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ON SUFFICIENT CONDITION OF STABILITY FOR THE FIRST ORDER DIFFERENTIAL EQUATION WITH RETARDED ARGUMENT

Abstract. The condition of admissibility of couples of spaces is obtained with the help of the $\mathcal{W}-$method for a differential equation with one concentrated delay. This test is similar to the well-known A. D. Myshkis test on asymptotic stability of such an equation.

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1. Introduction

In the article of A. D. Myshkis [1] for the scalar differential equation with concentrated delay

$$\begin{cases} y'(t) + p(t)y(\tau(t)) = 0, \quad t \geq 0, \\ y(\xi) = \varphi(\xi), \quad \text{if} \quad \xi < 0 \end{cases} \tag{1.1}$$

the following test of stability was obtained: if

$$\inf_{t\geq 0} p(t) > 0 \quad \text{and} \quad \sup_{t\geq 0} p(t) \sup_{t\geq 0} (t - \tau(t)) < 3/2 \quad (\leq 3/2), \tag{1.2}$$

then the trivial solution of the equation (1.2) is asymptotically stable (respectively, is stable in the sense of Lyapunov). The examples of unimprovability of the constant 3/2 were constructed in the same paper [1] (see also ([2], Ch. IV, §21; [3], Ch. VI, §38; [4], Ch. 5, 5.5).

This test was extended to the scalar equation with distributed delay

$$\begin{cases} y'(t) + \int\limits_{-\infty}^{t} y(s)d_s r(t,s) = 0, \quad t \geq 0, \\ y(\xi) = \varphi(\xi), \ \text{if} \ \xi < 0, \end{cases} \tag{1.3}$$

in the monographs ([2], Ch. IV, §21; [3], Ch. VI, §38) (see also [4], Ch. 5, 5.5; [5], Ch. 2, §1, 1.4; [6], Ch. 3, 3.4, Ex. 3.2)).

In her works ([7]–[10], [11], Ch. 3), V.V. Malygina proved that the strict inequality (1.2) and its analogs guarantee exponential estimates with negative indices for Cauchy functions of the equations (1.1), (1.3). She also proved that Cauchy functions of the equations (1.1), (1.2) are bounded in the triangle $0 \leq s \leq t \leq \infty$ in case of the equality in test (1.2) and in its analogs.

Besides, in the paper [10], for a differential equation with several delays it was constructed an unimprovable region of parameters of such equation that guarantees for the Cauchy function the exponential estimate with a negative index.

Note that similar conditions of stability and asymptotic stability were obtained for some classes of nonlinear differential equations with aftereffect (see [4], Ch. 5, 5.5; [12]–[16]).

2. Main Result

Hereinafter we will consider the nonhomogeneous scalar equation

$$\begin{cases} y'(t) + p(t)y(\tau(t)) = v(t), \quad t \in [0,\infty), \\ y(\xi) = \varphi(\xi), \quad \text{if} \quad \xi < 0. \end{cases} \tag{2.1}$$

Let $\tau : [0,\infty) \to \mathbf{R}$ be a measurable function satisfying $\tau(t) \leq t$ for all $t \geq 0$, $\varphi : (-\infty, 0) \to \mathbf{R}$ be a measurable and essentially bounded in essential function, and $p, v : [0,\infty) \to \mathbf{R}$ measurable and essentially bounded on

$[0,\infty)$ functions. By $\mathbf{L}_\infty$ denote the Banach space of such functions v and p with the norm $\|v\|_{\mathbf{L}_\infty} \stackrel{\text{def}}{=} \operatorname*{vrai\,sup}_{t\geq 0}|v(t)|$.

We will use the notation of the monographs [11], [17]:

$$y_\tau(t) \stackrel{\text{def}}{=} (\mathcal{S}_\tau y)(t) \stackrel{\text{def}}{=} \begin{cases} y(\tau(t)), & \text{if } \tau(t)\geq 0, \\ 0, & \text{if } \tau(t)<0; \end{cases}$$

$$\varphi^\tau(t) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } \tau(t)\geq 0, \\ \varphi(\tau(t)), & \text{if } \tau(t)<0. \end{cases}$$

The equation (2.1) can be rewritten as the equation $\mathcal{L}_\tau y = q$ with the linear operator $(\mathcal{L}_\tau y)(t) \stackrel{\text{def}}{=} y'(t) + p(t)(\mathcal{S}_\tau y)(t)$, defined on the space of locally absolutely continuous functions, and with the right hand side $q(t) = v(t) - p(t)\varphi^\tau(t)$.

Assume for simplicity that $\operatorname*{vrai\,inf}_{t\in\mathbf{R}} p(t) > 0$. In this case the equation (2.1) can be reduced to the equation

$$\begin{cases} \dot{x}(u) + x(h(u)) = w(u), & u\geq 0, \\ x(\zeta) = \psi(\zeta), \quad \text{if} \quad \zeta<0, \end{cases}$$

by the Kummer–Liouville transformation $u = g(t) \stackrel{\text{def}}{=} \int\limits_0^t p(s)\,ds$, $x(u) = y(g^{-1}(u))$, $\zeta = g(\xi)$, where

$$h(u) = u - \int\limits_{\tau(g^{-1}(u))}^{g^{-1}(u)} p(s)\,ds, \qquad w(u) = v(g^{-1}(u))/p(g^{-1}(u)),$$

$u\geq 0$, $\psi(\zeta) = \varphi(\xi)$, $\zeta<0$, $\xi<0$.

The last equation can be rewritten as

$$\mathcal{L}x = f \tag{2.2}$$

with the linear operator $(\mathcal{L}x)(u) \stackrel{\text{def}}{=} \dot{x}(u) + (\mathcal{S}_h x)(u)$, defined on the space of locally absolutely continuous functions, and with the right hand side $f(u) = w(u) - \psi^h(u)$. Hereinafter we will write the variable t in the equation (2.2) instead of the independent variable u.

The aim of this paper is to obtain a sufficient condition of $\mathbf{D}_0$−stability of the equation (2.2) with the help of the $\mathcal{W}$−method. This condition is similar to the A.D. Myshkis test. The description of the $\mathcal{W}$−method in stability theory of differential equations with aftereffect can be found in the monographs [11], [17].

Let us describe the $\mathcal{W}$−method on the example of the equation (2.2). We will take the equation

$$(\mathcal{L}_0 x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + x(t) = z(t), \quad t\in[0,\infty),$$

as a "model" equation. Let $\mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty)$ be the set of all solutions of this equation for all $z \in \mathbf{L}_\infty$. Every element x of the space $\mathbf{D}_0 \stackrel{\text{def}}{=} \mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty)$ admits the representation

$$x(t) = e^{-t} \int\limits_0^t e^s z(s)\, ds + x(0)e^{-t},$$

where $z \in \mathbf{L}_\infty$, $x(0) \in \mathbf{R}$. The norm of an element $x \in \mathbf{D}_0$ is defined by $\|x\|_{\mathbf{D}_0} \stackrel{\text{def}}{=} \|\dot{x} + x\|_{\mathbf{L}_\infty} + |x(0)|$. This norm is equivalent to the Sobolev norm $\|x\|_{\mathbf{W}^1_\infty} \stackrel{\text{def}}{=} \|\dot{x}\|_{\mathbf{L}_\infty} + \|x\|_{\mathbf{C}}$. Here and below $\mathbf{C}$ is the space of continuous and bounded functions $x : [0, \infty) \to \mathbf{R}$ with the norm $\|x\|_{\mathbf{C}} \stackrel{\text{def}}{=} \sup\limits_{t \geq 0} |x(t)|$.

Definition 2.1. We say that the equation (2.2) is $\mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty)-$ stable ($\mathbf{D}_0-$stable) if the set $\mathbf{D}(\mathcal{L}, \mathbf{L}_\infty)$ of all solutions of this equation for all $f \in \mathbf{L}_\infty$ coincides with the set $\mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty)$.

Let us remark, that for the equation (2.2) $\mathbf{D}_0-$stability is equivalent to the $\mathbf{C}-$stability. The latter means that all solutions of this equation are bounded on the semi-axis $[0, \infty)$ for every right hand side $f \in \mathbf{L}_\infty$.

Denote $h^+(t) = \max\{h(t), 0\}$. It takes place following

Theorem 2.1. *Let for some* $b \geq 0$ *and* $\Delta \in (0, 1)$ *the inequality*

$$\operatorname*{vrai\,sup}_{t \geq b} |t - h^+(t) - \Delta| < 1 - \frac{\Delta^2}{2} \tag{2.3}$$

be fulfilled. Then the equation (2.2) *is* $\mathbf{D}_0$*-stable.*

Remark 2.1. The inequality (2.3) can be rewritten for some $\varepsilon > 0$ and for almost all $t \geq b$ in the equivalent form

$$\frac{\Delta^2}{2} + \Delta - 1 + \varepsilon < t - h^+(t) < -\frac{\Delta^2}{2} + \Delta + 1 - \varepsilon.$$

Let us take Δ such that $\dfrac{\Delta^2}{2} + \Delta - 1 = 0$ and $\Delta \in (0, 1)$. Then

$$-\frac{\Delta^2}{2} + \Delta + 1 = 2(\sqrt{3} - 1).$$

Thus, for $\Delta = \sqrt{3} - 1$, the condition (2.3) has the form

$$\operatorname*{vrai\,sup}_{t \geq b} (t - h^+(t)) < 2(\sqrt{3} - 1).$$

Denote $\delta \stackrel{\text{def}}{=} 1 - \Delta$. Let us calculate

$$\frac{\Delta^2}{2} + \Delta - 1 + \varepsilon = \frac{1}{2} - \frac{\delta}{2}(4 - \delta) + \varepsilon, \quad -\frac{\Delta^2}{2} + \Delta + 1 - \varepsilon = \frac{3}{2} - \frac{\delta^2}{2} - \varepsilon$$

The inequalities

$$\frac{\Delta^2}{2}+\Delta-1+\varepsilon>\frac{1}{2},\quad -\frac{\Delta^2}{2}+\Delta+1-\varepsilon<\frac{3}{2}$$

take place for some δ and ε. Hence it follows that the inequalities

$$\operatorname*{vrai\,sup}_{t\geq b}\,(t-h^{+}(t))<\frac{3}{2},\quad \operatorname*{vrai\,inf}_{t\geq b}\,(t-h^{+}(t))>\frac{1}{2}$$

guarantee $\mathbf{D}_0$−stability of the equation (2.2).

Remark 2.2. For the equation (2.2) similar results were obtained with usage of the $\mathcal{W}$−method in the works of S. A. Gusarenko [18]–[20], S. A. Gusarenko and A.I. Domoshnitskiĭ [21].

Proof of Theorem 2.1. For $t\geq 0$ we make transformations

$$\dot{x}(t)+x_h(t)=\dot{x}(t)+x(t)-[x(t)-x_h(t)]=$$
$$=\dot{x}(t)+x(t)-\int\limits_{h^{+}(t)}^{t}\dot{x}(s)\,ds-\chi^{h}(t)x(0).$$

Here $\chi^{h}(t)\overset{\text{def}}{=}\begin{cases}0, & \text{if } h(t)>0,\\ 1, & \text{if } h(t)\leq 0.\end{cases}$ Further, by virtue of the equation (2.2) we get

$$\dot{x}(t)+x_h(t)=\dot{x}(t)+x(t)+\int\limits_{h^{+}(t)}^{t}x_h(s)\,ds-\int\limits_{h^{+}(t)}^{t}f(s)\,ds-\chi^{h}(t)x(0).\quad (2.4)$$

Denote $\Delta(t)\overset{\text{def}}{=}\min\{t,\Delta\}$ and $\nabla(t)\overset{\text{def}}{=}\max\{0,t-\Delta\}$. For some $\Delta\in(0,1)$ we write

$$\int\limits_{h^{+}(t)}^{t}x_h(s)\,ds=\int\limits_{h^{+}(t)}^{\nabla(t)}x_h(s)\,ds+\int\limits_{\nabla(t)}^{t}x_h(s)\,ds.$$

Further we transform the integral

$$J(t)\overset{\text{def}}{=}\int\limits_{\nabla(t)}^{t}x_h(s)\,ds=\int\limits_{\nabla(t)}^{t}\{x(t)-[x(t)-x_h(s)]\}\,ds=$$
$$=\Delta(t)x(t)-\int\limits_{\nabla(t)}^{t}\left[\int\limits_{h^{+}(s)}^{t}\dot{x}(\tau)\,d\tau+\chi^{h}(s)x(0)\right]ds=\Delta(t)x(t)+$$
$$+\int\limits_{\nabla(t)}^{t}\int\limits_{h^{+}(s)}^{t}x_h(\tau)\,d\tau ds-\int\limits_{\nabla(t)}^{t}\int\limits_{h^{+}(s)}^{t}f(\tau)\,d\tau ds-x(0)\int\limits_{\nabla(t)}^{t}\chi^{h}(s)\,ds=$$

$$= \Delta(t)x(t) + \int\limits_{\nabla(t)}^{t}\int\limits_{\nabla(t)}^{t} x_h(\tau)\,d\tau ds + \int\limits_{\nabla(t)}^{t}\int\limits_{h^+(s)}^{\nabla(t)} x_h(\tau)\,d\tau ds -$$
$$- \int\limits_{\nabla(t)}^{t}\int\limits_{h^+(s)}^{t} f(\tau)\,d\tau ds - x(0)\int\limits_{\nabla(t)}^{t} \chi^h(s)\,ds.$$

Since

$$\int\limits_{\nabla(t)}^{t}\int\limits_{\nabla(t)}^{t} x_h(\tau)\,d\tau ds = \Delta(t)J(t),$$

we have

$$J(t) = \frac{\Delta(t)}{1-\Delta(t)}x(t) +$$
$$+\frac{1}{1-\Delta(t)}\left[\int\limits_{\nabla(t)}^{t}\int\limits_{h^+(s)}^{\nabla(t)} x_h(\tau)\,d\tau ds - \int\limits_{\nabla(t)}^{t}\int\limits_{h^+(s)}^{t} f(\tau)\,d\tau ds - x(0)\int\limits_{\nabla(t)}^{t} \chi^h(s)\,ds\right].$$

If we substitute the latter expression in the formula (2.4), then we obtain

$$\dot{x}(t) + x_h(t) =$$
$$= \dot{x}(t) + x(t) + \frac{\Delta(t)}{1-\Delta(t)}x(t) + \frac{1}{1-\Delta(t)}\int\limits_{\nabla(t)}^{t}\int\limits_{h^+(s)}^{\nabla(t)} x_h(\tau)\,d\tau ds +$$
$$+ \int\limits_{h^+(t)}^{\nabla(t)} x_h(s)\,ds - \left[\frac{1}{1-\Delta(t)}\int\limits_{\nabla(t)}^{t} \chi^h(s)\,ds + \chi^h(t)\right]x(0) -$$
$$-\frac{1}{1-\Delta(t)}\int\limits_{\nabla(t)}^{t}\int\limits_{h^+(s)}^{t} f(\tau)\,d\tau ds - \int\limits_{h^+(t)}^{t} f(s)\,ds = f(t). \qquad (2.5)$$

Let the operator

$$(\mathcal{L}_\Delta x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + x(t) + \frac{\Delta(t)}{1-\Delta(t)}x(t) = \dot{x}(t) + \frac{1}{1-\Delta(t)}x(t)$$

be a model one. Apparently, for any $\Delta \in (0,1)$, the equality

$$\mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty) = \mathbf{D}(\mathcal{L}_\Delta, \mathbf{L}_\infty)$$

holds.

Let $\mathcal{W}_\Delta$ be the Cauchy operator for the model equation $\mathcal{L}_\Delta x = z$,

$$(\mathcal{W}_\Delta z)(t) = \int\limits_0^t W(t,s)z(s)\,ds, \quad \text{where} \quad W(t,s) = \exp\left[-\int\limits_s^t \frac{1}{1-\Delta(\tau)}\,d\tau\right]$$

the Cauchy function for this equation.

The equation (2.5) can be reduced to the equation

$$x(t) = (\mathcal{W}_\Delta\ \mathcal{M}x)(t) + \theta(t), \tag{2.6}$$

where

$$(\mathcal{M}x)(t) \stackrel{\text{def}}{=} -\frac{1}{1-\Delta(t)}\int\limits_{\nabla(t)}^{t}\int\limits_{h^+(s)}^{\nabla(t)} x_h(\tau)\,d\tau ds - \int\limits_{h^+(t)}^{\nabla(t)} x_h(s)\,ds,$$

$$\theta(t) \stackrel{\text{def}}{=} W(t,0)x(0) + (\mathcal{W}_\Delta\theta_1)(t),$$

$$\theta_1(t) \stackrel{\text{def}}{=} \left[\frac{1}{1-\Delta(t)}\int\limits_{\nabla(t)}^{t} \chi^h(s)\,ds + \chi^h(t)\right] x(0) +$$

$$+\frac{1}{1-\Delta(t)}\int\limits_{\nabla(t)}^{t}\int\limits_{h^+(s)}^{t} f(\tau)\,d\tau ds + \int\limits_{h^+(t)}^{t} f(s)\,ds + f(t).$$

If the inequality (2.3) guarantees the unique solvability of the equation (2.6) for any right hand side $\theta \in \mathbf{L}_\infty$, then the equation (2.2) will be $\mathbf{C}$-stable and moreover, $\mathbf{D}_0$-stable. Denote $\Omega \stackrel{\text{def}}{=} \mathcal{W}_\Delta\mathcal{M}$. The equation (2.6) can be rewritten as $x = \Omega x + \theta$, where $\Omega : \mathbf{C} \to \mathbf{C}$ is a linear bounded Volterra operator [11], [17].

By $\mathbf{C}_b \stackrel{\text{def}}{=} \mathbf{C}[0,b]$ denote the Banach space of continuous functions $x : [0,b] \to \mathbf{R}$ with the norm $||x||_{\mathbf{C}_b} \stackrel{\text{def}}{=} \max\limits_{t\in[0,b]} |x(t)|$; by $\mathbf{L}_{\infty,b} \stackrel{\text{def}}{=} \mathbf{L}_\infty[0,b]$ denote the Banach space of measurable and essentially bounded functions $z : [0,b] \to \mathbf{R}$ with the norm $||z||_{\mathbf{L}_{\infty,b}} \stackrel{\text{def}}{=} \operatorname{vrai\,sup}\limits_{t\in[0,b]}|z(t)|$; by $\mathbf{C}^b \stackrel{\text{def}}{=} \mathbf{C}[b,\infty)$ denote the Banach space of continuous and bounded functions $x : [b,\infty) \to \mathbf{R}$ with the norm $||x||_{\mathbf{C}^b} \stackrel{\text{def}}{=} \sup\limits_{t\geq b}|x(t)|$; by $\mathbf{L}^b_\infty \stackrel{\text{def}}{=} \mathbf{L}_\infty[b,\infty)$ denote the Banach space of measurable and essentially bounded functions $z : [b,\infty) \to \mathbf{R}$ with the norm $||z||_{\mathbf{L}^b_\infty} \stackrel{\text{def}}{=} \operatorname{vrai\,sup}\limits_{t\geq b}|z(t)|$. Let $(\mathbf{D}_0)_b \stackrel{\text{def}}{=} \mathbf{D}_0[0,b]$ be the Banach space of restrictions on the segment $[0,b]$ of all solutions of the equation (2.2) for all $z \in \mathbf{L}_\infty$. The following equivalent norms $||x||^*_{(\mathbf{D}_0)_b} \stackrel{\text{def}}{=} ||\dot{x}||_{\mathbf{L}_{\infty,b}} + |x(0)|$,

$\|x\|_{(\mathbf{D}_0)_b} \stackrel{\text{def}}{=} \|\dot{x}+x\|_{\mathbf{L}_{\infty,b}} + |x(0)|$, $\|x\|_{\mathbf{W}^1_\infty[0,b]} \stackrel{\text{def}}{=} \|\dot{x}\|_{\mathbf{L}_{\infty,b}} + \|x\|_{\mathbf{C}_b}$ can be used for an element $x \in (\mathbf{D}_0)_b$.

For any $x \in \mathbf{C}_b$ the operator $\Omega_b : \mathbf{C}_b \to \mathbf{C}_b$ is defined by $(\Omega_b x)(t) \stackrel{\text{def}}{=} (\Omega y_x)(t)$ for all $t \in [0,b]$, where $y_x \in \mathbf{C}$ is such a function that $y_x(t) \equiv x(t)$ for all $t \in [0,b]$. Similarly can be defined the operators $\mathcal{M}_b : \mathbf{C}_b \to \mathbf{L}_{\infty,b}$ and $(\mathcal{W}_\Delta)_b : \mathbf{L}_{\infty,b} \to \mathbf{C}_b$. Further, by virtue of the construction of the operator Ω, for any $x \in \mathbf{C}^b$ the operator $\Omega^b : \mathbf{C}^b \to \mathbf{C}^b$ is defined by $(\Omega^b x)(t) \stackrel{\text{def}}{=} (\Omega y_x \chi_{[b,\infty)})(t)$ for all $t \geq b$, where $y_x \in \mathbf{C}$ is such a function that $y_x(t) \equiv x(t)$ for all $t \geq b$, $\chi_{[b,\infty)}$ is the characteristic function of the semi-axis $[b,\infty)$. Similarly can be defined the operators $\mathcal{M}^b : \mathbf{C}^b \to \mathbf{L}^b_\infty$ and $\mathcal{W}^b_\Delta : \mathbf{L}^b_\infty \to \mathbf{C}^b$.

Prove that for any $b > 0$ the equation $x(t) - (\Omega_b x)(t) = \theta_b(t)$, $t \in [0,b]$, is uniquely solvable in the space $\mathbf{C}_b$ for every $\theta_b \in \mathbf{C}_b$. Indeed, by the conditions of Theorem 2.1 the operator $\Omega_b : \mathbf{C}_b \to \mathbf{C}_b$ is completely continuous as the direct product of the linear bounded operator $\mathcal{M}_b : \mathbf{C}_b \to \mathbf{L}_{\infty,b}$ and the linear completely continuous operator $(\mathcal{W}_\Delta)_b : \mathbf{L}_{\infty,b} \to \mathbf{C}_b$. The latter property follows from boundedness of the operator $(\mathcal{W}_\Delta)_b : \mathbf{L}_{\infty,b} \to (\mathbf{D}_0)_b$ and compactness of embedding of the space $(\mathbf{D}_0)_b$ in the space $\mathbf{C}_b$.

In the paper [22] it was proved that $\rho(\Lambda) = 0$ for any completely continuous linear Volterra operator $\Lambda : \mathbf{C}_b \to \mathbf{C}_b$ with the condition $(\Lambda x)(0) = 0$ for any $x \in \mathbf{C}_b$, $x(0) = 0$. Here by $\rho(\Lambda)$ denote the spectral radius of the operator Λ. Hence we obtain $\rho(\Omega_b) = 0$. As shown in the article [22], in that case for the unique solvability of the equation (2.6) it is enough to estimate $\rho(\Omega, +\infty)$. Denote by $\rho(\Omega, +\infty)$ the spectral radius of the operator Ω at the point $+\infty$. In the paper [22] the estimate $\rho(\Omega, +\infty) \leq \lim\limits_{b \to +\infty} \|\Omega^b\|_{\mathbf{C}^b \to \mathbf{C}^b}$ was obtained.

Prove that the inequality (2.3) guarantees the estimate $\|\Omega^b\|_{\mathbf{C}^b \to \mathbf{C}^b} < 1$. Thus we have

$$\int\limits_{\nabla(t)}^{t} \int\limits_{h^+(s)}^{\nabla(t)} x_h(\tau)\, d\tau ds = \int\limits_{\nabla(t)}^{t} \int\limits_{\nabla(s)}^{\nabla(t)} x_h(\tau)\, d\tau ds + \int\limits_{\nabla(t)}^{t} \int\limits_{h^+(s)}^{\nabla(s)} x_h(\tau)\, d\tau ds.$$

Let us estimate for $b \geq \Delta$ the norm of the operator $\mathcal{M}^b : \mathbf{C}^b \to \mathbf{L}^b_\infty$:

$$\|\mathcal{M}^b\|_{\mathbf{C}^b \to \mathbf{L}^b_\infty} \leq \frac{1}{1-\Delta} \left[\operatorname*{vrai\,sup}_{t \geq b} \int\limits_{t-\Delta}^{t} \int\limits_{s-\Delta}^{t-\Delta} d\tau ds + \operatorname*{vrai\,sup}_{t \geq b} \int\limits_{t-\Delta}^{t} \int\limits_{h^+(s)}^{s-\Delta} d\tau ds \right] + \\ + \operatorname*{vrai\,sup}_{t \geq b} \int\limits_{h^+(t)}^{t-\Delta} ds \leq$$

$$\leq \frac{1}{1-\Delta}\left\{\operatorname*{vrai\,sup}_{t\geq b}\int\limits_{t-\Delta}^{t}(t-s)\,ds+\operatorname*{vrai\,sup}_{t\geq b}\int\limits_{t-\Delta}^{t}[s-\Delta-h^{+}(s)]\,ds\right\}+$$
$$+\operatorname*{vrai\,sup}_{t\geq b}|t-\Delta-h^{+}(t)|\leq$$
$$\leq \frac{1}{1-\Delta}\left[\frac{\Delta^2}{2}+\Delta\operatorname*{vrai\,sup}_{s\geq b}|s-\Delta-h^{+}(s)|\right]+\operatorname*{vrai\,sup}_{t\geq b}|t-\Delta-h^{+}(t)|.$$

Calculate for $b\geq\Delta$ the norm of the operator $\mathcal{W}^b_\Delta:\mathbf{L}^b_\infty\to\mathbf{C}^b$:

$$\|\mathcal{W}^b_\Delta\|_{\mathbf{L}^b_\infty\to\mathbf{C}^b}=\operatorname*{vrai\,sup}_{t\geq b}\int\limits_{b}^{t}\exp\left[-\frac{1}{1-\Delta}(t-s)\right]ds=$$
$$=(1-\Delta)\operatorname*{vrai\,sup}_{t\geq b}\left[1-\exp\left(-\frac{t}{1-\Delta}\right)\right]=1-\Delta.$$

Calculate and estimate:

$$\|\Omega_b\|_{\mathbf{C}^b\to\mathbf{C}^b}=\|\mathcal{W}^b_\Delta\mathcal{M}^b\|_{\mathbf{C}^b\to\mathbf{C}^b}\leq$$
$$\leq(1-\Delta)\frac{1}{1-\Delta}\left[\frac{\Delta^2}{2}+\Delta\operatorname*{vrai\,sup}_{s\geq b}|s-\Delta-h^{+}(s)|\right]+$$
$$+(1-\Delta)\operatorname*{vrai\,sup}_{t\geq b}|t-\Delta-h^{+}(t)|=\frac{\Delta^2}{2}+\Delta\operatorname*{vrai\,sup}_{s\geq b}|s-\Delta-h^{+}(s)|+$$
$$+(1-\Delta)\operatorname*{vrai\,sup}_{t\geq 0}|t-\Delta-h^{+}(t)|=\frac{\Delta^2}{2}+\operatorname*{vrai\,sup}_{t\geq 0}|t-\Delta-h^{+}(t)|.$$

To conclude the proof, it remains to note that for $b\geq\Delta$ the condition (2.3) guarantees the estimate $\|\mathcal{W}^b_\Delta\mathcal{M}^b\|_{\mathbf{C}^b\to\mathbf{C}^b}<1$. Therefore, the equation (2.2) is $\mathbf{C}$-stable. Moreover, this equation is $\mathbf{D}_0$-stable.

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