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ON THE INVERSION AND CHARACTERIZATION OF THE RIESZ POTENTIALS IN THE WEIGHTED LEBESGUE SPACES

Abstract. The method of approximative inverse operators is applied to the inversion problem for the Riesz potentials $f = I^{\alpha}\varphi$, $0 < \operatorname{Re}\alpha < n$, and the characterization of the range $I^{\alpha}(L^p_w)$ with densities φ in the Lebesgue spaces $L^p_w(\mathbb{R}^n)$ and a Muckenhoupt weight w. The general situation is considered when potentials $f \in L^q_v(\mathbb{R}^n)$, $1 < p < \infty$, and $q \geq p$ and Muckenhoupt weights w and v are independent, being related to each other only by integral conditions.

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რეზიუმე. მიახლოებითი შებრუნებული ოპერატორების მეთოდი გამოყენებულია რისის პოტენციალების $f = I^{\alpha}\varphi$, $0 < \operatorname{Re}\alpha < n$, შებრუნების პრობლემის ამოსახსნელად და $I^{\alpha}(L^p_w)$ მნიშვნელობათა სიმრავლის დასახასიათებლად, სადაც φ სიმკვრივეები ლებეგის $L^p_w(\mathbb{R}^n)$ სივრცეებიდანაა მუკენჰაუპტის w წონით. განხილულია ზოგადი სიტუაცია, როდესაც $f \in L^q_v(\mathbb{R}^n)$, $1 < p < \infty$ და $q \geq p$, ხოლო მუკენჰაუპტის w და v წონები დამოუკიდებელია და ერთმანეთან მხოლოდ ინტეგრალური პირობებითაა დაკავშირებული.

1. Introduction

We consider the Riesz potential operator

$$f(x) = I^\alpha \varphi(x) = \frac{1}{\gamma(\alpha)} \int\limits_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-\alpha}}\, dy, \tag{1.1}$$

where, as usual,

$$\gamma(\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}, \tag{1.2}$$

as acting from a weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ into another such space $L^q_v(\mathbb{R}^n)$ with $q > p > 1$ and the general weight functions w and v of the Muckenhoupt type.

We admit complex values of α and assume that $0 < \operatorname{Re}\alpha < n$.

It is known ([18], Ch. 3 and Ch. 7; [19], Section 27) that in the case of real α, the operator (left) inverse to I^α has the form of a hypersingular operator

$$\varphi(x) = (I^\alpha)^{-1} f(x) = \mathbb{D}^\alpha f(x) \ := \frac{1}{d_{n,l}(\alpha)} \int\limits_{\mathbb{R}^n} \frac{(\Delta^l_y f)(x)}{|y|^{n+\alpha}}\, dy, \tag{1.3}$$

known also as the Riesz fractional derivative, where $(\Delta^\ell_y f)(x)$ is either a centered or non-centered finite difference of f of order ℓ ($\ell > \alpha$ or $\ell > 2\left[\frac{\alpha}{2}\right]$ depending on the type of the finite difference), and the integral in (1.3) is treated as convergent in the norm of the space of functions φ. This also works for complex α with $0 < \operatorname{Re}\alpha < 2$ and $\ell = 1$ (see [18] and [19] for details). The inversion of the potential I^α with densities $\varphi \in L^p(\mathbb{R}^n)$ and description of the range $I^\alpha[L^p(\mathbb{R}^n)]$ in terms of the construction (1.3) was given in [15] (see also [18], Theorems 3.22, 7.9 and 7.11). Similar results for the weighted spaces $L^p_w(\mathbb{R}^n)$ with the Muckenhoupt weight w were obtained in [13] and [12] (see [18], Theorem 7.36).

A modification of the method of hypersingular operators which works for all complex α with $0 < \operatorname{Re}\alpha < n$, but requires the generalized finite differences, may be found in [18], p. 83.

There exists also an alternative approach to the inversion of the Riesz potential operator based on the method of approximative inverse operators (AIO) which works well for all complex α in the strip $0 < \operatorname{Re}\alpha < n$. This approach, realized in [16] (see also [18], Ch. 11) for non-weighted spaces $L^p(\mathbb{R}^n)$, provides the construction of the inverse operator in the form

$$\mathbb{D}^\alpha f(x) = \lim_{\substack{\varepsilon\to 0\\ (L_p)}} T^\alpha_\varepsilon f, \qquad 0 < \operatorname{Re}\alpha < n, \qquad 1 < p < \frac{n}{\operatorname{Re}\alpha}, \tag{1.4}$$

where

$$T^\alpha_\varepsilon f = \varepsilon^{-n} \int\limits_{\mathbb{R}^n} h_\alpha(y) f(x-\varepsilon y)\, dy \tag{1.5}$$

and the kernel $h_\alpha(y) \in L^1(\mathbb{R}^n)$ has the property that its Fourier transform has the form

$$\hat{h}_\alpha(\xi) = |\xi|^\alpha \hat{k}(\xi) \tag{1.6}$$

with $k(x)$ any function such that

$$k(x) \in L^1(\mathbb{R}^n) \bigcap I^\alpha(L^1) \tag{1.7}$$

(see also a similar approach for the realization of fractional powers of operators in [17]). An extension of this alternative inversion of [16] to the case of weighted spaces with Muckenhoupt weight was given in [14]. Observe that relation (1.7) means that

$$h_\alpha(x) \in L^1(\mathbb{R}^n) \qquad \text{and} \qquad h_\alpha(x) = \mathbb{D}^\alpha k(x), \quad k \in L^1(\mathbb{R}^n), \tag{1.8}$$

so that

$$h_\alpha(x) \in L^1(\mathbb{R}^n) \qquad \text{and} \qquad I^\alpha h_\alpha(x) \in L^1(\mathbb{R}^n). \tag{1.9}$$

Some examples of functions $k(x)$ and $h_\alpha(x)$ satisfying the conditions (1.6)–(1.8) were given in [16] (see also [18], Sections 1.4–1.5 of Ch. 11).

The results obtained in [16] provide a characterization of the range $I^\alpha(L^p_w)$, in particular, in terms of its imbedding into the space $L^q_v(\mathbb{R}^n)$ with the Sobolev exponent $q = \frac{np}{n-\alpha p}$ (which assumes that $p < \frac{n}{\alpha}$) and weight $v = w^{\frac{q}{p}}$.

Meanwhile, it is actual to obtain a more general result for the densities $\varphi \in L^p_w(\mathbb{R}^n)$ and potentials $f \in L^q_v(\mathbb{R}^n)$, when $1 < p < \infty$ (not only $1 < p < \frac{n}{\operatorname{Re}\alpha}$) and $q \geq p$ (not only $q = \frac{np}{n-\operatorname{Re}\alpha p}$) and the weights w and v are independent, being related to each other only by integral inequalities (two-weight approach, see [5], [3], [4], [2]).

This goal is realized in this paper.

Notation:

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$;
for $E \subset \mathbb{R}^n$, by $|E|$ we denote the Lebesgue measure of E;
$B(x, r)$ is the ball of radius r centered at the point x;
$F\varphi(\xi) = \hat{\varphi}(\xi) = \int\limits_{\mathbb{R}^n} e^{i\xi y}\varphi(y)\, dy$;
$F^{-1}f(x) = \hat{f}(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{-ix\xi} f(\xi)\, d\xi$;
$\langle f, \omega \rangle = \int\limits_{\mathbb{R}^n} f(x)\overline{\omega(x)}\, dx$;
$\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions.

2. Preliminaries

a) On weights and weighted spaces. Let w be a locally integrable almost everywhere positive function called a weight on $\mathbb{R}^n$. As usual, by $L^p_w(\mathbb{R}^n)$ we denote the weighted Lebesgue space of all measurable functions

$f : \mathbb{R}^n \to \mathbb{R}^1$ with the finite norm

$$\|f\|_{L^p_w} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

Definition 2.1. Let $1 < p < \infty$. We say that a weight w belongs to A_p, if

$$\sup \left(\frac{1}{|B|} \int\limits_B w(x)\, dx \right) \left(\frac{1}{|B|} \int\limits_B w^{1-p'}(x)\, dx \right)^{p-1} < \infty, \qquad p' = \frac{p}{p-1},$$

where the supremum is taken over all balls $B, B \subset \mathbb{R}^n$.

As is well known ([11], [1]), the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{B \ni x} \int\limits_B |f(y)|\, dy$$

is bounded in the space $L^p_w(\mathbb{R}^n)$ if and only if $w \in A_p$.

It is known that

$$L^p_w(\mathbb{R}^n) \subset L^1_\rho(\mathbb{R}^n), \qquad \rho(x) = (1 + |x|)^{-n} \tag{2.1}$$

for any weight $w \in A_p$ and

$$w \in A_p \quad \Leftrightarrow \quad w^{1-p'} \in A_{p'} \tag{2.2}$$

for all $1 < p < \infty$.

We remind the definition of the Lizorkin class

$$\Phi = \{\varphi \in \mathcal{S} : \hat{\varphi} \in \Psi\}, \text{ where } \Psi = \{\psi \in \mathcal{S} : D^k \psi(0) = 0, \ |k| = 0, 1, 2, \dots\}$$

([7], [8], [9], see also [18], p.39), which is invariant with respect to the Riesz potential operator I^α.

The Riesz potential operator $I^{i\theta}$ of purely imaginary order $i\theta$ is defined by its Fourier multiplier $m(\xi) = |\xi|^{i\theta}$:

$$I^{i\theta}\varphi = F^{-1}|\xi|^{i\theta}F\varphi, \qquad \varphi \in \Phi, \qquad \theta \in \mathbb{R}^1, \tag{2.3}$$

which is well suited for the space $L^p_w(\mathbb{R}^n)$, $w \in A_p$, according to Theorem C given below.

Lemma 2.2. *The operator $I^{i\theta}$ is bounded in the space $L^p_w(\mathbb{R}^n), 1 < p < \infty$ for all $w \in A_p$*

The statement of the lemma is obtained by direct verification of the Mikhlin–Hörmander condition

$$\sup_{R>0} \left(R^{s|j|-n} \int\limits_{R<|\xi|<2R} |D^j m(\xi)|^s\, d\xi \right) < \infty, \qquad |j| \le n,$$

where $1 < s \le 2$, which is sufficient for $m(\xi)$ to be a Fourier multiplier in the weighted space $L^p_w(\mathbb{R}^n)$, $1 < p < \infty$, with $w \in A_p$, see [6], Theorem

2 (one may choose any $s \in (1,2]$ different from $\frac{n}{n-1}, \frac{n}{n-2}, \dots, \frac{n}{n-k}$, $k \le \frac{n}{2}$, when checking this condition for $m(\xi) = |\xi|^{i\theta}$).

Definition 2.3. Let μ be a measure on $\mathbb{R}^n$. We say that μ satisfies the doubling condition if there exists a positive constant b such that the inequality

$$\mu B(x, 2r) \le b\mu B(x, r)$$

holds for all the balls $B(x, r)$.

Definition 2.4. A measure μ on $\mathbb{R}^n$ satisfies the reverse doubling condition if there exists positive constants $\eta_1 > 1$ and $\eta_2 > 1$ such that

$$\mu B(x, \eta_1 r) \ge \eta_2 \mu B(x, r)$$

holds for all the balls $B(x, r)$.

The following statement is well known (see [21], page 11, Lemma 20).

Proposition A. *Let μ satisfy the doubling condition. Then μ satisfies the reverse doubling condition.*

In the sequel we denote $wE = \int\limits_E w(x)\, dx$ for any measurable set $E \subset \mathbb{R}^n$, where w is a weight. Note that this measure satisfies the reverse doubling condition if $w \in A_p$.

We will base ourselves on the following theorems.

Theorem A (see [4], p.116). *Let $1 < p < \infty$, $0 < \alpha < n$, and let w and v be weights on $\mathbb{R}^n$. Let the weights v and $w^{1-p'}$ satisfy the reverse doubling condition. Then the operator I^α is bounded from $L^p_w(\mathbb{R}^n)$ into $L^q_v(\mathbb{R}^n)$ if and only if*

$$\sup |B|^{\frac{\alpha}{n}-1} \left(\int\limits_B v(x)\, dx \right)^{\frac{1}{q}} \left(\int\limits_B w^{1-p'}(x)\, dx \right)^{\frac{1}{p'}} < \infty \tag{2.4}$$

where the supremum is taken over all the balls $B \subset \mathbb{R}^n$.

Remark 2.5. Let $1 < p < \infty$, let α be complex with $0 < \operatorname{Re}\alpha < n$ and let the weights v and $w^{1-p'}$ satisfy the reverse doubling condition. The operator I^α is bounded in the space $L^p_w(\mathbb{R}^n)$ if and only if the condition (2.4) is satisfied with $|B|^{\frac{\alpha}{n}-1}$ replaced by $|B|^{\frac{\operatorname{Re}\alpha}{n}-1}$.

Indeed, it suffices to observe that $I^\alpha\varphi = I^{i\theta} I^{\operatorname{Re}\alpha}\varphi$ for $\varphi \in \Phi$, where Φ is dense in $L^p_w(\mathbb{R}^n)$ by Theorem C given below and the operator $I^{i\theta}$ is boundedly invertible in $L^p_w(\mathbb{R}^n)$.

For the dilatation kernels

$$k_\varepsilon(x) = \frac{1}{\varepsilon^n} k\left(\frac{x}{\varepsilon}\right),$$

the following extension of Stein's theorem to weighted spaces was given in [12] (see also [18], Theorem 7.31).

Theorem B. a) *Let $k(x)$ have a non-increasing radial dominant $b(|x|) \in L_1(\mathbb{R}^n)$ and $f \in L^p_w,\ w \in A_p$. Then*

$$\sup_{\varepsilon>0} |(k_\varepsilon * f)(x)| \le c\|b\|_1 (Mf)(x), \tag{2.5}$$

where $(Mf)(x)$ is the Hardy–Littlewood maximal function.

b) *If in addition $\int\limits_{R^n} k(x)dx = 1$, then*

$$(k_\varepsilon * f)(x) \to f(x)$$

as $\varepsilon \to 0$ in the L^p_w-norm and almost everywhere.

Theorem C ([18], Theorem 7.34 and [13], Theorem 4.3). *The Lizorkin class Φ is dense in the weighted space $L^p_w(\mathbb{R}^n)$ for any weight $w \in A_p, 1 < p < \infty$.*

Theorem D ([10], [22]). *Let $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$. The operator I^α is bounded from $L^p(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ if and only if $I^\alpha v \in L^{p'}_{loc}$ and*

$$I^\alpha [I^\alpha v]^{p'}(x) \le c I^\alpha v(x) \quad \text{almost everywhere.} \tag{2.6}$$

Remark 2.6. Theorem D is also valid for complex α with $0 < \operatorname{Re}\alpha < n$, if condition (2.6) is replaced by

$$I^{\operatorname{Re}\alpha}[I^{\operatorname{Re}\alpha} v]^{p'}(x) \le c I^{\operatorname{Re}\alpha} v(x) \quad \text{almost everywhere} \tag{2.7}$$

(see the arguments in the proof of Corollary 2.5).

We will also need the condition dual to (2.7), namely

$$I^{\operatorname{Re}\alpha}[I^{\operatorname{Re}\alpha} w^{1-p'}]^{p}(x) \le c I^{\operatorname{Re}\alpha} w^{1-p'}(x) \quad \text{almost everywhere.} \tag{2.8}$$

Let $1 < p < q < p^*$, where $p^* = \frac{np}{n-\alpha p}$ and $\alpha < \min\left\{\frac{n}{p}, \frac{n}{q}\right\}$. Then a simple example of weight functions $w \in A_p$ and $v \in A_p$ for which condition (2.4) holds, is that of power functions:

$$w(x) = |x|^\beta, \qquad v(x) = |x|^\gamma, \tag{2.9}$$

where

$$\alpha p - n < \beta < n(p-1), \quad \gamma = q\left(\frac{n}{p} + \frac{\beta}{p} - \alpha\right) - n \tag{2.10}$$

(see Appendix). As to the conditions (2.7) and (2.8), they are valid for

$$\begin{gathered} v(x) = |x|^{-\operatorname{Re}\alpha p} \in A_p, \quad 0 < \operatorname{Re}\alpha < \frac{n}{p}, \qquad \text{and} \\ w(x) = |x|^{\operatorname{Re}\alpha p} \in A_p, \qquad 0 < \operatorname{Re}\alpha < \frac{n}{p'}, \end{gathered} \tag{2.11}$$

respectively

c) Appropriate kernels.

Definition 2.7. A kernel $h_\alpha(x) \in L^1(\mathbb{R}^n), 0 < \operatorname{Re}\alpha < n$, is called *appropriate* if it satisfies the assumption in (1.9),

$$\int\limits_{\mathbb{R}^n} (I^\alpha h_\alpha)(x)\, dx = 1,$$

and both $h_\alpha(x)$ and $I^\alpha h_\alpha(x)$ have integrable non-increasing radial dominants.

It is known that the following functions are examples of *appropriate* kernels:

$$1)\quad h_\alpha(x) = F^{-1}(|\xi|^\alpha e^{-|\xi|}) = \\ = \frac{\Gamma(n+\alpha)}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} F\left(\frac{n+\alpha}{2}, \frac{n+\alpha+1}{2}; \frac{n}{2}; -|x|^2\right), \tag{2.12}$$

where $F\left(\frac{n+\alpha}{2}, \frac{n+\alpha+1}{2}; \frac{n}{2}; z\right)$ is the Gauss hypergeometric function, and

$$2)\quad h_\alpha(x) = \frac{(-1)^m}{\gamma_n(2m-\alpha)}\ \Delta^m \left(\frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}-m}}\right) = \\ = \frac{1}{\gamma_n(-\alpha)}\left[\frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}}} + \sum_{k=1}^{n} \frac{(-1)^k c_{m,k}}{(1+|x|^2)^{\frac{n+\alpha}{2}+k}}\right], \tag{2.13}$$

where $c_{m,k} = \binom{m}{k} \frac{\left(\frac{n+1}{2}\right)_k}{\left(\frac{\alpha}{2}-m+1\right)_k}$ and m is any integer such that $m > \frac{\operatorname{Re}\alpha}{2}, \alpha \neq 2, 4, 6, \dots$ (see [18], Lemmas 11.7–11.8 and 11.13).

Obviously, the set of appropriate kernels is rich enough. Indeed, if $h_\alpha(x)$ is an appropriate kernel, then any convolution

$$\mathcal{K} * h_\alpha(x) = \int\limits_{\mathbb{R}^n} \mathcal{K}(x-y) h_\alpha(y)\, dy$$

with $\mathcal{K} \in L^1(\mathbb{R}^n)$ and $\int\limits_{\mathbb{R}^n} \mathcal{K}(y)\, dy = 1$, is also an appropriate kernel.

3. Statement of the Main Results

Our first theorem provides the following two-weighted result on the inversion of the Riesz potential operator.

Theorem 3.1. *Let $1 < p < \infty, 0 < \operatorname{Re}\alpha < n$ and $w \in A_p$. Assume that there exist $q, p < q < \infty$ and a weight function $v \in A_q$ such that (2.4) holds. Then the equality*

$$f = I^\alpha \varphi \qquad \textit{with} \qquad \varphi \in L^p_w(\mathbb{R}^n) \tag{3.1}$$

implies

$$\varphi = \lim_{\varepsilon\to 0} T_\varepsilon^\alpha f = \lim_{\varepsilon\to 0} \varepsilon^{-n} \int\limits_{\mathbb{R}^n} h_\alpha(y) f(x-\varepsilon y)\, dy, \tag{3.2}$$

where $h_\alpha(y)$ *is any appropriate kernel* (*see Definition* 2.7) *and the limit in* (3.2) *is taken in* L^p_w*-norm or almost everywhere.*

The next theorem gives the two-weighted description of the range of the Riesz potential.

Theorem 3.2. *Let* $1 < p < \infty, 0 < \operatorname{Re}\alpha < n$, $w \in A_p$, *and let there exist* $q, p < q < \infty$ *and* $v \in A_q$ *such that* (2.4) *holds. A function* f *belongs to the range* $I^\alpha(L^p_w)$ *if and only if*

i) $f \in L^q_v(\mathbb{R}^n)$,

ii) *one of the following two conditions is fulfilled:*

a) $\lim\limits_{\varepsilon\to 0} T_\varepsilon^\alpha f \in L^p_w(\mathbb{R}^n)$ *where* T_ε^α *is the operator* (1.5) *with any appropriate kernel* $h_\alpha(x)$ *and the limit is taken with respect to the* $L^p_w(\mathbb{R}^n)$*-norm;*

b) $\sup\limits_{\varepsilon>0} \|T_\varepsilon^\alpha f\|_{L^p_w} < \infty$.

The following theorem presents the corresponding inversion statement for the Riesz potential operators in the case where $1 < p < \frac{n}{\operatorname{Re}\alpha}$ and $w \equiv 1$. It is based on Theorem D.

Theorem 3.3. *Let* $1 < p < \infty$, $0 < \operatorname{Re}\alpha < \frac{n}{p}$ *and* $v \in A_p$. *Suppose that* (2.6) *holds. A function* f *belongs to the range* $I^\alpha(L^p)$ *if and only if*

i) $f \in L^p_v(\mathbb{R}^n)$,

ii) *one of the following two conditions is fulfilled:*

a) $\lim\limits_{\varepsilon\to 0} T_\varepsilon^\alpha f \in L^p(\mathbb{R}^n)$ *with any appropriate kernel* $h_\alpha(x)$ *in the operator* T_ε^α, *the limit being taken with respect to the* $L^p(\mathbb{R}^n)$*-norm;*

b) $\sup\limits_{\varepsilon>0} \|T_\varepsilon^\alpha f\|_{L^p} < \infty$.

Finally, the last two theorems give some statements dual to the situation considered in Theorem 3.3 and provide both the inversion statement and the characterization of the range.

Theorem 3.4. *Let* $1 < p < \infty$, $0 < \operatorname{Re}\alpha < \frac{n}{p'}$ *and* $w \in A_p$. *Suppose that* $I^\alpha(w^{1-p'}) \in L^p_{loc}$ *and (2.8) holds. If* $f = I^\alpha\varphi$ *with* $\varphi \in L^p_w(\mathbb{R}^n)$, *then*

$$\varphi = \lim_{\varepsilon\to 0} T_\varepsilon^\alpha f = \lim_{\varepsilon\to 0} \varepsilon^{-n} \int\limits_{\mathbb{R}^n} h_\alpha(y) f(x-\varepsilon y)\, dy, \tag{3.3}$$

where $h_\alpha(y)$ *is any appropriate kernel and the limit is taken in* L^p_w*-norm or almost everywhere.*

Theorem 3.5. *Let* $1 < p < \infty$, $0 < \operatorname{Re}\alpha < \frac{n}{p'}$ *and* $w \in A_p$. *Suppose that* $I^\alpha(w^{1-p'}) \in L^p_{loc}$ *and* (2.8) *holds. Then* $f \in I^\alpha(L^p_w)$ *if and only if*

i) $f \in L^p(\mathbb{R}^n)$,
ii) *one of the following two conditions is fulfilled:*
a) $\lim\limits_{\varepsilon\to 0} T_\varepsilon^\alpha f \in L_w^p(\mathbb{R}^n)$ *where* $\lim\limits_{\varepsilon\to 0} T_\varepsilon^\alpha$ *is the same as in* (3.3) *with any appropriate kernel* $h_\alpha(x)$ *and the limit being taken in the* $L_w^p(\mathbb{R}^n)$*-norm;*
b) $\sup\limits_{\varepsilon>0} \|T_\varepsilon^\alpha f\|_{L_w^p} < \infty$.

4. Proofs

The proofs of Theorems 3.1 and 3.2 represent a modification of the proofs of Theorems 3.1 and 3.2 from [14].

Proof of Theorem 3.1. For $\varphi \in \Phi$ there holds the equality

$$(T_\varepsilon^\alpha I^\alpha \varphi)(x) = \frac{1}{\varepsilon^n} k\left(\frac{x}{\varepsilon}\right) * \varphi \qquad \text{with} \qquad k(x) \in L^1(\mathbb{R}^n), \tag{4.1}$$

which follows via Fourier transforms from (1.5)–(1.7). Let us show that this relation remains valid for all $\varphi \in L_w^p(\mathbb{R}^n)$. Let ε be fixed and let $\varphi_0 \in L_w^p(\mathbb{R}^n)$. To show that (4.1) is valid for φ_0, we pass to the limit in (4.1) as $\Phi \ni \varphi \to \varphi_0$, but do this in different norms for the left-hand and right-hand sides of (4.1).

By Theorem C, there exists a sequence $\varphi_m \in \Phi$ such that $\varphi_m \to \varphi_0$ in the L_w^p-norm. The left-hand side operator

$$A_\varepsilon = T_\varepsilon^\alpha I^\alpha$$

is bounded from $L_w^p(\mathbb{R}^n)$ into $L_v^q(\mathbb{R}^n)$ by Theorem A (with Remark 2.5 taken into account), Theorem B, Proposition A and the fact that $w \in A_p$ and $v \in A_q$. Therefore,

$$A_\varepsilon \varphi_m \to A_\varepsilon \varphi_0 \qquad \text{in} \qquad L_v^q(\mathbb{R}^n). \tag{4.2}$$

On the other hand, the right-hand side operator

$$B_\varepsilon = \frac{1}{\varepsilon^n} k\left(\frac{x}{\varepsilon}\right) * \varphi$$

is bounded in the space $L_w^p(\mathbb{R}^n)$ by Theorem B and the fact that $w \in A_p$. Therefore,

$$B_\varepsilon \varphi_m \to B_\varepsilon \varphi_0 \qquad \text{in} \qquad L_w^p(\mathbb{R}^n). \tag{4.3}$$

From (4.2)–(4.3) it follows that there exists a subsequence φ_{m_k} such that

$$A_\varepsilon \varphi_{m_k} \to A_\varepsilon \varphi_0 \qquad \text{and} \qquad A_\varepsilon \varphi_{m_k} \to A_\varepsilon \varphi_0 \qquad \text{almost everywhere}$$

and we arrive at (4.1) for $\varphi_0 \in L_w^p(\mathbb{R}^n)$.

It remains to observe that by Theorem C and the condition $w \in A_p$, we have that $\frac{1}{\varepsilon^n} k\left(\frac{x}{\varepsilon}\right) * \varphi$ converges in $L_w^p(\mathbb{R}^n)$ as $\varepsilon \to 0$. Therefore, passing to the limit in (4.1) as $\varepsilon \to 0$, we obtain the desired relation (3.2).

Proof of Theorem 3.2. Necessity follows from Theorems A (with Remark 2.5 taken into account) and B, and the relation (4.1) proved for $f \in L_w^p(\mathbb{R}^n)$.

Let us prove the sufficiency. Let $f \in L^q_v(\mathbb{R}^n)$ and suppose that the condition a) of our theorem is satisfied. Let $\varphi = \lim_{\varepsilon\to 0} T^\alpha_\varepsilon f$, the limit being taken in the $L^p_w(\mathbb{R}^n)$-norm. The following relation is valid:

$$\langle f, \psi\rangle \;=\; \langle I^\alpha\varphi, \psi\rangle\,, \qquad \psi \in \Phi. \tag{4.4}$$

Indeed, for $\varphi \in \Phi$ we have

$$\langle I^\alpha\varphi, \psi\rangle \;=\; \langle \varphi, I^\alpha\psi\rangle \;=\; \Big\langle \lim_{\substack{\varepsilon\to 0\\ (L^p_w)}} T^\alpha_\varepsilon f, I^\alpha\psi \Big\rangle = \lim_{\varepsilon\to 0}\langle T^\alpha_\varepsilon f, I^\alpha\psi\rangle =$$

$$= \lim_{\varepsilon\to 0}\langle f, T^\alpha_\varepsilon I^\alpha\psi\rangle \;=\; \lim_{\varepsilon\to 0}\left\langle f, \frac{1}{\varepsilon^n}k\left(\frac{x}{\varepsilon}\right) * \psi\right\rangle \;=\; \langle f, \varphi\rangle\,.$$

Here the first equality follows from Fubini theorem which is justified with the aid of the Hölder inequality

$$|\,\langle I^\alpha\varphi, \psi\rangle\,| \le \|I^\alpha\varphi\|_{L^q_v}\|\psi\|_{L^{q'}_{v^{1-q'}}} < \infty$$

since $I^\alpha\varphi \in L^q_v(\mathbb{R}^n)$ by Theorem A. The third equality is obvious as the convergence in $L^p_w(\mathbb{R}^n)$ implies that in the space Φ'. The fourth equality follows from the Fubini theorem:

$$|\,\langle f, T^\alpha_\varepsilon I^\alpha\psi\rangle\,| \le \|f\|_{L^q_v}\|T^\alpha_\varepsilon I^\alpha\psi\|_{L^{q'}_{v^{1-q'}}} < \infty$$

(note that $I^\alpha\psi \in \Phi$ and by Theorem B $T^\alpha_\varepsilon I^\alpha\psi \in L^{q'}_{v^{1-q'}}$ because $v^{1-q'} \in A_{q'}$). The fifth equality, that is, the equality (4.1) has already been justified. The last equality is justified with the aid of the Hölder inequality and Theorem B since $\frac{1}{\varepsilon^n}k\left(\frac{x}{\varepsilon}\right) * \psi \to \psi$ almost everywhere and

$$\left|\left\langle f, \frac{1}{\varepsilon^n}k\left(\frac{x}{\varepsilon}\right) * \psi\right\rangle\right| \le \|f\|_{L^q_v}\left\|\frac{1}{\varepsilon^n}k\left(\frac{x}{\varepsilon}\right) * \psi\right\|_{L^{q'}_{v^{1-q'}}} \le c\|f\|_{L^q_v}.$$

From (4.4) it follows that

$$f(x) = (I^\alpha\varphi)(x) + P(x),$$

where $P(x)$ is a polynomial. By (2.1) we obtain that $P(x) \equiv 0$. Hence $f \in I^\alpha(L^p_w)$.

Now let $f \in L^q_v(\mathbb{R}^n)$ and suppose that the condition b) is satisfied. Since the space $L^p_w(\mathbb{R}^n)$ is reflexive, we have that the set $\{T^\alpha_\varepsilon f\}_{\varepsilon>0}$ is weakly compact. Hence there exists a subsequence $\left\{T^\alpha_{\varepsilon_k} f\right\}_{k=1}^\infty$ which weakly converges in $L^p_w(\mathbb{R}^n)$ to a function $\varphi \in L^p_w(\mathbb{R}^n)$. Arguing as above, we easily obtain that $f(x) = (I^\alpha\varphi)(x)$.

Proof of Theorem 3.3 is obtained by repeating the arguments of the proof of Theorem 3.2, but with reference to Theorems B,C and D this time.

Proof of Theorem 3.4 is similar to that of Theorem 3.1. We only note that, using duality arguments, by Theorem D (with Remark 2.6 taken into account) the operator I^α is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $I^\alpha w^{1-p'} \in L^p_{loc}$ and (2.8) holds.

Proof of Theorem 3.5 is similar to that of Theorem 3.1.

5. Appendix

Let us prove that the pair of weights from (2.9) governs two-weight inequality for the Riesz potentials.

Proposition 5.1. *Let $1 < p \leq q < p^*$, where $p^* = \frac{np}{n-\alpha p}$ and $\alpha < \frac{n}{q}$. Suppose that $\alpha p - n < \beta < n(p-1)$ and $\gamma = q\left(\frac{n}{p} + \frac{\beta}{p} - \alpha\right) - n$. Then $-n < \gamma < q(n-\alpha) < n(q-1)$ and the following inequality holds*

$$\Bigg(\int\limits_{\mathbb{R}^n} |x|^\gamma |I^\alpha f(x)|^q \, dx \Bigg)^{\frac{1}{q}} \leq c \Bigg(\int\limits_{\mathbb{R}^n} |x|^\beta |f(x)|^p \, dx \Bigg)^{\frac{1}{p}}. \tag{5.1}$$

Proof. Let $f \geq 0$. We have

$$\|I^\alpha f(x)\|_{L^q_{|x|^\gamma}} \leq c(I_1 + I_2 + I_3),$$

where

$$I_1 = \Bigg(\int\limits_{\mathbb{R}^n} |x|^\gamma \Bigg(\int\limits_{|y| \leq \frac{|x|}{2}} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy \Bigg)^q dx \Bigg)^{\frac{1}{q}},$$

$$I_2 = \Bigg(\int\limits_{\mathbb{R}^n} |x|^\gamma \Bigg(\int\limits_{\frac{|x|}{2} < |y| < 2|x|} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy \Bigg)^q dx \Bigg)^{\frac{1}{q}}$$

and

$$I_3 = \Bigg(\int\limits_{\mathbb{R}^n} |x|^\gamma \Bigg(\int\limits_{|y| > 2|x|} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy \Bigg)^q dx \Bigg)^{\frac{1}{q}}.$$

If $|y| \leq \frac{1}{2}|x|$, then $\frac{|x|}{2} \leq |x-y|$. Therefore using Hardy's two-weight inequality, we get

$$I_1 \leq c \Bigg(\int\limits_{\mathbb{R}^n} |x|^{\gamma + (\alpha - n)q} \Bigg(\int\limits_{|y| \leq |x|} f(y) \, dy \Bigg)^q dx \Bigg)^{\frac{1}{q}} \leq c \|f\|_{L^q_{|x|^\beta}}$$

since

$$\Bigg(\int\limits_{|x|>t} |x|^{\gamma + (\alpha-n)q} \, dx \Bigg)^{\frac{1}{q}} \Bigg(\int\limits_{|x|<t} |x|^{\beta(1-p')} \, dx \Bigg)^{\frac{1}{p'}} =$$

$$= c \Bigg(\int\limits_t^\infty \tau^{\gamma + (\alpha-n)q + n - 1} \, d\tau \Bigg)^{\frac{1}{q}} \Bigg(\int\limits_0^t \tau^{\beta(1-p') + n - 1} \, d\tau \Bigg)^{\frac{1}{p'}} =$$

$$= c\,t^{\frac{\gamma+(\alpha-n)q+n}{q}} \cdot t^{\frac{\beta(1-p')+n}{p'}} = c.$$

For I_3 we apply two-weight inequality for the operator adjoint to the Hardy operator. We have

$$I_3 \le c\Big(\int\limits_{\mathbb{R}^n} |x|^\gamma \Big(\int\limits_{|y|>2|x|} \frac{f(y)}{|y|^{n-\alpha}}\,dy\Big)^q dx\Big)^{\frac{1}{q}} \le \|f\|_{L^p_{|x|^\beta}}.$$

The last inequality holds because

$$\Big(\int\limits_{|x|<t} |x|^\gamma\,dx\Big)^{\frac{1}{q}}\Big(\int\limits_{|x|>t} |x|^{\beta(1-p')+(\alpha-n)p'}\,dx\Big)^{\frac{1}{p'}} =$$

$$= c\Big(\int\limits_0^t \tau^{\gamma+n-1}\,d\tau\Big)^{\frac{1}{q}}\Big(\int\limits_t^\infty \tau^{\beta(1-p')+(\alpha-n)p'+n-1}\,d\tau\Big)^{\frac{1}{p'}} =$$

$$= c\,t^{\frac{\gamma+n}{q}-\frac{\beta}{p}+\alpha-n+\frac{n}{p'}} = c.$$

Then, as $q < p^*$, we have $\frac{p^*}{q} > 1$. Applying Hölder's inequality with the exponent $\frac{p^*}{q}$, we obtain

$$I_2 = \int\limits_{\mathbb{R}^n} |x|^\gamma\Big(\int\limits_{\frac{|x|}{2}<|y|<2|x|} f(y)|x-y|^{\alpha-n}\,dy\Big)^q dx =$$

$$= \sum_k \int\limits_{2^k<|x|<2^{k+1}} |x|^\gamma\Big(\int\limits_{\frac{|x|}{2}<|y|<2|x|} f(y)|x-y|^{\alpha-n}\,dy\Big)^q dx \le$$

$$= \sum_k \Big(\int\limits_{2^k<|x|<2^{k+1}} |x|^{\gamma\,\frac{p^*}{p^*-q}}\,dx\Big)^{\frac{p^*-q}{p^*}} \times$$

$$\times \Big(\int\limits_{2^k<|x|<2^{k+1}} \Big(\int\limits_{\frac{|x|}{2}<|y|<2|x|} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy\Big)^{p^*} dx\Big)^{\frac{p}{p^*}} =$$

$$= \sum_k \Big(\int\limits_{2^k<|x|<2^{k+1}} |x|^{\gamma\,\frac{p^*}{p^*-q}}\,dx\Big)^{\frac{p^*-q}{p^*}} \times$$

$$\times \Big(\int\limits_{2^k<|x|<2^{k+1}} \Big(\int\limits_{\mathbb{R}^n} \frac{f(y)\chi_{2^{k-1}<|y|<2^{k+1}}}{|x-y|^{n-\alpha}}\,dy\Big)^{p^*} dx\Big)^{\frac{q}{p^*}}.$$

Applying Sobolev's inequality for the second factor, we obtain the estimate

$$I_2^q \le c\sum_k 2^{k(\gamma+\frac{(p^*-q)n}{p^*})}\Big(\int\limits_{2^{k-1}<|y|<2^{k+1}} (f(y))^p\,dy\Big)^{\frac{q}{p}} =$$

$$= c\sum_k 2^{\frac{k\beta q}{p}}\Bigg(\int\limits_{2^{k-1}<|y|<2^{k+1}} \big(f(y)\big)^p\,dy\Bigg)^{\frac{q}{p}} \le$$

$$\le c\sum_k \Bigg(\int\limits_{2^{k-1}<|y|<2^{k+1}} \big(f(y)\big)^p|y|^\beta\,dy\Bigg)^{\frac{q}{p}} \le c\bigg(\int\limits_{\mathbb{R}^n} \big(f(y)\big)^p|y|^\beta\,dy\bigg)^{\frac{q}{p}}.$$

Here the following implications were used:

$$\gamma + \frac{p^*-q}{p^*}\cdot n = \beta\,\frac{q}{p} \Longleftrightarrow \gamma + n - \frac{q(n-\alpha p)n}{np} = \beta\,\frac{q}{p} \Longleftrightarrow$$

$$\Longleftrightarrow \gamma + n\frac{qn}{p} + q\alpha = \beta\,\frac{q}{p} \Longleftrightarrow \gamma = q\Big(\frac{\beta}{p} + \frac{n}{p}\alpha\Big). \qquad \square$$

The inequality (5.1) was proved in [20], but for completeness we give its proof (different from that given in [20]).

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