subdomains of Riemannian manifolds, under minimal smoothness assumptions. This includes the Hodge–Laplacian, Dirac type operators, as well as Maxwell's equations and other related PDE's. The emphasis is on regularity (or lack thereof) of solutions, coefficients of the operators, and boundaries of the domains involved.

In this paper we continue this line of work by considering the following geometric context. Let $\mathcal{M}$ be a smooth Riemannian manifold, and denote by d, δ the exterior derivative operator and its adjoint, respectively, on $\mathcal{M}$. We shall find it convenient to further embed $\mathcal{M}$ into a larger 'space-time' manifold $\mathcal{M}\times\mathbb{R}$; throughout the paper all differential forms are defined on (subdomains of) $\mathcal{M}$ and with values in the Grassmann algebra of $\mathcal{M}\times\mathbb{R}$.

In this scenario, it is therefore natural to consider the Dirac type operator

$$\mathbb{D}_k := d+\delta+k\,dt \tag{1.1}$$

where k is a complex parameter (playing the role of the wave number), t is the 'time' variable and $dt\cdot$ acts as a Clifford algebra multiplier. The main goal is to study Poisson problems with half-Dirichlet boundary conditions of the type

$$(BVP)\;\left\{\begin{array}{l}\mathbb{D}_k u=\eta \text{ in } \Omega,\\ u_{\rm tan} \text{ or } u_{\rm nor} \text{ prescribed on } \partial\Omega.\end{array}\right. \tag{1.2}$$

Above, Ω is an arbitrary Lipschitz subdomain of $\mathcal{M}$ and $u_{\rm tan}$, $u_{\rm nor}$ are, respectively, the tangential and the normal component of u on $\partial\Omega$.

The case $k=0$ and $\eta\in L^p(\Omega)$ has been studied in [18]. In this scenario, there are intimate connections between (1.2) and classical topological invariants of the underlying domain. A thorough analysis of the case when $\eta=0$ and boundary data are from $L^p(\partial\Omega)$ can be found in [21]. There, the estimates for the solution u are in terms of the so called nontangential maximal function. For three-dimensional manifolds, the problem (1.2) with $\eta\in H^{s,p}(\Omega)$, the L^p-based Sobolev space of fractional index of smoothness s, has been treated in [22] for the optimal range of the parameters s, p.

While the corresponding situation in higher dimensions remains an open problem at the moment, here we consider the case when $\dim\mathcal{M}\geq 3$, $k\in\mathbf{C}$ and $\eta\in L^p(\Omega)$ for $p\in(2-\epsilon,2+\epsilon)$, where $\epsilon>0$ depends on the Lipschitz domain Ω. The spirit of our main result (cf. Theorem 8.3 for details) is that, in the aforementioned context, the problem (1.2) is well-posed, provided $\epsilon=\epsilon(\partial\Omega)$ is small enough (depending on the Lipschitz character of the domain Ω).

It is worth pointing out that (1.2) can be thought of as an 'elliptization' of (the standard, non-coercive) Maxwell's equations

$$\text{(Maxwell)} \begin{cases} dE - ikH = 0 \text{ in } \Omega, \\ \delta H + ikE = 0 \text{ in } \Omega, \\ E_{\tan} \text{ or } E_{\mathrm{nor}} \text{ prescribed on } \partial\Omega, \end{cases} \tag{1.3}$$

under the identification $u = H - i\, dt \cdot E$. This also better clarifies the role played by the parameter k in (1.1). In the L^p context, (1.3) has been solved in [20], [11].

For C^∞ structures and for sufficiently regular data, problems such as (1.2) are regular elliptic (cf. [25], [2]) and, hence, Fredholm solvable (i.e., well-posed modulo finite dimensional spaces). The standard approach in this scenario is via algebras of pseudo-differential operators.

Nonetheless, the nature of the problem at hand changes drastically when the smoothness assumptions are significantly relaxed. This is certainly the case for the situation we are interested in, i.e. Lipschitz boundaries, metric tensors with low regularity, and L^p data. In this context, the method of layer potentials (which we systematically employ in this paper) leads to considering operators which can only be described in terms of singular integrals (of Calderón–Zygmund type). Here we rely on harmonic analysis techniques which have been successful in the treatment of second-order, constant coefficient elliptic boundary problems in Lipschitz domains of the Euclidean space. The reader is referred to the excellent survey [15] for a more thorough discussion in this regard.

The organization of the paper is as follows. Section 2 contains a discussion of the geometrical set up. Several distinguished spaces of (boundary) differential forms – of Sobolev–Besov type – are introduced in Sections 3–4. Boundary integral operators of Cauchy type are defined in Section 5. The main issues studied here are boundedness and jump relations. As a preliminary step, half-Dirichlet boundary problems for (1.1) of finite L^2-energy are analyzed in Section 6. In turn, this well-posedness result translates into a Rellich type estimate to the effect that

$$\|u_{\tan}\| \approx \|u_{\mathrm{nor}}\|, \tag{1.4}$$

uniformly for L^2 null-solutions of $\mathbb{D}_k$. Here, the norms are taken with respect to certain subspaces of $B^{2,2}_{-1/2}(\partial\Omega)$, well adapted for the problem at hand; cf. Corollary 6.4 for a precise statement. A similar estimate at the $L^2(\partial\Omega)$ level has been proved in [21].

The Rellich estimate (1.4) is then utilized in Section 7 in order to show that the boundary versions of our Cauchy operators are isomorphisms at the level of Besov spaces. Such invertibility results are crucial in the treatment of the Poisson problem for (1.1), taken up in Section 8. Here we also discuss connections with Maxwell's system (1.3). In Section 9 and Section 10 we look at finer spectral properties of our Cauchy type integral operators in the context of boundary Besov spaces. In Section 9 we prove that their essential spectra are contained in $\mathbf{C}\setminus(-\frac{1}{2},\frac{1}{2})$. In this section we also discuss the well-posedness of a natural transmission problem for the operator (1.1) with Lipschitz interfaces. Finally, in Section 10, we show that the Fredholm spectra of these operators are contained in a certain hyperbola whose geometric characteristics depend (explicitly) on the underlying domain.

2. Geometrical Preliminaries

Let $\mathcal{M}$ be a smooth, oriented, connected, compact, boundaryless manifold of real dimension m. We equip $\mathcal{M}$ with a metric tensor

$$g=\sum_{i,j} g_{ij}dx_i\otimes dx_j, \qquad g_{ij}\in C^{1,1}, \quad \forall\, i,j. \tag{2.1}$$

As is customary, take $(g^{ij})_{i,j}$ to be the matrix inverse to $(g_{ij})_{ij}$, and set $g:=\det\left[(g_{ij})_{i,j}\right]$. Also, we let $d\mathcal{V}$ denote the corresponding volume element, so that, locally, $d\mathcal{V}=\sqrt{g}\,dx$.

Denote by $T\mathcal{M}$ the tangent bundle to $\mathcal{M}$ and by $\Lambda^\ell T\mathcal{M}$ its ℓ-th exterior power. Sections in this latter vector bundle are ℓ-differential forms and can be described in local coordinates $(x_1,\dots,x_m)$ as $u=\sum_{|I|=\ell}u_I\,dx^I$. Here the sum is performed over ordered ℓ-tuples $I=(i_1,\dots,i_\ell)$, $1\leq i_1<i_2<\cdots<i_\ell\leq m$ and, for each such I, $dx^I:=dx_{i_1}\wedge\cdots\wedge dx_{i_\ell}$. Also, the wedge stands for the usual exterior product of forms, while $|I|$ denotes the cardinality of I.

The Grassmann algebra on $\mathcal{M}$ is then defined as

$$\mathcal{G}_{\mathcal{M}}:=\oplus_{0\leq\ell\leq m}\Lambda^\ell T\mathcal{M}. \tag{2.2}$$

We shall not make any notational distinction between $\Lambda^\ell T\mathcal{M}$, $\mathcal{G}_{\mathcal{M}}$ and their respective complexified versions.

The Hermitian structure in the fibers on $T\mathcal{M}$ extends naturally to the cotangent bundle $T^*\mathcal{M}$ by setting $\langle dx_i,dx_j\rangle:=g^{ij}$. In turn, this induces a Hermitian structure on $\Lambda^\ell T\mathcal{M}$ by selecting $\{\omega^I\}_{|I|=\ell}$ to be an orthonormal frame in $\Lambda^\ell T\mathcal{M}$ provided $\{\omega_j\}_{1\leq j\leq m}$ is an orthonormal frame in $T^*\mathcal{M}$ (locally). Finally, the latter further induces a Hermitian structure on $\mathcal{G}_{\mathcal{M}}$, by insisting that the direct sum in (2.2) is also orthogonal. We denote by $\langle\cdot,\cdot\rangle$ the corresponding (pointwise) inner product in $\mathcal{G}_{\mathcal{M}}$.

Let us now embed the Riemannian manifold $\mathcal{M}$ into $\mathbb{R}\times\mathcal{M}$, equipped with the product metric. Here $\mathbb{R}$ is considered with the standard metric, i.e. if t denotes the generic variable in $\mathbb{R}$, then $|dt|=1$. In the sequel, it will be convenient to work in a (one codimensional) extension $\mathcal{E}$ of the Grassmann algebra $\mathcal{G}_{\mathcal{M}}$, consisting of forms of the type $f+dt\wedge g$, with $f,g\in\mathcal{G}_{\mathcal{M}}$. In other words,

$$\mathcal{G}_{\mathcal{M}}\hookrightarrow\mathcal{E}:=\mathcal{G}_{\mathcal{M}}\oplus(dt\wedge\mathcal{G}_{\mathcal{M}})\hookrightarrow\mathcal{G}_{\mathbb{R}\times\mathcal{M}}. \tag{2.3}$$

Moreover, the above direct sum is orthogonal, and we continue to denote by $\langle\cdot,\cdot\rangle$ the corresponding (pointwise) inner product on $\mathcal{E}$. In particular, for arbitrary one-forms α, β, and $u, w \in \mathcal{E}$, the following are true:

$$\alpha \wedge (\beta \vee u) + \beta \vee (\alpha \wedge u) = \langle \alpha, \beta \rangle u, \quad \langle \alpha \wedge u, w \rangle = \langle u, \alpha \vee w \rangle. \tag{2.4}$$

Also, throughout the paper, we shall let d stand for the (exterior) derivative operator on the manifold $\mathcal{M}$ and denote by δ its formal adjoint (with respect to the metric introduced above). Thus, $d(f + dt \wedge g) = df - dt \wedge dg$ and $\delta(f + dt \wedge g) = \delta f - dt \wedge \delta g$, for any $f, g \in \mathcal{G}_{\mathcal{M}}$. In particular, $d\,d = 0$, $\delta\,\delta = 0$, and

$$\Delta := -(d\delta + \delta d) \tag{2.5}$$

is the Hodge–Laplacian on $\mathcal{M}$. The bundle $\mathcal{E}$ has a natural Clifford algebra structure. In fact, if $\cdot$ stands for the Clifford algebra product, then

$$\alpha \cdot u = \alpha \wedge u - \alpha \vee u, \qquad \langle \alpha \cdot u, v \rangle = -\langle u, \alpha \cdot v \rangle. \tag{2.6}$$

for every $\alpha \in \Lambda^1$ and $u, v \in \mathcal{E}$.

Next, we introduce the family of Dirac type operators, indexed by $k \in \mathbf{C}$,

$$\mathbb{D}_k := d + \delta + k\,dt, \tag{2.7}$$

where dt acts as a Clifford algebra multiplier. Thus

$$\mathbb{D}_k : C^1(\mathcal{M}, \mathcal{E}) \to C^0(\mathcal{M}, \mathcal{E}) \tag{2.8}$$

is an elliptic first order differential operator. Some of its most immediate properties are:

$$\mathbb{D}_k^2 = -\Delta - k^2, \;\; \bar{\mathbb{D}}_k = -\mathbb{D}_k, \;\; \mathbb{D}_k^c = \mathbb{D}_{k^c}, \;\; \mathbb{D}_k^t = \mathbb{D}_{-k}, \;\; \mathbb{D}_k^* = \mathbb{D}_{-k^c}, \tag{2.9}$$

where $(\cdot)^c$ denotes complex conjugation, and Δ is the Hodge–Laplacian on $\mathcal{M}$.

A domain $\Omega \subset \mathcal{M}$ is called *Lipschitz* provided its boundary is given by graphs of Lipschitz functions in suitable local coordinates. Fix such a Lipschitz domain Ω and denote by $d\sigma$ its surface measure (inherited from the metric on $\mathcal{M}$), and by $\nu \in T^*\mathcal{M}$ its outward unit conormal, defined a.e. on $\partial\Omega$. Then, for each $u, v \in C^1(\bar{\Omega}, \mathcal{E})$, we have

$$\iint_{\Omega} \langle du, v \rangle \, d\mathcal{V} - \iint_{\Omega} \langle u, \delta v \rangle \, d\mathcal{V} = \int_{\partial\Omega} \langle \nu \wedge u, v \rangle \, d\sigma = \int_{\partial\Omega} \langle u, \nu \vee v \rangle \, d\sigma. \tag{2.10}$$

3. Sobolev Spaces of Differential Forms

Call a measurable section $f : \partial\Omega \to \mathcal{E}$ *tangential* if $\nu \vee f = 0$ a.e. on $\partial\Omega$, and *normal* if $\nu \wedge f = 0$ on $\partial\Omega$. We now define two closed subspaces of $L^p(\partial\Omega, \mathcal{E})$, i.e.

$$L^p_{\tan}(\partial\Omega, \mathcal{E}) := \{v \in L^p(\partial\Omega, \mathcal{E});\, \nu \vee v = 0\}, \tag{3.1}$$

$$L^p_{\rm nor}(\partial\Omega, \mathcal{E}) := \{v \in L^p(\partial\Omega, \mathcal{E});\, \nu \wedge v = 0\}. \tag{3.2}$$

Assume now that the form $f \in L^p_{\tan}(\partial\Omega, \mathcal{E})$, $1 < p < \infty$, has the property that there exists a constant $\kappa > 0$ so that

$$\Big| \int_{\partial\Omega} \langle d\psi, f\rangle \, d\sigma \Big| \le \kappa \|\psi\|_{L^q(\partial\Omega,\mathcal{E})} \quad \text{for any } \psi \in C^1(\mathcal{M}, \mathcal{E}), \tag{3.3}$$

where $1/p + 1/q = 1$. By Riesz's representation theorem, there exists a form in $L^p(\partial\Omega, \mathcal{E})$, which we denote by $\delta_\partial f$, so that

$$\int_{\partial\Omega} \langle d\psi, f\rangle \, d\sigma = \int_{\partial\Omega} \langle \psi, \delta_\partial f\rangle \, d\sigma, \quad \text{for any } \psi \in C^1(\mathcal{M}, \mathcal{E}). \tag{3.4}$$

We set

$$L^{p,\delta}_{\tan}(\partial\Omega, \mathcal{E}) := \{ f \in L^p_{\tan}(\partial\Omega, \mathcal{E});\ \delta_\partial f \in L^p(\partial\Omega, \mathcal{E}) \}, \tag{3.5}$$

and equip it with the natural norm

$$\|f\|_{L^{p,\delta}_{\tan}(\partial\Omega,\mathcal{E})} := \|f\|_{L^p(\partial\Omega,\mathcal{E})} + \|\delta_\partial f\|_{L^p(\partial\Omega,\mathcal{E})}. \tag{3.6}$$

See [19] for a more extensive discussion, and other references. It is not difficult to check that

$$\delta_\partial \delta_\partial f = 0 \ \text{ and } \ \nu \vee \delta_\partial f = 0 \ \text{ on } \ \partial\Omega \ \text{ for any } \ f \in L^{p,\delta}_{\tan}(\partial\Omega, \mathcal{E}). \tag{3.7}$$

For $f \in L^p_{\rm nor}(\partial\Omega, \mathcal{E})$, we define the distribution $d_\partial f$ by requiring that

$$\int_{\partial\Omega} \langle \delta\psi, f\rangle \, d\sigma = \int_{\partial\Omega} \langle \psi, d_\partial f\rangle \, d\sigma \quad \text{for any } \psi \in C^1(\mathcal{M}, \mathcal{E}). \tag{3.8}$$

Set

$$L^{p,d}_{\rm nor}(\partial\Omega, \mathcal{E}) := \{ f \in L^p_{\rm nor}(\partial\Omega, \mathcal{E});\ d_\partial f \in L^p(\partial\Omega, \mathcal{E}) \}, \tag{3.9}$$

equipped with the natural norm

$$\|f\|_{L^{p,d}_{\rm nor}(\partial\Omega,\mathcal{E})} := \|f\|_{L^p(\partial\Omega,\mathcal{E})} + \|d_\partial f\|_{L^p(\partial\Omega,\mathcal{E})}. \tag{3.10}$$

Analogously to (3.7), we have that

$$d_\partial d_\partial f = 0 \ \text{ and } \ \nu \wedge d_\partial f = 0 \ \text{ on } \ \partial\Omega \ \text{ for any } \ f \in L^{p,d}_{\rm nor}(\partial\Omega, \mathcal{E}). \tag{3.11}$$

4. Besov Spaces of Differential Forms

Let $H^{s,p}(\Omega)$ and $B^{p,q}_s(\Omega)$, $s > 0$, $1 < p < \infty$, stand, respectively, for the usual scales of Sobolev and Besov spaces on Ω. We shall also work with $B^{p,q}_s(\partial\Omega)$, $0 < |s| < 1$, $1 < p, q < \infty$, the class of Besov spaces on $\partial\Omega$. In particular, the trace map

$$\mathrm{Tr} : H^{s,p}(\Omega) \longrightarrow B^{p,p}_{s-1/p}(\partial\Omega) \tag{4.1}$$

is well-defined and bounded for $1 < p < \infty$, $1/p < s < 1 + 1/p$, and has a bounded right inverse (Gagliardo's lemma). More detailed accounts on these matters can be found in [1], [12], [24]. Finally, set $B^{p,q}_s(\Omega, \mathcal{E}) := B^{p,q}_s(\Omega) \otimes \mathcal{E}$, $B^{p,q}_s(\partial\Omega, \mathcal{E}) := B^{p,q}_s(\partial\Omega) \otimes \mathcal{E}$, etc.

Let us consider $d, \delta : L^p(\Omega, \mathcal{E}) \to L^p(\Omega, \mathcal{E})$ as unbounded operators with domains

$$\mathrm{Dom}(d; L^p(\Omega, \mathcal{E})) := \{u \in L^p(\Omega, \mathcal{E});\ du \in L^p(\Omega, \mathcal{E})\}, \tag{4.2}$$

$$\mathrm{Dom}(\delta; L^p(\Omega, \mathcal{E})) := \{u \in L^p(\Omega, \mathcal{E});\ \delta u \in L^p(\Omega, \mathcal{E})\}. \tag{4.3}$$

The action of d and δ on their respective domains is considered in the sense of distributions.

For $u \in \mathrm{Dom}(\delta; L^p(\Omega, \mathcal{E}))$, $1 < p < \infty$, we now define the distribution $\nu \vee u$ on $\partial\Omega$ by requiring that

$$\langle \nu \vee u,\, \varphi \rangle := -\iint\limits_{\Omega} \langle \delta u, v \rangle \, d\mathcal{V} + \iint\limits_{\Omega} \langle u, dv \rangle \, d\mathcal{V}, \tag{4.4}$$

for any $v \in H^{1,q}(\Omega, \mathcal{E})$, $1/p + 1/q = 1$, with $\mathrm{Tr}\, v = \varphi$. Thus, the right side of (4.4) is well defined for $\varphi \in B^{q,q}_{1/p}(\partial\Omega, \mathcal{E})$, independently of the choice of such v, so we have

$$\nu \vee u \in B^{p,p}_{-1/p}(\partial\Omega, \mathcal{E}) \tag{4.5}$$

with naturally accompanying estimates. Furthermore, if $u \in \mathrm{Dom}(d; L^p(\Omega, \mathcal{E}))$ then we can define the distribution $\nu \wedge u$ by a similar procedure. Once again,

$$\nu \wedge u \in B^{p,p}_{-1/p}(\partial\Omega, \mathcal{E}) \tag{4.6}$$

plus natural estimates. It follows that the mappings

$$\mathrm{Dom}(d; L^p(\Omega, \mathcal{E})) \ni u \mapsto \nu \wedge u \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega, \mathcal{E}), \tag{4.7}$$

$$\mathrm{Dom}(\delta; L^p(\Omega, \mathcal{E})) \ni u \mapsto \nu \vee u \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega, \mathcal{E}), \tag{4.8}$$

are well-defined and bounded, when the spaces in the left side are equipped with the natural graph norm.

Next, for $1 < p < \infty$ we introduce some distinguished subspaces of $B^{p,p}_{-1/p}(\partial\Omega, \mathcal{E})$. More concretely, we define

$$\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega) := \\ = \Big\{ f \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega, \mathcal{E});\ \exists u \in \mathrm{Dom}(d; L^p(\Omega, \mathcal{E}))) \text{ with } \nu \wedge u = f \Big\}, \tag{4.9}$$

equipped with the natural norm

$$\|f\|_{\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega)} := \\ = \inf\ \big\{ \|u\|_{L^p(\Omega,\mathcal{E})} + \|du\|_{L^p(\Omega,\mathcal{E})};\ u \in \mathrm{Dom}(d; L^p(\Omega, \mathcal{E}),\ \nu \wedge u = f \big\} \tag{4.10}$$

and

$$\mathcal{X}^p_{\tan}(\partial\Omega) := \\ = \Big\{ g \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega, \mathcal{E});\ \exists\, v \in \mathrm{Dom}(\delta; L^p(\Omega, \mathcal{E})) \text{ with } \nu \vee v = g \Big\}, \tag{4.11}$$

endowed with the norm

$$\|g\|_{\mathcal{X}^p_{\tan}(\partial\Omega)} := \\ = \inf\left\{\|v\|_{L^p(\Omega,\mathcal{E})} + \|\delta v\|_{L^p(\Omega,\mathcal{E})};\ v \in \mathrm{Dom}(\delta; L^p(\Omega,\mathcal{E})),\ \nu \vee v = g\right\}. \tag{4.12}$$

We also define the (bounded) operators

$$d_\partial : \mathcal{X}^p_{\rm nor}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\rm nor}(\partial\Omega), \tag{4.13}$$

$$\delta_\partial : \mathcal{X}^p_{\tan}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\tan}(\partial\Omega), \tag{4.14}$$

by setting

$$d_\partial(\nu \wedge u) := -\nu \wedge du, \quad u \in \mathrm{Dom}(d; L^p(\Omega,\mathcal{E})), \tag{4.15}$$

$$\delta_\partial(\nu \vee v) := -\nu \vee \delta v, \quad v \in \mathrm{Dom}(\delta; L^p(\Omega,\mathcal{E})). \tag{4.16}$$

The following proposition, proved in [18], collects some of the main properties of the spaces introduced above. To state it, for each $\Omega \subset \mathcal{M}$ we set $\Omega_+ := \Omega$, $\Omega_- := \mathcal{M} \setminus \bar{\Omega}$.

Proposition 4.1. *Let Ω be a Lipschitz subdomain of $\mathcal{M}$. Then, for each $1 < p < \infty$, the following hold.*

(i) $\mathcal{X}^p_{\rm nor}(\partial\Omega_+) \equiv \mathcal{X}^p_{\rm nor}(\partial\Omega_-)$, *in the sense that the two spaces coincide as sets and their respective norms are equivalent.*

(ii) $\mathcal{X}^p_{\rm nor}(\partial\Omega)$ *is a reflexive Banach space and* $\mathcal{X}^p_{\rm nor}(\partial\Omega) \hookrightarrow B^{p,p}_{-\frac{1}{p}}(\partial\Omega,\mathcal{E})$ *continuously.*

(iii) *Both inclusions* $\nu \wedge C^1(\bar{\Omega},\mathcal{E})|_{\partial\Omega} \hookrightarrow \mathcal{X}^p_{\rm nor}(\partial\Omega)$ *and* $L^{p,d}_{\rm nor}(\partial\Omega,\mathcal{E}) \hookrightarrow \mathcal{X}^p_{\rm nor}(\partial\Omega)$ *are continuous and with dense range.*

(iv) *The class* $\{\mathcal{X}^p_{\rm nor}(\partial\Omega)\}_{1<p<\infty}$ *is a complex interpolation scale. That is, for each* $0 < \theta < 1$, $1 < p_0, p_1 < \infty$ *and* $1/p := (1-\theta)/p_0 + \theta/p_1$, *we have*

$$[\mathcal{X}^{p_0}_{\rm nor}(\partial\Omega), \mathcal{X}^{p_1}_{\rm nor}(\partial\Omega)]_\theta = \mathcal{X}^p_{\rm nor}(\partial\Omega). \tag{4.17}$$

(v) *If the metric tensor is smooth and* $\partial\Omega \in C^\infty$ *then, for each* $1 < p < \infty$,

$$\mathcal{X}^p_{\rm nor}(\partial\Omega) \equiv \left\{f \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega,\mathcal{E});\ \nu \wedge f = 0 \ \text{ and }\ d_\partial f \in B^{p,p}_{-\frac{1}{p}}(\partial\Omega,\mathcal{E})\right\} \tag{4.18}$$

in the sense that the two spaces coincide as sets and

$$\|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)} \approx \|f\|_{B^{p,p}_{-\frac{1}{p}}(\partial\Omega,\mathcal{E})} + \|d_\partial f\|_{B^{p,p}_{-\frac{1}{p}}(\partial\Omega,\mathcal{E})}. \tag{4.19}$$

(vi) *The scales* $\mathcal{X}^p_{\tan}(\partial\Omega)$ *enjoy similar properties as* (i) $--$(v) *above.*

(vii) *For each* $1 < p, q < \infty$ *conjugate exponents, the mapping*

$$\nu \wedge \cdot\ : \mathcal{X}^p_{\tan}(\partial\Omega) \longrightarrow (\mathcal{X}^p_{\rm nor}(\partial\Omega))^* \tag{4.20}$$

defined by

$$\langle \nu \wedge f, g\rangle := \iint_\Omega \langle u, dw\rangle\, d\mathcal{V} - \iint_\Omega \langle \delta u, w\rangle\, d\mathcal{V} \tag{4.21}$$

for $u \in \mathrm{Dom}(\delta; L^q(\Omega,\mathcal{E}))$*, with* $f = \nu \vee u$ *and* $w \in \mathrm{Dom}(d; L^p(\Omega,\mathcal{E}))$ *with* $g = \nu \wedge w$ *is well-defined and bounded. In fact, this map is an isomorphism. Its adjoint is*

$$\nu \vee \cdot\, : \mathcal{X}^p_{\mathrm{nor}}(\partial\Omega) \longrightarrow (\mathcal{X}^p_{\mathrm{tan}}(\partial\Omega))^* \tag{4.22}$$

whose action is defined similarly.

5. Integral Operators

The Hodge–Laplacian $\Delta : H^{1,2}(\mathcal{M},\mathcal{G}_M) \to H^{-1,2}(\mathcal{M},\mathcal{G}_M)$ is a bounded, negative, formally self-adjoint operator. Since $(\Delta - \lambda I)^{-1}$ gives rise to a negative, self-adjoint, compact operator on $L^2(\mathcal{M},\mathcal{G}_M)$ for $\lambda \in \mathbf{R}$ with $|\lambda|$ large, it follows that there exists $\mathrm{Spec}\,(\Delta) \subseteq (-\infty, 0]$, a discrete set (which accumulates only at $-\infty$) so that

$$z \notin \mathrm{Spec}\,(\Delta) \Rightarrow (\Delta - zI) : H^{1,2}(\mathcal{M},\mathcal{G}_M) \to H^{-1,2}(\mathcal{M},\mathcal{G}_M) \text{ is invertible.} \tag{5.1}$$

Select an increasing sequence $k_j \in [0,\infty)$ so that

$$\{-(k_j)^2\}_j = \mathrm{Spec}\,(\Delta). \tag{5.2}$$

For $k \notin \{\pm k_j\}_j$, we let $\Gamma_k(x,y)$ be the Schwartz kernel of $\Delta + k^2$. In particular, the fact that Δ commutes with d and δ translated into

$$\delta_x(\Gamma_k(x,y)) = d_y(\Gamma_k(x,y)), \qquad d_x(\Gamma_k(x,y)) = \delta_y(\Gamma_k(x,y)). \tag{5.3}$$

Next, fix a Lipschitz domain $\Omega \subset \mathcal{M}$ with outward unit normal ν and surface measure $d\sigma$, and denote by $\mathcal{S}_k$ the single layer potential operator on $\partial\Omega$ with kernel $\Gamma_k(x,y)$, i.e.

$$\mathcal{S}_k f(x) := \int_{\partial\Omega} \langle \Gamma_k(x,y), f(y)\rangle\, d\sigma(y), \quad x \in \mathcal{M} \setminus \partial\Omega, \tag{5.4}$$

where $f \in L^p(\partial\Omega,\mathcal{E})$, $1 < p < \infty$. Note that $(\Delta + k^2)\mathcal{S}_k f = 0$ in $\mathcal{M} \setminus \partial\Omega$. Also, set

$$S_k f := \mathcal{S}_k f|_{\partial\Omega}. \tag{5.5}$$

Going further, let us introduce the principal value singular integral operators

$$M_k f(x) := p.v. \int_{\partial\Omega} \langle \nu(x) \vee d_x\Gamma_k(x,y), f(y)\rangle\, d\sigma(y), \quad x \in \partial\Omega, \tag{5.6}$$

and

$$N_k f(x) := p.v. \int_{\partial\Omega} \langle \nu(x) \wedge \delta_x\Gamma_k(x,y), f(y)\rangle\, d\sigma(y), \quad x \in \partial\Omega. \tag{5.7}$$

Here, p.v. $\int_{\partial\Omega} \dots$ is taken in the sense of removing geodesic balls (with respect to some smooth background metric); see [23] for more details.

These operators are the higher degree analogue of the so-called *magnetostatic* and *electrostatic* operators arising in scattering theory in $\mathbf{R}^3$ (cf., e.g., [6]). At the level of L^p spaces, these have been studied in detail in [19].

For further reference, some basic properties are collected in the theorems stated below. For proofs, the reader is referred to [19], [24]. Here, we only want to mention that all restrictions to the boundary of $\partial\Omega$ are taken in the pointwise nontangential sense. That is,

$$u\Big|_{\partial\Omega}(x) := \lim_{y\in\gamma(x),\, y\to x} u(y), \quad x \in \partial\Omega, \tag{5.8}$$

where $\gamma(x) \subseteq \Omega$ is an appropriate nontangential approach region. Also, $\mathcal{N}$ is going to denote the nontangential maximal operator defined on some section u in Ω by

$$(\mathcal{N}u)(x) := \sup\{|u(y)|;\, y \in \gamma(x)\}, \quad x \in \partial\Omega. \tag{5.9}$$

Finally, recall the convention that $\Omega_+ := \Omega$, $\Omega_- := \mathcal{M} \setminus \bar{\Omega}$.

Theorem 5.1. *Let $\Omega \subset \mathcal{M}$ be a Lipschitz domain. Also, fix $k \in \mathbf{C} \setminus \{\pm k_j\}_j$. Then for each $1 < p < \infty$ we have:*

(i) *There exists $C = C(\partial\Omega, p) > 0$ so that*

$$\begin{aligned} &\|\mathcal{N}(S_k f)\|_{L^p(\partial\Omega)},\ \|\mathcal{N}(d\mathcal{S}_k f)\|_{L^p(\partial\Omega)}, \\ &\quad \|\mathcal{N}(\delta\mathcal{S}_k f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega,\mathcal{E})}, \end{aligned} \tag{5.10}$$

uniformly for $f \in L^p(\partial\Omega, \mathcal{E})$, and

$$\|\mathcal{N}(S_k f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{H^{-1,p}(\partial\Omega,\mathcal{E})}, \tag{5.11}$$

uniformly for $f \in H^{-1,p}(\partial\Omega, \mathcal{E}) = (H^{1,q}(\partial\Omega, \mathcal{E}))^*$, $1/p + 1/q = 1$.

(ii) *The following jump-relations are valid*

$$\begin{aligned} \nu \vee d\mathcal{S}_k f\Big|_{\partial\Omega_\pm} &= \mp\tfrac{1}{2}(\nu \vee (\nu \wedge f)) + M_k f, \\ \nu \wedge \delta\mathcal{S}_k f\Big|_{\partial\Omega_\pm} &= \pm\tfrac{1}{2}(\nu \wedge (\nu \vee f)) + N_k f \end{aligned} \tag{5.12}$$

a.e. on $\partial\Omega$ for each $f \in L^p(\partial\Omega, \mathcal{E})$ and $1 < p < \infty$.

(ii) *The following intertwining identities are valid*

$$\delta\mathcal{S}_k f = \mathcal{S}_k(\delta_\partial f), \qquad \forall\, f \in L^{p,\delta}_{\tan}(\partial\Omega, \mathcal{E}), \tag{5.13}$$

$$d\mathcal{S}_k g = \mathcal{S}_k(d_\partial g), \qquad \forall\, g \in L^{p,d}_{\mathrm{nor}}(\partial\Omega, \mathcal{E}). \tag{5.14}$$

(iii) *The adjoint of M_k acting on $L^p_{\tan}(\partial\Omega, \mathcal{E})$ is the operator M^t_k acting on $L^q_{\tan}(\partial\Omega, \mathcal{E})$, with $1/p + 1/q = 1$, given by*

$$M^t_k = \nu \vee N_k(\nu \wedge \cdot). \tag{5.15}$$

(iv) *For $0 \leq s \leq 1$, the operator*

$$\mathcal{S}_k : H^{-s,p}(\partial\Omega, \mathcal{E}) \longrightarrow B^{p,p^\#}_{1-s+1/p}(\Omega, \mathcal{E}) \tag{5.16}$$

is well defined and bounded. Hereafter, we set

$$p^\# := \max\{p, 2\}. \tag{5.17}$$

We now aim at extending the action of the operators M_k and N_k to the spaces $\mathcal{X}^p_{\rm tan}(\partial\Omega)$ and $\mathcal{X}^p_{\rm nor}(\partial\Omega)$, respectively. As a preliminary step, we first analyze the action of $\mathcal{S}_k$ on these spaces.

Lemma 5.2. *Assume that $\Omega\subset\mathcal{M}$ is a Lipschitz domain. Then, for $k\notin\{\pm k_j\}_j$ and $1<p<\infty$, the following hold.*

(i) *The operators*

$$\mathcal{S}_k:\mathcal{X}^p_{\rm nor}(\partial\Omega)\longrightarrow H^{1,p}(\Omega,\mathcal{E}), \tag{5.18}$$

$$\mathcal{S}_k:\mathcal{X}^p_{\rm tan}(\partial\Omega)\longrightarrow H^{1,p}(\Omega,\mathcal{E}) \tag{5.19}$$

are well-defined and bounded. In particular, $S_k=Tr\circ\mathcal{S}_k$ and

$$\begin{array}{c} S_k:\mathcal{X}^p_{\rm nor}(\partial\Omega)\longrightarrow B^{p,p}_{1-1/p}(\partial\Omega,\mathcal{E}),\\ S_k:\mathcal{X}^p_{\rm nor}(\partial\Omega)\longrightarrow B^{p,p}_{1-1/p}(\partial\Omega,\mathcal{E}) \end{array} \tag{5.20}$$

are bounded. In fact, so are the operators

$$\nu\wedge S_k:\mathcal{X}^p_{\rm tan}(\partial\Omega)\longrightarrow\mathcal{X}^p_{\rm nor}(\partial\Omega), \tag{5.21}$$

$$\nu\vee S_k:\mathcal{X}^p_{\rm nor}(\partial\Omega)\longrightarrow\mathcal{X}^p_{\rm tan}(\partial\Omega). \tag{5.22}$$

(ii) *The nontangential maximal function estimates*

$$\|\mathcal{N}(\mathcal{S}_kf)\|_{L^p(\partial\Omega)}\leq C\|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)}, \tag{5.23}$$

$$\|\mathcal{N}(\mathcal{S}_kg)\|_{L^p(\partial\Omega)}\leq C\|g\|_{\mathcal{X}^p_{\rm tan}(\partial\Omega)} \tag{5.24}$$

hold uniformly in f, g.

(iii) *The intertwining identities*

$$d\mathcal{S}_kf=\mathcal{S}_k(d_\partial f),\quad \forall\, f\in\mathcal{X}^p_{\rm nor}(\partial\Omega) \tag{5.25}$$

$$\delta\mathcal{S}_kg=\mathcal{S}_k(\delta_\partial g),\quad \forall\, g\in\mathcal{X}^p_{\rm tan}(\partial\Omega) \tag{5.26}$$

hold.

We next turn our attention to the operators M_k, N_k. In analogy with (5.12) we define

$$\left(\mp\tfrac12 I+M_k\right)f:=\nu\vee(d\,\mathcal{S}_kf|_{\Omega_\pm}),\qquad f\in\mathcal{X}^p_{\rm tan}(\partial\Omega), \tag{5.27}$$

$$\left(\pm\tfrac12 I+N_k\right)g:=\nu\wedge(\delta\,\mathcal{S}_kg|_{\Omega_\pm}),\qquad g\in\mathcal{X}^p_{\rm nor}(\partial\Omega). \tag{5.28}$$

Proposition 5.3. *Let Ω be a Lipschitz domain and let $k\notin\{\pm k_j\}_j$. Then for any $1<p<\infty$ the operators $\pm\frac12 I+N_k$, originally defined on $L^p(\partial\Omega,\mathcal{E})$ have bounded extensions to $\mathcal{X}^p_{\rm nor}(\partial\Omega)$, i.e.*

$$\pm\tfrac12 I+N_k:\mathcal{X}^p_{\rm nor}(\partial\Omega)\longrightarrow\mathcal{X}^p_{\rm nor}(\partial\Omega). \tag{5.29}$$

In fact, similar results are also valid for the operators

$$\pm\tfrac12 I+M_k:\mathcal{X}^p_{\rm tan}(\partial\Omega)\longrightarrow\mathcal{X}^p_{\rm tan}(\partial\Omega). \tag{5.30}$$

Moreover, if $1/p+1/q=1$, the diagrams

$$\begin{array}{ccc} \mathcal{X}^p_{\tan}(\partial\Omega) & \xrightarrow[\sim]{\nu\wedge\cdot} & (\mathcal{X}^q_{\rm nor}(\partial\Omega))^* \\ {\scriptstyle \pm\frac{1}{2}I+M_k}\downarrow & & \downarrow{\scriptstyle (\pm\frac{1}{2}I+N_k)^t} \\ \mathcal{X}^p_{\tan}(\partial\Omega) & \xrightarrow[\sim]{\nu\wedge\cdot} & (\mathcal{X}^q_{\rm nor}(\partial\Omega))^* \end{array} \tag{5.31}$$

and

$$\begin{array}{ccc} \mathcal{X}^q_{\rm nor}(\partial\Omega) & \xrightarrow[\sim]{\nu\vee\cdot} & (\mathcal{X}^p_{\tan}(\partial\Omega))^* \\ {\scriptstyle \pm\frac{1}{2}I+N_k}\downarrow & & \downarrow{\scriptstyle (\pm\frac{1}{2}I+M_k)^t} \\ \mathcal{X}^q_{\rm nor}(\partial\Omega) & \xrightarrow[\sim]{\nu\vee\cdot} & (\mathcal{X}^p_{\tan}(\partial\Omega))^* \end{array} \tag{5.32}$$

are commutative.

Our next task is to define certain Cauchy type operators which are well adapted to the problems we intend to study in this paper. Concretely, for $k\in\mathbf{C}\setminus\{\pm k_j\}_j$, introduce

$$\mathcal{C}_k f(x) := \int_{\partial\Omega} \langle \mathbb{D}_{k,x}\Gamma_k(x,y), f(y)\rangle\, d\sigma(y), \qquad x\notin\partial\Omega, \tag{5.33}$$

and

$$C_k f(x) = \text{p.v.}\int_{\partial\Omega} \langle \mathbb{D}_{k,x}\Gamma_k(x,y), f(y)\rangle\, d\sigma(y), \qquad x\in\partial\Omega, \tag{5.34}$$

if $f\in L^p(\partial\Omega,\mathcal{E})$, $1<p<\infty$. It follows that

$$C_k : L^p(\partial\Omega,\mathcal{E})\longrightarrow L^p(\partial\Omega,\mathcal{E}) \tag{5.35}$$

is well-defined and bounded for each $1<p<\infty$; cf. [19], which further builds on [5]. Also, for $1<p<\infty$,

$$\|\mathcal{N}(\mathcal{C}_k f)\|_{L^p(\partial\Omega)} \le \kappa \|f\|_{L^p(\partial\Omega,\mathcal{E})}, \tag{5.36}$$

uniformly for $f\in L^p(\partial\Omega,\mathcal{E})$. For further reference, we also note that

$$\mathcal{C}_k f = \mathbb{D}_k \mathcal{S}_k f = d\mathcal{S}_k f + \delta\mathcal{S}_k f + k\,dt\cdot\mathcal{S}_k f. \tag{5.37}$$

In particular, from this and Theorem 5.1,

$$\mathcal{C}_k f|_{\partial\Omega_\pm} = \mp\tfrac{1}{2}\nu\cdot f + C_k f, \tag{5.38}$$

a.e. on $\partial\Omega$, for each $f\in L^p(\partial\Omega,\mathcal{E})$, $1<p<\infty$.

The first order of business is to extend the action of these operators to $\mathcal{X}^p_{\rm nor}(\partial\Omega)$ and to $\mathcal{X}^p_{\rm tan}(\partial\Omega)$. More specifically, we shall eventually prove that the operators

$$\mathcal{C}_k,\, d\mathcal{C}_k,\, \delta\mathcal{C}_k : \mathcal{X}^p_{\rm nor}(\partial\Omega) \longrightarrow L^p(\Omega,\mathcal{E}), \tag{5.39}$$

$$\mathcal{C}_k,\, d\mathcal{C}_k,\, \delta\mathcal{C}_k : \mathcal{X}^p_{\rm tan}(\partial\Omega) \longrightarrow L^p(\Omega,\mathcal{E}), \tag{5.40}$$

and

$$\pm\tfrac{1}{2}I + \nu\wedge C_k : \mathcal{X}^p_{\rm nor}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\rm nor}(\partial\Omega), \tag{5.41}$$

$$\pm\tfrac{1}{2}I + \nu\vee C_k : \mathcal{X}^p_{\rm tan}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\rm tan}(\partial\Omega), \tag{5.42}$$

are well-defined and bounded for each $1<p<\infty$.

With this aim in mind, fix an arbitrary $f\in\mathcal{X}^p_{\rm nor}(\partial\Omega)$ and define $u := \mathcal{C}_k f = \mathbb{D}_k\mathcal{S}_k f$ in Ω. Our immediate aim is to show that, for each $1<p<\infty$,

$$\|u\|_{L^p(\Omega,\mathcal{E})} + \|du\|_{L^p(\Omega,\mathcal{E})} + \|\delta u\|_{L^p(\Omega,\mathcal{E})} \leq \kappa\|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)}, \tag{5.43}$$

uniformly in f. First, the point (i) in Lemma 5.2 gives that $u\in L^p(\Omega,\mathcal{E})$ and $\|u\|_{L^p(\Omega,\mathcal{E})}\leq\kappa\|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)}$. Going further, we write

$$\begin{aligned} du &= d\delta\mathcal{S}_k f - k\,dt\cdot d\mathcal{S}_k f = \\ &= -\delta d\mathcal{S}_k f + k^2\mathcal{S}_k f - k\,dt\cdot d\mathcal{S}_k f = \\ &= -\delta\mathcal{S}_k(d_\partial f) + k^2\mathcal{S}_k f - k\,dt\cdot d\mathcal{S}_k f, \end{aligned} \tag{5.44}$$

where, for the last equality, we have used the fact that $\mathcal{S}_k$ intertwines d and d_∂; cf. (5.25). Now, (5.44) in concert with (i) in Lemma 5.2, gives that $du\in L^p(\Omega,\mathcal{E})$ plus a natural estimate. The case of δu is similar and this concludes the proof of (5.43). This, in turn, entails the boundedness of the operators in (5.39)–(5.40).

Next, in analogy with (5.38) we define

$$\left(\tfrac{1}{2}I + \nu\wedge C_k\right) f := \nu\wedge\mathcal{C}_k f \in \mathcal{X}^p_{\rm nor}(\partial\Omega), \quad \forall\, f\in\mathcal{X}^p_{\rm nor}(\partial\Omega), \tag{5.45}$$

$$\left(\tfrac{1}{2}I + \nu\vee C_k\right) g := \nu\vee\mathcal{C}_k g \in \mathcal{X}^p_{\rm tan}(\partial\Omega), \quad \forall\, g\in\mathcal{X}^p_{\rm tan}(\partial\Omega). \tag{5.46}$$

Thanks to (5.39)–(5.40) and the above definitions, it follows that, for each $1<p<\infty$, the operators (5.41)–(5.42) are well defined and bounded, and that their action is compatible with that on the space $L^p(\partial\Omega,\mathcal{E})$.

Next, we elaborate on connections between the Cauchy operators just introduced and the operators M_k, N_k.

Proposition 5.4. *Assume that $k\in\mathbf{C}\setminus\{\pm k_j\}_j$, and fix a Lipschitz domain $\Omega\subset\mathcal{M}$. Then for each $\lambda\in\mathbf{C}$,*

$$(\lambda I + \nu\wedge C_k)f = (\lambda I + N_k)f + \nu\wedge S_k(d_\partial f) - k\,dt\cdot(\nu\wedge S_k f) \tag{5.47}$$

for any $f\in\mathcal{X}^p_{\rm nor}(\partial\Omega)$, $1<p<\infty$. Also,

$$(\lambda I + \nu\vee C_k)g = (\lambda I + M_k)g + \nu\vee S_k(\delta_\partial g) - k\,dt\cdot(\nu\vee S_k g) \tag{5.48}$$

for any $g\in\mathcal{X}^p_{\rm tan}(\partial\Omega)$, $1<p<\infty$.

Proof. As far as (5.47) is concerned, obviously, it suffices to consider the case $\lambda = \frac{1}{2}$. In this situation the identity (5.47) follows from (5.37) and the jump relations of (the derivatives of) the single layer, at least if f is in $L^p_{\tan}(\partial\Omega, \mathcal{E})$. The extension to $f \in \mathcal{X}^p_{\tan}(\partial\Omega, \mathcal{E})$ is then done by density. The case of (5.48) is similar. $\square$

Proposition 5.5. *Retain the same assumptions made in Proposition* 5.4. *Then, for each* $1 < p < \infty$*, the operators*

$$\nu \wedge S_k : \mathcal{X}^p_{\mathrm{nor}}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\mathrm{nor}}(\partial\Omega), \tag{5.49}$$

$$\nu \vee S_k : \mathcal{X}^p_{\tan}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\tan}(\partial\Omega) \tag{5.50}$$

are compact. Furthermore, for $k_1, k_2 \in \mathbf{C} \setminus \{\pm k_j\}_j$*, so are*

$$\nu \wedge C_{k_1} - \nu \wedge C_{k_2} : \mathcal{X}^p_{\mathrm{nor}}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\mathrm{nor}}(\partial\Omega), \tag{5.51}$$

$$\nu \vee C_{k_1} - \nu \vee C_{k_2} : \mathcal{X}^p_{\tan}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\tan}(\partial\Omega). \tag{5.52}$$

Proof. Indeed, let $(f_\alpha)_\alpha$ be a bounded sequence in $\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega)$. The aim is to prove that, by eventually restricting to a subsequence, $\{\nu \wedge S_k f_\alpha\}_\alpha$ converges in $\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega)$. In turn, since

$$\begin{aligned} \|\nu \wedge S_k f_\alpha - \nu \wedge S_k f_\beta\|_{\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega)} &\leq \|\mathcal{S}_k(f_\alpha - f_\beta)\|_{L^p(\Omega,\mathcal{E})} + \\ &+ \|d\mathcal{S}_k(f_\alpha - f_\beta)\|_{L^p(\Omega,\mathcal{E})} = \\ = \|\mathcal{S}_k(f_\alpha - f_\beta)\|_{L^p(\Omega,\mathcal{E})} &+ \|\mathcal{S}_k(d_\partial f_\alpha - d_\partial f_\beta)\|_{L^p(\Omega,\mathcal{E})} \end{aligned} \tag{5.53}$$

it suffices to show that $\mathcal{S}_k : B^{p,p}_{-1/p}(\partial\Omega, \mathcal{E}) \to L^p(\Omega, \mathcal{E})$ is a compact operator. This, however, is an immediate consequence of (5.16) and Rellich's selection lemma.

This proves that the operator (5.49) is compact. The case of (5.50) is similar.

As for the second part of the proposition, from what we have proved so far and (5.47)–(5.48), it suffices to show that $M_{k_1} - M_{k_2}$ and $N_{k_1} - N_{k_2}$ are compact operators on $\mathcal{X}^p_{\tan}(\partial\Omega)$ and $\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega)$, respectively.

In turn, this claim follows from the fact that the main singularity in $\Gamma_k(x, y)$ is actually independent of k. The idea is that this implies that operators with integral kernels $\Gamma_{k_1}(x, y) - \Gamma_{k_2}(x, y)$, $d\Gamma_{k_1}(x, y) - d\Gamma_{k_2}(x, y)$, $\delta\Gamma_{k_1}(x, y) - \delta\Gamma_{k_2}(x, y)$, are only weekly singular; cf. [19], [18] for details. $\square$

6. Finite Energy Solutions for Half-Dirichlet Problems

In this section we initiate the study of (1.2), by considering the case when $\eta = 0$ and when solutions have finite energy. That is, we shall work with the class of functions u satisfying

$$\iint_\Omega \left[|u|^2 + |du|^2 + |\delta u|^2\right] d\mathrm{Vol} < +\infty. \tag{6.1}$$

Thus, the context is somewhat analogous to that for Maxwell's equations in the L^2 setting (cf. [7], Vol. I, pp. 96-97 or Vol. III, p. 240, [4], p. 68 and [25], Vol. I, p. 169). Our approach is variational, and the departure point is the following general functional analytic result.

Proposition 6.1. *Let H be a Hilbert space and suppose that $T : Dom\,(T)(\subseteq H) \longrightarrow H$ is a linear, unbounded closed, densely defined operator, which also satisfies $T^2 = 0$.*

Then $T + T^*$ is self-adjoint, its spectrum, $Spec\,(T + T^*)$, is a subset of $\mathbb{R}$ and, for each complex number $z \notin Spec\,(T + T^*)$, there holds

$$\|w\|_H + \|Tw\|_H + \|T^*w\|_H \leq C_z\|(zI - T - T^*)w\|_H, \tag{6.2}$$

uniformly for $w \in Dom\,(T + T^*)$.

Proof. For starters, we note that $\mathrm{Dom}\,(T + T^*) = \mathrm{Dom}\,(T) \cap \mathrm{Dom}\,(T^*)$ is dense in H, as a consequence of (6.5) below. Next, since $(T^*)^* = \bar{T} = T$, it follows that $T + T^* = T^* + (T^*)^* \subseteq (T + T^*)^*$, i.e. $T + T^*$ is symmetric (cf., e.g., p. 191 in [26], or [14]).

To prove the opposite inclusion, fix some arbitrary $u \in \mathrm{Dom}\,((T + T^*)^*)$. It follows that there exists $w \in H$ so that

$$\langle (T+T^*)v, u\rangle = \langle v, w\rangle, \quad \forall\, v \in \mathrm{Dom}\,(T+T^*) = \mathrm{Dom}\,(T)\cap\mathrm{Dom}\,(T^*). \tag{6.3}$$

We aim at showing that $u \in \mathrm{Dom}\,((T^*)^*) = \mathrm{Dom}\,(\bar{T}) = \mathrm{Dom}\,(T)$. To this end, it suffices to prove that

$$|\langle T^*\phi, u\rangle| \leq C\|\phi\|_H, \qquad \text{uniformly for } \phi \in \mathrm{Dom}\,(T^*). \tag{6.4}$$

At this stage, we recall a lemma due to M. P. Gaffney [9] (cf. also Proposition 1.3.8 in [8]) which states that, under the current hypotheses on T,

$$S := I + TT^* + T^*T \text{ is a self-adjoint operator on } H, \tag{6.5}$$

with a bounded, everywhere defined inverse

$$S^{-1} = (I + TT^*)^{-1} + (I + T^*T)^{-1} - I.$$

All operators above are considered in the sense of composition of unbounded operators. For further reference, let us also note that $S \geq I \Rightarrow S^{-1} \leq I$; in particular,

$$\|S^{-1}\| \leq 1. \tag{6.6}$$

With an eye toward (6.4), fix an arbitrary $\phi \in \mathrm{Dom}\,(T^*)$ and set $\psi := S^{-1}\phi$. It follows that $\psi \in \mathrm{Dom}\,(S)$ and $\|\psi\|_H \leq \|\phi\|_H$. Also, simple calculations give

$$\begin{gathered}\langle \phi - \psi, \psi\rangle = \|\psi\|_H^2 + \|T^*\psi\|_H^2 + \|T\psi\|_H^2 \quad \text{and} \\ \|\phi - \psi\|_H^2 = \|TT^*\psi\|_H^2 + \|T^*T\psi\|_H^2.\end{gathered} \tag{6.7}$$

Thus, $\|TT^*\psi\|_H$, $\|T^*\psi\|_H \leq C\|\phi\|_H$. Next, we claim that $TT^*\psi \in \mathrm{Dom}\,(T + T^*)$ and note that, granted $T^2 = 0$, this comes down to checking the membership of $TT^*\psi$ to $\mathrm{Dom}\,(T^*)$. Nonetheless, this is a direct consequence of $\phi = \psi + TT^*\psi + T^*T\psi$, $\phi \in \mathrm{Dom}\,(T^*)$, and the fact that $(T^*)^2 = 0$. Thus, using $v := TT^*\psi$ in (6.3) and observing that $(T+T^*)v = T^*(TT^*\psi) = T^*\phi - T^*\psi$, yields

$$\langle T^*\phi, u\rangle = \langle (T+T^*)v, u\rangle + \langle T^*\psi, u\rangle = \langle TT^*\psi, w\rangle + \langle T^*\psi, u\rangle. \tag{6.8}$$

With this at hand and our previous estimates on $TT^*\psi$, $T^*\psi$, (6.4) readily follows. Thus, $u \in \mathrm{Dom}\,(T)$. The same reasoning but with T replaced by T^* also gives that $u \in \mathrm{Dom}\,(T^*)$ so that, ultimately, $u \in \mathrm{Dom}\,(T+T^*)$. Thus, $\mathrm{Dom}\,((T+T^*)^*) \subseteq \mathrm{Dom}\,(T+T^*)$ so that $(T+T^*)^* = T+T^*$, as desired.

We are left with proving (6.2). Fix $z \notin \mathrm{Spec}\,(T+T^*)$, $w \in \mathrm{Dom}\,(T+T^*)$, and set $u := (zI - T - T^*)w \in H$. Then, it follows that

$$\|w\|_H = \|(zI - T - T^*)^{-1}u\|_H \leq C_z\|u\|_H. \tag{6.9}$$

Also, a straightforward calculation (which utilizes the fact that $T^2 = 0$) gives

$$\|Tw\|_H^2 + \|T^*w\|_H^2 = \|u\|_H^2 - |z|^2\|w\|_H^2 + 4(\mathrm{Re}\,z)(\mathrm{Re}\,\langle Tw, w\rangle). \tag{6.10}$$

Since, for each $\epsilon > 0$, the right hand side above is $\leq \|u\|_H^2 + C_{z,\epsilon}\|w\|_H^2 + \epsilon\|Tw\|_H^2$, for some appropriately large constant $C_{z,\epsilon}$, the estimate (6.2) follows from this and (6.9). $\square$

Our next theorem deals with the half-Dirichlet boundary value problem for the Dirac operator $\mathbb{D}_k$ in the class of forms of finite (global) L^2-energy.

Theorem 6.2. *For each arbitrary Lipschitz domain Ω in $\mathcal{M}$ there exists a sequence of real numbers $\{k_j^\wedge\}_j$ which does not accumulate in $\mathbb{R}$ and with the following significance. For any complex number $k \notin \{k_j^\wedge\}_j$ the boundary problem*

$$\begin{cases} \mathbb{D}_k u = 0 \ \text{ in } \Omega, \\ \nu \wedge u = f \in B^{2,2}_{-\frac{1}{2}}(\partial\Omega, \mathcal{E}) \end{cases} \tag{6.11}$$

is solvable in the class of forms of (global) L^2-energy, i.e. satisfying u, du, $\delta u \in L^2(\Omega, \mathcal{E})$, if and only if $f \in \mathcal{X}^2_{\rm nor}(\partial\Omega)$. In this latter case, the solution is unique and

$$\|u\|_{L^2(\Omega)} + \|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} \approx \|f\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \tag{6.12}$$

uniformly in f.

Furthermore, there exists a sequence of real numbers $\{k_j^\vee\}_j$ (with no finite accumulation point) so that for any complex number $k \notin \{k_j^\wedge\}_j$ the dual problem

$$\begin{cases} \mathbb{D}_k u = 0 \ \text{ in } \ \Omega, \\ \nu \vee u = g \in B^{2,2}_{-\frac{1}{2}}(\partial\Omega, \mathcal{E}) \end{cases} \tag{6.13}$$

is well-posed in the class of forms satisfying u, du, $\delta u \in L^2(\Omega, \mathcal{E})$, if and only if $g \in \mathcal{X}^2_{\rm tan}(\partial\Omega)$. In this latter situation, we also have

$$\|u\|_{L^2(\Omega)} + \|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} \approx \|g\|_{\mathcal{X}^2_{\rm tan}(\partial\Omega)} \tag{6.14}$$

uniformly in g.

In the proof of this theorem (as well as later on), the following regularity result from [19], [24] is going to be important.

Theorem 6.3. *For any Ω arbitrary Lipschitz domain in $\mathcal{M}$ there exists $\epsilon = \epsilon(\Omega) > 0$ with the following significance. Assume that $2 - \epsilon < p < 2 + \epsilon$ and that the form $u \in L^p(\Omega, \mathcal{E})$ has, in the sense of distributions, $du \in L^p(\Omega, \mathcal{E})$ and $\delta u \in L^p(\Omega, \mathcal{E})$. Then the following are equivalent:*

(i) *$\nu \vee u$, initially considered as a distribution, actually belongs to $L^p(\partial\Omega, \mathcal{E})$ (thus, in fact, to $L^p_{\tan}(\partial\Omega, \mathcal{E})$);*

(ii) *$\nu \wedge u$, initially considered as a distribution, actually belongs to $L^p(\partial\Omega, \mathcal{E})$ (hence, in fact, to $L^p_{\rm nor}(\partial\Omega, \mathcal{E})$).*

Moreover, if (i) *or* (ii) *above is valid, then $u \in B^{p,p^\#}_{1/p}(\Omega, \mathcal{E})$ (recall that $p^\# := \max\{p, 2\}$). Also, naturally accompanying estimates are valid in each case.*

We now turn to the task of presenting the

Proof of Theorem 6.2. Consider δ as a densely defined, closed, unbounded operator on $L^2(\Omega, \mathcal{E})$, with domain $\mathrm{Dom}(\delta; L^2(\Omega, \mathcal{E}))$, and set $T := -dt \cdot \delta = \delta(dt \cdot)$. Since $T^2 = 0$, Proposition 6.1 applies and gives that $T + T^*$ is self-adjoint on $L^2(\Omega, \mathcal{E})$. Note that the domain of $T^* = \delta^*(dt\cdot) = -dt \cdot \delta^*$ is $\{u \in \mathrm{Dom}(d; L^2(\Omega, \mathcal{E}));\ \nu \wedge u = 0\}$. Thus,

$$\mathrm{Dom}(T + T^*) = \\ = \{u \in L^2(\Omega, \mathcal{E}); du, \delta u \in L^2(\Omega, \mathcal{E}), \nu \wedge u = 0\} \hookrightarrow B^{2,2}_{1/2}(\Omega, \mathcal{E}), \qquad (6.15)$$

where the last inclusion follows from Theorem 6.3. In particular, $\mathrm{Dom}\,(T + T^*) \hookrightarrow L^2(\Omega, \mathcal{E})$ is compact, by Rellich's selection lemma. In this scenario, it is well-known that $\mathrm{Spec}\,(T + T^*)$ consists only of real eigenvalues of finite multiplicity and which accumulate only at $\pm\infty$. Denote by $\{k_j^\wedge\}_j$ this set. Hence, if $\mathbb{D}_{k,\wedge} := dt \cdot (kI - T - T^*) = \delta^* + \delta + k\,dt\cdot$, in the sense of unbounded operators, we have that $(\mathbb{D}_{k,\wedge})^{-1}$ exists and is a bounded operator on $L^2(\Omega, \mathcal{E})$ for each complex number $k \notin \{k_j^\wedge\}_j$.

Let us now turn our attention to the boundary problems in the statement of the theorem under discussion. The fact that $f \in \mathcal{X}^2_{\rm nor}(\partial\Omega)$ is a necessary condition for the solvability of (6.11) in the class of finite L^2-energy forms is clear from definitions. Conversely, let $f \in \mathcal{X}^2_{\rm nor}(\partial\Omega)$ be arbitrary and fix some $k \notin \{k_j^\wedge\}_j$. Then there exists $w \in \mathrm{Dom}(d; L^2(\Omega))$ so that $f = \nu \wedge w$ and $\|f\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \approx \|w\|_{L^2(\Omega,\mathcal{E})} + \|dw\|_{L^2(\Omega,\mathcal{E})}$.

Indeed, we claim that w can be chosen with the additional property that $\delta w = 0$. To see this, we invoke the Hodge decomposition $w = d\alpha + \delta\beta + \gamma$, where $\alpha \in \mathrm{Dom}\,(d; L^2(\Omega; \mathcal{E}))$, $\nu \wedge \alpha = 0$, $\beta \in \mathrm{Dom}\,(\delta; L^2(\Omega; \mathcal{E}))$ and $\gamma \in \mathrm{Dom}\,(d; L^2(\Omega; \mathcal{E})) \cap \mathrm{Dom}\,(\delta; L^2(\Omega; \mathcal{E}))$ satisfies $\nu \wedge \gamma = 0$; see Theorem 6.1 in [18]. Then, the fact that $w \in \mathrm{Dom}\,(d; L^2(\Omega; \mathcal{E}))$ entails $\delta\beta \in \mathrm{Dom}\,(d; L^2(\Omega; \mathcal{E}))$. Also, since $\nu \wedge d\alpha = -d_\partial(\nu \wedge \alpha) = 0$, it follows that $f = \nu \wedge w = \nu \wedge (\delta\beta)$. Redenoting $\delta\beta$ by w shows that there is no loss of generality in assuming that $\delta w = 0$.

If that this is the case, then a solution of (6.11) is given by

$$u := w - (\mathbb{D}_{k,\wedge})^{-1}(\mathbb{D}_k w). \tag{6.16}$$

Notice that u has finite L^2-energy and, by (6.2) in Proposition 6.1,

$$\begin{aligned} &\|(\mathbb{D}_{k,\wedge})^{-1}(\mathbb{D}_k w)\|_{L^2(\Omega)}+ \\ &+\|d(\mathbb{D}_{k,\wedge})^{-1}(\mathbb{D}_k w)\|_{L^2(\Omega)} + \|\delta(\mathbb{D}_{k,\wedge})^{-1}(\mathbb{D}_k w)\|_{L^2(\Omega)} \leq \\ &\leq C\|\mathbb{D}_k w\|_{L^2(\Omega)} \leq C(\|w\|_{L^2(\Omega)} + \|dw\|_{L^2(\Omega)}) \leq C\|f\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)}. \end{aligned} \tag{6.17}$$

From this, the estimate (6.12) follows easily. Finally, uniqueness follows from our assumption that $k \notin \operatorname{Spec}(T + T^*)$. The proof of the portion of the theorem concerning (6.11) is therefore finished.

As for (6.13), define this time $\mathbb{D}_{k,\vee}$, analogously to the operator $\mathbb{D}_{k,\wedge}$. Once again, $(\mathbb{D}_{k,\vee})^{-1}$ exists and is a bounded operator on $L^2(\Omega, \mathcal{E})$ for any complex number k, except for a subset $\{k_j^{\vee}\}_j$ of $\mathbb{R}$ without finite accumulation points. The rest is much as before. □

A very useful consequence of this theorem is recorded below.

Corollary 6.4. *Let Ω be a Lipschitz domain in $\mathcal{M}$ and fix a complex number*

$$k \notin \{k_j^{\wedge}\}_j \cup \{k_j^{\vee}\}_j. \tag{6.18}$$

Then

$$\|\nu \wedge u\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \approx \|\nu \vee u\|_{\mathcal{X}^2_{\rm tan}(\partial\Omega)} \tag{6.19}$$

uniformly for forms u satisfying

$$u,\, du,\, \delta u \in L^2(\Omega, \mathcal{E}), \quad \textit{and} \;\; \mathbb{D}_k u = 0 \;\; \textit{in} \;\; \Omega. \tag{6.20}$$

Proof. Our assumption on k ensures that $(\mathbb{D}_{k,\wedge})^{-1}$ and $(\mathbb{D}_{k,\vee})^{-1}$ exist and are bounded operators on $L^2(\Omega, \mathcal{E})$. From Theorem 6.2 it follows that, under the current hypotheses,

$$\|\nu\wedge u\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \approx \|u\|_{L^2(\Omega)} + \|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} \approx \|\nu\vee u\|_{\mathcal{X}^2_{\rm tan}(\partial\Omega)} \tag{6.21}$$

uniformly in u satisfying (6.20). □

7. Inverting Boundary Layer Operators

Our next goal is to show that the operators (5.41)–(5.42) are actually *isomorphisms* for all p's in some open interval containing 2 (further restrictions on the complex parameter k are also needed).

Theorem 7.1. *For each Lipschitz domain in $\Omega \subseteq \mathcal{M}$ there exists a discrete set $\mathcal{U} \subseteq \mathbb{R}$ with no finite accumulation point and $\epsilon > 0$ with the following property. If $2 - \epsilon < p < 2 + \epsilon$ and $k \in \mathbf{C} \setminus \mathcal{U}$, then the operators in* (5.41)–(5.42) *are in fact isomorphisms.*

Proof. We shall only consider the case of (5.41) since the proof for (5.42) is very similar.

Let us assume for a moment that the operators in (5.41) are invertible for $p := 2$ and $k := k_o$, some fixed, purely imaginary complex number. Then the extension to the more general situation described in the statement of the theorem is accomplished as follows. To begin with, since by Proposition 4.1 the family $\{\mathcal{X}^p_{\rm nor}(\partial\Omega)\}_{1<p<\infty}$ is a complex interpolation scale, it follows that there exists $\epsilon = \epsilon(\Omega) > 0$ so that the operators in (5.41) are isomorphisms if $2 - \epsilon < p < 2 + \epsilon$ and $k = k_o$. This is a consequence of known stability results (cf., e.g., [13] for a discussion).

Extending further the aforementioned invertibility result to arbitrary wave numbers $k \in \mathbf{C} \setminus \mathcal{U}$, requires two other ingredients. First, recall that $\nu \wedge C_k - \nu \wedge C_{k_o}$ is a compact operator on $\mathcal{X}^p_{\rm nor}(\partial\Omega)$ for any $1 < p < \infty$ and $k \in \mathbf{C} \setminus \{\pm k_j\}$. In particular, if $2 - \epsilon < p < 2 + \epsilon$, the operators in (5.41) are Fredholm with index zero for any k as above. The second ingredient is a general stability result (cf. [13]) to the effect that if a linear operator T, mapping a complex interpolation scale $\mathcal{X}^p$ (of quasi-Banach spaces) boundedly into itself, is Fredholm with index zero on $\mathcal{X}^p$ for $2-\epsilon < p < 2+\epsilon$ and is invertible on $\mathcal{X}^2$, then T is actually invertible on $\mathcal{X}^p$ for each $2-\epsilon < p < 2+\epsilon$.

Summarizing, at this stage it suffices to prove that the operators in (5.41) are isomorphisms if $p = 2$ and $k \in \mathbf{C} \setminus \mathcal{U}$ for some appropriate discrete set $\mathcal{U} \subseteq \mathbb{R}$. For the time being, suppose that $\mathcal{U}$ contains $\{\pm k_j^\wedge\}_j$, $\{\pm k_j^\vee\}_j$ and Spec (Δ). Assuming that this is the case, for $f \in \mathcal{X}^2_{\rm nor}(\partial\Omega)$ arbitrary, we set

$$u := \mathbb{D}_k \mathcal{S}_k f = \mathcal{S}_k(d_\partial f) + \delta \mathcal{S}_k f + k\, dt \cdot \mathcal{S}_k f, \quad \text{in} \ \ \Omega_\pm. \tag{7.1}$$

It follows that u, du, $\delta u \in L^2(\Omega_\pm, \mathcal{E})$ and $\mathbb{D}_k u = 0$ both in Ω_+ and in Ω_-. Consequently, by Corollary 6.4 and our assumptions on k,

$$\|\nu \wedge (u|_{\Omega_\pm})\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \approx \|\nu \vee (u|_{\Omega_\pm})\|_{\mathcal{X}^2_{\rm tan}(\partial\Omega)}. \tag{7.2}$$

The claim we make at this stage is that

$$\nu \vee (u|_{\Omega_+}) = \nu \vee (u|_{\Omega_-}) \quad \text{in} \quad \mathcal{X}^2_{\rm tan}(\partial\Omega). \tag{7.3}$$

In order to see this, we note that the applications

$$\mathcal{X}^2_{\rm nor}(\partial\Omega) \ni f \mapsto \nu \vee (u|_{\Omega_\pm}) \in \mathcal{X}^2_{\rm tan}(\partial\Omega) \tag{7.4}$$

are continuous. Hence, by (iii) in Proposition 4.1 (with $p = 2$), it suffices to prove (7.3) when $f \in L^{2,d}_{\rm nor}(\partial\Omega, \mathcal{E})$. In this case, based on the results of Section 5 we see that, even in the sense of nontangential convergence to the boundary,

$$\begin{aligned} \nu \vee u|_{\partial\Omega_\pm} &= \nu \vee S_k(d_\partial f) + \nu \vee \left[\pm\tfrac{1}{2}\nu \vee f + \delta S_k f\right] + k\, \nu \vee (dt \cdot S_k f) = \\ &= \nu \vee S_k(d_\partial f) + \nu \vee \delta S_k f + k\, \nu \vee (dt \cdot S_k f). \end{aligned} \tag{7.5}$$

From this, (7.3) follows.

Next, notice that (5.45) gives

$$\nu \wedge (u|_{\Omega_\pm}) = \left(\pm\tfrac{1}{2} I + \nu \wedge C_k\right) f. \tag{7.6}$$

Armed with (7.3) and (7.6) we are finally ready to tackle the issue of invertibility of the operators (5.41) when $p = 2$. Based on these and (7.2) we may write:

$$\begin{aligned}
\|f\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} &\leq \big\|\big(\tfrac{1}{2}I + \nu\wedge C_k\big)f\big\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} + \big\|\big(-\tfrac{1}{2}I + \nu\wedge C_k\big)f\big\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} = \\
&= \|\nu\wedge(u|_{\Omega_-})\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} + \|\nu\wedge(u|_{\Omega_+})\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \leq \\
&\leq C\|\nu\vee(u|_{\Omega_-})\|_{\mathcal{X}^2_{\rm tan}(\partial\Omega)} + C\|\nu\vee(u|_{\Omega_+})\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \leq \\
&\leq C\min\big\{\|\nu\vee(u|_{\Omega_+})\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)}\,,\,\|\nu\vee(u|_{\Omega_-})\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)}\big\} \leq \\
&\leq C\min\big\{\|\nu\wedge(u|_{\Omega_+})\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)}\,,\,\|\nu\wedge(u|_{\Omega_-})\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)}\big\} = \\
&= C\min\big\{\big\|\big(\tfrac{1}{2}I + \nu\wedge C_k\big)f\big\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)}\,,\,\big(-\tfrac{1}{2}I + \nu\wedge C_k\big)f\big\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)}\big\}.
\end{aligned} \tag{7.7}$$

That is,

$$\|f\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \leq C\big\|\big(\pm\tfrac{1}{2}I + \nu\wedge C_k\big)f\big\|_{\mathcal{X}^2_{\rm nor}(\partial\Omega)} \quad \text{uniformly for } f\in\mathcal{X}^2_{\rm nor}(\partial\Omega). \tag{7.8}$$

In particular, $\pm\frac{1}{2}I + \nu\wedge C_k : \mathcal{X}^2_{\rm nor}(\partial\Omega) \to \mathcal{X}^2_{\rm nor}(\partial\Omega)$ are one-to-one and with closed range.

At this point we need to recall a result from [21] according to which $\pm\frac{1}{2}I + \nu\wedge C_k$ are isomorphisms of $L^{2,d}_{\rm nor}(\partial\Omega,\mathcal{E})$ for all complex k's, except a discrete subset of the real line. In concert with (iii) in Proposition 4.1, this gives that the operators under discussion have also dense ranges when acting on $\mathcal{X}^2_{\rm nor}(\partial\Omega)$. Thus, by eventually enlarging $\mathcal{U}$ we may conclude that the operators (5.41) are indeed isomorphisms of $\mathcal{X}^2_{\rm nor}(\partial\Omega)$, as desired.

The remaining cases are treated in a similar fashion, and this completes the proof of the theorem. □

Next, for $1 < p < \infty$, introduce

$$\mathcal{Y}^p_{\rm nor}(\partial\Omega) := L^p_{\rm nor}(\partial\Omega,\mathcal{E}) \cap \mathcal{X}^p_{\rm nor}(\partial\Omega), \tag{7.9}$$

$$\mathcal{Z}^p_{\rm nor}(\partial\Omega) := \{f \in \mathcal{X}^p_{\rm nor}(\partial\Omega);\, d_\partial f \in L^p(\partial\Omega,\mathcal{E})\} \tag{7.10}$$

equipped with the norms

$$\|f\|_{\mathcal{Y}^p_{\rm nor}(\partial\Omega)} := \|f\|_{L^p(\partial\Omega,\mathcal{E})} + \|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)}, \tag{7.11}$$

$$\|f\|_{\mathcal{Z}^p_{\rm nor}(\partial\Omega)} := \|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)} + \|d_\partial f\|_{L^p(\partial\Omega,\mathcal{E})}. \tag{7.12}$$

Clearly, this makes $\mathcal{Y}^p_{\rm nor}(\partial\Omega)$ and $\mathcal{Z}^p_{\rm nor}(\partial\Omega)$ Banach spaces. Analogously, we introduce

$$\mathcal{Y}^p_{\rm tan}(\partial\Omega) := L^p_{\rm tan}(\partial\Omega,\mathcal{E}) \cap \mathcal{X}^p_{\rm tan}(\partial\Omega), \tag{7.13}$$

$$\mathcal{Z}^p_{\rm tan}(\partial\Omega) := \{f \in \mathcal{X}^p_{\rm tan}(\partial\Omega);\, \delta_\partial f \in L^p(\partial\Omega,\mathcal{E})\} \tag{7.14}$$

and equip them with the natural norms.

Theorem 7.2. *Let Ω be a Lipschitz domain in $\mathcal{M}$ and let k be as in Theorem 7.1. Then the operators*

$$\pm\tfrac{1}{2}I + \nu \wedge C_k : \mathcal{Y}^p_{\rm nor}(\partial\Omega) \longrightarrow \mathcal{Y}^p_{\rm nor}(\partial\Omega), \tag{7.15}$$

$$\pm\tfrac{1}{2}I + \nu \wedge C_k : \mathcal{Z}^p_{\rm nor}(\partial\Omega) \longrightarrow \mathcal{Z}^p_{\rm nor}(\partial\Omega) \tag{7.16}$$

are well defined and bounded for each $1 < p < \infty$. Furthermore, there exists $\epsilon = \epsilon(\Omega) > 0$ so that for $2 - \epsilon < p < 2 + \epsilon$ they are in fact isomorphisms.

Similar conclusions are valid for the operators

$$\pm\tfrac{1}{2}I + \nu \vee C_k : \mathcal{Y}^p_{\rm tan}(\partial\Omega) \longrightarrow \mathcal{Y}^p_{\rm tan}(\partial\Omega), \tag{7.17}$$

$$\pm\tfrac{1}{2}I + \nu \vee C_k : \mathcal{Z}^p_{\rm tan}(\partial\Omega) \longrightarrow \mathcal{Z}^p_{\rm tan}(\partial\Omega). \tag{7.18}$$

Proof. That the actions of $\nu \wedge C_k$ on $L^p_{\rm nor}(\partial\Omega, \mathcal{E})$ and on $\mathcal{X}^p_{\rm nor}(\partial\Omega)$ agree on the intersection, can be seen via a routine limiting argument. Also, the boundedness of $\frac{1}{2}I + \nu \wedge C_k$ on $\mathcal{Y}^p_{\rm nor}(\partial\Omega, \mathcal{E})$ follows from that of $\frac{1}{2}I + \nu \wedge C_k$ on $L^p_{\rm nor}(\partial\Omega, \mathcal{E})$ and on $\mathcal{X}^p_{\rm nor}(\partial\Omega)$, separately.

Next, we consider the operators (7.16). Let $f \in \mathcal{Z}^p_{\rm nor}(\partial\Omega)$ and set $u := \mathbb{D}_k \mathcal{S}_k f$. Then

$$\begin{aligned} d_\partial\left(\tfrac{1}{2}I + \nu \wedge C_k\right) f &= d_\partial(\nu \wedge u) = \\ &= -\nu \wedge du = \\ &= \nu\wedge\Big(\delta\mathcal{S}_k(d_\partial f)\Big) - k^2\nu \wedge S_k f + k\,\nu \wedge \Big(dt \cdot \mathcal{S}_k(d_\partial f)\Big) = \\ &= \left(\tfrac{1}{2}I + \nu \wedge C_k\right)(d_\partial f) - k^2\nu \wedge S_k f. \end{aligned} \tag{7.19}$$

In particular, by Lemma 5.2 plus the fact that $\nu \wedge C_k$ is a bounded mapping of $\mathcal{X}^p_{\rm nor}(\partial\Omega)$ and of $L^p(\partial\Omega, \mathcal{E})$ for each $1 < p < \infty$, we see that the operators (7.16) are indeed well-defined and bounded.

Next, consider the issue of the invertibility of the operators (7.15)–(7.16) when p is close to 2. First, injectivity on $\mathcal{X}^p_{\rm nor}(\partial\Omega)$ clearly entails injectivity on $\mathcal{Y}^p_{\rm nor}(\partial\Omega, \mathcal{E})$ and $\mathcal{Z}^p_{\rm nor}(\partial\Omega, \mathcal{E})$. To show ontoness, in the light of Theorem 7.1, it suffices to prove the implications

$$\left.\begin{array}{l} f \in \mathcal{X}^p_{\rm nor}(\partial\Omega) \\ \left(\tfrac{1}{2}I + \nu \wedge C_k\right) f \in L^p_{\rm nor}(\partial\Omega, \mathcal{E}) \end{array}\right\} \Rightarrow f \in L^p_{\rm nor}(\partial\Omega, \mathcal{E}), \tag{7.20}$$

$$\left.\begin{array}{l} f \in \mathcal{X}^p_{\rm nor}(\partial\Omega) \\ d_\partial\left(\tfrac{1}{2}I + \nu \wedge C_k\right) f \in L^p_{\rm nor}(\partial\Omega, \mathcal{E}) \end{array}\right\} \Rightarrow d_\partial f \in L^p_{\rm nor}(\partial\Omega, \mathcal{E}). \tag{7.21}$$

To this end, let f be as in the left side of (7.20) and set $u := \mathbb{D}_k \mathcal{S}_k f$ in $\Omega_\pm$. As before,

$$\left\{\begin{array}{l} u,\, du,\, \delta u \in L^p(\Omega_\pm, \mathcal{E}), \\ \nu \wedge (u|_{\Omega_+}) = \left(\tfrac{1}{2}I + \nu \wedge C_k\right) f \in L^p_{\rm nor}(\partial\Omega, \mathcal{E}). \end{array}\right. \tag{7.22}$$

Theorem 6.3 applied to u in Ω_+ then implies that

$$\nu \vee (u|_{\Omega_+}) \in L^p_{\tan}(\partial\Omega, \mathcal{E}). \tag{7.23}$$

Now, as in (7.3),

$$\nu \vee (u|_{\Omega_+}) = \nu \vee (u|_{\Omega_-}) \tag{7.24}$$

so that, by (7.23),

$$\nu \vee (u|_{\Omega_-}) \in L^p_{\tan}(\partial\Omega, \mathcal{E}). \tag{7.25}$$

In turn, the first condition in (7.22) and (7.25) together with the same regularity result (i.e. Theorem 6.3), applied this time to u in Ω_-, yield

$$\nu \wedge (u|_{\Omega_-}) \in L^p_{\rm nor}(\partial\Omega, \mathcal{E}). \tag{7.26}$$

With this at hand and using the fact that, by definition,

$$\nu \wedge (u|_{\Omega_\pm}) = \big(\mp\tfrac{1}{2}I + \nu \wedge C_k\big)\, f \tag{7.27}$$

we arrive at the conclusion that

$$\begin{aligned} f &= \big(\tfrac{1}{2}I + \nu \wedge C_k\big)\, f - \big(-\tfrac{1}{2}I + \nu \wedge C_k\big)\, f = \\ &= \nu \wedge (u|_{\Omega_+}) - \nu \wedge (u|_{\Omega_-}) \in L^p_{\rm nor}(\partial\Omega, \mathcal{E}). \end{aligned} \tag{7.28}$$

This proves (7.20) and concludes the proof of the part in Theorem 7.2 which refers to the operators (7.15).

As for (7.21), note that if f is as in the left side of (7.21) then (7.19) and Lemma 5.2 imply that $d_\partial f \in \mathcal{X}^p_{\rm nor}(\partial\Omega)$ has the property that

$$\big(\tfrac{1}{2}I + \nu \wedge C_k\big)\,(d_\partial f) \in L^p_{\rm nor}(\partial\Omega, \mathcal{E}).$$

Thus, by our assumptions on k and (7.20), it follows that $d_\partial f \in L^p(\partial\Omega, \mathcal{E})$, as wanted. This finishes the proof of the claim made in the statement of the theorem for the operators (7.16).

Finally, the last part of the statement of the theorem follows in a similar manner. $\square$

8. The Poisson Problem for Dirac Operators

In this section we study the Poisson problem (1.2). The departure point is the L^p-version of the half-Dirichlet problem (6.11), in the theorem below.

Theorem 8.1. *Let Ω be a Lipschitz domain in $\mathcal{M}$. Then there exists a discrete set of real numbers $\mathcal{U}$ without finite accumulation points and $\epsilon = \epsilon(\Omega) > 0$ so that the following holds.*

If $2-\epsilon < p < 2+\epsilon$ and the complex number k satisfies $k \notin \mathcal{U}$, the Dirac boundary value problem

$$\left\{\begin{array}{l} u,\, du,\, \delta u \in L^p(\Omega, \mathcal{E}), \\ \mathbb{D}_k u = 0 \ \textit{in}\ \Omega, \\ \nu \wedge u = f \in \mathcal{X}^p_{\rm nor}(\partial\Omega), \end{array}\right. \tag{8.1}$$

is uniquely solvable. Also, there exists $C > 0$ so that the solution u of (8.1) *satisfies*

$$\|u\|_{L^p(\Omega)} + \|du\|_{L^p(\Omega)} + \|\delta u\|_{L^p(\Omega)} \leq C\|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)}. \tag{8.2}$$

Moreover, the following regularity statements are true:

(i) $\mathcal{N}(u) \in L^p(\partial\Omega) \Leftrightarrow f \in \mathcal{Y}^p_{\rm nor}(\partial\Omega)$. *If one* (*and, hence, both*) *of these conditions is true then, actually, $u \in B^{p,p^\#}_{1/p}(\Omega, \mathcal{E})$;*

(ii) $\mathcal{N}(du) \in L^p(\partial\Omega) \Leftrightarrow f \in \mathcal{Z}^p_{\rm nor}(\partial\Omega)$. *If one* (*and, hence, both*) *of these conditions is valid then, in fact, $du \in B^{p,p^\#}_{1/p}(\Omega, \mathcal{E})$;*

(iii) $\mathcal{N}(u), \mathcal{N}(du), \mathcal{N}(\delta u) \in L^p(\partial\Omega) \Leftrightarrow f \in L^{p,d}_{\rm nor}(\partial\Omega, \mathcal{E})$. *If one* (*and, hence, both*) *of these conditions holds then*

$$\|\mathcal{N}(u)\|_{L^p(\Omega)} + \|\mathcal{N}(du)\|_{L^p(\Omega)} + \|\mathcal{N}(\delta u)\|_{L^p(\Omega)} \leq C\|f\|_{L^{p,d}_{\rm nor}(\partial\Omega,\mathcal{E})}. \tag{8.3}$$

Finally, similar results hold for the dual problem, i.e.

$$\begin{cases} u,\, du,\, \delta u \in L^p(\Omega, \mathcal{E}), \\ \mathbb{D}_k u = 0 \ \textit{in}\ \Omega, \\ \nu \vee u = g \in \mathcal{X}^p_{\rm tan}(\partial\Omega). \end{cases} \tag{8.4}$$

Proof. Let $\epsilon > 0$ and $\mathcal{U} \subseteq \mathbb{R}$ be so that the conclusions of Theorem 7.1 and Theorem 7.2 are valid for each $2 - \epsilon < p < 2 + \epsilon$ and $k \in \mathbf{C} \setminus \mathcal{U}$. Granted this, a solution to (8.1) can be expressed in the form

$$u := \mathbb{D}_k \mathcal{S}_k \Big[\big(\tfrac{1}{2} I + \nu \wedge C_k\big)^{-1} f \Big] \quad \text{in} \quad \Omega. \tag{8.5}$$

From this (and the mapping properties of the operators involved), it is clear that u satisfies the desired L^p estimates.

Turning our attention to the uniqueness part, assume that u is a null-solution for (8.1). Then Theorem 6.3 applied to u and du (note that $\nu \wedge du = -d_\partial(\nu \wedge u) = 0$) gives that $u, du \in B^{p,p^\#}_{1/p}(\Omega, \mathcal{E})$. Since $0 = \mathbb{D}_k u = du + \delta u + k\, dt \cdot u = 0$, we also see that $\delta u \in B^{p,p^\#}_{1/p}(\Omega, \mathcal{E})$. Consequently, by standard embedding results, we see that u is also a null-solution for the $L^{p+\gamma}$ version of (8.1) for some $\gamma > 0$. This is an improvement over the original regularity assumptions on u. Iterating this procedure finitely many times yields that u is a null-solution for the L^2 version of (8.1). At this stage, granted that $k \notin \{k_j^\wedge\}_j$, which we can and will assume, we may conclude (based on Theorem 6.2) that $u = 0$, as wanted.

Next, we consider the regularity statements. Clearly, the fact that $f \in \mathcal{Y}^p_{\rm nor}(\partial\Omega)$ entails $\mathcal{N}(u) \in L^p(\partial\Omega)$. The converse implication follows from the fact that $(\Delta + k^2)u = 0$, $\mathcal{N}(u) \in L^p(\partial\Omega) \Rightarrow \exists\, u|_{\partial\Omega} \in L^p(\partial\Omega, \mathcal{E})$ in the sense of the nontangential convergence; cf. [19]. Note that, granted the membership of f to $\mathcal{Y}^p_{\rm nor}(\partial\Omega)$, (8.5) entails the fact that u belongs to $B^{p,p^\#}_{1/p}(\Omega, \mathcal{E})$. This proves (i).

The left-to-right implication in (ii) is seen from the identity

$$\nu \wedge du = -d_\partial(\nu \wedge u) = -d_\partial f \tag{8.6}$$

and the fact that $(\Delta + k^2)(du) = 0$, $\mathcal{N}(du) \in L^p(\partial\Omega)$ imply $(du)|_{\partial\Omega} \in L^p(\partial\Omega, \mathcal{E})$. To see the opposite implication in (ii) first we note that, by Theorem 7.2, $g := \left(\mp\frac{1}{2}I + \nu \wedge C_k\right)^{-1} f \in \mathcal{Z}^p_{\rm nor}(\partial\Omega)$. Then, the desired conclusion follows from the identity

$$du = -\delta\mathcal{S}_k(d_\partial g) + k^2\mathcal{S}_k g - k\, dt \cdot \mathcal{S}_k(d_\partial g), \tag{8.7}$$

and (ii) in Lemma 5.2. That any of the two conditions in (ii) implies the membership of du to $B^{p,p^\#}_{1/p}(\Omega, \mathcal{E})$ is also seen from (8.7). Finally, (iii) is a direct consequence of (i) and (ii). □

Corollary 8.2. *Let Ω be a Lipschitz domain and let $k \in \mathbf{C}$, $\epsilon = \epsilon(\Omega) > 0$ be as in Theorems 7.1–7.2. Then, for $2 - \epsilon < p < 2 + \epsilon$, consider the normal-to-tangential operator*

$$\mathrm{NT}_k : \mathcal{X}^p_{\rm nor}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\rm tan}(\partial\Omega) \tag{8.8}$$

given by

$$\mathrm{NT}_k(f) := \nu \vee u, \tag{8.9}$$

where u is the solution of the boundary problem (8.1) with boundary datum f. Then, for k as in the statement of Theorem 8.1, the following hold.

(i) NT_k *is an isomorphism of $\mathcal{X}^p_{\rm nor}(\partial\Omega)$ onto $\mathcal{X}^p_{\rm tan}(\partial\Omega)$;*
(ii) NT_k *maps $\mathcal{Y}^p_{\rm nor}(\partial\Omega, \mathcal{E})$ isomorphically onto $\mathcal{Y}^p_{\rm tan}(\partial\Omega)$;*
(iii) NT_k *maps $\mathcal{Z}^p_{\rm nor}(\partial\Omega, \mathcal{E})$ isomorphically onto $\mathcal{Z}^p_{\rm tan}(\partial\Omega)$.*

Furthermore, the operators

$$\nu \wedge C_k : \mathcal{X}^p_{\rm tan}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\rm nor}(\partial\Omega) \tag{8.10}$$

$$\nu \wedge C_k : \mathcal{Y}^p_{\rm tan}(\partial\Omega) \longrightarrow \mathcal{Y}^p_{\rm nor}(\partial\Omega) \tag{8.11}$$

$$\nu \wedge C_k : \mathcal{Z}^p_{\rm tan}(\partial\Omega) \longrightarrow \mathcal{Z}^p_{\rm nor}(\partial\Omega) \tag{8.12}$$

are isomorphisms. Finally, similar results are valid for the operator $\nu \vee C_k$.

Proof. The first part is an immediate consequence of Theorem 8.1. Also, the second part follows easily from Theorems 7.1–7.2, with the aid of the identity $\nu \wedge C_k = -[\frac{1}{2}I + \nu \wedge C_k] \circ \mathrm{NT}_k^{-1}$ (cf. also [21]). □

Next we discuss the general L^p-Poisson boundary problem for the Maxwell-Dirac operator $\mathbb{D}_k$ with half-Dirichlet boundary conditions.

Theorem 8.3. *For any Ω Lipschitz domain there exists $\epsilon = \epsilon(\Omega) > 0$ and a discrete subset $\mathcal{U} \subseteq \mathbb{R}$ with the following significance. For any $2-\epsilon < p <$*

$2+\epsilon$ and $k \in \mathbf{C}\setminus\mathcal{U}$, the L^p-Poisson boundary problem for the Maxwell-Dirac operator $\mathbb{D}_k$ with half-Dirichlet boundary conditions:

$$\begin{cases} \mathbb{D}_k u = \eta \in L^p(\Omega, \mathcal{E}), \\ u,\, du,\, \delta u \in L^p(\Omega, \mathcal{E}), \\ \nu \wedge u = f \in \mathcal{X}^p_{\rm nor}(\partial\Omega) \end{cases} \tag{8.13}$$

has a unique solution which also satisfies

$$\|u\|_{L^p(\Omega)} + \|du\|_{L^p(\Omega)} + \|\delta u\|_{L^p(\Omega)} \leq C(\|\eta\|_{L^p(\Omega)} + \|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)}). \tag{8.14}$$

Furthermore, a similar set of conclusions holds true when the dual boundary condition is emphasized, i.e. for

$$\begin{cases} \mathbb{D}_k v = \xi \in L^p(\Omega, \mathcal{E}), \\ v,\, dv,\, \delta v \in L^p(\Omega, \mathcal{E}), \\ \nu \vee v = g \in \mathcal{X}^p_{\rm tan}(\partial\Omega). \end{cases} \tag{8.15}$$

Proof. Assume that k and $\epsilon > 0$ are as in Theorems 7.1–7.2, and introduce the Newtonian potential

$$\Pi_k u(x) := \iint_\Omega \langle \Gamma_k(x,y), u(y)\rangle \, d\mathcal{V}(y), \quad x \in \Omega. \tag{8.16}$$

As in the classical setting of the Euclidean space, for each $1 < p < \infty$,

$$\Pi_k : L^p(\Omega, \mathcal{E}) \longrightarrow H^{2,p}(\Omega, \mathcal{E}) \tag{8.17}$$

is well-defined and bounded.

We look for a solution u of (8.13) expressed in the form

$$u := \mathbb{D}_k \Pi_k \eta + \mathcal{C}_k g, \tag{8.18}$$

where $g \in \mathcal{X}^p_{\rm nor}(\partial\Omega)$ is yet to be determined. Note that, by Lemma 5.2, u satisfies

$$\|u\|_{L^p(\Omega)} + \|du\|_{L^p(\Omega)} + \|\delta u\|_{L^p(\Omega)} \leq C(\|\eta\|_{L^p(\Omega)} + \|g\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)}), \tag{8.19}$$

as well as all the conditions in (8.13) except for the requirement that $\nu \wedge u = f$. However, choosing

$$g := \left(\tfrac{1}{2}I + \nu \wedge C_k\right)^{-1} \Big(f - \nu \wedge \mathrm{Tr}\,(\mathbb{D}_k \Pi_k \eta)\Big) \in \mathcal{X}^p_{\rm nor}(\partial\Omega) \tag{8.20}$$

takes care of this as well. Moreover, for this choice,

$$\|g\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)} \leq C(\|\eta\|_{L^p(\Omega)} + \|f\|_{\mathcal{X}^p_{\rm nor}(\partial\Omega)}), \tag{8.21}$$

so that (8.21) and (8.19) yield (8.14). Uniqueness follows from the corresponding uniqueness part in Theorem 8.1. This concludes the proof of the first part of the theorem. Finally, the argument for the dual problem is similar and we omit it. $\square$

We conclude this section by attempting to rephrase the L^p-Poisson problem (with half-Dirichlet boundary conditions) for the Maxwell-Dirac operator $\mathbb{D}_k$ in terms of differential forms on $\mathcal{M}$. Thus, by 'eliminating' dt in (8.13), the latter becomes

$$\begin{cases} E,\, dE,\, \delta E \in L^p(\Omega, \mathcal{E}), \\ H,\, dH,\, \delta H \in L^p(\Omega, \mathcal{E}), \\ \delta E + dE - ikH = J \in L^p(\Omega, \mathcal{E}), \\ \delta H + dH + ikE = K \in L^p(\Omega, \mathcal{E}), \\ \nu \wedge E = f \in \mathcal{X}^p_{\mathrm{nor}}(\partial\Omega, \mathcal{E}), \\ \nu \wedge H = g \in \mathcal{X}^p_{\mathrm{tan}}(\partial\Omega, \mathcal{E}). \end{cases} \tag{8.22}$$

This is the L^p-Poisson problem for the elliptic version of the Maxwell system. Theorem 8.3 then immediately translates into the following.

Theorem 8.4. *For any Lipschitz subdomain Ω of $\mathcal{M}$ there exists some $\epsilon = \epsilon(\Omega) > 0$ and a discrete subset $\mathcal{U} \subseteq \mathbb{R}$ (with no finite accumulation points) having the following significance. For any $2 - \epsilon < p < 2 + \epsilon$ and $k \in \mathbf{C} \setminus \mathcal{U}$, the Poisson problem for the generalized Maxwell system* (8.22) *is well-posed.*

The system (8.22) should be compared with the L^p-Poisson problem for the ordinary Maxwell system, i.e.

$$\begin{cases} E, H \in L^p(\Omega, \mathcal{E}), \\ dE - ikH = J \in L^p(\Omega, \mathcal{E}), \\ \delta H + ikE = K \in L^p(\Omega, \mathcal{E}), \\ \nu \wedge E = f \in \mathcal{X}^p_{\mathrm{nor}}(\partial\Omega, \mathcal{E}). \end{cases} \tag{8.23}$$

For a treatment of (8.23) see [18]. Notice that (8.23) splits into a direct sum of boundary problems according to the degrees of the differential forms involved.

Theorem 8.5. *Retain the assumptions and notation in the previous theorem. Then, when $J = K = 0$,* (8.22) *and* (8.23) *are equivalent if and only if*

$$g = -ik^{-1} d_\partial f. \tag{8.24}$$

Proof. This is seen as in [21], granted the results of this section. We leave the details to the interested reader. □

9. Spectral Radius Estimates

In this section we study finer spectral properties of the operators $\nu \wedge C_k$, $\nu \vee C_k$, M_k and N_k. Fix an arbitrary Lipschitz domain Ω and, for k as in

Sections 5–8, consider the operators

$$\begin{aligned} &T_k : \mathcal{X}^p_{\tan}(\partial\Omega,\mathcal{E}) \longrightarrow \mathcal{X}^p_{\rm nor}(\partial\Omega,\mathcal{E}), \\ &T_k f := \nu \wedge \big(\delta d \mathcal{S}_k f|_\Omega\big), \quad f \in \mathcal{X}^p_{\tan}(\partial\Omega,\mathcal{E}), \end{aligned} \tag{9.1}$$

and

$$\begin{aligned} &R_k : \mathcal{X}^p_{\rm nor}(\partial\Omega,\mathcal{E}) \longrightarrow \mathcal{X}^p_{\tan}(\partial\Omega,\mathcal{E}), \\ &R_k g := \nu \vee \big(d\delta \mathcal{S}_k g|_\Omega\big), \quad g \in \mathcal{X}^p_{\rm nor}(\partial\Omega,\mathcal{E}). \end{aligned} \tag{9.2}$$

Their main properties are summarized below.

Proposition 9.1. *Let Ω be a Lipschitz subdomain of $\mathcal{M}$ and assume that the complex parameter k is such that all results in Sections 5–8 hold. Then the following are true.*

(i) *The operators T_k, R_k are well-defined and bounded for each $1 < p < \infty$ and $k \in \mathbf{C}$.*

(ii) *The following identities hold:*

$$k^2\left(-\tfrac{1}{2}I + M_k\right)\left(\tfrac{1}{2}I + M_k\right) = R_k \circ T_k, \tag{9.3}$$

$$k^2\left(-\tfrac{1}{2}I + N_k\right)\left(\tfrac{1}{2}I + N_k\right) = T_k \circ R_k, \tag{9.4}$$

and

$$T_k \circ M_k = N_k \circ T_k, \qquad R_k \circ N_k = M_k \circ R_k. \tag{9.5}$$

(iii) *There exists $\epsilon > 0$ so that for each $2 - \epsilon < p < 2 + \epsilon$ the operators* (9.1), (9.2)*, are Fredholm with index zero for any $k \in \mathbf{C}$. In fact, except for k in a discrete subset of $\mathbb{R}$, these operators are in fact isomorphisms* (*for p near* 2)*.*

(iv) *There holds*

$$\mathrm{NT}_k = T_k \circ \left(-\tfrac{1}{2}I + M_k\right)^{-1}. \tag{9.6}$$

(v) *For any $f \in \mathcal{X}^p_{\tan}(\partial\Omega,\mathcal{E})$ and $g \in \mathcal{X}^q_{\tan}(\partial\Omega,\mathcal{E})$ with $1/p + 1/q = 1$, we have*

$$\langle T_k f, \nu \wedge g\rangle = \langle \nu \wedge f, T_k g\rangle, \tag{9.7}$$

where the pairings are understood in the sense of (vii) *in Proposition* 4.1*. A similar identity holds for the operator R_k.*

Proof. For starters, note that

$$T_k f = -\nu \wedge \big(d\mathcal{S}_k(\delta_\partial f)|_\Omega\big) + k^2\,\nu \wedge S_k f = d_\partial(\nu \wedge S_k(\delta_\partial f)) + k^2\,\nu \wedge S_k f. \tag{9.8}$$

This and (5.21) then justify (i).

Next, we observe that if $(\Delta + k^2)u = 0$ in Ω, then the Green's integral representation formula

$$u = -d\mathcal{S}_k(\nu \vee u) + \delta\mathcal{S}_k(\nu \wedge u) + \mathcal{S}_k(\nu \wedge \delta u) - \mathcal{S}_k(\nu \vee du) \tag{9.9}$$

holds whenever $u, du, \delta u \in \mathrm{Dom}(d; L^p(\Omega,\mathcal{E})) \cap \mathrm{Dom}(\delta; L^p(\Omega,\mathcal{E}))$; cf. [11] for a similar identity in a slightly different context.

Fix now $f \in \mathcal{X}^p_{\tan}(\partial\Omega, \mathcal{E})$ and write (9.9) for $u := d\mathcal{S}_k f$. Then the first identity in (9.5) follows after some straightforward algebra. If, on the other hand, we utilize $u := \delta\mathcal{S}_k g$, for $g \in \mathcal{X}^p_{\rm nor}(\partial\Omega, \mathcal{E})$ then we arrive at the second identity in (9.5).

To see (9.3), we use (9.9) with $u := d\mathcal{S}_k f$, $f \in \mathcal{X}^p_{\tan}(\partial\Omega, \mathcal{E})$ and then apply $\nu \vee d\delta$ to both sides. Similarly, (9.4) follows by writing (9.9) for $u := \delta\mathcal{S}_k g$, $g \in \mathcal{X}^p_{\rm nor}(\partial\Omega, \mathcal{E})$ and applying $\nu \wedge \delta d$ to both sides.

Next, we note that if $E := d\mathcal{S}_k f$ and $H := \delta E$ in $\Omega_\pm$, then (E, H) solve Maxwell's equations (1.3). The fact that $T_k f = \nu \wedge (H|_\Omega)$ in concert with $\mathrm{NT}_k\big(\nu \vee (E|_\Omega)\big) = \nu \wedge (H|_\Omega)$ translates precisely into (iv).

Consider now (v). By a standard density argument (cf. (iii) in Proposition 4.1) we may take $f \in \nu \wedge C^1(\mathcal{M}, \mathcal{E})|_{\partial\Omega}$ and $g \in \nu \vee C^1(\mathcal{M}, \mathcal{E})|_{\partial\Omega}$. Assuming that this is the case we write

$$\begin{aligned}\langle \nu \wedge dS_k(\delta_\partial f), \nu \wedge g\rangle &= \langle dS_k(\delta_\partial f), g\rangle = \langle S_k(\delta_\partial f), \delta_\partial g\rangle = \\ &= \langle \delta_\partial f, S_k(\delta_\partial g)\rangle = \langle f, dS_k(\delta_\partial g)\rangle = \\ &= \langle \nu \wedge f, \nu \wedge dS_k(\delta_\partial g)\rangle. \end{aligned} \tag{9.10}$$

In the light of (9.1), the identity (9.7) follows. □

Theorem 9.2. *Let Ω be a Lipschitz domain in $\mathcal{M}$ and let k be as in Theorem 7.1. Then there exists $\epsilon = \epsilon(\Omega) > 0$ so that for each $2-\epsilon < p < 2+\epsilon$ the operators*

$$\lambda I + \nu \wedge C_k : \mathcal{X}^p_{\tan}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\tan}(\partial\Omega) \tag{9.11}$$

$$\lambda I + \nu \vee C_k : \mathcal{X}^p_{\rm nor}(\partial\Omega) \longrightarrow \mathcal{X}^p_{\rm nor}(\partial\Omega) \tag{9.12}$$

are Fredholm with index zero for each $\lambda \in \mathbf{C} \setminus (-\frac{1}{2}, \frac{1}{2})$. In particular,

$$\textit{the essential spectral radius of } \nu \vee C_k \textit{ on } \mathcal{X}^p_{\tan}(\partial\Omega) \textit{ is } < \tfrac{1}{2}, \tag{9.13}$$

and

$$\textit{the essential spectral radius of } \nu \wedge C_k \textit{ on } \mathcal{X}^p_{\rm nor}(\partial\Omega) \textit{ is } < \tfrac{1}{2}. \tag{9.14}$$

Proof. Since $\mathcal{X}^p_{\tan}(\partial\Omega)$ is a complex interpolation scale, it suffices to consider the case $p = 2$ only. The extension to $p \in (2-\epsilon, 2+\epsilon)$ then follows from the stability of the property of being Fredholm; cf. the discussion in [13]. Also, there is no loss of generality in assuming that $k \in i\mathbb{R} \setminus 0$, which we shall do for the duration of this proof. In particular, $k^c = -k$.

For an arbitrary, fixed $f \in \mathcal{X}^2_{\tan}(\partial\Omega)$ consider $E := d\mathcal{S}_k f$ and $H := \delta E$ in $\Omega_\pm$. Thus, $dE = \delta H = 0$, $(\Delta + k^2)E = (\Delta + k^2)H = 0$ and $dH = k^2 E$ in $\Omega_\pm$. Furthermore, $T_k f = \nu \wedge (H|_{\Omega_+}) = \nu \wedge (H|_{\Omega_-})$ and $f = \nu \vee (E|_{\Omega_-}) - \nu \vee (E|_{\Omega_+})$.

Consequently,

$$\begin{aligned}\langle f, \nu \vee T_k f^c\rangle &= \langle \nu \vee (E|_{\Omega_-}), [\nu \vee (\nu \wedge (H|_{\Omega_-}))]^c\rangle - \\ &\quad -\langle \nu \vee (E|_{\Omega_+}), [\nu \vee (\nu \wedge (H|_{\Omega_+}))]^c\rangle = \\ &= -\iint\limits_{\Omega_-} \Big\{\langle E, [dH]^c\rangle - \langle \delta E, H^c\rangle\Big\} d\mathcal{V} - \\ &\quad - \iint\limits_{\Omega_+} \Big\{\langle E, [dH]^c\rangle - \langle \delta E, H^c\rangle\Big\} d\mathcal{V} = \\ &= \iint\limits_{\mathcal{M}} \Big\{(-k^2)^c|E|^2 + |H|^2\Big\} d\mathcal{V}, \qquad (9.15)\end{aligned}$$

since the outward unit conormal to Ω_- is $-\nu$. Recalling that we are currently assuming that $k \in i\mathbb{R} \setminus 0$, we may therefore write

$$\begin{aligned}\|f\|^2_{\mathcal{X}^2_{\mathrm{nor}}(\partial\Omega)} &\leq \|\nu \vee (E|_{\Omega_-})\|^2_{\mathcal{X}^2_{\mathrm{nor}}(\partial\Omega)} + \|\nu \vee (E|_{\Omega_+})\|^2_{\mathcal{X}^2_{\mathrm{nor}}(\partial\Omega)} \leq \\ &\leq C\iint\limits_{\Omega_-} \Big\{|E|^2 + |\delta E|^2\Big\} d\mathcal{V} + C\iint\limits_{\Omega_+} \Big\{|E|^2 + |\delta E|^2\Big\} d\mathcal{V} \leq \\ &\leq C\iint\limits_{\mathcal{M}} \Big\{(-k^2)^c|E|^2 + |H|^2\Big\} d\mathcal{V} = \\ &= \langle f, \nu \vee T_k f^c\rangle. \qquad (9.16)\end{aligned}$$

Next, we make the claim that

$$\langle M_k f, \nu \vee T_k f^c\rangle \in \mathbb{R}, \qquad \forall\, f \in \mathcal{X}^2_{\tan}(\partial\Omega). \qquad (9.17)$$

To see this, we use the commutativity of the diagram (5.31) along with the first intertwining identity in (9.5) to write

$$\begin{aligned}\langle M_k f, \nu \vee T_k f^c\rangle &= \langle f, \nu \vee N_k T_k f^c\rangle = \langle f, \nu \vee T_k M_k f^c\rangle = \\ &= \langle \nu \vee T_k f, M_k f^c\rangle = [\langle M_k f, \nu \vee T_k f^c\rangle]^c; \qquad (9.18)\end{aligned}$$

(recall that $k \in i\mathbb{R}$). This justifies (9.17).

At this stage, we compute

$$\begin{aligned}\Big|\mathrm{Im}\,\langle(\lambda I + M_k f), \nu \vee T_k f^c\rangle\Big| &= \Big|\mathrm{Im}\,\lambda\Big|\langle f, \nu \vee T_k f^c\rangle \geq \\ &\geq C\Big|\mathrm{Im}\,\lambda\Big|\|f\|^2_{\mathcal{X}^2_{\tan}(\partial\Omega)}, \qquad (9.19)\end{aligned}$$

by (9.16). From this, it readily follows that

$$\lambda \in \mathbf{C} \setminus \mathbb{R},\ k \in i\mathbb{R} \setminus 0 \Longrightarrow \|(\lambda I + M_k) f\|_{\mathcal{X}^2_{\tan}(\partial\Omega)} \geq C_\lambda \|f\|_{\mathcal{X}^2_{\tan}(\partial\Omega)}. \qquad (9.20)$$

In particular,

$$\lambda \in \mathbf{C} \setminus \mathbb{R},\ k \in i\mathbb{R} \setminus 0 \Longrightarrow \lambda I + M_k \tag{9.21}$$
$$\text{Fredholm with index zero on } \mathcal{X}^2_{\tan}(\partial\Omega).$$

Next, if $\lambda \in (\frac{1}{2}, \infty)$, we write

$$\langle(\lambda I + M_k f), \nu \vee T_k f^c\rangle = \langle(\tfrac{1}{2} I + M_k f), \nu \vee T_k f^c\rangle + \\ + \Big(\lambda - \tfrac{1}{2}\Big)\langle f, \nu \vee T_k f^c\rangle =: I + II. \tag{9.22}$$

Now,

$$\begin{aligned} I &= \langle \nu \vee (E|_{\Omega_-}), \nu \vee (\nu \wedge (H_{\Omega_-}))^c\rangle = \\ &= -\iint\limits_{\Omega_-} \Big\{\langle E, [dH]^c\rangle - \langle \delta E, [H]^c\rangle\Big\} d\mathcal{V} = \\ &= \iint\limits_{\Omega_-} \Big\{(-k^2)^c |E|^2 + |H|^2\Big\} d\mathcal{V} \geq 0, \end{aligned} \tag{9.23}$$

whereas $II \geq C_\lambda \|f\|^2_{\mathcal{X}^2_{\tan}(\partial\Omega)}$, by (9.16). At this point we may therefore conclude that $\lambda I + M_k$ is bounded from below on $\mathcal{X}^2_{\tan}(\partial\Omega)$ whenever $\lambda \in (\frac{1}{2}, \infty)$ and $k \in i\mathbb{R}$. Thus, in this setting, $\lambda I + M_k$ is Fredholm with index zero.

When $\lambda \in (-\infty, -\frac{1}{2})$ and $k \in i\mathbb{R}$ we write

$$\langle(\lambda I + M_k f), \nu \vee T_k f^c\rangle = \langle(-\tfrac{1}{2} I + M_k f), \nu \vee T_k f^c\rangle + \\ + \Big(\lambda + \tfrac{1}{2}\Big)\langle f, \nu \vee T_k f^c\rangle =: III + IV. \tag{9.24}$$

This time,

$$\begin{aligned} III &= \langle \nu \vee (E|_{\Omega_+}), \nu \vee (\nu \wedge (H|_{\Omega_+}))^c\rangle = \\ &= \iint\limits_{\Omega_+} \Big\{\langle E, [dH]^c\rangle - \langle \delta E, [H]^c\rangle\Big\} d\mathcal{V} = \\ &= -\iint\limits_{\Omega_+} \Big\{(-k^2)^c |E|^2 + |H|^2\Big\} d\mathcal{V} \leq 0, \end{aligned} \tag{9.25}$$

and $IV \leq C_\lambda \|f\|^2_{\mathcal{X}^2_{\tan}(\partial\Omega)}$ for some positive constant C_λ. Nonetheless, in this setting, it follows once again that $\lambda I + M_k$ is bounded from below on $\mathcal{X}^2_{\tan}(\partial\Omega)$. Accordingly, $\lambda I + M_k$ is Fredholm with index zero for $\lambda \in (-\infty, -\frac{1}{2})$ and $k \in i\mathbb{R}$, as well.

Summarizing, at this point we have proved that $\lambda I + M_k$ is Fredholm with index zero on $\mathcal{X}^2_{\tan}(\partial\Omega)$ for each $\lambda \in \mathbf{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$ and each $k \in i\mathbb{R} \setminus 0$. In the light of Proposition 5.4 and Proposition 5.5, this further implies that

$\lambda I + \nu \wedge C_k$ is Fredholm with index zero on $\mathcal{X}^2_{\tan}(\partial\Omega)$ for each $\lambda \in \mathbf{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$ and each k as in Theorem 7.1.

In concert with Theorem 7.1 (where the endpoints $\lambda = \pm\frac{1}{2}$ have been treated) this ultimately proves the claim made in the statement of Theorem 9.2 about the operator (9.11). The case of the operator (9.12) follows from what we have proved up to this point and duality (or by proceeding in a similar manner). □

We now state a simple lemma to the effect that the spectral radius of a linear, bounded operator acting on a complex interpolation scale is logarithmically convex.

Lemma 9.3. *Assume that $T : X_i \to X_i$, $i = 0, 1$, is a bounded, linear operator between two pairs of Banach spaces. Denote by $r(T; X)$ the spectral radius of T on X. Then*

$$r(T; [X_0, X_1]_\theta) \leq r(T; X_0)^{1-\theta} r(T; X_1)^\theta \tag{9.26}$$

for each $0 \leq \theta \leq 1$.

Proof. Let us Denote by $\|T\|_{\mathcal{L}(X)}$ the operator norm of T on X. Then, for each positive integer n, the interpolation inequality

$$\|T\|_{\mathcal{L}([X_0,X_1]_\theta)} \leq \|T\|^{1-\theta}_{\mathcal{L}(X_0)} \|T\|^\theta_{\mathcal{L}(X_1)} \tag{9.27}$$

holds for each fixed $\theta \in [0, 1]$. Taking the n-th root of both sides and letting $n \to \infty$ yields (9.26). □

Theorem 9.4. *If $k \in \mathbf{C}$ is such that $|\mathrm{Im}\, k| > |\mathrm{Re}\, k|$ then the spectral radius of M_k on $\mathcal{X}^p_{\tan}(\partial\Omega)$ is $< \frac{1}{2}$ for each $2 - \epsilon < p < 2 + \epsilon$. A similar result holds for N_k on $\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega)$.*

Proof. We know, from Lemma 9.3, that the property that $r(T; X) < \frac{1}{2}$ is amenable to interpolation; hence, by (iv) in Proposition 4.1 it suffices to treat the case $p = 2$ only.

In this later scenario, the estimate

$$\|f\|_{\mathcal{X}^2_{\tan}(\partial\Omega)} \leq C_\lambda \|(\lambda I + M_k f)\|_{\mathcal{X}^2_{\tan}(\partial\Omega)}, \qquad \lambda \in \mathbf{C},\ |\lambda| > \tfrac{1}{2}, \tag{9.28}$$

has been established in the proof of Theorem 9.2 when $k \in i\mathbb{R}$. In fact, much as in Lemma 7.3, p. 949-950 of [11], the same conclusion holds when $k \in \mathbf{C}$ is such that $|\mathrm{Im}\, k| > |\mathrm{Re}\, k|$.

This easily leads to the desired conclusion for the operator M_k. The case of N_k is similar (alternatively, one can use duality; cf. Proposition 5.3). □

For each parameter $\mu \in \mathbf{C}$, consider the Dirac transmission boundary value problem

$$\begin{cases} u,\, du,\, \delta u \in L^p(\Omega_+, \mathcal{E}), \\ w,\, dw,\, \delta w \in L^p(\Omega_-, \mathcal{E}), \\ \mathbb{D}_k u = \eta \in L^p(\Omega_+, \mathcal{E}), \\ \mathbb{D}_k w = \zeta \in L^p(\Omega_-, \mathcal{E}), \\ \nu \wedge u - \mu\, \nu \wedge w = f \in \mathcal{X}^p_{\rm nor}(\partial\Omega), \\ \nu \vee u - \nu \vee w = g \in \mathcal{X}^p_{\rm tan}(\partial\Omega). \end{cases} \tag{9.29}$$

The main goal is to discuss the well-posedness of (9.29). Eventually, we shall prove the following.

Theorem 9.5. *For each Lipschitz domain $\Omega \subset \mathcal{M}$ there exists $\epsilon > 0$ and a discrete set of numbers $\mathcal{U} \subset \mathbb{R}$ without finite accumulation points so that the following holds.*

For each transmission parameter $\mu \in \mathbf{C} \setminus (-\infty, 0)$, $\mu \neq 1$, the problem (9.29) *is Fredholm solvable, with index zero, whenever $2 - \epsilon < p < 2 + \epsilon$ and $k \notin \mathcal{U}$.*

If, in fact, $|\mathrm{Im}\, k| > |\mathrm{Re}\, k|$ and $\mu > 0$, $\mu \neq 1$, then the problem (9.29) *is actually is uniquely solvable, with natural estimates.*

For the time being we study the uniqueness issue.

Lemma 9.6. *Suppose that $|p - 2|$ is sufficiently small and that $\mu > 0$, $\mu \neq 1$. Also, assume that k satisfies $|\mathrm{Im}\, k| > |\mathrm{Re}\, k|$. Then the boundary problem* (9.29) *has at most one solution.*

Proof. By linearity we may assume that $f = 0$, $g = 0$, and $\eta = 0$, $\zeta = 0$; our goal is to prove that $u = 0$ and $w = 0$. From Theorem 8.1, we know that any null-solutions u, w of $\mathbb{D}_k$ which also satisfy the conditions in the first two lines of (9.29) admit the integral representation formula $u = \mathcal{C}_k(h_1)$, $w = \mathcal{C}_k(h_2)$, for some $h_1, h_2 \in \mathcal{X}^p_{\rm nor}(\partial\Omega)$. The fact that $\nu \vee u = \nu \vee w$ then forces $\nu \vee C_k(h_1 - h_2) = 0$ and, ultimately, $h_1 = h_2 =: h$, provided k is as in Sections 7–8.

Our immediate priority is to show that h is more regular than arbitrary sections in $\mathcal{X}^p_{\rm nor}(\partial\Omega)$. Indeed, the Cauchy integral representation formulas for u and w give that $0 = \nu \wedge u - \mu\, \nu \wedge w = (\mu - 1)(\lambda I + \nu \wedge C_k)h$, where $\lambda := (\mu + 1)/[2(\mu - 1)] \in \mathbb{R} \setminus (-\frac{1}{2}, \frac{1}{2})$. Consequently, Proposition 5.4 and Theorem 9.4 eventually give $h \in L^{2,d}_{\rm nor}(\partial\Omega, \mathcal{E})$ (assuming that p was sufficiently close to 2, to begin with). Thus,

$$\mathcal{N}(u),\, \mathcal{N}(du),\, \mathcal{N}(\delta u) \in L^2(\partial\Omega) \text{ and } \mathcal{N}(u),\, \mathcal{N}(du),\, \mathcal{N}(\delta u) \in L^2(\partial\Omega). \tag{9.30}$$

Having established the regularity statement (9.30) allows us to justifies the integration by parts identity

$$\iint_\Omega \left\{ |du|^2 + |\delta u|^2 \right\} d\mathcal{V} = \iint_\Omega \langle u, -\Delta u^c \rangle \, d\mathcal{V} +$$

$$+\int_{\partial\Omega}\langle\nu\wedge u, du^c\rangle\,d\sigma-\int_{\partial\Omega}\langle\nu\vee u,\delta u^c\rangle\,d\sigma. \tag{9.31}$$

Going further, we replace $-\Delta u^c$ by $(k^2)^c u^c$ and du^c, δu^c by $-\delta u^c - k^c dt\cdot u^c$ and $-du^c - k^c dt\cdot u^c$, respectively. A useful identity at this stage is $\mathrm{Re}\langle\nu\wedge u, dt\cdot u^c\rangle = \mathrm{Re}\langle\nu\vee u, dt\cdot u^c\rangle$, which can be seen with the help of (2.6). Thus, after taking the real parts of both sides and cancelling the terms involving dt, the identity (9.31) becomes

$$\iint_{\Omega}\big\{|du|^2+|\delta u|^2-\mathrm{Re}(k^2)\,|u|^2\big\}\,d\mathcal{V}=$$

$$=\mathrm{Re}\int_{\partial\Omega}\langle\nu\vee u, du^c\rangle\,d\sigma-\mathrm{Re}\int_{\partial\Omega}\langle\nu\wedge u,\delta u^c\rangle\,d\sigma=$$

$$=-\mathrm{Re}\int_{\partial\Omega}\langle\nu\vee u,\nu\vee(d_{\partial}(\nu\wedge u))^c\rangle\,d\sigma+$$

$$+\mathrm{Re}\int_{\partial\Omega}\langle\nu\wedge u,\nu\wedge(\delta_{\partial}(\nu\vee u))^c\rangle\,d\sigma. \tag{9.32}$$

Utilizing the transmission boundary conditions in the last two integrals and then retracing essentially the integration by parts formulas (with w, Ω_-, $-\nu$ in place of u, Ω_+, ν), we arrive at

$$\iint_{\Omega_+}\big\{|du|^2+|\delta u|^2-\mathrm{Re}(k^2)\,|u|^2\big\}\,d\mathcal{V}=$$

$$=-\mu\iint_{\Omega_-}\big\{|dw|^2+|\delta w|^2-\mathrm{Re}(k^2)\,|w|^2\big\}\,d\mathcal{V}. \tag{9.33}$$

Since, by assumption, $|\mathrm{Im}\,k|>|\mathrm{Re}\,k|$ and $\mu>0$, this last identity ultimately forces $u=w=0$, as desired. □

We now temporarily digress in order to comment on the invertibility of the operators (9.11)–(9.12). At the level of L^2 spaces, such a result has been proved in [21].

Proposition 9.7. *Let Ω be a Lipschitz domain in $\mathcal{M}$ and assume that k satisfies $|\mathrm{Im}\,k|>|\mathrm{Re}\,k|$. Then there exists $\epsilon=\epsilon(\Omega)>0$ so that for each $2-\epsilon<p<2+\epsilon$ the operators*

$$\lambda I+\nu\wedge C_k:\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega)\longrightarrow\mathcal{X}^p_{\mathrm{nor}}(\partial\Omega), \tag{9.34}$$

$$\lambda I+\nu\vee C_k:\mathcal{X}^p_{\mathrm{tan}}(\partial\Omega)\longrightarrow\mathcal{X}^p_{\mathrm{tan}}(\partial\Omega) \tag{9.35}$$

are invertible for each $\lambda\in\mathbf{R}\setminus(-\frac12,\frac12)$.

Proof. From Theorem 9.2 we know that the operators in question are Fredholm with index zero. Therefore, there remains to establish that they are also one-to-one. Now, an inspection of the proof of Lemma 9.6 reveals that uniqueness for (9.29) (for all μ's positive, $\neq 1$) is *equivalent* with the fact that (9.11)–(9.12) are one-to-one (for all real λ's outside $(-\frac{1}{2},\frac{1}{2})$). Thus, by Lemma 9.6, the operators under discussion are also one-to-one (in the current setting); the desired conclusion follows.

Parenthetically, we would like to point out that instead of Lemma 9.6, we could have used (7.76) in [21]. □

We are now ready to present the final details in the

Proof of Theorem 9.5. By the results in Section 8, there exists a discrete set $\mathcal{U}\subset\mathbb{R}$ such that, whenever $k\in\mathbf{C}\setminus\mathcal{U}$, the following is true: any u, w satisfying the conditions in the first four lines of (9.29) can be uniquely represented as

$$u=\mathbb{D}_k\Pi_k\eta+\mathcal{C}_kh_1,\quad w=\mathbb{D}_k\Pi_k\zeta+\mathcal{C}_kh_2,\qquad\text{for some } h_1,h_2\in\mathcal{X}^p_{\rm nor}(\partial\Omega).\tag{9.36}$$

Corresponding to these integral representation formulas, the boundary conditions in (9.29) come down to

$$\nu\vee C_k(h_1-h_2)=g-\nu\vee\mathbb{D}_k\Pi_k\eta+\nu\vee\mathbb{D}_k\Pi_k\zeta,\tag{9.37}$$

$$\begin{aligned}(\tfrac{1}{2}I+\nu\wedge C_k)h_1-\mu(-\tfrac{1}{2}I+\nu\wedge C_k)h_2=\\=f-\nu\wedge\mathbb{D}_k\Pi_k\eta+\mu\,\nu\wedge\mathbb{D}_k\Pi_k\zeta.\end{aligned}\tag{9.38}$$

Solving for h_2 in (9.37) and substituting its expression in (9.38) yields an equation of the form $(\lambda I+\nu\wedge C_k)h_1=F$, where $\lambda:=(\mu+1)/[2(\mu-1)]\in\mathbf{C}\setminus(-\frac{1}{2},\frac{1}{2})$ and $F\in\mathcal{X}^p_{\rm nor}(\partial\Omega)$ has an explicit (linear) formula in terms of f,g,η,ζ. In particular, the index of the problem (9.29) equals $\mathrm{Index}\,(\lambda I+\nu\wedge C_k)=0$.

Given Lemma 9.6, the proof of the theorem is therefore finished. □

10. Further Spectral Analysis of the Operator C_k

In this section we shall work in the Euclidean context, i.e. $\mathcal{M}\equiv\mathbb{R}^m$. Let $\{e_j\}_j$ be the canonical orthonormal basis in $\mathbb{R}^m$ and set $e_{m+1}:=dt$. The starting point is the following Rellich type identity from [16]. To state it, we define $(u)_0$ as the scalar part of the $\oplus_\ell\Lambda^\ell$-valued function u. Also, set $u_\pm:=\{u\pm\bar{u}\}/2$.

Theorem 10.1. *Let $\Omega\subset\mathbb{R}^m$ be a Lipschitz domain and assume that $k\in i\mathbb{R}$ and $\mathbb{D}_ku=u\mathbb{D}_k=0$ in Ω. Then the following identities hold:*

$$\begin{aligned}\int_{\partial\Omega}|u|^2\nu_m\,d\sigma&=\pm2\,\mathrm{Re}\Big(\int_{\partial\Omega}e_m\,u\,(\nu u)^c_\pm\,d\sigma\Big)_0=\pm2\,\mathrm{Re}\Big(\int_{\partial\Omega}(u\nu)^c_\pm u\,e_m\,d\sigma\Big)_0=\\&=\pm2\,\mathrm{Re}\Big(\int_{\partial\Omega}(e_m\,u)^c_\pm\,\bar{\nu}\,\bar{u}\,d\sigma\Big)_0=\pm2\,\mathrm{Re}\Big(\int_{\partial\Omega}\bar{u}\,\bar{\nu}\,(u\,e_m)^c_\pm\,d\sigma\Big)_0.\end{aligned}$$

We now discuss a more elaborated version of Theorem 10.1. Define the tangent vector field t to $\partial\Omega$ by setting at almost every point on $\partial\Omega$

$$t := e_m - \langle e_m, \nu\rangle n = e_m - \nu_m \nu.$$

Theorem 10.2. *Assume the same hypotheses as in the previous theorem and, besides, that the function u is $\Lambda^{\ell+1}$-valued. Then*

$$\begin{aligned}\int_{\partial\Omega}(-\nu_m)|\nu\wedge u|^2\,d\sigma-\int_{\partial\Omega}(-\nu_m)|\nu\vee u|^2\,d\sigma&=2\mathrm{Re}\int_{\partial\Omega}\langle t\vee u,\nu\vee u^c\rangle\,d\sigma=\\&=-2\mathrm{Re}\int_{\partial\Omega}\langle t\wedge u,\nu\wedge u^c\rangle\,d\sigma.\end{aligned}$$

Proof. Let ϵ stand for the sign of $(-1)^{\frac{\ell(\ell+1)}{2}}$. From the easily checked identities

$$(\nu u)_+ = \begin{cases}\nu\wedge u, \text{ if } \frac{\ell(\ell+1)}{2} \text{ is odd,}\\[2ex] -\nu\vee u, \text{ if } \frac{\ell(\ell+1)}{2} \text{ is even,}\end{cases} \tag{10.1}$$

$$(\nu u)_- = \begin{cases}\nu\wedge u, \text{ if } \frac{\ell(\ell+1)}{2} \text{ is even,}\\[2ex] -\nu\vee u, \text{ if } \frac{\ell(\ell+1)}{2} \text{ is odd.}\end{cases} \tag{10.2}$$

we know that $(\nu u)_\epsilon = -\nu\vee u$. Consequently,

$$(\nu u\,(\nu u)^c_\epsilon)_0 = (-1)^{\frac{\ell(\ell+1)}{2}}\,|\nu\vee u|^2.$$

Recall from Theorem 10.1 that

$$\int_{\partial\Omega}|u|^2\,\nu_m\,d\sigma = 2(-1)^{\frac{\ell(\ell+1)}{2}}\,\mathrm{Re}\left(\int_{\partial\Omega}e_m\,u\,(\nu u)^c_\epsilon\,d\sigma\right)_0.$$

In the left-hand side, we use

$$|u|^2 = |\nu u|^2 = |\nu\vee u|^2 + |\nu\wedge u|^2.$$

As for the right-hand side, we use the fact that $e_m u = tu + \nu_m \nu u$, so that

$$\begin{aligned}(e_m\,u(\nu u)^c_\epsilon)_0 &= ((tu)(\nu u)^c_\epsilon)_0 - (-1)^{\frac{\ell(\ell+1)}{2}}(-\nu_m)|\nu\vee u|^2 =\\&= -(-1)^{\frac{\ell(\ell+1)}{2}}\langle t\vee u,\nu\vee u^c\rangle - (-1)^{\frac{\ell(\ell+1)}{2}}(-\nu_m)|\nu\vee u|^2.\end{aligned}$$

Combining all these observations, we arrive at the desired identity. □

Remark . Set $u_{nor} := \nu\wedge(\nu\vee u)$. Later, we shall need the fact that $t\wedge u$ and $t\wedge u_{nor}$ are congruent modulo tangential forms. Indeed, this is seen from

$$\begin{aligned}t\wedge u - t\wedge u_{nor} &= t\wedge(u-\nu\wedge(\nu\vee u)) = t\wedge(\nu\vee(\nu\wedge u)) =\\&= -\nu\vee(t\wedge(\nu\wedge u)) + \langle t,\nu\rangle(\nu\wedge u) =\\&= -\nu\vee(t\wedge(\nu\wedge u)),\end{aligned}$$

since $\langle t, \nu\rangle = 0$. Now $-\nu \vee (t \wedge (\nu \wedge u))$ is tangential and the conclusion follows.

We shall use the identity in the previous theorem in the following context. For some arbitrary, fixed $B \in L^{2,d}_{nor}(\partial\Omega, \Lambda^{\ell+1})$ we set

$$u := \tfrac{1}{ik} d\delta \mathcal{S}_k B + i(-1)^{\ell+1} \delta \mathcal{S}_k B \, e_{m+1} \tag{10.3}$$

in $\mathbb{R}^m \setminus \partial\Omega$. Set $\Omega_i := \Omega$, $\Omega_e := \mathbb{R}^m \setminus \bar{\Omega}$ and denote by $\nu^{i,e}$ their outward unit normal vectors. Clearly, $\nu^i = n = -\nu^e$. We denote by $u^{i,e}$ the nontangential boundary traces of u on $\partial\Omega_{i,e}$, respectively. One can easily check that u satisfies the hypotheses of the Theorem 10.2 both in Ω_i and in Ω_e, so that

$$\int_{\partial\Omega} (-\nu^{i,e}_m)|\nu \wedge u^{i,e}|^2 \, d\sigma - \int_{\partial\Omega} (-\nu^{i,e}_m)|\nu \vee u^{i,e}|^2 \, d\sigma = \tag{10.4}$$
$$= -2\mathrm{Re} \int_{\partial\Omega} \langle t \wedge u^{i,e}, \nu^{i,e} \wedge (u^{i,e})^c\rangle \, d\sigma.$$

Next, for a fixed, arbitrary real number λ, we multiply the two identities corresponding to writing (10.4) with boundary traces taken from the interior and from the exterior, respectively, by $\lambda \pm \frac{1}{2}$ and then add them up. By an earlier discussion, $\nu \vee u$ does not jump across $\partial\Omega$, i.e. $\nu \vee u^i = \nu \vee u^e$. Also, (by the remark following the proof of the Theorem 10.2) the jump of $t \wedge u$ is in the 'tangential direction', i.e. $t \wedge u^i - t \wedge u^e$ is a tangential form. Therefore, the resulting equality reads

$$\int_{\partial\Omega} (-\nu_m)|\nu \vee u|^2 \, d\sigma + \tag{10.5}$$
$$+ \int_{\partial\Omega} (-\nu_m) \left((\lambda + \tfrac{1}{2})|\nu \wedge u^i|^2 - (\lambda - \tfrac{1}{2})|\nu \wedge u^e|^2 \right) \, d\sigma =$$
$$= -2\mathrm{Re} \int_{\partial\Omega} \langle t \wedge u \, , \, (\lambda + \tfrac{1}{2})\nu \wedge (u^i)^c - (\lambda - \tfrac{1}{2})\nu \wedge (u^e)^c\rangle \, d\sigma.$$

A rather lengthy, yet straightforward, calculation based on the trace formulas

$$\nu \wedge u^{i,e} = -\tfrac{1}{ik} \left(d_\partial (\pm \tfrac{1}{2} I + N_k) B \right) + i(-1)^{\ell+1} (\pm \tfrac{1}{2} I + N_k) B \, e_{m+1}$$

allows us to transform (10.5) into

$$0 = \left(\tfrac{1}{4} - \lambda^2\right) \int_{\partial\Omega} (-\nu_m) \left(|B|^2 + \tfrac{1}{|k|^2} |d_\partial B|^2 \right) \, d\sigma +$$
$$+ \int_{\partial\Omega} (-\nu_m) \left(|(\lambda I + N_k) B|^2 + \tfrac{1}{|k|^2} |d_\partial (\lambda I + N_k) B|^2 \right) d\sigma -$$
$$- \int_{\partial\Omega} (-\nu_m) |\nu \wedge u|^2 d\sigma + \int_{\partial\Omega} 2\mathrm{Re} \, \langle t \wedge u \, , \, \big(i(-1)^{\ell+1} (\lambda I + N_k) B e_{m+1} -$$

$$-\tfrac{1}{ik}d_\partial(\lambda I+N_k)B\big)^c\rangle\,d\sigma. \tag{10.6}$$

This technical result is one of the key ingredients in our analysis of the semi-Fredholm spectrum of N_k, which we now commence. The main estimate on which our analysis of the spectrum of the operator N_k rests is contained in the following theorem.

Theorem 10.3. *Let $k\in i\mathbb{R}_+$. Consider Ω the Lipschitz domain in $\mathbb{R}^m$ above the graph of a Lipschitz function $\varphi:\mathbb{R}^{m-1}\to\mathbb{R}$ with Lipschitz constant ω. Also, let $\mathcal{H}_\omega\subseteq\mathbf{C}$ be the interior of the hyperbola in $\mathbb{R}^2\equiv\mathbf{C}$ having vertices at*

$$\left(\pm\frac{1}{2}\frac{\omega}{\sqrt{1+\omega^2}}\,,\,0\right)$$

and asymptotes with slopes $\pm\frac{1}{\omega}$.

Then, for any complex number $\lambda\in\mathcal{H}_\omega$, there exists a positive constant C, depending on λ and ω but independent of k, such that, for any $A\in L^{2,d}_{nor}(\partial\Omega,\Lambda^\ell)$ we have

$$\begin{aligned}\|A\|_{L^2(\partial\Omega)}+\tfrac{1}{|k|}\|d_\partial A\|_{L^2(\partial\Omega)}\leq&\\ \leq C\Big(\|(\lambda I+N_k)A\|_{L^2(\partial\Omega)}+\tfrac{1}{|k|}\|d_\partial(\lambda I+N_k)A\|_{L^2(\partial\Omega)}\Big).&\end{aligned} \tag{10.7}$$

Based on the estimate (10.7) and elementary functional analysis, we readily obtain the following.

Corollary 10.4. *With the above hypotheses, we have that the operators*

$$\lambda I+N_k:L^{2,d}_{nor}(\partial\Omega,\Lambda^\ell)\longrightarrow L^{2,d}_{nor}(\partial\Omega,\Lambda^\ell), \tag{10.8}$$

$$\lambda I+\nu\wedge C_k:L^{2,d}_{nor}(\partial\Omega,\oplus_\ell\Lambda^\ell)\longrightarrow L^{2,d}_{nor}(\partial\Omega,\oplus_\ell\Lambda^\ell) \tag{10.9}$$

are Fredholm with index zero for each $\lambda\in\mathbf{C}\setminus\mathcal{H}_\omega$.

Proof of Theorem 10.3. We shall first prove that

$$\begin{aligned}\big(\lambda^2-\tfrac14\big)\int_{\partial\Omega}(-\nu_m)|A|^2+\tfrac{1}{|k|^2}\big(\lambda^2-\tfrac14\big)\int_{\partial\Omega}(-\nu_m)|d_\partial A|^2\leq&\\ \leq(1+\omega^2)\int_{\partial\Omega}(-\nu_m)|(\lambda I+N_k)A|^2+&\\ +(1+\omega^2)\tfrac{1}{|k|^2}\int_{\partial\Omega}(-\nu_m)|d_\partial(\lambda I+N_k)A|^2&\end{aligned} \tag{10.10}$$

for any real λ with $|\lambda|>\frac12$. Before proceeding with the proof of (10.10) we first indicate how this estimate can be used to finish the proof of (10.7). To this end, we note that simple applications of Hölder's and Minkowski's inequalities show that for any $\mu\in C$ the right-hand side of (10.10) is majorized by

$$(1+\omega^2)\int_{\partial\Omega}(-\nu_m)|(\mu I+N_k)A|^2+(1+\omega^2)\tfrac{1}{|k|^2}\int_{\partial\Omega}(-\nu_m)|d_\partial(\mu I+N_k)A|^2+$$

$$+(1+\omega^2)|\mu-\lambda|^2\int_{\partial\Omega}(-\nu_m)|A|^2+\tfrac{1}{|k|^2}(1+\omega^2)|\mu-\lambda|^2\int_{\partial\Omega}(-\nu_m)|d_\partial A|^2+$$
$$+\mathcal{O}\left(\|A\|_{L^2(\partial\Omega)}\|(\mu I+N_k)A\|_{L^2(\partial\Omega)}\right)+$$
$$+\tfrac{1}{|k|^2}\mathcal{O}\left(\|d_\partial A\|_{L^2(\partial\Omega)}\|d_\partial(\mu I+N_k)A\|_{L^2(\partial\Omega)}\right).$$

Hence, the usual argument yields (10.7) (with μ in place of λ), provided $\mu\in\mathbf{C}$ is so that there exists $\lambda\in\mathbb{R}\setminus(-\frac{1}{2},\frac{1}{2})$ such that

$$\left(\lambda^2-\tfrac{1}{4}\right)--(1+\omega^2)\,|\mu-\lambda|^2>0.$$

Now a straightforward calculation shows that this is equivalent to the membership of μ to $\mathcal{H}_\omega$ and the theorem follows.

We are thus left with showing (10.10). The idea of proof is to further refine the estimates used to establish (10.6). Let us first estimate the last integral in (10.6). Recall the definition of u in (10.3) and the remark made immediately after the proof of Theorem 10.2 to the effect that $t\wedge u$ and $t\wedge(\nu\wedge(\nu\vee u))$ differ only by a tangential form. Also, since the decomposition $e_m=t+\nu_m\nu$ is orthogonal, we have $|t|^2=1-\nu_m^2=\frac{|\nabla\varphi|^2}{1+|\nabla\varphi|^2}$. In particular, $|t|\leq(-\nu_m)\,\omega$. Consequently, using the estimate

$$|t\vee(\nu\vee(\nu\wedge u))|\leq|t\cdot(\nu\vee(\nu\wedge u))|=|t||\nu\vee(\nu\wedge u)|=$$
$$=|t||\nu\wedge u|\leq\omega\,(-\nu_m)|\nu\wedge u|,$$

and Hölder's inequality, we get

$$\left|\int_{\partial\Omega}2\mathrm{Re}\,\langle t\wedge u\,,\,\left(i(-1)^{\ell+1}(\lambda I+N_k)A\,e_{m+1}-\tfrac{1}{ik}d_\partial(\lambda I+N_k)A\right)^c\rangle\,d\sigma\right|\leq$$
$$\leq 2\int_{\partial\Omega}|t\wedge(\nu\wedge(\nu\vee u))|\left|i(-1)^{\ell+1}(\lambda I+N_k)Ae_{m+1}-\tfrac{1}{ik}d_\partial(\lambda I+N_k)A\right|d\sigma\leq$$
$$\leq 2\omega\left(\int_{\partial\Omega}(-\nu_m)|\nu\wedge u|^2\,d\sigma\right)^{\frac{1}{2}}\times$$
$$\times\left(\int_{\partial\Omega}(-\nu_m)\left(|(\lambda I+N_k)A|^2+\tfrac{1}{|k|^2}|d_\partial(\lambda I+N_k)A|^2\right)\,d\sigma\right)^{\frac{1}{2}}.$$

With some self-explanatory notation, the structure of this inequality is

$$|X|\leq 2\omega Y^{1/2}Z^{1/2}$$

(e.g., $X:=\int_{\partial\Omega}2\mathrm{Re}\,\langle\cdot,\cdot\rangle\,d\sigma$, etc). Recall from (10.6), written with A in place of B, that the left-hand side (LHS) in (10.10) satisfies

$$LHS=Z-Y+X.$$

Hence,

$$LHS \leq Z - Y + 2\omega Y^{1/2} Z^{1/2} \leq Z + \omega^2 Z = (1+\omega^2)Z$$

which is precisely (10.10). The proof of the theorem is therefore complete. $\square$

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