Memoirs on Differential Equations and Mathematical Physics
Volume 25, 2002, 1–104

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DIFFERENTIAL AND INTEGRAL EQUATIONS IN THE SPACE OF REGULATED FUNCTIONS

Abstract. ???.

2000 Mathematics Subject Classification. ???.
Key words and phrases: ???.

Preface

This text is intended as a self-contained exposition of generalized linear differential and integral equations whose solutions are in general regulated functions (i.e. functions which can have only discontinuities of the first kind). In particular, the problems studied below cover as their special cases linear problems for systems with impulses. (For representative surveys of results concerning systems with impulses see e.g. [4], [6] or [58].) Essentially, the text is a collection of the papers [48], [49], [50], [51] and [52]. In comparison with the original versions of these papers, the notation used was unified and the common preliminaries were summarized in Chapter 1. Furthermore, some minor improvements of the exposition and corrections of several misprints were included.

Chapter 2 is a compilation of the papers [48] and [49]. In this chapter the properties of the Perron-Stieltjes integral with respect to regulated functions are investigated. It is shown that linear continuous functionals on the spaces $\mathbb{G}_{\mathrm{L}}[a,b]$ of functions regulated on $[a,b]$ and left-continuous on (a,b) and $\mathbb{G}_{\mathrm{reg}}[a,b]$ of functions regulated on $[a,b]$ and regular on (a,b) may be represented in the form

$$\Phi(x) = q\,x(a) + \int_a^b p(t)\,\mathrm{d}[x(t)],$$

where $q \in \mathbb{R}$ and $p(t)$ is a function of bounded variation on $[a,b]$. Some basic theorems (e.g. integration-by-parts formula, substitution theorem) known for the Perron-Stieltjes integral with respect to functions of bounded variation are extended to a more general case.

In Chapter 3 (cf. [52]) the continuous dependence of solutions to linear generalized differential equations of the form

$$x(t) = x(0) + \int_0^t \mathrm{d}[A_k(s)]\,x(s), \quad t \in [0,1]$$

on a parameter $k \in \mathbb{N}$ is discussed. In particular, known results due to Š. Schwabik [41] and M. Ashordia [1] are extended or amended.

Boundary value problems of the form

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\,x(s) = f(t) - f(0), \quad t \in [0,1],$$
$$M\,x(0) + \int_0^1 K(\tau)\,\mathrm{d}[x(\tau)] = r$$

and the corresponding controllability problems are dealt with in Chapter 4. (This chapter is based on the paper [49].) The adjoint problems are given in such a way that the usual duality theorems are valid. As a special case the interface boundary value problems are included. In contrast to the earlier papers (cf. e.g. [54], [46], [47], [43], [44] and the monograph [45]) the right-hand side of the generalized differential equation as well as the solutions of this equation can be in general regulated functions (not necessarily of bounded variation). Similar problems in the space of regulated functions were treated e.g. by Ch. S. Hönig [15], [17], [16], L. Fichmann [9] and L. Barbanti [5], who made use of the interior (Dushnik) integral. In our case the integral is the Perron-Stieltjes (Kurzweil) integral.

In Chapter 5 (cf. [51]) we investigate systems of linear integral equations in the space $\mathbb{G}_L^n$ of n-vector valued functions which are regulated on the closed interval $[0,1]$ and left-continuous on its interior $(0,1)$. In particular, we are interested in systems of the form

$$x(t) - A(t)\,x(0) - \int_0^1 B(t,s)\,\mathrm{d}[x(s)] = f(t),$$

where the n-vector valued function f and an $n \times n$-matrix valued function A are regulated on $[0,1]$ and left-continuous on $(0,1)$ and the entries of $B(t,.)$ have a bounded variation on $[0,1]$ for any $t \in [0,1]$ and the mapping $t \in [0,1] \mapsto B(t,.)$ is regulated on $[0,1]$ and left-continuous on $(0,1)$ as the mapping with values in the space of $n \times n$-matrix valued functions of bounded variation on $[0,1]$. We prove basic existence and uniqueness results for the given equation and obtain the explicit form of its adjoint equation. A special attention is paid to the Volterra (causal) type case. It is shown that in that case the given equation possesses a unique solution for any right-hand side from $\mathbb{G}_L^n$, and its representation by means of resolvent operators is given. The results presented cover e.g. the results known for systems of linear Stieltjes integral equations

$$x(t) - \int_0^1 \mathrm{d}_s[K(t,s)]\,x(s) = g(t) \quad \text{or} \quad x(t) - \int_0^t \mathrm{d}_s[K(t,s)]\,x(s) = g(t).$$

The study of such equations in the space of functions of bounded variation was initiated mainly by Š. Schwabik (see [35], [38] and [45]).

Chapter 1

Preliminaries

1.1. Basic notions

Throughout this text we denote by $\mathbb{N}$ the set of positive integers, $\mathbb{R}$ is the space of real numbers, $\mathbb{R}^{m\times n}$ is the space of real $m\times n$-matrices, $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ stands for the space of real column n-vectors and $\mathbb{R}^{1\times 1} = \mathbb{R}^1 = \mathbb{R}$.

For a matrix $A \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A)$ denotes its rank and A^{T} is its transpose. Furthermore, the elements of A are usually denoted by $a_{i,j}$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, and the norm $|A|$ of A is defined by

$$|A| = \max_{j=1,2,\dots,n} \sum_{i=1}^{m} |a_{i,j}|.$$

We have

$$A = (a_{i,j})_{\substack{i=1,2,\dots,m\\ j=1,2,\dots,n}} \quad \text{and} \quad A^{\mathrm{T}} = (a_{j,i})_{\substack{j=1,2,\dots,n\\ i=1,2,\dots,m}} \quad \text{for} \quad A \in \mathbb{R}^{m\times n}$$

and

$$|x| = \sum_{i=1}^{n} |x_i|, \;\; x^{\mathrm{T}} = (x_1, x_2, \dots, x_n) \;\; \text{and} \;\; |x^{\mathrm{T}}| = \max_{j=1,\dots,n} |x_j| \;\; \text{for} \;\; x \in \mathbb{R}^n.$$

Furthermore, for a matrix $A \in \mathbb{R}^{m\times n}$, its columns are denoted by $a^{[j]}$ $(A = (a^{[j]})_{j=1,2,\dots,n})$. Obviously,

$$|A| = \max_{j=1,2,\dots,n} |a^{[j]}| \quad \text{for all} \quad A \in \mathbb{R}^{m\times n}.$$

The symbols I and 0 stand respectively for the identity and the zero matrix of the proper type. For an $n\times n$-matrix A, $\det(A)$ denotes its determinant.

If $-\infty < a < b < \infty$, then $[a,b]$ and (a,b) denote the corresponding closed and open intervals, respectively. Furthermore, $[a,b)$ and $(a,b]$ are the corresponding half-open intervals.

The sets $d = \{t_0, t_1, \dots, t_m\}$ of points in the closed interval $[a, b]$ such that $a = t_0 < t_1 < \dots < t_m = b$ are called ***divisions*** of $[a, b]$. Given a division d of $[a, b]$, its elements are usually denoted by $t_0, t_1, \dots, t_m$. The set of all divisions of the interval $[a, b]$ is denoted by $\mathcal{D}[a, b]$.

Given $M \subset \mathbb{R}$, χ_M denotes its characteristic function ($\chi_M(t) = 1$ if $t \in M$ and $\chi_M(t) = 0$ if $t \notin M$.)

Finally, if $\mathbb{X}$ is a Banach space and $M \subset \mathbb{X}$, then $\mathrm{cl}(M)$ stands for the closure of M in $\mathbb{X}$.

1.2. Functions

Regulated functions 1.2.1. A function $F : [a, b] \mapsto \mathbb{R}^{m\times n}$ which has limits

$$F(t+) = \lim_{\tau\to t+} F(\tau) \in \mathbb{R}^{m\times n} \quad \text{and} \quad F(s-) = \lim_{\tau\to s-} F(\tau) \in \mathbb{R}^{m\times n}$$

for all $t \in [a, b)$ and all $s \in (a, b]$ is said to be ***regulated*** on $[a, b]$. The set of all $m \times n$-matrix valued regulated functions on $[a, b]$ is denoted by $\mathbb{G}^{m\times n}[a, b]$. For $F \in \mathbb{G}^{m\times n}[a, b]$, we put $F(a-) = F(a)$ and $F(b+) = F(b)$. Furthermore, for any $t \in [a, b]$ we define

$$\Delta^+ F(t) = F(t+) - F(t), \ \Delta^- F(t) = F(t) - F(t-) \\ \text{and} \quad \Delta F(t) = F(t+) - F(t-).$$

(In particular, we have $\Delta^- F(a) = \Delta^+ F(b) = 0$, $\Delta F(a) = \Delta^+ F(a)$ and $\Delta F(b) = \Delta^- F(b)$.) We shall write $\mathbb{G}^n[a, b]$ instead of $\mathbb{G}^{n\times 1}[a, b]$, $\mathbb{G}^{1\times 1}[a, b] = \mathbb{G}[a, b]$. Obviously, $F \in \mathbb{G}^{m\times n}[a, b]$ if and only if all its components $f_{ij} : [a, b] \mapsto \mathbb{R}$ are regulated on $[a, b]$ ($f_{ij} \in \mathbb{G}[a, b]$ for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$).

It is known (cf. [15, Corollary 3.2a]) that if $F \in \mathbb{G}^{m\times n}[a, b]$, then for any $\varepsilon > 0$ the set of points $t \in [a, b]$ such that $|\Delta^+ F(t)| > \varepsilon$ or $|\Delta^- F(t)| > \varepsilon$ is finite. Consequently, for any $F \in \mathbb{G}[a, b]$ the set of its discontinuities in $[a, b]$ is countable. The subset of $\mathbb{G}^{m\times n}[a, b]$ consisting of all functions regulated on $[a, b]$ and left-continuous on (a, b) will be denoted by $\mathbb{G}_{\mathrm{L}}^{m\times n}[a, b]$.

The set of all functions $F \in \mathbb{G}^{m\times n}[a, b]$ which are ***regular*** on (a, b), i.e.

$$2\,F(t) = F(t-) + F(t+) \quad \text{for all} \quad t \in (a, b),$$

will be denoted by $\mathbb{G}_{\mathrm{reg}}^{m\times n}[a, b]$.

We define

$$\|F\| = \sup_{t\in[a,b]} |F(t)| \quad \text{for} \quad F \in \mathbb{G}^{m\times n}[a, b].$$

Clearly, $\|F\| < \infty$ for any $F \in \mathbb{G}^{m\times n}[a, b]$ and when endowed with this norm, $\mathbb{G}^{m\times n}[a, b]$ becomes a Banach space (cf. [15, Theorem 3.6]). As $\mathbb{G}_{\mathrm{L}}^{m\times n}[a, b]$ and $\mathbb{G}_{\mathrm{reg}}^{m\times n}[a, b]$ are closed in $\mathbb{G}^{m\times n}[a, b]$, they are also Banach spaces.

Functions of bounded variation 1.2.2. For $F:[a,b]\mapsto\mathbb{R}^{m\times n}$ its ***variation*** $\operatorname{var}_a^b F$ on the interval $[a,b]$ is defined by

$$\operatorname{var}_a^b F = \sup_{d\in\mathcal{D}[a,b]} \sum_{j=1}^{m} |F(t_j)-F(t_{j-1})|$$

(the supremum is taken over all divisions $d=\{t_0,t_1,\dots,t_m\}\in\mathcal{D}[a,b]$ of $[a,b]$). If $\operatorname{var}_a^b F<\infty$, we say that the function F has a ***bounded variation*** on the interval $[a,b]$. $\mathbb{BV}^{m\times n}[a,b]$ denotes the Banach space of $m\times n$-matrix valued functions of bounded variation on $[a,b]$ equipped with the norm

$$F\in\mathbb{BV}^{m\times n}[a,b]\mapsto \|F\|_{\mathbb{BV}} = |F(a)|+\operatorname{var}_a^b F.$$

Similarly as in the case of regulated functions, We shall write $\mathbb{BV}^n[a,b]$ instead of $\mathbb{BV}^{n\times 1}[a,b]$ and $\mathbb{BV}[a,b]$ instead of $\mathbb{BV}^{1\times 1}[a,b]$. $F\in\mathbb{BV}^{m\times n}[a,b]$ if and only if $f_{ij}\in\mathbb{BV}[a,b]$ for all $i=1,2,\dots,m$ and $j=1,2,,\dots,n$.

A function $f:[a,b]\mapsto\mathbb{R}$ is called a ***finite step function*** on $[a,b]$ if there exists a division $\{t_0,t_1,\dots,t_m\}$ of $[a,b]$ such that f is constant on every open interval (t_{j-1},t_j), $j=1,2,\dots,m$. The set of all finite step functions on $[a,b]$ is denoted by $\mathbb{S}[a,b]$. It is known that $\mathbb{S}[a,b]$ is dense in $\mathbb{G}[a,b]$ (cf. [15, Theorem 3.1]). It means that $f:[a,b]\mapsto\mathbb{R}$ is regulated if and only if it is a uniform limit on $[a,b]$ of a sequence of finite step functions.

A function $f:[a,b]\mapsto\mathbb{R}$ is called a ***break function*** on $[a,b]$ if there exist sequences

$$\{t_k\}_{k=1}^{\infty}\subset[a,b],\quad \{c_k^-\}_{k=1}^{\infty}\subset\mathbb{R}\quad\text{and}\quad \{c_k^+\}_{k=1}^{\infty}\subset\mathbb{R}$$

such that $t_k\neq t_j$ for $k\neq j$, $c_k^-=0$ if $t_k=a$, $c_k^+=0$ if $t_k=b$,

$$\sum_{k=1}^{\infty}(|c_k^-|+|c_k^+|)<\infty$$

and

$$f(t)=\sum_{t_k\le t} c_k^- + \sum_{t_k<t} c_k^+ \qquad \text{for}\quad t\in[a,b] \tag{1.2.1}$$

or equivalently

$$f(t)=\sum_{k=1}^{\infty} c_k^-\,\chi_{[t_k,b]}(t)+c_k^+\,\chi_{(t_k,b]}(t)\quad\text{for}\quad t\in[a,b].$$

Clearly, if f is given by (1.2.1), then

$$\Delta^+f(t_k)=c_k^+\quad\text{and}\quad\Delta^-f(t_k)=c_k^-\quad\text{for}\quad k\in\mathbb{N}.$$

Furthermore, for any such function we have

$$f(a)=0,\quad f(t-)=f(t)=f(t+)\ \text{if}\ \ t\in[a,b]\setminus\{t_k\}_{k=1}^{\infty}$$

and

$$\operatorname{var}_a^b f = \sum_{k=1}^{\infty} (|c_k^-| + |c_k^+|).$$

The set of all break functions on $[a,b]$ is denoted by $\mathbb{B}[a,b]$. Notice that neither $\mathbb{S}[a,b]$ nor $\mathbb{B}[a,b]$ are closed in $\mathbb{G}[a,b]$.

It is known that for any $f \in \mathbb{BV}[a,b]$ there exist uniquely determined functions $f^{\mathrm{C}} \in \mathbb{BV}[a,b]$ and $f^{\mathrm{B}} \in \mathbb{BV}[a,b]$ such that f^{C} is continuous on $[a,b]$, f^{B} is a break function on $[a,b]$ and $f(t) = f^{\mathrm{C}}(t) + f^{\mathrm{B}}(t)$ on $[a,b]$ (the ***Jordan decomposition*** of $f \in \mathbb{BV}[a,b]$). In particular, if $W = \{w_k\}_{k\in\mathbb{N}}$ is the set of discontinuities of f in $[a,b]$, then

$$f^{\mathrm{B}}(t) = \sum_{k=1}^{\infty} \big(\Delta^- f(w_k)\, \chi_{[w_k,b]}(t) + \Delta^+ f(w_k)\, \chi_{(w_k,b]}(t)\big) \quad \text{on} \quad [a,b]. \tag{1.2.2}$$

Moreover, if we define

$$f_n^{\mathrm{B}}(t) = \sum_{k=1}^{n} \big(\Delta^- f(w_k)\, \chi_{[w_k,b]}(t) + \Delta^+ f(w_k)\, \chi_{(w_k,b]}(t)\big) \tag{1.2.3}$$

for $t \in [a,b]$ and $n \in \mathbb{N}$ then

$$\lim_{n\to\infty} \|f_n^{\mathrm{B}} - f^{\mathrm{B}}\|_{\mathbb{BV}} = 0$$

(cf. e.g. [45, the proof of Lemma I.4.23]). Obviously,

$$\mathbb{S}[a,b] \subset \mathbb{B}[a,b] \subset \mathbb{BV}[a,b] \subset \mathbb{G}[a,b].$$

For more details concerning regulated functions or functions of bounded variation see the monographs by G. Aumann [3], T. H. Hildebrandt [14] and Ch. S. Hönig [15] and the papers by D. Fraňková [10] and [11].

1.2.3. As usual, the space of $m \times n$-matrix valued functions continuous on $[a,b]$ is denoted by $\mathbb{C}^{m\times n}[a,b]$ and the space of $m \times n$-matrix valued functions Lebesgue integrable on $[a,b]$ is denoted by $\mathbb{L}_1^{m\times n}[a,b]$. For given $F \in \mathbb{L}_1^{m\times n}[a,b]$ and $G \in \mathbb{C}^{m\times n}[a,b]$, the corresponding norms are defined by

$$\|F\|_{\mathbb{L}_1} = \int_a^b |F(t)|\, \mathrm{d}t \quad \text{and} \quad \|G\|_{\mathbb{C}} = \|g\| = \sup_{t\in[a,b]} |G(t)|.$$

Again, $\mathbb{C}^{n\times 1}[a,b] = \mathbb{C}^n[a,b]$, $\mathbb{C}^{1\times 1}[a,b] = \mathbb{C}[a,b]$, $\mathbb{L}_1^{n\times 1}[a,b] = \mathbb{L}_1^n[a,b]$ and $\mathbb{L}_1^{1\times 1}[a,b] = \mathbb{L}_1[a,b]$. Moreover, the space of $m \times n$-matrix valued functions absolutely continuous on $[a,b]$ is denoted by $\mathbb{AC}^{m\times n}[a,b]$, $\mathbb{AC}^{n\times 1}[a,b] = \mathbb{AC}^n[a,b]$, $\mathbb{AC}^{1\times 1}[a,b] = \mathbb{AC}[a,b]$, and

$$\|F\|_{\mathbb{AC}} = |F(a)| + \|F'\|_{\mathbb{L}_1} \quad \text{for} \quad F \in \mathbb{AC}^{n\times n}[a,b].$$

Notation 1.2.4. If $[a,b]=[0,1]$, we write simply $\mathbb{G}$ instead of $\mathbb{G}[0,1]$. Similar abbreviations are used for all the other symbols for function spaces introduced in this chapter.

Functions of two real variables 1.2.5. Let $-\infty < a < b < \infty$, $-\infty < c < d < \infty$ and let $F : [c,d] \times [a,b] \mapsto \mathbb{R}^{m\times n}$. If $t \in [c,d]$ and $s \in [a,b]$ are given, then the symbols $\operatorname{var}_a^b F(t,.)$ and $\operatorname{var}_c^d F(.,s)$ denote the variations of the functions

$$F(t,.) : \tau \in [a,b] \mapsto F(t,\tau) \in \mathbb{R}^{m\times n}$$

and

$$F(.,s) : \tau \in [c,d] \mapsto F(\tau,s) \in \mathbb{R}^{m\times n},$$

respectively. Furthermore, for $s \in [a,b]$ we put

$$\Delta_1^- F(\tau,s) = F(\tau,s) - F(\tau-,s) \text{ if } \tau \in (c,d], \quad \Delta_1^- F(c,s) = 0$$

and

$$\Delta_1^+ F(\tau,s) = F(\tau+,s) - F(\tau,s) \text{ if } \tau \in [c,d), \quad \Delta_1^+ F(d,s) = 0.$$

Similarly, for $t \in [c,d]$ we put

$$\Delta_2^- F(t,\sigma) = F(t,\sigma) - F(t,\sigma-) \text{ if } \sigma \in (a,b], \quad \Delta_2^- F(t,a) = 0$$

and

$$\Delta_2^+ F(t,\sigma) = F(t,\sigma+) - F(t,\sigma) \text{ if } \sigma \in [a,b), \quad \Delta_2^+ F(t,b) = 0.$$

The symbol $\mathrm{v}_{[c,d]\times[a,b]}(F)$ stands for the ***Vitali variation*** of F on $[c,d]\times[a,b]$ defined by

$$\mathrm{v}_{[c,d]\times[a,b]}(F) =$$

$$= \sup_D \sum_{i,j=1}^m |F(t_i,s_j) - F(t_{i-1},s_j) - F(t_i,s_{j-1}) + F(t_{i-1},s_{j-1})| < \infty,$$

where the supremum is taken over all net subdivisions

$$D = \{c = t_0 < t_1 < \cdots < t_m = d;\, a = s_0 < s_1 < \cdots < s_m = b\}$$

of the interval $[c,d]\times[a,b]$. We say that the function F has a bounded Vitali variation on $[c,d]\times[a,b]$ if $\mathrm{v}_{[c,d]\times[a,b]}(F) < \infty$. Moreover, F is said to be of ***strongly bounded variation*** on $[c,d]\times[a,b]$ if

$$\mathrm{v}_{[c,d]\times[a,b]}(F) + \operatorname{var}_a^b F(c,.) + \operatorname{var}_c^d F(.,a) < \infty.$$

The set of $n\times n$-matrix valued functions of strongly bounded variation on $[c,d]\times[a,b]$ is denoted by $\mathcal{SBV}^{n\times n}([c,d]\times[a,b])$.

If no misunderstanding can arise, instead of $\mathrm{v}_{[c,d]\times[a,b]}(F)$ we shall write simply $\mathrm{v}(F)$ and instead of $\mathcal{SBV}^{n\times n}([0,1]\times[0,1])$ we shall write $\mathcal{SBV}^{n\times n}$. (For the basic properties of the Vitali variation and of the set $\mathcal{SBV}$, see [14, Section III.4] and [45, Section I.6].)

1.3. Integrals and operators

Perron-Stieltjes integral 1.3.1. The integrals which occur in this text are Perron - Stieltjes integrals. We will work with the following definition which is a special case of the definition due to J. Kurzweil [19]:

Let $-\infty < a < b < \infty$. The couples $D = (d, \xi)$, where $d = \{t_0, t_1, \ldots, t_m\} \in \mathcal{D}[a,b]$ is a division of $[a,b]$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m$ is such that

$$t_{j-1} \le \xi_j \le t_j \quad \text{for all} \quad j = 1, 2, \ldots, m$$

are called ***partitions*** of $[a,b]$. The set of all partitions of the interval $[a,b]$ is denoted by $\mathcal{P}[a,b]$. An arbitrary positive valued function $\delta : [a,b] \mapsto (0, \infty)$ is called a ***gauge*** on $[a,b]$. Given a gauge δ on $[a,b]$, the partition (d, ξ) of $[a,b]$ is said to be δ***-fine*** if

$$[t_{j-1}, t_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for any} \quad j = 1, 2, \ldots, m.$$

For given functions $f, g : [a,b] \mapsto \mathbb{R}$ and a partition $D = (d, \xi) \in \mathcal{P}[a,b]$ of $[a,b]$ let us define

$$S_D(f\,\Delta g) = \sum_{j=1}^{m} f(\xi_j)\,[g(t_j) - g(t_{j-1})].$$

We say that $I \in \mathbb{R}$ is the ***Kurzweil integral*** of f with respect to g from a to b and denote

$$I = \int_a^b f(t)\,\mathrm{d}[g(t)] \quad \text{or} \quad I = \int_a^b f\,\mathrm{d}g$$

if for any $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$|I - S_D(f\,\Delta g)| < \varepsilon \text{ for all } \ \delta - \text{fine partitions } \ D \text{ of } \ [a,b].$$

For the definition of the Kurzweil integral it is necessary to mention the fundamental fact that given an arbitrary gauge δ on $[a,b]$, the set of all δ-fine partitions of $[a,b]$ is non-empty (Cousin's lemma). The ***Perron-Stieltjes integral*** with respect to a function not necessarily of bounded variation was defined by A. J. Ward [55] (cf. also S. Saks [32, Chapter VI]). It can be shown that the Kurzweil integral is equivalent to the Perron-Stieltjes integral (cf. [36, Theorem 2.1], where the assumption $g \in \mathbb{BV}[a,b]$ is not used in the proof and may be omitted). Consequently, the Riemann-Stieltjes integral (both of the norm type and of the σ-type, cf. [14]) is its special case. The relationship between the Kurzweil integral, the σ-Young-Stieltjes integral and the Perron-Stieltjes integral was described by Š. Schwabik (cf. [36] and [37]).

It is well known (cf. e.g. [45, Theorems I.4.17, I.4.19 and Corollary I.4.27] that if $f \in \mathbb{G}[a,b]$ and $g \in \mathbb{BV}[a,b]$, then the integral

$$\int_a^b f\,\mathrm{d}g$$

exists and the inequality

$$\left| \int_a^b f \, \mathrm{d}g \right| \le \|f\| \operatorname{var}_a^b g$$

is true. The Kurzweil integral is an additive function of interval and possesses the usual linearity properties. For the proofs of these assertions and some more details concerning the Kurzweil integral with respect to functions of bounded variation see e.g. [19], [21], [40] and [45].

For matrix valued functions $F : [a, b] \mapsto \mathbb{R}^{p \times q}$ and $G : [a, b] \mapsto \mathbb{R}^{q \times n}$ such that all integrals

$$\int_a^b f_{i,k}(t) \, \mathrm{d}[g_{k,j}(t)] \quad (i = 1, 2, \dots, p;\ k = 1, 2, \dots, q;\ j = 1, 2, \dots, n)$$

exist (i.e. they have finite values), the symbol

$$\int_a^b F(t) \, \mathrm{d}[G(t)] \quad \text{(or more simply} \quad \int_a^b F \, \mathrm{d}G)$$

stands for the $p \times n-$matrix M with the entries

$$m_{i,j} = \sum_{k=1}^{q} \int_a^b f_{i,k} \, \mathrm{d}[g_{k,j}] \quad (i = 1, 2, \dots, p; j = 1, 2, \dots, n).$$

The integrals

$$\int_a^b \mathrm{d}[F] \, G \quad \text{and} \quad \int_a^b F \, \mathrm{d}[G] \, H$$

for matrix valued functions F, G and H of proper types are defined analogously.

Linear operators 1.3.2. For linear spaces $\mathbb{X}$ and $\mathbb{Y}$, the symbols $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ and $\mathcal{K}(\mathbb{X}, \mathbb{Y})$ denote the linear space of linear bounded mappings of $\mathbb{X}$ into $\mathbb{Y}$ and the linear space of linear compact mappings of $\mathbb{X}$ into $\mathbb{Y}$, respectively. If $\mathbb{X} = \mathbb{Y}$ we write $\mathcal{L}(\mathbb{X})$ and $\mathcal{K}(\mathbb{X})$. If $\mathscr{A} \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$, then $\mathscr{R}(\mathscr{A})$, $\mathscr{N}(\mathscr{A})$ and $\mathscr{A}^*$ denote its range, null space and adjoint operator, respectively. For $P \subset \mathbb{Y}$ and $\mathscr{A} \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$, the symbol $\mathscr{A}_{-1}(P)$ denotes the set of all $x \in \mathbb{X}$ for which $\mathscr{A}\, x \in P$.

Chapter 2

Regulated Functions and the Perron-Stieltjes Integral

2.1. Introduction

This chapter deals with the space $\mathbb{G}[a,b]$ of regulated functions on a compact interval $[a,b]$. It is known that when equipped with the supremal norm $\mathbb{G}[a,b]$ becomes a Banach space, and linear bounded functionals on its subspace $\mathbb{G}_{\mathrm{L}}[a,b]$ of functions regulated on $[a,b]$ and left-continuous on (a,b) can be represented by means of the ***Dushnik-Stieltjes (interior) integral*** . This result is due to H. S. Kaltenborn [18], cf. also Ch. S. Hönig [15, Theorem 5.1]. Together with the known relationship between the Dushnik-Stieltjes integral, the σ***-Young-Stieltjes integral*** and the Perron-Stieltjes integral (cf. Ch. S. Hönig [16] and Š. Schwabik [36], [37]) this enables us to see that Φ is a linear bounded functional on $\mathbb{G}_{\mathrm{L}}[a,b]$ if and only if there exists a real number q and a function $p(t)$ of bounded variation on $[a,b]$ such that

$$\Phi(x) = q\,x(a) + \int_a^b p(t)\,\mathrm{d}[x(t)] \quad \text{for any} \quad x \in \mathbb{G}_{\mathrm{L}}[a,b],$$

where the integral is the Perron-Stieltjes integral. We will give here the proof of this fact based only on the properties of the Perron-Stieltjes integral. To this aim, the proof of existence of the integral

$$\int_a^b f(t)\,\mathrm{d}[g(t)]$$

for any function f of bounded variation on $[a,b]$ and any function g regulated on $[a,b]$ is crucial. Furthermore, we will prove extensions of some theorems (e.g. integration-by-parts and substitution theorems) needed for dealing with generalized differential equations and Volterra-Stieltjes integral equations in the space $\mathbb{G}[a,b]$. Finally, a representation of a general linear bounded functional on the space of regular regulated functions on a compact interval is given.

Since we will make use of some of the properties of the ***σ-Riemann-Stieltjes integral*** , let us indicate here the proof that this integral is included in the Kurzweil integral. (For the definition of the σ-Riemann-Stieltjes integral, see e.g. [14, Sec. II.9].)

2.2 . Preliminaries

Proposition 2.2.1. *Let $f, g : [a,b] \mapsto \mathbb{R}$ and let $I \in \mathbb{R}$ be such that the σ-Riemann-Stieltjes integral*

$$\sigma \int_a^b f \, \mathrm{d}g \quad \textit{exists and equals } \ I.$$

Then the Perron-Stieltjes integral

$$\int_a^b f \, \mathrm{d}g \quad \textit{exists and equals } \ I,$$

as well.

Proof. Let

$$\sigma \int_a^b f \, \mathrm{d}g = I \in \mathbb{R},$$

i.e., for any $\varepsilon > 0$ there is a division $d_0 = \{s_0, s_1 \ldots, s_{m_0}\} \in \mathcal{D}[a,b]$ of $[a,b]$ such that

$$|S_D(f\,\Delta g) - I| < \varepsilon$$

is true for any partition $D = (d, \xi) \in \mathcal{P}[a,b]$ of $[a,b]$ such that $d \in \mathcal{D}[a,b]$ is a refinement of d_0. Let us define

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2} \min\{|t - s_j|; j = 0, 1, \ldots, m_0\} & \text{if } \ t \notin d_0, \\ \varepsilon & \text{if } \ t \in d_0. \end{cases}$$

Let a partition $D=(d,\xi)\in\mathcal{P}[a,b]$, $d=\{t_0,t_1,\ldots,t_m\}$, $\xi=(\xi_1,\xi_2,\ldots,\xi_m)^{\mathrm{T}}$, be given. Then D is $\delta_\varepsilon-$fine only if for any $j = 1, 2, \ldots, m_0$ there is an index i_j such that $s_j = \xi_{i_j}$. Furthermore, we have

$$S_D(f\,\Delta g) = \sum_{j=1}^{m} \big[[f(\xi_j)\,[g(t_j) - g(\xi_j)] + f(\xi_j)\,[g(\xi_j) - g(t_{j-1})]\big].$$

Consequently, for any δ_ε−fine partition $D = (d, \xi)$ of $[a, b]$ the corresponding integral sum $S_D(f\,\Delta g)$ equals the integral sum $S_{D'}(f\,\Delta g)$ corresponding to a partition $D' = (d', \xi')$, where d' is a division of $[a, b]$ such that $d_0 \subset d'$. Hence

$$|S_{D'}(f\,\Delta g) - I| < \varepsilon.$$

This means that the Kurzweil integral $\int_a^b f\,\mathrm{d}g$ exists and

$$\int_a^b f\,\mathrm{d}g = \sigma \int_a^b f\,\mathrm{d}g = I$$

is true. □

To prove the existence of the Perron-Stieltjes integral $\int_a^b f\,\mathrm{d}g$ for any $f \in \mathbb{BV}[a, b]$ and any $g \in \mathbb{G}[a, b]$ in Theorem 2.3.8 the following assertion is helpful.

Proposition 2.2.2. *Let $f \in \mathbb{BV}[a, b]$ be continuous on $[a, b]$ and let $g \in \mathbb{G}[a, b]$, then both the σ-Riemann-Stieltjes integrals $\sigma \displaystyle\int_a^b f\,\mathrm{d}g$ and $\sigma \int_a^b g\,\mathrm{d}f$ exist.*

Proof. Let $f \in \mathbb{BV}[a, b]$ which is continuous on $[a, b]$ and $g \in \mathbb{G}[a, b]$ be given. According to the integration-by-parts formula [14, II.11.7] for σ-Riemann-Stieltjes integrals, to prove the lemma it is sufficient to show that the integral $\sigma\int_a^b g\,\mathrm{d}f$ exists.

First, let us assume that an arbitrary $\tau \in [a, b]$ is given and $g = \chi_{[a,\tau]}$. Let us put

$$d_0 = \begin{cases} \{a, b\} & \text{if } \tau = a \text{ or } \tau = b, \\ \{a, \tau, b\} & \text{if } \tau \in (a, b). \end{cases}$$

It is easy to see that then for any partition $D = (d, \xi)$ such that

$$d_0 \subset d = \{t_0, t_1, \ldots, t_m\}$$

we have $\tau = t_k$ for some $k \in \{0, 1, \ldots, m\}$ and

$$S_D(g\,\Delta f) = \begin{cases} f(\tau) - f(a) & \text{if } \xi_{k+1} > \tau, \\ f(t_{k+1}) - f(a) & \text{if } \xi_{k+1} = \tau. \end{cases}$$

Since f is assumed to be continuous, it is easy to show that for a given $\varepsilon > 0$ there exists a division d_* of $[a, b]$ such that $d_0 \subset d_*$ and

$$|S_D(g\,\Delta f) - [f(\tau) - f(a)]| < \varepsilon$$

is true for any partition $D = (d, \xi)$ of $[a, b]$ with $d_* \subset d$, i.e.

$$\sigma \int_a^b \chi_{[a,\tau]} \, \mathrm{d}f = f(\tau) - f(a) \quad \text{for all} \quad \tau \in [a, b].$$

By a similar argument we could also show the following relations:

$$\sigma \int_a^b \chi_{[a,\tau)} \, \mathrm{d}f = f(\tau) - f(a) \quad \text{for all} \quad \tau \in (a, b],$$

$$\sigma \int_a^b \chi_{[\tau,b]} \, \mathrm{d}f \; = f(b) - f(\tau) \quad \text{for all} \quad \tau \in [a, b],$$

and

$$\sigma \int_a^b \chi_{(\tau,b]} \, \mathrm{d}f \; = f(b) - f(\tau) \quad \text{for all} \quad \tau \in [a, b).$$

Since any finite step function is a linear combination of functions $\chi_{[\tau,b]}$ $(a \le \tau \le b)$ and $\chi_{(\tau,b]}$ $(a \le \tau < b)$, it follows that the integral

$$\sigma \int_a^b g \, \mathrm{d}f$$

exists for any $f \in \mathbb{BV}[a, b]$ continuous on $[a, b]$ and any $g \in \mathbb{S}[a, b]$.

Now, if $g \in \mathbb{G}[a, b]$ is arbitrary, then there exists a sequence $\{g_n\}_{k=1}^{\infty} \subset \mathbb{S}[a, b]$ such that

$$\lim_{n\to\infty} \|g_n - g\| = 0.$$

Since by the preceding part of the proof of the lemma all the integrals $\sigma\int_a^b g_n \, \mathrm{d}f$ have a finite value, by means of the convergence theorem [14, Theorem II.15.1] valid for σ-Riemann-Stieltjes integrals we obtain that the integral $\sigma\int_a^b g \, \mathrm{d}f$ exists and the relation

$$\lim_{n\to\infty} \sigma \int_a^b g_n \, \mathrm{d}f = \sigma \int_a^b g \, \mathrm{d}f \in \mathbb{R}$$

holds. This completes the proof. □

A direct corollary of Proposition 2.2.2 and of [14, Theorem II.13.17] is the following assertion which will be helpful for the proof of the integration-by-parts formula, Theorem 2.3.15. (Of course, we could prove it as well by an argument similar to that used in the proof of Proposition 2.2.2.)

Corollary 2.2.3. *Let* $f \in \mathbb{BV}[a, b]$ *and* $g \in \mathbb{G}[a, b]$. *Let*

$$\Delta^+ f(t)\, \Delta^+ g(t) = \Delta^- f(t)\, \Delta^- g(t) = 0 \quad \textit{for all} \quad t \in (a, b).$$

Then both the σ-Riemann-Stieltjes integrals

$$\sigma\int_a^b f\,\mathrm{d}g \quad and \quad \sigma\int_a^b g\,\mathrm{d}f$$

exist.

2.3. Perron-Stieltjes integral with respect to regulated functions

In this section we deal with the integrals

$$\int_a^b f(t)\,\mathrm{d}[g(t)] \quad \text{and} \quad \int_a^b g(t)\,\mathrm{d}[f(t)],$$

where $f \in \mathbb{BV}[a,b]$ and $g \in \mathbb{G}[a,b]$. We prove some basic theorems (integration-by-parts formula, convergence theorems, substitution theorem and unsymmetric Fubini theorem) needed in the theory of Stieltjes integral equations in the space $\mathbb{G}[a,b]$. However, our first task is the proof of existence of the integral $\int_a^b f\,\mathrm{d}g$ for any $f \in \mathbb{BV}[a,b]$ and any $g \in \mathbb{G}[a,b]$. First, we will consider some simple special cases.

Proposition 2.3.1. *Let $g \in \mathbb{G}[a,b]$ and $\tau \in [a,b]$. Then*

$$\int_a^b \chi_{[a,\tau]}\,\mathrm{d}g = g(\tau+) - g(a), \tag{2.3.1}$$

$$\int_a^b \chi_{[a,\tau)}\,\mathrm{d}g = g(\tau-) - g(a), \tag{2.3.2}$$

$$\int_a^b \chi_{[\tau,b]}\,\mathrm{d}g = g(b) - g(\tau-), \tag{2.3.3}$$

$$\int_a^b \chi_{(\tau,b]}\,\mathrm{d}g = g(b) - g(\tau+) \tag{2.3.4}$$

and

$$\int_a^b \chi_{[\tau]}\,\mathrm{d}g = g(\tau+) - g(\tau-), \tag{2.3.5}$$

where $\chi_{[a)}(t) \equiv \chi_{(b]}(t) \equiv 0$ and the convention $g(a-) = g(a)$, $g(b+) = g(b)$ is used.

Proof. Let $g \in \mathbb{G}[a,b]$ and $\tau \in [a,b]$ be given.

a) Let $f = \chi_{[a,\tau]}$. It follows immediately from the definition that

$$\int_a^\tau f\,\mathrm{d}g = g(\tau) - g(a).$$

In particular, (2.3.1) holds in the case $\tau = b$. Assume $\tau \in [a, b)$. Let $\varepsilon > 0$ be given and let

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2}\,|\tau - t| & \text{if } \ \tau < t \le b, \\ \varepsilon & \text{if } \ t = \tau. \end{cases}$$

It is easy to see that any δ_ε–fine partition $D = (d, \xi)$ of $[\tau, b]$ must satisfy

$$\xi_1 = t_0 = \tau, \quad t_1 < \tau + \varepsilon \quad \text{and} \quad S_D(f\,\Delta g) = g(t_1) - g(\tau).$$

Consequently,

$$\int_\tau^b f\,\mathrm{d}g = g(\tau+) - g(\tau)$$

and

$$\begin{aligned} \int_a^b f\,\mathrm{d}g &= \int_a^\tau f\,\mathrm{d}g + \int_\tau^b f\,\mathrm{d}g \\ &= g(\tau) - g(a) + g(\tau+) - g(\tau) = g(\tau+) - g(a), \end{aligned}$$

i.e. the relation (2.3.1) is true for every $\tau \in [a, b]$.

b) Let $f = \chi_{[a,\tau)}$. If $\tau = a$, then $f \equiv 0$, $g(\tau-) - g(a) = 0$ and (2.3.2) is trivial. Let $\tau \in (a, b]$. For a given $\varepsilon > 0$, let us define a gauge δ_ε on $[a, \tau]$ by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{2}\,|\tau - t| & \text{if } \ a \le t < \tau, \\ \varepsilon & \text{if } \ t = \tau. \end{cases}$$

Then for any δ_ε–fine partition $D = (d, \xi)$ of $[a, \tau]$ we have

$$t_m = \xi_m = \tau, \quad t_{m-1} < \tau - \varepsilon \quad \text{and} \quad S_D(f\,\Delta g) = g(t_{m-1}) - g(a).$$

It follows immediately that

$$\int_a^\tau f\,\mathrm{d}g = g(\tau-) - g(a)$$

and in view of the obvious identity

$$\int_\tau^b f\,\mathrm{d}g = 0,$$

this implies (2.3.2).

c) The remaining relations follow from (2.3.1), (2.3.2) and from the equalities

$$\chi_{[\tau,b]} = \chi_{[a,b]} - \chi_{[a,\tau)}, \quad \chi_{(\tau,b]} = \chi_{[a,b]} - \chi_{[a,\tau]} \quad \text{and}$$
$$\chi_{[\tau]} = \chi_{[a,\tau]} - \chi_{[a,\tau)}.$$

□

Remark 2.3.2. Since any finite step function is a linear combination of functions $\chi_{[\tau,b]}$ $(a \le \tau \le b)$ and $\chi_{(\tau,b]}$ $(a \le \tau < b)$, it follows immediately from Proposition 2.3.1 that the integral $\int_a^b f\,\mathrm{d}g$ exists for any $f \in \mathbb{S}[a,b]$ and any $g \in \mathbb{G}[a,b]$.

Other simple cases are covered by

Proposition 2.3.3. *Let $\tau \in [a,b]$. Then for any function $f: [a,b] \mapsto \mathbb{R}$ the following relations are true*

$$\int_a^b f\,\mathrm{d}\chi_{[a,\tau]} = \begin{cases} -f(\tau) & \text{if } \tau < b, \\ 0 & \text{if } \tau = b, \end{cases} \tag{2.3.6}$$

$$\int_a^b f\,\mathrm{d}\chi_{[a,\tau)} = \begin{cases} -f(\tau) & \text{if } \tau > a, \\ 0 & \text{if } \tau = a, \end{cases} \tag{2.3.7}$$

$$\int_a^b f\,\mathrm{d}\chi_{[\tau,b]} = \begin{cases} f(\tau) & \text{if } \tau > a, \\ 0 & \text{if } \tau = a, \end{cases} \tag{2.3.8}$$

$$\int_a^b f\,\mathrm{d}\chi_{(\tau,b]} = \begin{cases} f(\tau) & \text{if } \tau < b, \\ 0 & \text{if } \tau = b \end{cases} \tag{2.3.9}$$

and

$$\int_a^b f\,\mathrm{d}\chi_{[\tau]} = \begin{cases} -f(a) & \text{if } \tau = a, \\ 0 & \text{if } a < \tau < b, \\ f(b) & \text{if } \tau = b, \end{cases} \tag{2.3.10}$$

where $\chi_{[a)}(t) \equiv \chi_{(b]}(t) \equiv 0$ and the convention $g(a-) = g(a)$, $g(b+) = g(b)$ is used.

For the *proof* see [45, I.4.21 and I.4.22].

Corollary 2.3.4. *Let $W = \{w_1, w_2, \dots, w_n\} \subset [a,b]$, $c \in \mathbb{R}$ and $h : [a,b] \mapsto \mathbb{R}$ be such that*

$$h(t) = c \quad \text{for all} \quad t \in [a,b] \setminus W. \tag{2.3.11}$$

Then

$$\int_a^b f\,\mathrm{d}h = f(b)\,[h(b) - c] - f(a)\,[h(a) - c] \tag{2.3.12}$$

is true for any function $f : [a,b] \mapsto \mathbb{R}$.

Proof. A function $h : [a,b] \mapsto \mathbb{R}$ fulfils (2.3.11) if and only if

$$h(t) = c + \sum_{k=1}^{n} [h(w_j) - c]\,\chi_{[w_j]}(t) \quad \text{on} \quad [a,b].$$

Thus the formula (2.3.12) follows from (2.3.6) (with $\tau = b$) and from (2.3.10). $\square$

Remark 2.3.5. It is well known (cf. [45, I.4.17] or [40, Theorem 1.22]) that if $g \in \mathbb{BV}[a,b]$, $h : [a,b] \mapsto \mathbb{R}$ and $h_n : [a,b] \mapsto \mathbb{R}$, $n \in \mathbb{N}$, are such that $\int_a^b h_n \,\mathrm{d}g$ exist for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|h_n - h\| = 0$, then $\int_a^b h \,\mathrm{d}g$ exists and

$$\lim_{n\to\infty} \int_a^b h_n \,\mathrm{d}g = \int_a^b h \,\mathrm{d}g \tag{2.3.13}$$

is true. To prove an analogous assertion for the case $g \in \mathbb{G}[a,b]$ we need the following auxiliary assertion.

Lemma 2.3.6. *Let $f \in \mathbb{BV}[a,b]$ and $g \in \mathbb{G}[a,b]$. Then the inequality*

$$|S_D(f\,\Delta g)| \le \big(f(a)| + |f(b)| + \operatorname{var}_a^b f\big)\,\|g\| \tag{2.3.14}$$

is true for an arbitrary partition D of $[a,b]$.

Proof. For an arbitrary partition $D = (d,\xi)$ of $[a,b]$ we have (putting $\xi_0 = a$ and $\xi_{m+1} = b$)

$$\begin{aligned}
|S_D(f\,\Delta g)| &= |f(b)\,g(b) - f(a)\,g(a) - \sum_{j=1}^{m+1} [f(\xi_j) - f(\xi_{j-1})]\,g(t_{j-1})| \\
&\le \Big(|f(b)| + |f(a)| + \sum_{j=1}^{m+1} |f(\xi_j) - f(\xi_{j-1})|\Big)\,\|g\| \\
&\le \big(|f(a)| + |f(b)| + \operatorname{var}_a^b f\big)\,\|g\|.
\end{aligned}$$

$\square$

Theorem 2.3.7. *Let $g \in \mathbb{G}[a,b]$ and let $h_n, h\colon\ [a,b] \mapsto \mathbb{R}$ be such that*

$$\int_a^b h_n \,\mathrm{d}g \quad \textit{exists for any} \;\; n \in \mathbb{N} \quad \textit{and} \quad \lim_{n\to\infty} \|h_n - h\|_{\mathbb{BV}} = 0.$$

Then $\int_a^b h \,\mathrm{d}g$ *exists and* (2.3.13) *is true.*

Proof. Since

$$|f(b)| \le |f(a)| + |f(b) - f(a)| \le |f(a)| + \operatorname{var}_a^b f,$$

we have by (2.3.14)

$$|S_D((h_m - h_k)\,\Delta g)| \le 2\,\|h_m - h_k\|_{\mathbb{BV}}\,\|g\|$$

for all $m, k \in \mathbb{N}$ and all partitions D of $[a, b]$. Consequently,

$$\left| \int_a^b (h_m - h_k)\, \mathrm{d}g \right| \le 2\, \|h_m - h_k\|_{\mathbb{BV}}\, \|g\|$$

holds for all $m, k \in \mathbb{N}$. This immediately implies that there is an $I \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \int_a^b h_n\, \mathrm{d}g = I.$$

It remains to show that

$$I = \int_a^b h\, \mathrm{d}g. \tag{2.3.15}$$

For a given $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that

$$\left| \int_a^b h_{n_0}\, \mathrm{d}g - I \right| < \varepsilon \quad \text{and} \quad \|h_{n_0} - h\|_{\mathbb{BV}} < \varepsilon, \tag{2.3.16}$$

and let δ_ε be such a gauge on $[a, b]$ that

$$\left| S_D(h_{n_0}\, \Delta g) - \int_a^b h_{n_0}\, \mathrm{d}g \right| < \varepsilon \tag{2.3.17}$$

for all $\delta_\varepsilon-$fine partitions D of $[a, b]$. Given an arbitrary $\delta_\varepsilon-$fine partition D of $[a, b]$, we have by (2.3.16), (2.3.17) and Lemma 2.3.6

$$\begin{aligned} |I - S_D(h\, \Delta g)| \le \left| I - \int_a^b h_{n_0}\, \mathrm{d}g \right| + \left| \int_a^b h_{n_0}\, \mathrm{d}g - S_D(h_{n_0}\, \Delta g) \right| \\ + \left| S_D(h_{n_0}\, \Delta g) - S_D(h\, \Delta g) \right| \le 2\,\varepsilon + |S_D([h_{n_0} - h]\, \Delta g)| \\ \le 2\,\varepsilon + 2\, \|h_{n_0} - h\|_{\mathbb{BV}}\, \|g\| \le 2\,\varepsilon\,(1 + \|g\|) \end{aligned}$$

wherefrom the relation (2.3.15) immediately follows. This completes the proof of the theorem. □

Now we can prove

Theorem 2.3.8. *Let $f \in \mathbb{BV}[a, b]$ and $g \in \mathbb{G}[a, b]$. Then the integral $\int_a^b f\, \mathrm{d}g$ exists and the inequality*

$$\left| \int_a^b f\, \mathrm{d}g \right| \le \big(|f(a)| + |f(b)| + \mathrm{var}_a^b\, f \big)\, \|g\| \tag{2.3.18}$$

is true.

Proof. Let $f \in \mathbb{BV}[a,b]$ and $g \in \mathbb{G}[a,b]$ be given. Let $W = \{w_k\}_{k\in\mathbb{N}}$ be the set of discontinuities of f in $[a,b]$ and let $f = f^{\mathrm{C}} + f^{\mathrm{B}}$ be the Jordan decomposition of f (i.e., f^{C} is continuous on $[a,b]$ and f^{B} is given by (1.2.2)). We have

$$\lim_{n\to\infty} \|f_n^{\mathrm{B}} - f^{\mathrm{B}}\|_{\mathbb{BV}} = 0$$

for f_n^{B}, $n \in \mathbb{N}$, given by (1.2.3). By (2.3.3) and (2.3.4),

$$\int_a^b f_n^{\mathrm{B}}\,\mathrm{d}g = \\ = \sum_{k=1}^{n} [\Delta^+ f(w_k)\,(g(b) - g(w_k+)) + \Delta^- f(w_k)\,(g(b) - g(w_k-))] \tag{2.3.19}$$

holds for any $n \in \mathbb{N}$. Thus according to Theorem 2.3.7 the integral $\int_a^b f^{\mathrm{B}}\,\mathrm{d}g$ exists and

$$\int_a^b f^{\mathrm{B}}\,\mathrm{d}g = \lim_{n\to\infty} \int_a^b f_n^{\mathrm{B}}\,\mathrm{d}g. \tag{2.3.20}$$

The integral $\int_a^b f^{\mathrm{C}}\,\mathrm{d}g$ exists as the σ-Riemann-Stieltjes integral by Proposition 2.2.2. This means that $\int_a^b f\,\mathrm{d}g$ exists and

$$\int_a^b f\,\mathrm{d}g = \int_a^b f^{\mathrm{C}}\,\mathrm{d}g + \int_a^b f^{\mathrm{B}}\,\mathrm{d}g = \int_a^b f^{\mathrm{C}}\,\mathrm{d}g + \lim_{n\to\infty} \int_a^b f_n^{\mathrm{B}}\,\mathrm{d}g.$$

The inequality (2.3.18) follows immediately from Lemma 2.3.6. $\square$

Remark 2.3.9. Since

$$\sum_{k=1}^{\infty} \big|[\Delta^+ f(w_k)\,(g(b) - g(w_k+)) + \Delta^- f(w_k)\,(g(b) - g(w_k-))]\big| \\ \le 2\,\|g\| \sum_{k=1}^{\infty} (|\Delta^+ f(w_k)| + |\Delta^- f(w_k)|) \le 2\,\|g\|\,(\mathrm{var}_a^b\, f) < \infty,$$

we have in virtue of (2.3) and (2.3.20)

$$\int_a^b f^{\mathrm{B}}\,\mathrm{d}g = \\ = \sum_{k=1}^{\infty} [\Delta^+ f(w_k)\,(g(b) - g(w_k+)) + \Delta^- f(w_k)\,(g(b) - g(w_k-))]. \tag{2.3.21}$$

As a direct consequence of Theorem 2.3.8 we obtain

Corollary 2.3.10. *Let $h_n \in \mathbb{G}[a,b]$, $n \in \mathbb{N}$, and let $h \in \mathbb{G}[a,b]$ be such that*

$$\lim_{n\to\infty} \|h_n - h\| = 0.$$

Then for any $f \in \mathbb{BV}[a,b]$ the integrals

$$\int_a^b f\,\mathrm{d}h \quad \text{and} \quad \int_a^b f\,\mathrm{d}h_n,\ n \in \mathbb{N}$$

exist and

$$\lim_{n\to\infty} \int_a^b f\,\mathrm{d}h_n = \int_a^b f\,\mathrm{d}h.$$

Lemma 2.3.11. *Let $h : [a,b] \mapsto \mathbb{R}$, $c \in \mathbb{R}$ and $W = \{w_k\}_{k\in\mathbb{N}} \subset [a,b]$ be such that* (2.3.11) *and*

$$\sum_{k=1}^{\infty} |h(w_k) - c| < \infty \tag{2.3.22}$$

hold. Furthermore, for $n \in \mathbb{N}$, let us put $W_n = \{w_1, w_2, \dots, w_n\}$ and

$$h_n(t) = \begin{cases} c & \text{if } t \in [a,b] \setminus W_n, \\ h(t) & \text{if } t \in W_n. \end{cases} \tag{2.3.23}$$

Then $h_n \in \mathbb{BV}[a,b]$ for any $n \in \mathbb{N}$, $h \in \mathbb{BV}[a,b]$ and

$$\lim_{n\to\infty} \|h_n - h\|_{\mathbb{BV}} = 0. \tag{2.3.24}$$

Proof. The functions h_n, $n \in \mathbb{N}$, and h evidently have a bounded variation on $[a,b]$. For a given $n \in \mathbb{N}$, we have

$$h_n(t) - h(t) = \begin{cases} 0 & \text{if } t \in W_n \text{ or } t \in [a,b] \setminus W \\ c - h_n(t) & \text{if } t = w_k \text{ for some } k > n. \end{cases}$$

Thus,

$$\lim_{n\to\infty} h_n(t) = h(t) \quad \text{on} \quad [a,b] \tag{2.3.25}$$

and, moreover,

$$\sum_{j=1}^{m} \big|(h_n(t_j) - h(t_j)) - (h_n(t_{j-1}) - h(t_{j-1}))\big| \le 2 \sum_{k=n+1}^{\infty} |h(w_k) - c|$$

holds for any $n \in \mathbb{N}$ and any division $\{t_0, t_1, \dots, t_m\}$ of $[a,b]$. Consequently,

$$\operatorname{var}_a^b (h_n - h) \le 2 \sum_{k=n+1}^{\infty} |h(w_k) - c| \tag{2.3.26}$$

is true for any $n \in \mathbb{N}$. In virtue of the assumption (2.3.22) the right-hand side of (2.3.26) tends to 0 as $n \to \infty$. Hence (2.3.24) follows from (2.3.25) and (2.3.26). □

Proposition 2.3.12. *Let* $h : [a,b] \mapsto \mathbb{R}$, $c \in \mathbb{R}$ *and* $W = \{w_k\}_{k\in\mathbb{N}}$ *be such that* (2.3.11) *and* (2.3.22) *hold. Then*

$$\int_a^b h \, \mathrm{d}g = \sum_{k=1}^{\infty} [h(w_k) - c] \, \Delta g(w_k) + c \, [g(b) - g(a)]$$

is true for any $g \in \mathbb{G}[a,b]$.

Proof. Let $g \in \mathbb{G}[a,b]$ be given. Let $W_n = \{w_1, w_2, \dots, w_n\}$ for $n \in \mathbb{N}$ and let the functions h_n, $n \in \mathbb{N}$, be given by (2.3.23). Given an arbitrary $n \in \mathbb{N}$, then (2.3.1) (with $\tau = b$) and (2.3.5) from Proposition 2.3.1 imply

$$\int_a^b h_n \, \mathrm{d}g = \sum_{k=1}^{n} [h(w_k) - c] \, \Delta g(w_k) + c \, [g(b) - g(a)].$$

Since (2.3.22) yields

$$\sum_{k=1}^{n} |[h(w_k) - c] \, \Delta g(w_k)| \le 2 \, \|g\| \sum_{k=1}^{\infty} |h(w_k) - c| < \infty$$

and Lemma 2.3.11 implies

$$\lim_{n\to\infty} \|h_n - h\|_{\mathbb{BV}} = 0,$$

we can use Theorem 2.3.7 to prove that

$$\int_a^b h \, \mathrm{d}g = \lim_{n\to\infty} \int_a^b h_n \, \mathrm{d}g = \sum_{k=1}^{\infty} [h(w_k) - c] \, \Delta g(w_k) + c \, [g(b) - g(a)].$$ □

Proposition 2.3.13. *Let* $h : [a,b] \mapsto \mathbb{R}$, $c \in \mathbb{R}$ *and* $W = \{w_k\}_{k\in\mathbb{N}}$ *fulfil* (2.3.11). *Then*

$$\int_a^b f \, \mathrm{d}h = f(b) \, [h(b) - c] - f(a) \, [h(a) - c] \tag{2.3.27}$$

is true for any $f \in \mathbb{BV}[a,b]$.

Proof. Let $f \in \mathbb{BV}[a,b]$. For a given $n \in \mathbb{N}$, let $W_n = \{w_1, w_2, \dots, w_n\}$ and let h_n be given by (2.3.23). Then

$$\lim_{n\to\infty} \|h_n - h\| = 0. \tag{2.3.28}$$

Indeed, let $\varepsilon > 0$ be given and let $n_0 \in \mathbb{N}$ be such that $k \geq n_0$ implies

$$|h(w_k) - c| < \varepsilon. \tag{2.3.29}$$

(Such an n_0 exists since $|h(w_k) - c| = |\Delta^- h(w_k)| = |\Delta^+ h(w_k)|$ for any $k \in \mathbb{N}$ and the set of those $k \in \mathbb{N}$ for which the inequality (2.3.29) does not hold may be only finite.) Now, for any $n \geq n_0$ and any $t \in [a,b]$ such that $t = w_k$ for some $k > n$ ($t \in W \setminus W_n$) we have

$$|h_n(t) - h(t)| = |h_n(w_k) - h(w_k)| = |c - h(w_k)| < \varepsilon.$$

Since $h_n(t) = h(t)$ for all the other $t \in [a,b]$ ($t \in \big([a,b] \setminus W\big) \cup W_n$), it follows that $|h_n(t) - h(t)| < \varepsilon$ on $[a,b]$, i.e.

$$\|h_n - h\| < \varepsilon.$$

This proves the relation (2.3.28).

By Corollary 2.3.4 we have for any $n \in \mathbb{N}$

$$\int_a^b f \, \mathrm{d}h_n = f(b)\,[h(b) - c] - f(a)\,[h(a) - c].$$

Making use of (2.3.28) and Corollary 2.3.10 we obtain

$$\int_a^b f \, \mathrm{d}h = \lim_{n\to\infty} \int_a^b f \, \mathrm{d}h_n = f(b)\,[h(b) - c] - f(a)\,[h(a) - c].$$

□

Corollary 2.3.14. *Let $h \in \mathbb{BV}[a,b]$, $c \in \mathbb{R}$ and $W = \{w_k\}_{k\in\mathbb{N}}$ fulfil* (2.3.11)*. Then* (2.3.27) *is true for any $f \in \mathbb{G}[a,b]$.*

Proof. By Proposition 2.3.12, (2.3.27) is true for any $f \in \mathbb{BV}[a,b]$. Making use of the density of $\mathbb{S}[a,b] \subset \mathbb{BV}[a,b]$ in $\mathbb{G}[a,b]$ and of the convergence theorem mentioned in Remark 2.3.5 we complete the proof of this assertion. □

Theorem 2.3.15. (Integration-by-parts) *Let $f \in \mathbb{BV}[a,b]$ and $g \in \mathbb{G}[a,b]$. Then both the integrals $\int_a^b f \, \mathrm{d}g$ and $\int_a^b g \, \mathrm{d}f$ exist and*

$$\begin{aligned}\int_a^b f \, \mathrm{d}g + \int_a^b g \, \mathrm{d}f &= f(b)\,g(b) - f(a)\,g(a) \\ &\quad + \sum_{t\in[a,b]} [\Delta^- f(t)\,\Delta^- g(t) - \Delta^+ f(t)\,\Delta^+ g(t)].\end{aligned} \tag{2.3.30}$$

Proof. The existence of the integral $\int_a^b g\,\mathrm{d}f$ is well known and the existence of $\int_a^b g\,\mathrm{d}f$ is guaranteed by Theorem 2.3.8. Furthermore,

$$\int_a^b f\,\mathrm{d}g + \int_a^b g\,\mathrm{d}f = \int_a^b f(t)\,\mathrm{d}[g(t)+\Delta^+ g(t)] + \int_a^b g(t)\,\mathrm{d}[f(t)-\Delta^- f(t)]$$
$$- \int_a^b f(t)\,\mathrm{d}[\Delta^+ g(t)] + \int_a^b g(t)\,\mathrm{d}[\Delta^- f(t)].$$

It is easy to verify that the function $h(t) = \Delta^+ g(t)$ fulfils the relation (2.3.11) with $c = 0$ and $h(b) = 0$. Consequently, Proposition 2.3.13 yields

$$\int_a^b f(t)\,\mathrm{d}[\Delta^+ g(t)] = -f(a)\,\Delta^+ g(a).$$

Similarly, by Corollary 2.3.14 we have

$$\int_a^b g(t)\,\mathrm{d}[\Delta^- f(t)] = \Delta^- f(b)\,g(b).$$

Hence

$$\int_a^b f\,\mathrm{d}g + \int_a^b g(t)\,\mathrm{d}f = \int_a^b f(t)\,\mathrm{d}[g(t+)] + \int_a^b g(t)\,\mathrm{d}[f(t-)]$$
$$+ f(a)\,\Delta^+ g(a) + \Delta^- f(b)\,g(b). \tag{2.3.31}$$

The first integral on the right-hand side can be modified in the following way:

$$\int_a^b f(t)\,\mathrm{d}[g(t+)] = \int_a^b f(t-)\,\mathrm{d}[g(t+)] + \int_a^b \Delta^- f(t)\,\mathrm{d}[g(t+)]. \tag{2.3.32}$$

Making use of Proposition 2.3.12 and taking into account that $\Delta g_1(t) = \Delta g(t)$ on $[a,b]$ for the function g_1 defined by $g_1(t) = g(t+)$ on $[a,b]$, we further obtain

$$\int_a^b \Delta^- f(t)\,\mathrm{d}[g(t+)] = \sum_{t\in[a,b]} \Delta^- f(t)\,\Delta g(t). \tag{2.3.33}$$

Similarly,

$$\int_a^b g(t)\,\mathrm{d}[f(t-)] = \int_a^b g(t+)\,\mathrm{d}f(t-) - \int_a^b \Delta^+ g(t)\,\mathrm{d}[f(t-)]$$
$$= \int_a^b g(t+)\,\mathrm{d}[f(t-)] - \sum_{t\in[a,b]} \Delta^+ g(t)\,\Delta f(t). \tag{2.3.34}$$

The function $f(t-)$ is left-continuous on $[a,b]$, while $g(t+)$ is right-continuous on $[a,b)$. It means that both the integrals

$$\int_a^b f(t-)\,\mathrm{d}[g(t+)] \quad \text{and} \quad \int_a^b g(t+)\,\mathrm{d}[f(t-)]$$

exist as σ-Riemann-Stieltjes integrals (cf. Corollary 2.2.3), and making use of the integration-by-parts theorem for these integrals (cf. [14, Theorem II.11.7]) we get

$$\int_a^b f(t-)\mathrm{d}[g(t+)] + \int_a^b g(t+)\mathrm{d}[f(t-)] = f(b-)g(b) - f(a)g(a+). \qquad (2.3.35)$$

Inserting (2.3.32)-(2.3.35) into (2.3.31) we obtain

$$\begin{aligned}\int_a^b f\,\mathrm{d}g + \int_a^b g\,\mathrm{d}f &= f(b-)\,g(b) - f(a)\,g(a+)\\ &\quad + \sum_{t\in[a,b]} \Delta^- f(t)\,[\Delta^- g(t) + \Delta^+ g(t)]\\ &\quad - \sum_{t\in[a,b]} [\Delta^- f(t) + \Delta^+ f(t)]\,\Delta^+ g(t) + f(a)\,\Delta^+ g(a) + \Delta^- f(b)\,g(b)\\ &= f(b)\,g(b) - f(a)\,g(a) + \sum_{t\in[a,b]} [\Delta^- f(t)\,\Delta^- g(t) - \Delta^+ f(t)\,\Delta^+ g(t)]\end{aligned}$$

and this completes the proof. □

The following proposition describes some properties of indefinite Perron-Stieltjes integrals.

Proposition 2.3.16. *Let $f : [a,b] \mapsto \mathbb{R}$ and $g : [a,b] \mapsto \mathbb{R}$ be such that $\int_a^b f\,\mathrm{d}g$ exists. Then the function*

$$h(t) = \int_a^t f\,\mathrm{d}g$$

is defined on $[a,b]$ and
(i) *if $g \in \mathbb{G}[a,b]$, then $h \in \mathbb{G}[a,b]$ and*

$$\Delta^+ h(t) = f(t)\,\Delta^+ g(t), \quad \Delta^- h(t) = f(t)\,\Delta^- g(t) \quad \text{on} \quad [a,b]; \qquad (2.3.36)$$

(ii) *if $g \in \mathbb{BV}[a,b]$ and f is bounded on $[a,b]$, then $h \in \mathbb{BV}[a,b]$.*

Proof. The former assertion follows from [19, Theorem 1.3.5]. The latter follows immediately from the inequality

$$\sum_{j=1}^m \left| \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g \right| \le \sum_{j=1}^m \big[\|f\|\,(\mathrm{var}_{t_{j-1}}^{t_j} g)\big]b = \|f\|\,(\mathrm{var}_a^b\, g)$$

which is valid for any division $\{t_0, t_1, \dots, t_m\}$ of $[a,b]$. □

In the theory of generalized differential equations the ***substitution formula***

$$\int_a^b h(t)\,\mathrm{d}\Big[\int_a^t f(s)\,\mathrm{d}[g(s)]\Big] = \int_a^b h(t)f(t)\,\mathrm{d}[g(t)] \tag{2.3.37}$$

is often needed. In [14, II.19.3.7] this formula is proved for the σ-Young-Stieltjes integral under the assumption that $g \in \mathbb{G}[a,b]$, h is bounded on $[a,b]$ and the integral $\int_a^b f\,\mathrm{d}g$ as well as one of the integrals in (2.3.37) exist. In [45, Theorem I.4.25] this assertion was proved for the Kurzweil integral. Though it was assumed there that $g \in \mathbb{BV}[a,b]$, this assumption was not used in the proof. We will give here a slightly different proof based on the ***Saks-Henstock lemma*** (cf.e.g.[40, Lemma 1.11]).

Lemma 2.3.17. (Saks-Henstock) *Let $f,g : [a,b] \mapsto \mathbb{R}$ be such that the integral $\int_a^b f\,\mathrm{d}g$ exists. Let $\varepsilon > 0$ be given and let δ be a gauge on $[a,b]$ such that*

$$\Big|S_D(f\,\Delta g) - \int_a^b f\,\mathrm{d}g\Big| < \varepsilon$$

is true for any $\delta-$fine partition D of $[a,b]$. Then for an arbitrary system $\{([\beta_i,\gamma_i],\sigma_i),\ i=1,2,\dots,k\}$ of intervals and points such that

$$a \le \beta_1 \le \sigma_1 \le \gamma_1 \le \beta_2 \le \dots \le \beta_k \le \sigma_k \le \gamma_k \le b \tag{2.3.38}$$

and

$$[\beta_i,\gamma_i] \subset [\sigma_i - \delta(\sigma_i), \sigma_i + \delta(\sigma_i)], \quad i = 1,2,\dots,k,$$

the inequality

$$\Big|\sum_{i=1}^k \Big[f(\sigma_i)\,[g(\gamma_i) - g(\beta_i)] - \int_{\beta_i}^{\gamma_i} f\,\mathrm{d}g\Big]\Big| < \varepsilon \tag{2.3.39}$$

is true.

Making use of Lemma 2.3.17 we can prove the following useful assertion.

Lemma 2.3.18. *If $f : [a,b] \mapsto \mathbb{R}$ and $g : [a,b] \mapsto \mathbb{R}$ are such that $\int_a^b f\,\mathrm{d}g$ exists, then for any $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that*

$$\sum_{j=1}^m \Big|f(\xi_j)\,[g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g\Big| < \varepsilon \tag{2.3.40}$$

is true for any $\delta-$fine partition (d,ξ) of $[a,b]$.

Proof. Let $\delta : [a,b] \mapsto (0,\infty)$ be such that

$$\Big|S_D(f\,\Delta g) - \int_a^b f\,\mathrm{d}g\Big| = \Big|\sum_{j=1}^m f(\xi_j)\,[g(t_j) - g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g\Big| < \frac{\varepsilon}{2}$$

for all $\delta-$fine partitions $D=(d,\xi)$ of $[a,b]$. Let us choose an arbitrary $\delta-$fine partition $D=(d,\xi)$ of $[a,b]$. Let $\gamma_i=t_{p_i}$ and $\beta_i=t_{p_i-1}$, $i=1,2,\dots,k$, be all the points of the division d such that

$$f(\xi_{p_i})\,[g(\gamma_i)-g(\beta_i)]-\int_{\beta_i}^{\gamma_i} f\,\mathrm{d}g\geq 0.$$

Then the system $\{([\beta_i,\gamma_i],\sigma_i),\ i=1,2,\dots,k\}$, where $\sigma_i=\xi_{p_i}$, fulfils (2.3.38) and (2.3.39) and hence we can use Lemma 2.3.17 to prove that the inequality

$$\sum_{i=1}^{k}\left|f(\xi_{p_i})\,[g(\gamma_i)-g(\beta_i)]-\int_{\beta_i}^{\gamma_i} f\,\mathrm{d}g\right|<\frac{\varepsilon}{2}$$

is true. Similarly, if $\omega_i=t_{q_i}$ and $\theta_i=t_{q_i-1}$, $i=1,2,\dots,r$ are all points of the division d such that

$$f(\xi_{q_i})\,[g(\omega_i)-g(\theta_i)]-\int_{\theta_i}^{\omega_i} f\,\mathrm{d}g\leq 0,$$

then the inequality

$$\sum_{i=1}^{r}\left|f(\xi_{q_i})\,[g(\omega_i)-g(\theta_i)]-\int_{\theta_i}^{\omega_i} f\,\mathrm{d}g\right|<\frac{\varepsilon}{2}$$

follows from Lemma 2.3.17, as well. Summarizing, we conclude that

$$\begin{aligned}
&\sum_{j=1}^{m}\left|f(\xi_j)\,[g(t_j)-g(t_{j-1})]-\int_{t_{j-1}}^{t_j} f\,\mathrm{d}g\right|\\
&\qquad=\sum_{i=1}^{k}\left|f(\xi_{p_i})\,[g(\gamma_i)-g(\beta_i)]-\int_{\beta_i}^{\gamma_i} f\,\mathrm{d}g\right|\\
&\qquad+\sum_{i=1}^{r}\left|f(\xi_{q_i})\,[g(\omega_i)-g(\theta_i)]-\int_{\theta_i}^{\omega_i} f\,\mathrm{d}g\right|\\
&\qquad<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

This completes the proof. □

Theorem 2.3.19. (Substitution) *Let $h:[a,b]\mapsto\mathbb{R}$ be bounded on $[a,b]$ and let $f,g:[a,b]\mapsto\mathbb{R}$ be such that the integral $\displaystyle\int_a^b f\,\mathrm{d}g$ exists. Then the integral*

$$\int_a^b h(t)\,f(t)\,\mathrm{d}[g(t)]$$

exists if and only if the integral

$$\int_a^b h(t)\,\mathrm{d}\Big[\int_a^t f(s)\,\mathrm{d}[g(s)]\Big]$$

exists, and in this case the relation (2.3.37) *is true.*

Proof. Let $|h(t)| \le C < \infty$ on $[a,b]$. Let us assume that the integral

$$\int_a^b h(t)f(t)\,\mathrm{d}[g(t)]$$

exists and let $\varepsilon > 0$ be given. There exists a gauge δ_1 on $[a,b]$ such that

$$\Big|\sum_{j=1}^m h(\xi_k)f(\xi_j)\,[g(t_j)-g(t_{j-1})] - \int_a^b h(t)f(t)\,\mathrm{d}[g(t)]\Big| < \frac{\varepsilon}{2}$$

is satisfied for any δ_1-fine partition (d,ξ) of $[a,b]$. By Lemma 2.3.18 there exists a gauge δ on $[a,b]$ such that $\delta(t) \le \delta_1(t)$ on $[a,b]$ and

$$\sum_{j=1}^m \Big| f(\xi_j)\,[g(t_j)-g(t_{j-1})] - \int_{t_{j-1}}^{t_j} f\,\mathrm{d}g\Big| < \frac{\varepsilon}{2C}$$

is true for any $\delta-$fine partition (d,ξ) of $[a,b]$. Let us denote

$$k(t) = \int_a^t f\,\mathrm{d}g \quad \text{for} \quad t \in [a,b].$$

Then for any $\delta-$fine partition $D=(d,\xi)$ of $[a,b]$ we have

$$\begin{aligned}
&\Big|S_D(h\,\Delta k) - \int_a^b h(t)f(t)\,\mathrm{d}[g(t)]\Big| \\
&\quad = \Big|\sum_{j=1}^m h(\xi_j)\int_{t_{j-1}}^{t_j} f\,\mathrm{d}g - \sum_{j=1}^m h(\xi_j)f(\xi_j)\,[g(t_j)-g(t_{j-1})] \\
&\qquad + \sum_{j=1}^m h(\xi_j)f(\xi_j)\,[g(t_j)-g(t_{j-1})] - \int_a^b h(t)f(t)\,\mathrm{d}[g(t)]\Big| \\
&\quad \le \Big|\sum_{j=1}^m h(\xi_j)\Big[\int_{t_{j-1}}^{t_j} f\,\mathrm{d}g - f(\xi_j)\,[g(t_j)-g(t_{j-1})]\Big]\Big| \\
&\qquad + \Big|\sum_{j=1}^m h(\xi_j)\,f(\xi_j)\,[g(t_j)-g(t_{j-1})] - \int_a^b h\,f\,\mathrm{d}g\Big| < \varepsilon.
\end{aligned}$$

This implies the existence of the integral $\displaystyle\int_a^b h\,\mathrm{d}k$ and the relation (2.3.37).
The second implication can be proved in an analogous way. □

The convergence result 2.3.10 enables us to extend the known theorems on the change of the integration order in iterated integrals

$$\int_c^d g(t)\,\mathrm{d}\Big[\int_a^b h(t,s)\,\mathrm{d}[f(s)]\Big],\quad \int_a^b \Big(\int_c^d g(t)\,\mathrm{d}_t[h(t,s)]\Big)\,\mathrm{d}[f(s)],\qquad (2.3.41)$$

where $-\infty < c < d < \infty$ and h is of strongly bounded variation on $[c,d]\times[a,b]$ (cf. 1.2.5).

Theorem 2.3.20. (Unsymmetric Fubini Theorem) *Let* $h:\ [c,d]\times[a,b]\mapsto\mathbb{R}$ *be such that*

$$\mathrm{v}\,(h) + \mathrm{var}_a^b\, h(c,.) + \mathrm{var}_c^d h(.,a) < \infty.$$

Then for any $f\in\mathbb{BV}[a,b]$ *and any* $g\in\mathbb{G}(c,d)$ *both the integrals* (2.3.41) *exist and*

$$\int_c^d g(t)\,\mathrm{d}\Big[\int_a^b h(t,s)\,\mathrm{d}[f(s)]\Big] = \int_a^b \Big(\int_c^d g(t)\,\mathrm{d}_t[h(t,s)]\Big)\,\mathrm{d}[f(s)].\qquad (2.3.42)$$

Proof. Let us notice that by [45, Theorem I.6.20] our assertion is true if g is also supposed to be of bounded variation. In the general case of $g\in\mathbb{G}[a,b]$ there exists a sequence $\{g_n\}_{n=1}^{\infty}\subset\mathbb{S}[a,b]$ such that $\lim_{n\to\infty}\|g-g_n\|=0$. Then, since the function

$$v(t) = \int_a^b h(t,s)\,\mathrm{d}[f(s)]$$

is of bounded variation on $[c,d]$ (cf. the first part of the proof of [45, Theorem I.6.20]), the integral on the left-hand side of (2.3.42) exists and by Corollary 2.3.10 and [45, Theorem I.6.20] we have

$$\begin{aligned}\int_c^d g(t)\,\mathrm{d}\Big[\int_a^b h(t,s)\,\mathrm{d}[f(s)]\Big] &= \lim_{n\to\infty}\int_c^d g_n(t)\,\mathrm{d}\Big[\int_a^b h(t,s)\,\mathrm{d}[f(s)]\Big]\\ &= \lim_{n\to\infty}\int_a^b \Big(\int_c^d g_n(t)\,\mathrm{d}_t[h(t,s)]\Big)\,\mathrm{d}[f(s)].\qquad (2.3.43)\end{aligned}$$

Let us denote

$$w_n(t) = \int_c^d g_n(t)\,\mathrm{d}_t[h(t,s)]\quad\text{for}\quad s\in[a,b]\quad\text{and}\quad n\in\mathbb{N}.$$

Then $w_n\in\mathbb{BV}[a,b]$ for any $n\in\mathbb{N}$ (cf. [45, Theorem I.6.18]) and by [45, Theorem I.4.17] mentioned here in Remark 2.3.5 we obtain

$$\lim_{n\to\infty} w_n(s) = \int_c^d g_n(t)\,\mathrm{d}_t[h(t,s)] := w(s)\ \text{ on }\ [a,b].$$

As

$$|w_n(s)-w(s)| \le \|g_n-g\|\big(\operatorname{var}_c^d h(.,s)\big) \le \|g_n-g\|\big(\operatorname{v}(h)+\operatorname{var}_c^d h(.,a)\big)$$

for any $s\in[a,b]$ (cf. [45, Lemma I.6.6]), we have

$$\lim_{n\to\infty}\|w_n-w\|=0.$$

It means that $w\in\mathbb{G}[a,b]$ and by Theorem 2.3.8 the integral

$$\int_a^b w(s)\,\mathrm{d}[f(s)] = \int_a^b \left(\int_c^d g(t)\,\mathrm{d}_t[h(t,s)]\right)\mathrm{d}[f(s)]$$

exists as well. Since obviously

$$\begin{aligned}\lim_{n\to\infty}\int_a^b \left(\int_c^d g_n(t)\,\mathrm{d}_t[h(t,s)]\right)\mathrm{d}[f(s)] &= \lim_{n\to\infty}\int_a^b w_n(s)\,\mathrm{d}[f(s)]\\ &= \int_a^b w(s)\,\mathrm{d}[f(s)] = \int_a^b \left(\int_c^d g(t)\,\mathrm{d}_t[h(t,s)]\right)\mathrm{d}[f(s)],\end{aligned}$$

the relation (2.3.42) follows from (2.3). □

2.4. Linear bounded functionals on the space of left-continuous regulated functions

By Theorem 2.3.8 the expression

$$\Phi_\eta(x) = q\,x(a) + \int_a^b p\,\mathrm{d}x \tag{2.4.1}$$

is defined for any $x\in\mathbb{G}[a,b]$ and any $\eta=(p,q)\in\mathbb{BV}[a,b]\times\mathbb{R}$. Moreover, for any $\eta\in\mathbb{BV}[a,b]\times\mathbb{R}$, the relation (2.4.1) defines a linear bounded functional on $\mathbb{G}_{\mathrm{L}}[a,b]$.

Proposition 2.3.3 immediately implies

Lemma 2.4.1. *Let $\eta=(p,q)\in\mathbb{BV}[a,b]\times\mathbb{R}$. Then*

$$\begin{aligned}&\Phi_\eta(\chi_{[a,b]}) = q,\\ &\Phi_\eta(\chi_{(\tau,b]}) = p(\tau) \quad \textit{for all} \quad \tau\in[a,b),\\ &\Phi_\eta(\chi_{[b]}) \;\; = p(b).\end{aligned} \tag{2.4.2}$$

Corollary 2.4.2. *Let $\eta=(p,q)\in\mathbb{BV}[a,b]\times\mathbb{R}$ and $\Phi_\eta(x)=0$ for all $x\in\mathbb{S}[a,b]$ which are left-continuous on (a,b). Then $p(t)\equiv 0$ on $[a,b]$ and $q=0$.*

Lemma 2.4.3. *Let $x \in \mathbb{G}[a,b]$ and $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$. Then*

$$\begin{aligned}
&\Phi_\eta(x) = x(a) \quad \text{if } p \equiv 0 \text{ on } [a,b] \text{ and } q = 1, \\
&\Phi_\eta(x) = x(b) \quad \text{if } p \equiv 1 \text{ on } [a,b] \text{ and } q = 1, \\
&\Phi_\eta(x) = x(\tau-) \text{ if } p = \chi_{[a,\tau)} \text{ on } [a,b],\ \tau \in (a,b] \text{ and } q = 1, \\
&\Phi_\eta(x) = x(\tau+) \text{ if } p = \chi_{[a,\tau]} \text{ on } [a,b],\ \tau \in [a,b) \text{ and } q = 1.
\end{aligned} \tag{2.4.3}$$

Proof follows from Proposition 2.3.1.

Corollary 2.4.4. *Let $x \in \mathbb{G}[a,b]$ and $\Phi_\eta(x) = 0$ for all $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$. Then*

$$x(a) = x(a+) = x(\tau-) = x(\tau+) = x(b-) = x(b) \tag{2.4.4}$$

holds for any $\tau \in (a,b)$. In particular, if $x \in \mathbb{G}_L[a,b]$ (x is left-continuous on (a,b)) and $\Phi_\eta(x) = 0$ for all $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$, then $x(t) \equiv 0$ on $[a,b]$.

Remark 2.4.5. The space $\mathbb{BV}[a,b] \times \mathbb{R}$ is supposed to be equipped with the usual norm ($\|\eta\|_{\mathbb{BV}\times\mathbb{R}} = |q| + \|p\|_{\mathbb{BV}}$ for $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$). Obviously, $\mathbb{BV}[a,b] \times \mathbb{R}$ is a Banach space with respect to this norm.

Proposition 2.4.6. *The spaces $\mathbb{G}_L[a,b]$ and $\mathbb{BV}[a,b] \times \mathbb{R}$ form a dual pair with respect to the bilinear form*

$$x \in \mathbb{G}_L[a,b],\ \eta \in \mathbb{BV}[a,b] \times \mathbb{R} \mapsto \Phi_\eta(x), \tag{2.4.5}$$

i.e.

$$\Phi_\eta(x) = 0 \text{ for all } x \in \mathbb{G}_L[a,b] \implies \eta = 0 \in \mathbb{BV}[a,b] \times \mathbb{R}$$

and

$$\Phi_\eta(x) = 0 \text{ for all } \eta \in \mathbb{BV}[a,b] \times \mathbb{R} \implies x = 0 \in \mathbb{G}_L[a,b].$$

Proof follows from Corollaries 2.4.2 and 2.4.4.

On the other hand, we have

Lemma 2.4.7. *Let Φ be a linear bounded functional on $\mathbb{G}_L[a,b]$ and let*

$$p(t) = \begin{cases} \Phi(\chi_{(t,b]}) & \text{if } t \in [a,b), \\ \Phi(\chi_{[b]}) & \text{if } t = b. \end{cases} \tag{2.4.6}$$

Then $p \in \mathbb{BV}[a,b]$ and

$$|p(a)| + |p(b)| + \operatorname{var}_a^b p \le 2\,\|\Phi\|, \tag{2.4.7}$$

where

$$\|\Phi\| = \sup_{x \in \mathbb{G}_L[a,b],\, \|x\| \le 1} |\Phi(x)|.$$

Proof is analogous to that of part c (i) of [15, Theorem 5.1]. Indeed, for an arbitrary division $\{t_0, t_1, \dots, t_m\}$ of $[a,b]$ we have

$$\begin{aligned}
&\sup_{|c_j|\le 1, c_j\in\mathbb{R}} \Big|p(a)\,c_0 + p(b)\,c_{m+1} + \sum_{j=1}^{m}[p(t_j)-p(t_{j-1})]\,c_j\Big| \\
&\quad = \sup_{|c_j|\le 1, c_j\in\mathbb{R}} \Big|\varPhi(c_0\,\chi_{(a,b]} + c_{m+1}\,\chi_{[b]} - \sum_{j=1}^{m-1} c_j\,\chi_{(t_{j-1},t_j]} + c_m\,\chi_{(t_{m-1},b)})\Big| \\
&\quad \le \sup_{\|h\|\le 2, h\in\mathbb{G}_{\mathrm{L}}[a,b]} |\varPhi(h)| = 2\,\|\varPhi\|.
\end{aligned}$$

In particular, for $c_0 = \operatorname{sgn} p(a)$, $c_{m+1} = \operatorname{sgn} p(b)$ and $c_j = \operatorname{sgn}(p(t_j) - p(t_{j-1}))$, $j = 1, 2, \dots, m$, we get

$$|p(a)| + |p(b)| + \sum_{j=1}^{m}|p(t_j) - p(t_{j-1})| \le 2\,\|\varPhi\|,$$

and the inequality (2.4.7) immediately follows.

Using the ideas from the proof of [15, Theorem 5.1] we may now prove the following representation theorem.

Theorem 2.4.8. *$\varPhi$ is a linear bounded functional on $\mathbb{G}_{L}[a,b]$ ($\varPhi \in \mathbb{G}^*_{L}(a,b)$) if and only if there is an $\eta = (p,q) \in \mathbb{BV}[a,b]\times\mathbb{R}$ such that*

$$\varPhi(x) = \varPhi_\eta(x)\Big(:= q\,x(a) + \int_a^b p\,\mathrm{d}x\Big) \quad \text{for any} \quad x \in \mathbb{G}_{L}[a,b]. \tag{2.4.8}$$

The mapping

$$\Xi:\ \eta \in \mathbb{BV}[a,b]\times\mathbb{R} \mapsto \varPhi_\eta \in \mathbb{G}^*_{L}(a,b)$$

is an isomorphism.

Proof. Let a linear bounded functional $\varPhi$ on $\mathbb{G}_{\mathrm{L}}[a,b]$ be given and let us put

$$q = \varPhi(\chi_{[a,b]}) \quad \text{and} \quad p(t) = \begin{cases} \varPhi(\chi_{(t,b]}) & \text{if } t \in [a,b), \\ \varPhi(\chi_{[b]}) & \text{if } t = b. \end{cases} \tag{2.4.9}$$

Then Lemma 2.3.6 implies $\eta = (p,q) \in \mathbb{BV}[a,b]\times\mathbb{R}$ and by Lemma 2.4.1 we have

$$\begin{gathered}
\varPhi(\chi_{[a,b]}) = \varPhi_\eta(\chi_{[a,b]}),\ \varPhi(\chi_{[b]}) = \varPhi_\eta(\chi_{[b]}) \quad \text{and} \\
\varPhi(\chi_{(t,b]}) = \varPhi_\eta(\chi_{(t,b]}) \quad \text{for all} \quad t \in [a,b).
\end{gathered}$$

Since all functions from $\mathbb{S}[a,b]\cap\mathbb{G}_{\mathrm{L}}[a,b]$ obviously are finite linear combinations of the functions

$$\chi_{[a,b]},\ \chi_{(\tau,b]},\ \tau \in [a,b),\ \chi_{[b]},$$

it follows that $\Phi(x) = \Phi_\eta(x)$ is true for any $x \in \mathbb{S}[a,b] \cap \mathbb{G}_{\rm L}[a,b]$. Now, the density of $\mathbb{S}[a,b] \cap \mathbb{G}_{\rm L}[a,b]$ in $\mathbb{G}_{\rm L}[a,b]$ implies that

$$\Phi(x) = \Phi_\eta(x) \quad \text{for all} \quad x \in \mathbb{G}_{\rm L}[a,b].$$

This completes the proof of the first assertion of the theorem.

Lemma 2.3.6 yields that

$$|\Phi_\eta(x)| \le \big(|p(a)| + |p(b)| + \operatorname{var}_a^b p + |q|\big)\, \|x\|$$

is true for any $x \in \mathbb{G}_{\rm L}[a,b]$ and, consequently,

$$\|\Phi_\eta\| \le |p(a)| + |p(b)| + \operatorname{var}_a^b p + |q| \le 2\,\big(\|p\|_{\mathbb{BV}} + |q|\big) = 2\,\|\eta\|_{\mathbb{BV}\times\mathbb{R}}.$$

On the other hand, according to Lemma 2.4.7 we have

$$\|p\|_{\mathbb{BV}} \le (|p(a)| + |p(b)| + \operatorname{var}_a^b p) \le 2\,\|\Phi\|.$$

Furthermore, in virtue of (2.4.9) we have $|q| \le \|\Phi\|$ and hence

$$\|\eta\|_{\mathbb{BV}\times\mathbb{R}} = \|p\|_{\mathbb{BV}} + |q| \le 2\,\|\Phi\|.$$

It means that

$$\frac{1}{2}\,\|\Phi\| \le \|\eta\|_{\mathbb{BV}\times\mathbb{R}} \le 3\,\|\Phi\|$$

and this completes the proof of the theorem. □

2.5. Linear bounded functionals on the space of regular regulated functions

Recall that the subspace of $\mathbb{G}[a,b]$ consisting of all functions regulated on $[a,b]$ and such that

$$f(t) = \frac{1}{2}[f(t-) + f(t+)] \quad \text{for all} \quad t \in (a,b)$$

is denoted by $\mathbb{G}_{\rm reg}[a,b]$ and the functions belonging to $\mathbb{G}_{\rm reg}[a,b]$ are usually said to be *regular* on (a,b).

In this section we shall show that linear bounded functionals on $\mathbb{G}_{\rm reg}[a,b]$ may be represented in the form (2.4.8), as well. To this aim the following lemmas will be helpful.

Lemma 2.5.1. *A function $f : [a,b] \mapsto \mathbb{R}$ is a finite step function on $[a,b]$ which is regular on (a,b) ($f \in \mathbb{S}[a,b] \cap \mathbb{G}_{reg}[a,b]$) if and only if there* *are real numbers* $\alpha_1, \alpha_2, \dots, \alpha_m$ *and a division* $d = \{t_0, t_1, \dots, t_m\}$ *of* $[a,b]$ *such that*

$$f(t) = \sum_{j=0}^{N} \alpha_j\, h_j(t) \quad \textit{on} \quad [a,b],$$

where

$$h_0 = 1,\ h_1 = \chi_{(a,b]},\ h_j = \frac{1}{2}\chi_{[t_j]} + \chi_{(t_j,b]}$$
$$\textit{for}\ \ j = 2,3,\dots,m-1 \ \textit{and}\ \ h_m = \chi_{[b]}.$$

Proof. Obviously a function $f : [a,b] \mapsto \mathbb{R}$ belongs to $\mathbb{S}[a,b] \cap \mathbb{G}_{\text{reg}}[a,b]$ if and only if there are real numbers $c_0, c_1, \dots, c_{N+1}$ and a division $d = \{t_0, t_1, \dots, t_N\}$ of $[a,b]$ such that

$$f(t) = \begin{cases} c_0 & \text{if } \ t = a, \\ c_j & \text{if } \ t \in (t_{j-1}, t_j) \quad \text{for some } \ j = 1,2,\dots,m, \\ \frac{c_j + c_{j+1}}{2}, & \text{if } \ t = t_j \qquad\qquad \text{for some } \ j = 1,2,\dots,m-1, \\ c_{m+1} & \text{if } \ t = b. \end{cases}$$

i.e.

$$\begin{aligned} f(t) = c_0\, \chi_{[a]}(t) &+ \sum_{j=1}^{m} c_j\, \chi_{(t_{j-1},t_j)}(t) \\ &+ \frac{1}{2}\Big(\sum_{j=1}^{m-1} (c_j + c_{j+1})\, \chi_{[t_j]}(t)\Big) \\ &+ c_{m+1}\, \chi_{[b]}(t) \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{2.5.1}$$

It is easy to verify that the right-hand side of (2.5.1) may be rearranged as

$$\begin{aligned} f = {} & c_0\, \chi_{[a,b]} - c_0\, \chi_{(a,b]} + \sum_{j=1}^{m} c_j\, \chi_{(t_{j-1},b]} - \sum_{j=1}^{m-1} c_j\, \chi_{(t_j,b]} \\ - {} & \frac{1}{2} \sum_{j=1}^{m-1} c_j\, \chi_{[t_j]} - c_m\, \chi_{[b]} + \frac{1}{2} \sum_{j=1}^{m-1} c_{j+1}\, \chi_{[t_j]} + c_{m+1}\, \chi_{[b]} \\ = {} & c_0\, \chi_{[a,b]} - c_0\, \chi_{(a,b]} + \sum_{j=0}^{m-1} c_{j+1}\, \chi_{(t_j,b]} - \sum_{j=1}^{m-1} c_j\, \chi_{(t_j,b]} \\ - {} & \frac{1}{2} \sum_{j=1}^{m-1} c_j\, \chi_{[t_j]} + \frac{1}{2} \sum_{j=1}^{m-1} c_{j+1}\, \chi_{[t_j]} + c_{m+1}\, \chi_{[b]} - c_m\, \chi_{[b]} \\ = {} & c_0\, \chi_{[a,b]} + (c_1 - c_0)\, \chi_{(a,b]} + \sum_{j=1}^{m-1} (c_{j+1} - c_j)\, \big(\chi_{(t_j,b]} + \frac{1}{2}\, \chi_{[t_j]}\big) \\ + {} & (c_{m+1} - c_m)\, \chi_{[b]}, \end{aligned}$$

wherefrom the assertion of the lemma immediately follows. □

Lemma 2.5.2. *The set* $\mathbb{S}[a,b] \cap \mathbb{G}_{reg}[a,b]$ *is dense in* $\mathbb{G}_{reg}[a,b]$.

Proof. Let $x \in \mathbb{G}_{\text{reg}}[a,b]$ and $\varepsilon > 0$ be given. Since $\text{cl}(\mathbb{S}[a,b]) = \mathbb{G}[a,b]$, there exists a $\xi \in \mathbb{S}[a,b]$ such that $|x(t) - \xi(t)| < \varepsilon$ is true for any $t \in [a,b]$. Consequently, we have

$$\begin{gathered} |x(t-) - \xi(t-)| < \varepsilon \text{ and } \ |x(s+) - \xi(s+)| < \varepsilon \\ \text{for } \ t \in [a,b),\ s \in (a,b]. \end{gathered} \tag{2.5.2}$$

Let us put

$$\xi^*(t) = \begin{cases} \xi(a) & \text{if } \ t = a, \\ \frac{1}{2}\left(\xi(t+) + \xi(t-)\right) & \text{if } \ t \in (a,b), \\ \xi(b) & \text{if } \ t = b. \end{cases}$$

Obviously $\xi^*(t-) = \xi(t-)$ and $\xi^*(s+) = \xi(s+)$ for all $t \in (a,b]$ and $s \in [a,b)$, respectively. In particular, $\xi^*(t) = \xi(t)$ for any point t of continuity of ξ. It follows that $\xi^* \in \mathbb{S}[a,b] \cap \mathbb{G}_{\text{reg}}[a,b]$. Furthermore, in virtue of (2.5.2) we have for any $t \in (a,b)$

$$|x(t) - \xi^*(t)| = \frac{1}{2}\left|[x(t-) - \xi(t-)] + [x(t+) - \xi(t+)]\right| < \varepsilon,$$

wherefrom the assertion of the lemma immediately follows. □

Lemma 2.5.3. *Let Φ be an arbitrary linear bounded functional on* $\mathbb{G}_{reg}[a,b]$. *Let us define*

$$p(t) = \begin{cases} \Phi(\chi_{(a,b]}), & \textit{if } \ t = a, \\ \Phi(\frac{1}{2}\,\chi_{[t]} + \chi_{(t.b]}), & \textit{if } \ t \in (a,b), \\ \Phi(\chi_{[b]}), & \textit{if } \ t = b. \end{cases} \tag{2.5.3}$$

Then

$$\operatorname{var}_a^b p \le \|\Phi\| = \sup_{x \in \mathbb{G}_{reg}[a,b],\, \|x\| \le 1} |\Phi(x)|$$

(*i.e.* $p \in \mathbb{BV}[a,b]$).

Proof. Let $d = \{t_0, t_1, \dots, t_m\}$ be an arbitrary division of $[a,b]$ and let $\alpha_j \in \mathbb{R}$, $j = 1, 2, \dots, m$, be such that $|\alpha_j| \le 1$ for all $j = 1, 2, \dots, m$. Then

$$\sum_{j=1}^{m} \alpha_j [p(t_j) - p(t_{j-1})] = \alpha_1 \big[\Phi(\frac{1}{2}\chi_{[t_1]} + \chi_{(t_1,b]}) - \Phi(\chi_{(a,b]})\big] \tag{2.5.4}$$

$$+\sum_{j=2}^{m-1}\alpha_j\big[\varPhi(\frac{1}{2}\chi_{[t_j]}+\chi_{(t_j,b]})-\varPhi(\frac{1}{2}\chi_{[t_{j-1}]}+\chi_{(t_{j-1},b]})\big]$$
$$+\alpha_m\big[\varPhi(\chi_{[b]})-\varPhi(\frac{1}{2}\chi_{[t_{m-1}]}+\chi_{(t_{m-1},b]})\big]=\varPhi(h),$$

where

$$\begin{aligned}
h = \;& \alpha_1\,\big[\frac{1}{2}\,\chi_{[t_1]}+\chi_{(t_1,b]}-\chi_{(a,b]}\big]+\alpha_m\,\big[\chi_{[b]}-\frac{1}{2}\,\chi_{[t_{m-1}]}-\chi_{(t_{m-1},b]}\big]\\
&+\sum_{j=2}^{m-1}\alpha_j\,\big[\frac{1}{2}\,\chi_{[t_j]}+\chi_{(t_j,b]}-\frac{1}{2}\,\chi_{[t_{j-1}]}-\chi_{(t_{j-1},b]}\big]\\
= \;& \alpha_1\,\big[\frac{1}{2}\,\chi_{[t_1]}-\chi_{(a,t_1]}\big]-\alpha_m\,\big[\frac{1}{2}\,\chi_{[t_{m-1}]}+\chi_{(t_{m-1},b)}\big]\\
&+\sum_{j=2}^{m-1}\alpha_j\,\big[\frac{1}{2}\,\chi_{[t_j]}-\chi_{(t_{j-1},t_j]}-\frac{1}{2}\,\chi_{[t_{j-1}]}\big]\\
= \;& -\alpha_1\,\big[\frac{1}{2}\,\chi_{[t_1]}+\chi_{(a,t_1)}\big]-\alpha_m\big[\frac{1}{2}\,\chi_{[t_{m-1}]}+\chi_{(t_{m-1},b)}\big]\\
&-\frac{1}{2}\sum_{j=2}^{m-1}\alpha_j\,\chi_{[t_j]}-\frac{1}{2}\sum_{j=2}^{m-1}\alpha_j\,\chi_{[t_{j-1}]}-\sum_{j=2}^{m-1}\alpha_j\,\chi_{(t_{j-1},t_j)}\\
= \;& -\frac{1}{2}\sum_{j=1}^{m-1}\alpha_j\,\chi_{[t_j]}-\frac{1}{2}\sum_{j=2}^{m}\alpha_j\,\chi_{[t_{j-1}]}-\sum_{j=1}^{m}\alpha_j\,\chi_{(t_{j-1},t_j)}\\
= \;& -\sum_{j=1}^{m-1}\frac{\alpha_j+\alpha_{j+1}}{2}\,\chi_{[t_j]}-\sum_{j=1}^{m}\alpha_j\,\chi_{(t_{j-1},t_j)}\\
= \;& -\alpha_1\,\chi_{(a,t_1)}-\sum_{j=2}^{m-1}\big(\frac{\alpha_j+\alpha_{j+1}}{2}\,\chi_{[t_j]}+\alpha_j\,\chi_{(t_{j-1},t_j)}\big)-\alpha_m\,\chi_{(t_{m-1},b)}.
\end{aligned}$$

It is easy to see that $h\in\mathbb{S}[a,b]\cap\mathbb{G}_{\rm reg}[a,b]$ and $|h(t)|\le 1$ for all $t\in[a,b]$. Consequently, by (2.5.4), we have that

$$\sup_{|\alpha_j|\le 1, j=1,2,\dots,m}\big|\sum_{j=1}^{m}\alpha_j\,[p(t_j)-p(t_{j-1})]\big|\le\sup_{x\in\mathbb{G}_{\rm reg}[a,b],\|x\|\le 1}\|\varPhi(x)\|$$

is true for any division $d=\{t_0,t_1,\dots,t_m\}$ of $[a,b]$. In particular, choosing

$$\alpha_j=\operatorname{sgn}[p(t_j)-p(t_{j-1})]\quad\text{for}\quad j=1,2,\dots,m,$$

we get

$$\sum_{j=1}^{m}|p(t_j)-p(t_{j-1})|\le\sup_{x\in\mathbb{G}_{\rm reg}[a,b],\|x\|\le 1}\|\varPhi(x)\|<\infty$$

and this yields $\operatorname{var}_a^b p\le\|\varPhi\|<\infty.$ $\square$

Lemma 2.5.4. *Let $\varPhi$ be an arbitrary linear bounded functional on $\mathbb{G}_{reg}[a,b]$ and let $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$ be given by* (2.5.3) *and $q = \varPhi(\chi_{[a,b]})$. Let us define*

$$\varPhi_\eta(x) = q\,x(a) + \int_a^b p\,\mathrm{d}x \quad \textit{for} \quad x \in \mathbb{G}[a,b]. \tag{2.5.5}$$

Then $\varPhi_\eta$ is a linear bounded functional on $\mathbb{G}[a,b]$,

$$\varPhi_\eta(x) = \varPhi(x) \quad \textit{for all} \quad x \in \mathbb{G}_{reg}[a,b] \tag{2.5.6}$$

and

$$\sup_{x\in\mathbb{G}[a,b],\|x\|\le 1} |\varPhi_\eta(x)| \le |q| + 2\,(|p(a)| + \operatorname{var}_a^b p). \tag{2.5.7}$$

Proof. By Theorem 2.3.8, $\varPhi_\eta(x)$ is defined and

$$|\varPhi_\eta(x)| \le \big(|q| + |p(a)| + |p(b)| + \operatorname{var}_a^b p\big)\,\|x\| \quad \text{for all} \quad x \in \mathbb{G}[a,b]. \tag{2.5.8}$$

It means that $\varPhi_\eta$ is a linear bounded functional on $\mathbb{G}[a,b]$ and the inequality (2.5.7) is true. It is easy to verify that the relation (2.5.6) holds for any function h from the set

$$\left\{\chi_{[a,b]}, \chi_{(a,b]}, \frac{1}{2}\chi_{[\tau]} + \chi_{(\tau,b]}, \chi_{[b]}; \tau \in (a,b)\right\}.$$

According to Lemmas 2.5.1 and 2.5.2 this implies that (2.5.6) holds for all $x \in \mathbb{G}_{\mathrm{reg}}[a,b]$. □

Lemma 2.5.5. *Let $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$. Then $\varPhi_\eta(x) = 0$ for all $x \in \mathbb{S}[a,b] \cap \mathbb{G}_{reg}[a,b]$ if and only if $q = 0$ and $p(t) \equiv 0$ on $[a,b]$.*

Proof. Let $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$ and let $\varPhi_\eta(x) = 0$ for all $x \in \mathbb{S}[a,b] \cap \mathbb{G}_{\mathrm{reg}}[a,b]$. Then $\varPhi(\chi_{[a,b]}) = q = 0$. Furthermore, by Proposition 2.3.3 we have

$$\varPhi_\eta(\chi_{(a,b]}) = p(a) = 0,$$
$$\varPhi_\eta(\frac{1}{2}\,\chi_{[\tau]} + \chi_{(\tau,b]}) = p(\tau) = 0 \quad \text{for} \quad \tau \in (a,b)$$

and

$$\varPhi_\eta(\chi_{[b]}) = p(b) = 0.$$

By Lemma 2.5.1 this completes the proof. □

Remark 2.5.6. Let us notice that if $x \in \mathbb{G}_{\mathrm{reg}}[a,b]$, then $\varPhi_\eta(x) = 0$ for all $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$ if and only if $x(t) \equiv 0$ on $[a,b]$. In fact, let $x \in \mathbb{G}[a,b]$ and let $\varPhi_\eta(x) = 0$ for all $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$. Then by Corollary 2.4.4 we have

$$x(a) = x(a+) = x(t-) = x(t+) = x(b-) = x(b) = 0 \quad \text{for all} \quad t \in (a,b).$$

In particular, if $x \in \mathbb{G}_{\mathrm{reg}}[a,b]$, then $x(t) = 0$ for any $t \in [a,b]$.

Theorem 2.5.7. *A mapping* $\Phi : \mathbb{G}_{reg}[a,b] \mapsto \mathbb{R}$ *is a linear bounded functional on* $\mathbb{G}_{reg}[a,b]$ $(\Phi \in \mathbb{G}^*_{reg}[a,b])$ *if and only if there is an* $\eta = (p,q) \in \mathbb{BV}[a,b] \times \mathbb{R}$ *such that* $\Phi = \Phi_\eta$, *where* Φ_η *is given by* (2.5.5). *The mapping* $\Xi : \eta \in \mathbb{BV}[a,b] \times \mathbb{R} \mapsto \Phi_\eta \in \mathbb{G}^*_{reg}[a,b]$ *generates an isomorphism between* $\mathbb{BV}[a,b] \times \mathbb{R}$ *and* $\mathbb{G}^*_{reg}[a,b]$.

Proof. By Lemmas 2.5.4 and 2.5.5 and by the inequality (2.5.7) the mapping Ξ is a bounded linear one-to-one mapping of $\mathbb{BV}[a,b] \times \mathbb{R}$ onto $\mathbb{G}^*_{\mathrm{reg}}[a,b]$. Consequently, by the Bounded Inverse Theorem, the mapping Ξ^{-1} is bounded, as well. $\square$

Chapter 3

Initial Value Problems for Linear Generalized Differential Equations

3.1. Introduction

This chapter deals with the ***initial value problem*** for the linear homogeneous generalized differential equation

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) = 0, \quad t \in [0,1], \quad x(0) = \widetilde{x}, \tag{3.1.1}$$

where $A \in \mathbb{BV}^{n\times n}$ and $\widetilde{x} \in \mathbb{R}^n$ are given and solutions are functions $x : [0,1] \mapsto \mathbb{R}^n$ with bounded variation on $[0,1]$ ($x \in \mathbb{BV}^n$).

The basic properties of the Perron-Stieltjes integral with respect to scalar regulated functions were described in Chapter 2. The extension of these results to vector or matrix valued functions is obvious and hence for the basic facts concerning integrals we shall refer to the corresponding assertions from Chapter 2.

Let $P_k \in \mathbb{L}_1^{n\times n}$ for $k \in \mathbb{N}\cup\{0\}$ and let $X_k \in \mathbb{AC}^{n\times n}$ be the corresponding fundamental matrices, i.e.

$$X_k(t) = \mathrm{I} + \int_0^t P_k(s)\, X_k(s)\, \mathrm{d}s \quad \text{on} \quad [0,1] \quad \text{for} \quad k \in \mathbb{N}\cup\{0\}.$$

The following two assertions are representative examples of theorems on the continuous dependence of solutions of linear ordinary differential equations on a parameter.

Theorem 3.1.1. *If*

$$\lim_{k\to\infty}\int_0^1 |P_k(s)-P_0(s)|\,\mathrm{d}s = 0,$$

then

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{uniformly on} \;\; [0,1].$$

Theorem 3.1.2. (Kurzweil & Vorel, [22]) *Let there exist $m \in \mathbb{L}_1$ such that*

$$|P_k(t)| \le m(t) \quad \textit{a.e. on} \;\; [0,1] \quad \textit{for all} \;\; k \in \mathbb{N} \tag{3.1.2}$$

and let

$$\lim_{k\to\infty}\int_0^t P_k(s)\,\mathrm{d}s = \int_0^t P_0(s)\,\mathrm{d}s \quad \textit{uniformly on} \;\; [0,1].$$

Then

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{uniformly on} \;\; [0,1].$$

Remark 3.1.3. For $t \in [0,1]$ and $k \in \mathbb{N}\cup\{0\}$ denote

$$A_k(t) = \int_0^t P_k(s)\,\mathrm{d}s.$$

Then the assumptions of Theorem 3.1.2 can be reformulated for A_k as follows:

$$\begin{aligned}
&A_k \in \mathbb{AC}^{n\times n} \text{ for all } \; k \in \mathbb{N}\cup\{0\},\\
&\sup_{k\in\mathbb{N}} \|A_k'\|_{\mathbb{L}_1} < \infty,\\
&\lim_{k\to\infty} A_k(t) = A_0(t) \text{ uniformly on } \; [0,1].
\end{aligned}$$

Besides, the assumption (3.1.2) means that there exists a nondecreasing function $h_0 \in \mathbb{AC}$ such that

$$|A_k(t_2)-A_k(t_1)| \le |h_0(t_2)-h_0(t_1)| \quad \text{for all} \qquad t_1,t_2 \in [0,1].$$

In fact, we can put

$$h_0(t) = \int_0^t m(s)\mathrm{d}s \quad \text{on} \;\; [0,1].$$

3.2. A survey of known results

The following basic existence result for the initial value problem (3.1.1) may be found e.g. in [45] (cf. Theorem III.1.4) or in [41] (cf. Theorem 6.13).

Theorem 3.2.1. *Let $A \in \mathbb{BV}^{n\times n}$ be such that*

$$\det[\mathrm{I} - \Delta^- A(t)] \neq 0 \quad \textit{for all} \;\; t \in (0,1]. \tag{3.2.1}$$

Then there exists a unique $X \in \mathbb{BV}^{n\times n}$ such that

$$X(t) = \mathrm{I} + \int_0^t \mathrm{d}[A(s)]\,X(s) \quad \textit{on} \;\; [0,1]. \tag{3.2.2}$$

Definition 3.2.2. For a given $A \in \mathbb{BV}^{n\times n}$, the $n \times n$-matrix valued function $X \in \mathbb{BV}^{n\times n}$ fulfilling (3.2.2) is called the ***fundamental matrix*** corresponding to A.

When restricted to the linear case, Theorem 8.8 from [41], which describes the dependence of solutions of generalized differential equations on a parameter, reads as follows.

Theorem 3.2.3. *Let $A_0 \in \mathbb{BV}^{n\times n}$ satisfy* (3.2.1) *and let X_0 be the corresponding fundamental matrix. Let $A_k \in \mathbb{BV}^{n\times n}$, $k \in \mathbb{N}$, and scalar nondecreasing and left-continuous on $(0,1]$ functions h_k, $k \in \mathbb{N} \cup \{0\}$, be given such that h_0 is continuous on $[0,1]$ and*

$$\lim_{k\to\infty} A_k(t) = A_0(t) \quad \textit{on} \;\; [0,1], \tag{3.2.3}$$

$$|A_k(t_2) - A_k(t_1)| \leq |h_k(t_2) - h_k(t_1)| \qquad \textit{for all} \;\; t_1, t_2 \in [0,1] \;\textit{and}\; k \in \mathbb{N} \cup \{0\}, \tag{3.2.4}$$

$$\limsup_{k\to\infty} [h_k(t_2) - h_k(t_1)] \leq h_0(t_2) - h_0(t_1) \qquad \textit{whenever} \;\; 0 \leq t_1 \leq t_2 \leq 1. \tag{3.2.5}$$

Then for any $k \in \mathbb{N}$ sufficiently large the fundamental matrix X_k corresponding to A_k exists and

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{uniformly on} \;\; [0,1].$$

Lemma 3.2.4. *Under the assumptions of Theorem* 3.2.3 *we have*

$$\sup_{k\in\mathbb{N}} \operatorname{var} A_k < \infty \tag{3.2.6}$$

and

$$\lim_{k\to\infty} [A_k(t) - A_k(0)] = A_0(t) - A_0(0) \quad \textit{uniformly on} \;\; [0,1]. \tag{3.2.7}$$

Proof. [1] i) By (3.2.5) there is $k_0 \in \mathbb{N}$ such that

$$h_k(1) - h_k(0) \leq h_0(1) - h_0(0) + 1 \quad \text{for all} \quad k \geq k_0.$$

[1]The author is indebted to Ivo Vrkoč for his suggestions which led to a considerable simplification of this proof.

Hence for any $k \in \mathbb{N}$ we have

$$\operatorname{var} A_k \le \alpha_0 = \max\left(\left\{\operatorname{var} A_k;\ k \le k_0\right\} \cup \left\{h_0(1) - h_0(0) + 1\right\}\right) < \infty.$$

Thus we conclude that (3.2.6) is true.

ii) Suppose that

$$\lim_{k\to\infty} A_k(t) = A_0(t) \quad \text{uniformly on} \quad [0,1] \tag{3.2.8}$$

is not valid. Then there is $\widetilde{\varepsilon} > 0$ such that for any $\ell \in \mathbb{N}$ there exist $m_\ell \ge \ell$ and $t_\ell \in [0,1]$ such that

$$|A_{m_\ell}(t_\ell) - A_0(t_\ell)| \ge \widetilde{\varepsilon}. \tag{3.2.9}$$

We may assume that $m_{\ell+1} > m_\ell$ for any $\ell \in \mathbb{N}$ and

$$\lim_{\ell\to\infty} t_\ell = t_0 \in [0,1]. \tag{3.2.10}$$

Let $t_0 \in (0,1)$ and $\varepsilon > 0$ be given. Since h_0 is continuous, we may choose $\eta > 0$ in such a way that $t_0 - \eta \in [0,1]$, $t_0 + \eta \in [0,1]$ and

$$h_0(t_0 + \eta) - h_0(t_0 - \eta) < \varepsilon. \tag{3.2.11}$$

Furthermore, by (3.2.3) there is $\ell_1 \in \mathbb{N}$ such that

$$|A_{m_\ell}(t_0) - A_0(t_0)| < \varepsilon \quad \text{for all} \quad \ell \ge \ell_1$$

and by (3.2.4), (3.2.5) and (3.2.11) there is $\ell_2 \in \mathbb{N}$, $\ell_2 \ge \ell_1$, such that

$$\begin{aligned} &|A_{m_\ell}(\tau_2) - A_{m_\ell}(\tau_1)| \le h_0(t_0 + \eta) - h_0(t_0 - \eta) + \varepsilon < 2\,\varepsilon \\ &\quad \text{whenever} \quad \tau_1, \tau_2 \in (t_0 - \eta, t_0 + \eta) \text{ and } \ell \ge \ell_2. \end{aligned} \tag{3.2.12}$$

The relations (3.2.3) and (3.2.12) imply immediately that

$$\begin{aligned} |A_0(\tau_2) - A_0(\tau_1)| = \lim_{\ell\to\infty} |A_{m_\ell}(\tau_2) - A_{m_\ell}(\tau_1)| \le 2\,\varepsilon \\ \text{whenever} \quad \tau_1, \tau_2 \in (t_0 - \eta, t_0 + \eta). \end{aligned} \tag{3.2.13}$$

Finally, let $\ell_3 \in \mathbb{N}$ be such that $\ell_3 \ge \ell_2$ and

$$|t_\ell - t_0| < \eta \quad \text{for all} \quad \ell \ge \ell_3, \tag{3.2.14}$$

then in virtue of the relations (3.2.10)-(3.2.14) we have

$$\begin{aligned} &|A_{m_\ell}(t_\ell) - A_0(t_\ell)| \\ &\quad \le |A_{m_\ell}(t_\ell) - A_{m_\ell}(t_0)| + |A_{m_\ell}(t_0) - A_0(t_0)| + |A_0(t_0) - A_0(t_\ell)| \le 5\,\varepsilon. \end{aligned}$$

Hence, choosing $\varepsilon < \frac{1}{5}\,\widetilde{\varepsilon}$, we obtain by (3.2.9) that

$$\widetilde{\varepsilon} > |A_{m_\ell}(t_\ell) - A_0(t_\ell)| \ge \widetilde{\varepsilon}.$$

This being impossible, the relation (3.2.8) has to be true. The modification of the proof in the cases $t_0 = 0$ or $t_0 = 1$ and the extension of (3.2.8) to (3.2.7) is obvious. $\square$

Lemma 3.2.4 shows that Theorem 3.2.3 is a special case of the following result due to M. Ashordia (cf. [1]).

Theorem 3.2.5. *Let* $A_0 \in \mathbb{BV}^{n\times n}$ *satisfy* (3.2.1), *let* X_0 *be the corresponding fundamental matrix and let* $\{A_k\}_{k=1}^{\infty} \subset \mathbb{BV}^{n\times n}$ *be such that* (3.2.6) *and* (3.2.7) *are true. Then for any* $k \in \mathbb{N}$ *sufficiently large the fundamental matrix* X_k *corresponding to* A_k *exists and*

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{uniformly on} \quad [0,1].$$

Remark 3.2.6. Under the assumptions of Theorem 3.2.5 we have

$$\lim_{k\to\infty} A_k(t-) = A_0(t-) \text{ and } \lim_{k\to\infty} A_k(s+) = A_0(s+)$$

for all $t \in (0,1]$ and all $s \in [0,1)$, respectively. Thus Theorem 3.2.5 cannot cover the case when there is $t_0 \in (0,1]$ such that

$$A_k(t_0-) = A_k(t_0) \quad \text{for all} \quad k \in \mathbb{N} \quad \text{while} \quad A_0(t_0-) \neq A_0(t_0).$$

In particular, Theorem 3.2.5 does not apply to the following simple example.

Example 3.2.7. Consider the sequence of initial value problems

$$x_k' = a_k'(t)\, x_k \quad \text{on} \quad [-1,1], \quad x(-1) = \widetilde{x},$$

where

$$a_k(t) = \begin{cases} 0 & \text{if } t \le \alpha_k, \\ \dfrac{t-\alpha_k}{\beta_k-\alpha_k} & \text{if } t \in (\alpha_k,\beta_k), \\ 1 & \text{if } t \ge \beta_k; \end{cases}$$

$\{\alpha_k\}_{k=1}^{\infty}$ is an arbitrary increasing sequence in $[-1,0)$ tending to 0; $\{\beta_k\}_{k=1}^{\infty}$ is an arbitrary decreasing sequence in $(0,1]$ tending to 0 and

$$\lim_{k\to\infty} \frac{\alpha_k}{\alpha_k-\beta_k} = \varkappa \in [0,1).$$

For the corresponding solutions we have

$$x_k(t) = \left\{ \begin{array}{ll} \widetilde{x} & \text{if } \ t \le \alpha_k, \\ \mathrm{e}^{\frac{t-\alpha_k}{\beta_k-\alpha_k}}\, \widetilde{x} & \text{if } \ t \in (\alpha_k,\beta_k), \\ \mathrm{e}\, \widetilde{x} & \text{if } \ t \ge \beta_k, \end{array} \right\} \quad \text{and}$$

$$x_0(t) = \lim_{k\to\infty} x_k(t) = \begin{cases} \widetilde{x} & \text{if } \ t < 0, \\ \mathrm{e}^{\varkappa}\, \widetilde{x} & \text{if } \ t = 0, \\ \mathrm{e}\, \widetilde{x} & \text{if } \ t > 0, \end{cases}$$

while the unique solution $x(t)$ of the "limit" equation

$$x(t) = \widetilde{x} + \int_{-1}^{t} \mathrm{d}[a(s)]\, x(s),\ t \in [-1,1],$$

where

$$a(t) = \lim_{k\to\infty} a_k(t) = \begin{cases} 0 & \text{if } \ t < 0, \\ \varkappa & \text{if } \ t = 0, \\ 1 & \text{if } \ t > 0 \end{cases}$$

is given by

$$x(t) = \begin{cases} \widetilde{x} & \text{if } \ t < 0, \\ \dfrac{1}{1-\varkappa}\,\widetilde{x} & \text{if } \ t = 0, \\ \dfrac{2-\varkappa}{1-\varkappa}\,\widetilde{x} & \text{if } \ t > 0, \end{cases}$$

i.e. $x(t) \neq x_0(t)$ on $[-1,1]$.

On the other hand, x_0 is a solution to

$$x_0(t) = \widetilde{x} + \int_{-1}^{t} \mathrm{d}[a_0(t)]\, x_0(s)\, t \in [-1,1],$$

where

$$a_0(t) = \begin{cases} 0 & \text{if } \ t < 0, \\ 1-\mathrm{e}^{-\varkappa} & \text{if } \ t = 0, \\ (\mathrm{e}-1)\,\mathrm{e}^{-\varkappa} & \text{if } \ t > 0 \end{cases}$$

and a_k tends to a_0 in the following sense:

(a) *given arbitrary* $\alpha \in (-1,0)$ *and* $\beta \in (0,1)$, $\lim_{k\to\infty} a_k(t) = a_0(t)$ *uniformly on* $[-1,\alpha]$ *and* $\lim_{k\to\infty}[a_k(t)-a_k(\beta)] = a_0(t)-a_0(\beta)$ *uniformly on* $[\beta,1]$;

(b) $\lim_{k\to\infty} a_k(t) = a_0(t) + \widetilde{a}_0(t)$, *where*

$$\widetilde{a}_0(t) = \begin{cases} 0 & \textit{if } \ t < 0, \\ \varkappa + \mathrm{e}^{-\varkappa} - 1 & \textit{if } \ t = 0, \\ 1 - \mathrm{e}^{1-\varkappa} + \mathrm{e}^{-\varkappa} & \textit{if } \ t > 0; \end{cases}$$

(c) *for any* $z \in \mathbb{R}$ *and* $\varepsilon > 0$, *there is* $\delta > 0$ *such that for any* $\delta' \in (0, \delta)$ *there is* $k_0 \in \mathbb{N}$ *such that for any* $k \geq k_0$ *we have* $\alpha_k \geq -\delta'$, $\beta_k \leq \delta'$ *and the relations*

$$\Big| y_k(0) - y_k(-\delta') - \frac{\Delta^- a_0(0)\, z}{1 - \Delta^- a_0(0)} \Big| < \varepsilon \quad \textit{and}$$
$$|z_k(\delta') - z_k(0) - \Delta^+ a_0(0)\, z| < \varepsilon$$

are satisfied for any solution y_k *on* $[-\delta', 0]$ *of*

$$y_k' = a_k'(t)\, y_k \quad \textit{with} \;\; y_k(-\delta') \in (z - \delta, z + \delta)$$

and any solution z_k *on* $[0, \delta']$ *of*

$$z_k' = a_k'(t)\, z_k \quad \textit{with} \;\; z_k(0) \in (z - \delta, z + \delta).$$

In fact, for given $z \in \mathbb{R}$, $\delta' > 0$ and $k \in \mathbb{N}$ such that $\alpha_k \geq -\delta'$ we have

$$y_k(t) = \mathrm{e}^{\frac{t - \alpha_k}{\beta_k - \alpha_k}}\, y_k(-\delta') \;\; \text{on} \;\; [\alpha_k, 0]$$

and thus

$$\Big| y_k(0) - y_k(-\delta') - \frac{\Delta^- a_0(0)\, z}{1 - \Delta^- a_0(0)} \Big| = \Big| \big(\mathrm{e}^{\frac{-\alpha_k}{\beta_k - \alpha_k}} - 1\big)\, y_k(-\delta') - \big(\mathrm{e}^{\varkappa} - 1\big)\, z \Big|$$
$$\leq \Big| \mathrm{e}^{\frac{-\alpha_k}{\beta_k - \alpha_k}} - \mathrm{e}^{\varkappa} \Big|\, |z| + \Big| \mathrm{e}^{\frac{-\alpha_k}{\beta_k - \alpha_k}} - 1 \Big|\, \big| z - y_k(-\delta') \big|,$$

where

$$\lim_{k \to \infty} \Big| \mathrm{e}^{\frac{-\alpha_k}{\beta_k - \alpha_k}} - \mathrm{e}^{\varkappa} \Big| = 0, \quad \Big| \mathrm{e}^{\frac{-\alpha_k}{\beta_k - \alpha_k}} - 1 \Big| \leq 2 \quad \text{and} \quad \big| z - y_k(-\delta') \big| \leq \delta.$$

Analogously, if $k \in \mathbb{N}$ is such that $\beta_k \leq \delta'$, we have

$$z_k(t) = \mathrm{e}^{\frac{\beta_k}{\beta_k - \alpha_k}}\, z_k(0) \;\; \text{on} \;\; [0, \delta']$$

and thus

$$\Big| z_k(\delta') - z_k(0) - \Delta^+ a_0(0)\, z \Big| = \Big| \big(\mathrm{e}^{\frac{\beta_k}{\beta_k - \alpha_k}} - 1\big)\, z_k(-\delta') - \big(\mathrm{e}^{1-\varkappa} - 1\big)\, z \Big|$$
$$\leq \Big| \mathrm{e}^{\frac{\beta_k}{\beta_k - \alpha_k}} - \mathrm{e}^{1-\varkappa} \Big|\, |z| + \Big| \mathrm{e}^{\frac{\beta_k}{\beta_k - \alpha_k}} - 1 \Big|\, \big| z - z_k(0) \big|,$$

where

$$\lim_{k \to \infty} \Big| \mathrm{e}^{\frac{\beta_k}{\beta_k - \alpha_k}} - \mathrm{e}^{1-\varkappa} \Big| = 0, \quad \Big| \mathrm{e}^{\frac{\beta_k}{\beta_k - \alpha_k}} - 1 \Big| \leq 2 \quad \text{and} \quad \big| z - z_k(0) \big| \leq \delta.$$

Notice that if

$$x_0(t) = \widetilde{x} + \int_{-1}^{t} \mathrm{d}[a_0(t)]\, x_0(s) \;\; \text{on} \;\; [-1, 1],$$

then

$$\Delta^- x_0(0) = \Big(\frac{1}{1-\Delta^- a_0(0)} - 1\Big)\, x_0(0-) = \frac{\Delta^- a_0(0)}{1-\Delta^- a_0(0)}\, x_0(0-).$$

The convergence described in Example 3.2.7 is closely related to the notion of the ***emphatic convergence*** introduced by J. Kurzweil (cf. [20, Definition 4.1]).

Definition 3.2.8. A sequence $\{A_k\}_{k=1}^{\infty} \subset \mathbb{BV}^{n\times n}$ converges emphatically to $A_0 \in \mathbb{BV}^{n\times n}$ on $[0,1]$ if

(i) there exist nondecreasing functions $h_k : [0,1] \mapsto \mathbb{R}$, $k \in \mathbb{N}\cup\{0\}$, which are left-continuous on $(0,1]$ and such that

$$|A_k(t_2) - A_k(t_1)| \le |h_k(t_2) - h_k(t_1)|$$

for all $k \in \mathbb{N}\cup\{0\}$ and $t_1, t_2 \in [0,1]$;

(ii) $\limsup_{k\to\infty} \big[h_k(t_2) - h_k(t_1)\big] \le \big[h_0(t_2) - h_0(t_1)\big]$ whenever $0 \le t_1 \le t_2 \le 1$ and h_0 is continuous at t_1 and t_2;

(iii) there is $\widetilde{A}_0 \in \mathbb{BV}^{n\times n}$ such that $\lim_{k\to\infty} A_k(t) = A_0(t) + \widetilde{A}_0(t)$ whenever $h_0(t) = h_0(t+)$ and $|\widetilde{A}_0(t_2) - \widetilde{A}_0(t_1)| \le |\widetilde{h}_0(t_2) - \widetilde{h}_0(t_1)|$ for all $t_1, t_2 \in [0,1]$, where $\widetilde{h}_0$ stands for the break part of h_0;

(iv) if $h_0(t_0+) > h_0(t_0)$, then for any $z \in \mathbb{R}^n$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\delta' \in (0,\delta)$ there is $k_0 \in \mathbb{N}$ such that

$$|y_k(t_0+\delta') - y_k(t_0-\delta') - \Delta^+ A_0(t_0)\, z| \le \varepsilon$$

holds for any $k \ge k_0$, any $\widetilde{y}_k \in \mathbb{R}^n$ such that $|z - \widetilde{y}_k| \le \delta$ and any solution y_k of the equation

$$y_k(t) = \widetilde{y}_k + \int_{t_0-\delta'}^{t} \mathrm{d}[A_k(s)]\, y_k(s) \quad \text{on} \quad [t_0-\delta', t_0+\delta'].$$

The following assertion is a restriction of Theorem 4.1 from [20] to the linear case.

Theorem 3.2.9. *Let A_k converge emphatically on $[0,1]$ to A_0 and let the sequence $\{X_k\}_{k=1}^{\infty} \subset \mathbb{BV}^{n\times n}$ of the fundamental matrices corresponding respectively to A_k, $k \in \mathbb{N}$, be uniformly bounded on $[0,1]$ and such that*

$$\lim_{k\to\infty} X_k(t) = Z_0(t) \quad \text{on} \quad [0,1] \quad \text{whenever} \quad h_0(t+) = h_0(t).$$

Then $Z_0 \in \mathbb{BV}^{n\times n}$ and the function X_0 defined by

$$X_0(t) = \begin{cases} Z_0(t) & \text{if} \quad h_0(t+) = h_0(t), \\ Z_0(t-) & \text{otherwise} \end{cases}$$

is the fundamental matrix corresponding to A_0.

Remark 3.2.10. Let us notice that necessary and sufficient conditions ensuring the uniform convergence of fundamental matrices X_k corresponding to A_k, $k \in \mathbb{N}$, to the fundamental matrix X_0 corresponding to A_0 may be found in the paper [2] by M. Ashordia.

Results related to Theorem 3.2.9 obtained by the method of "*prolongation*" of functions of bounded variation to continuous functions along monotone functions and using the concept of *convergence under substitution* instead of the emphatic convergence were also obtained by D. Fraňková in [10] (cf. also [11]).

3.3. Emphatic convergence

Notation 3.3.1. For a given function $F \in \mathbb{BV}^{n\times n}$, the symbol $\mathscr{S}(F)$ stands for the set of the points of discontinuity of F in $[0,1]$, while

$$\mathscr{S}^+(F) = \{t \in [0,1); \Delta^+ F(t) \neq 0\} \text{ and } \mathscr{S}^-(F) = \{t \in [0,1); \Delta^- F(t) \neq 0\}.$$

If the set $\mathscr{S}(F) = \mathscr{S}^+(F) \cup \mathscr{S}^-(F)$ is finite, then for an arbitrary compact set M such that

$$M = \bigcup_{j=1}^{m} [\alpha_j, \beta_j] \subset [0,1] \setminus \mathscr{S}(F) \quad \text{and}$$
$$[\alpha_j, \beta_j] \cap [\alpha_k, \beta_k] = \emptyset \quad \text{for} \quad j \neq k, \tag{3.3.1}$$

we define

$$F^M(t) = F(t) - F(\alpha_j) \quad \text{if} \;\; t \in [\alpha_j, \beta_j].$$

Provided the set $\mathscr{S}(A_0)$ contains at most a finite number of elements, we can extend Theorem 3.2.9 to the case that the functions A_k, $k \in \mathbb{N} \cup \{0\}$, need not be left-continuous on $(0,1]$ in the following way.

Theorem 3.3.2. *Let* $A_0 \in \mathbb{BV}^{n\times n}$, $\mathscr{S}(A_0) = \{\tau_j\}_{j=1}^m$,

$$\det\left[\mathrm{I} - \Delta^- A_0(t)\right] \neq 0 \quad \textit{on} \quad (0,1]$$

and let X_0 *be the fundamental matrix corresponding to* A_0. *Assume that the sequence* $\{A_k\}_{k=1}^\infty \subset \mathbb{BV}^{n\times n}$ *is such that*

(i) $\sup_k \operatorname{var} A_k < \infty$ *and* $\det\left[\mathrm{I} - \Delta^- A_k(t)\right] \neq 0$ *on* $(0,1]$ *for all* $k \in \mathbb{N}$*;*

(ii) $\lim_{k\to\infty} A_k^M(s) = A_0^M(s)$ *uniformly on* M *for any* $M \subset [0,1] \setminus \mathscr{S}(A_0)$ *fulfilling* (3.3.1)*;*

(iii) *if* $\tau \in \mathscr{S}(A_0)$ *then for any* $z \in \mathbb{R}^n$ *and any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $\delta' \in (0,\delta)$ *there is* $k_0 \in \mathbb{N}$ *such that the relations*

$$\begin{aligned}&\left|y_k(\tau) - y_k(\tau-\delta') - \Delta^- A_0(\tau)\left[\mathrm{I} - \Delta^- A_0(\tau)\right]^{-1} z\right| \le \varepsilon,\\ &\left|z_k(\tau+\delta') - z_k(\tau) - \Delta^+ A_0(\tau)\, z\right| \le \varepsilon\end{aligned}$$

are satisfied for any $k \ge k_0$ *and* y_k *and* z_k *such that* $|z - y_k(\tau-\delta')| \le \delta$, $|z - z_k(\tau)| \le \delta$ *and*

$$\begin{aligned}y_k(t) &= y_k(\tau-\delta') + \int_{\tau-\delta'}^t \mathrm{d}[A_k(s)]\, y_k(s) \quad \text{on} \quad [\tau-\delta',\tau],\\ z_k(t) &= z_k(\tau) \qquad\quad + \int_{\tau}^t \mathrm{d}[A_k(s)]\, z_k(s) \quad\ \ \text{on} \quad [\tau,\tau+\delta'].\end{aligned}$$

Then for any $k \in \mathbb{N}$ *sufficiently large the fundamental matrix* X_k *corresponding to* A_k *is defined on* $[0,1]$ *and*

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \text{on} \quad [0,1].$$

Proof. Without any loss of generality we can restrict ourselves to the case that $m = 1$, i.e. let $\mathscr{S}(A_0) = \{\tau\}$, where $\tau \in (0,1)$. Assume that $\widetilde{x} \in \mathbb{R}^n$ is given and let x_k for any $k \in \mathbb{N} \cup \{0\}$ denote the solution to the equation

$$x_k(t) = \widetilde{x} + \int_0^t \mathrm{d}[A_k(s)]\, x_k(s) \quad \text{on} \quad [0,1].$$

By Theorem 3.2.5, our assumptions (i) and (ii) imply that for any $\alpha \in (0,\tau)$ we have

$$\lim_{k\to\infty} x_k(t) = x_0(t) \quad \text{uniformly on} \quad [0,\alpha].$$

Consequently,

$$\lim_{k\to\infty} x_k(t) = x_0(t) \quad \text{for all} \quad t \in [0,\tau). \tag{3.3.2}$$

Furthermore, for any $\delta' \in (0,\tau)$ and $k \in \mathbb{N}$ we have

$$\begin{aligned}|x_0(\tau) - x_k(\tau)| \le{}& \left|x_0(\tau) - x_0(\tau-\delta') - \Delta^- A_0(\tau)\,[\mathrm{I} - \Delta^- A_0(\tau)]^{-1}\, x_0(\tau-)\right|\\ &+ \left|\Delta^- A_0(\tau)\,[\mathrm{I} - \Delta^- A_0(\tau)]^{-1}\, x_0(\tau-) - (x_k(\tau) - x_k(\tau-\delta'))\right|\\ &+ |x_0(\tau-\delta') - x_k(\tau-\delta')|.\end{aligned} \tag{3.3.3}$$

Let an arbitrary $\varepsilon > 0$ be given. By the assumption (iii) there exists $\delta \in (0,\varepsilon)$ such that for all $\delta' \in (0,\delta)$ there exists $k_1 = k_1(\delta') \in \mathbb{N}$ such that for any $k \ge k_1$ and for any solution y_k of the equation

$$y_k(t) = y_k(\tau-\delta') + \int_{\tau-\delta'}^t \mathrm{d}[A_k(s)]\, y_k(s) \quad \text{on} \quad [\tau-\delta',\tau]$$

such that $|y_k(\tau-\delta')-x_0(\tau-)|<\delta$ we have

$$\left|y_k(\tau)-y_k(\tau-\delta')-\Delta^- A_0(\tau)\,[\mathrm{I}-\Delta^- A_0(\tau)]^{-1}\,x_0(\tau-)\right|<\varepsilon. \qquad (3.3.4)$$

Let us choose $\delta'\in(0,\delta)$ in such a way that

$$|x_0(\tau-)-x(\tau-\delta')|<\frac{\delta}{2}$$

is true. Furthermore, according to (3.3.2) there is $k_0\in\mathbb{N}$ such that $k_0\ge k_1$ and

$$|x_0(\tau-\delta')-x_k(\tau-\delta')|<\frac{\delta}{2}\quad\text{for all}\quad k\ge k_0. \qquad (3.3.5)$$

In particular, for $k\ge k_0$ we have

$$|x_0(\tau-)-x_k(\tau-\delta')|<\delta.$$

Thus, if we put $y_k(t)=x_k(t)$ on $[\tau-\delta',\tau]$, then the relation (3.3.4) will be satisfied for any $k\ge k_0$, i.e., we have

$$\left|x_k(\tau)-x_k(\tau-\delta')-\Delta^- A_0(\tau)\,[\mathrm{I}-\Delta^- A_0(\tau)]^{-1}\,x_0(\tau-)\right|<\varepsilon \qquad (3.3.6)$$

for all $k\ge k_0$. Now, inserting (3.3.5)-(3.3.6) into (3.3.3), we obtain that

$$|x_k(\tau)-x_0(\tau)|<\frac{\delta}{2}+\frac{\delta}{2}+\varepsilon<2\,\varepsilon$$

is satisfied for any $k\ge k_0$, i.e.

$$\lim_{k\to\infty}x_k(\tau)=x_0(\tau). \qquad (3.3.7)$$

Further, we will prove that there is $\eta>0$ such that

$$\lim_{k\to\infty}x_k(t)=x_0(t)$$

is true on $(\tau,\tau+\eta)$ as well. To this aim, let $\varepsilon>0$ be given and let $\eta_0\in(0,\varepsilon)$ be such that

$$|x_0(s)-x_0(\tau+)|<\varepsilon\quad\text{for all}\ \ s\in(\tau,\tau+\eta_0). \qquad (3.3.8)$$

By the assumption (iii) there exists $\eta\in(0,\eta_0)$ such that for any $\eta'\in(0,\eta)$ there is $\ell_1=\ell_1(\eta')\in\mathbb{N}$ such that for any $k\ge\ell_1$ and for any solution z_k of the equation

$$z_k(t)=z_k(\tau)+\int_\tau^t \mathrm{d}[A_k(s)]\,z_k(s)\ \text{on}\ \ [\tau,\tau+\eta']$$

such that $|z_k(\tau) - x_0(\tau)| < \eta$ we have

$$\big|z_k(\tau + \eta') - z_k(\tau) - \Delta^+ A_0(\tau)\, x_0(\tau)\big| < \varepsilon. \tag{3.3.9}$$

Let us choose $\eta' \in (0, \eta)$ arbitrarily. By (3.3.8), we have

$$|x_0(\tau - \eta') - x_0(\tau+)| < \varepsilon. \tag{3.3.10}$$

Furthermore, by (3.3.7) there is $\ell_0 \in \mathbb{N}$ such that $\ell_0 \geq \ell_1$ and

$$|x_k(\tau) - x_0(\tau)| < \eta \quad \text{for all} \quad k \geq \ell_0.$$

Thus, by (3.3.9), for any $k \geq \ell_0$ we have

$$|x_k(\tau + \eta') - x_k(\tau) - \Delta^+ A_0(\tau)\, x_0(\tau)| < \varepsilon. \tag{3.3.11}$$

Making use of (3.3.10)-(3.3.11) we finally get for any $k \geq k_0$

$$\begin{aligned}
&|x_k(\tau + \eta') - x_0(\tau + \eta')| \\
&\quad \leq |x_k(\tau + \eta') - x_k(\tau) - x_0(\tau+) + x_0(\tau)| \\
&\quad + |x_0(\tau + \eta') - x_0(\tau+)| + |x_k(\tau) - x_0(\tau)| \\
&\quad = |x_k(\tau + \eta') - x_k(\tau) - \Delta^+ A_0(\tau)\, x_0(\tau)| \\
&\quad + |x_0(\tau+) - x_0(\tau + \eta')| + |x_k(\tau) - x_0(\tau)| < 3\,\varepsilon,
\end{aligned}$$

i.e. $\lim_{k\to\infty} x_k(t) = x_0(t)$ for all $t \in (\tau, \tau + \eta)$.

The proof of the theorem can be completed by using Theorem 3.2.5 and taking into account that $\widetilde{x} \in \mathbb{R}^n$ was chosen arbitrarily. The extension to the general case $m \in \mathbb{N}$ is obvious. $\square$

Remark 3.3.3. Obviously, if we did not restrict ourselves to the case of only a finite number of discontinuities of A_0, we should replace the assumptions (i)-(ii) in Theorem 3.3.2 by assumptions of the form (i)-(ii) from Definition 3.2.8.

Remark 3.3.4. The following concept due to M. Pelant (cf. [27]) leads to another interesting convergence effect which most probably cannot be explained by Theorem 3.3.2.

Let $A \in \mathbb{BV}^{n\times n}$ and let the divisions $D_k = \{0 = t_0^k < \cdots < t_{p_k}^k = 1\}$, $k \in \mathbb{N}$, of $[0,1]$ be such that $D_k \supset \{t \in [0,1];\, t = \frac{i}{2^k}, i = 0, 1, \ldots 2^k\} \cup \{t \in (0,1];\, |\Delta^- A(t)| \geq \frac{1}{k}\} \cup \{t \in [0.1);\, |\Delta^+ A(t)| \geq \frac{1}{k}\}$.

For a given $k \in \mathbb{N}$, let us put

$$A_k(t) = \begin{cases} A(t) & \text{if } \ t \in D_k, \\[2mm] A(t_{i-1}^k) + \dfrac{A(t_i^k) - A(t_{i-1}^k)}{t_i^k - t_{i-1}^k}\,(t - t_{i-1}^k) & \text{if } \ t \in (t_{i-1}^k, t_i^k). \end{cases}$$

Then we say that the sequence $\{A_k, D_k\}_{k=1}^\infty$ *piecewise linearly approximates A.*

Furthermore, for a given $A \in \mathbb{BV}^{n\times n}$, let us define A_0 on $[0,1]$ by

$$\begin{aligned}A_0(t) = A(t) &- \sum_{s\in\mathscr{S}^-(A)} \Delta^- A(s)\,\chi_{[s,1]}(t) - \sum_{s\in\mathscr{S}^+(A)} \Delta^+ A(s)\,\chi_{(s,1]}(t)\\ &+ \sum_{s\in\mathscr{S}^-(A)} \Big(\mathrm{I} - \big[\exp\big(\Delta^- A(s)\big)\big]^{-1}\Big)\,\chi_{[s,1]}(t)\\ &+ \sum_{s\in\mathscr{S}^+(A)} \Big(\exp\big(\Delta^+ A(s)\big) - \mathrm{I}\Big)\,\chi_{(s,1]}(t). \end{aligned} \tag{3.3.12}$$

Then $\det\big[\mathrm{I} - \Delta^- A_0(t)\big] \neq 0$ on $(0,1]$ and the following assertion may be proved (cf. [27]).

Let $A \in \mathbb{BV}^{n\times n}$, *let* A_0 *be given by* (3.3.12), *let* $\{A_k, D_k\}_{k=1}^{\infty}$ *piecewise linearly approximate* A *and let for a given* $k \in \mathbb{N}$, X_k *denote the fundamental matrix corresponding to* A_k. *Then*

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \text{for all} \quad t \in [0,1].$$

Furthermore, if $A \in \mathbb{BV}^{n\times n}$ is such that the relations

$$\begin{aligned}&\det[\mathrm{I} - \Delta^- A(t)] \neq 0 \quad \text{on} \quad (0,1] \quad \text{and}\\ &\det[\mathrm{I} + \Delta^+ A(t)] \neq 0 \quad \text{on} \quad [0,1)\end{aligned} \tag{3.3.13}$$

are true, then for $t \in [0,1]$ we can define

$$\begin{aligned}A_0^*(t) = A(t) &- \sum_{s\in\mathscr{S}^-(A)} \Delta^- A(s)\,\chi_{[s,1]}(t) - \sum_{s\in\mathscr{S}^+(A)} \Delta^+ A(s)\,\chi_{(s,1]}(t)\\ &+ \sum_{s\in\mathscr{S}^-(A)} \ln\big[\mathrm{I} - \Delta^- A(s)\big]^{-1}\,\chi_{[s,1]}(t)\\ &+ \sum_{s\in\mathscr{S}^+(A)} \ln\big[\mathrm{I} + \Delta^+ A(s)\big]\,\chi_{(s,1]}(t)\end{aligned} \tag{3.3.14}$$

and the following assertion is an immediate corollary of the above mentioned result of M. Pelant.

Theorem 3.3.5. *Let* $A \in \mathbb{BV}^{n\times n}$ *be such that* (3.3.13) *holds and let* X *be the fundamental matrix corresponding to* A. *Let* A_0^* *be given by* (3.3.14), *let* $\{A_k, D_k\}_{k=1}^{\infty}$ *piecewise linearly approximate* A_0^* *and let for a given* $k \in \mathbb{N}$, X_k *denote the fundamental matrix corresponding to* A_k. *Then*

$$\lim_{k\to\infty} X_k(t) = X(t) \quad \textit{for all} \quad t \in [0,1].$$

Chapter 4

Linear Boundary Value Problems for Generalized Differential Equations

4.1. Introduction

This chapter is devoted to linear ***boundary value problems*** for generalized linear differential equations

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) = f(t) - f(0), \quad t \in [0,1], \tag{4.1.1}$$

$$M\, x(0) + \int_0^1 K(s)\, \mathrm{d}[x(s)] = r \quad (\in \mathbb{R}^m) \tag{4.1.2}$$

and the corresponding controllability problems. In particular, we obtain the adjoints to these problems in such a way that the usual duality theory can be extended to them. In contrast to the earlier papers (cf. e.g. [54], [46], [47], [43], [44] and the monograph [45]) the right-hand side of the equation (4.1.1) can be in general a regulated function (not necessarily of bounded variation). Similar problems in the space of regulated functions were treated e.g. by Ch. S. Hönig [15], [17], [16], L. Fichmann [9] and L. Barbanti [5], where the interior (Dushnik) integral was used. Let us notice that by Theorem 2.4.8 the left-hand side of the additional condition (4.1.2) represents the general form of a linear bounded mapping of the space of functions regulated on the closed interval $[0,1]$ and left-continuous on its interior $(0,1)$, equipped with the supremal norm, into $\mathbb{R}^n$.

The basic properties of the Perron-Stieltjes integral with respect to scalar regulated functions were described in Chapter 2. The extension of these results to vector valued or matrix valued functions is obvious (they are used componentwise in these situations) and hence for the basic facts concerning integrals with respect to regulated functions we shall refer to the corresponding assertions from Chapter 2.

4.2. Auxiliary lemma

The following property of the functions of strongly bounded variation (cf. 1.2.5) will be helpful later.

Lemma 4.2.1. *Let* $W : [0,1] \times [0,1] \mapsto \mathbb{R}^{n\times n}$ *be such that*

$$v(W) + \mathrm{var}_0^1 W(0,.) < \infty. \tag{4.2.1}$$

Then for any $g \in \mathbb{G}^n$*, the function*

$$w(t) = \int_0^1 \mathrm{d}_s[W(t,s)]\, g(s), \quad t \in [0,1] \tag{4.2.2}$$

is defined and has a bounded variation on $[0,1]$ *and*

$$\begin{aligned} w(t+) &= \int_0^1 \mathrm{d}_s[W(t+,s)]\, g(s) \quad \text{if } \ t \in [0,1), \\ w(t-) &= \int_0^1 \mathrm{d}_s[W(t-,s)]\, g(s) \quad \text{if } \ t \in (0,1]. \end{aligned} \tag{4.2.3}$$

Proof. Let $g \in \mathbb{G}^n$ be given. Since (4.2.1) implies that $\mathrm{var}_0^1 W(t,.) < \infty$ for any $t \in [0,1]$ (cf. e.g. Lemma I.6.6 in [45]), the function (4.2.2) is defined for any $t \in [0,1]$. Furthermore, let an arbitrary division $d = \{t_0, t_1, \ldots, t_k\}$ of $[0,1]$ be given. Then by Lemmas I.4.16 and I.6.13 of [45] we have

$$\sum_{j=1}^k |w(t_j) - w(t_{j-1})| \le \sum_{j=1}^k \mathrm{var}_0^1(W(t_j,.) - W(t_{j-1},.))\, \|g\| \le v(W)\, \|g\|,$$

and consequently

$$\mathrm{var}_0^1\, w \le v(W)\, \|g\| < \infty.$$

In particular, $w \in \mathbb{G}^n$. Moreover, by [45, Lemma I.6.14]) all the functions

$$W(t+,.) \quad \text{and} \quad W(s-,0), \quad t \in [0,1),\, s \in (0,1]$$

are of bounded variation on $[0,1]$. Thus the integrals on the right-hand sides of (4.2.3) are well defined. As g is on $[0,1]$ a uniform limit of a sequence of finite step functions and any finite step function on $[0,1]$ is a linear combination of simple jump functions on $[0,1]$

$$\chi_{[0,\sigma]},\ \chi_{[\sigma,1]}, \quad \sigma \in [0,1], \tag{4.2.4}$$

it is sufficient to verify the relations (4.2.3) for the case that g is a simple jump function of the type (4.2.4). Let $g = \chi_{[0,\sigma]}$, where $\sigma \in [0,1]$. Then for any $t \in [0,1]$ we have

$$w(t) = \int_0^\sigma \mathrm{d}_s[W(t,s)] + (W(t,\sigma+) - W(t,\sigma)) = W(t,\sigma+) - W(t,\sigma).$$

Consequently,

$$w(t+) = W(t+,\sigma+) - W(t+,0) \text{ if } \ t \in [0,1)$$

and

$$w(t-) = W(t-,\sigma+) - W(t-,0) \text{ if } \ t \in (0,1].$$

On the other hand, we have

$$\int_0^1 \mathrm{d}_s[W(t+,s)]\, g(s) = W(t+,\sigma+) - W(t+,0) \text{ if } \ t \in [0,1)$$

and

$$\int_0^1 \mathrm{d}_s[W(t-,s)]\, g(s) = W(t-,\sigma+) - W(t-,0) \text{ if } \ t \in (0,1].$$

This means that the function $g = \chi_{[0,\sigma]}$ satisfies the relations (4.2.3) for any $\sigma \in [0,1)$. Similarly we could verify that the function $g = \chi_{[\sigma,1]}$ satisfies (4.2.3) for any $\sigma \in [0,1]$, and this completes the proof. $\square$

4.3. Boundary value problem

We will consider the ***boundary value problem*** of determining a function $x : [0,1] \mapsto \mathbb{R}^n$ fulfilling the generalized differential equation (4.1.1) and the additional condition (4.1.2).

Throughout this chapter we assume

$$A \in \mathbb{BV}^{n\times n},\ A(0+) = A(0),\ A(t-) = A(t) \ \text{ on } \ (0,1], \tag{4.3.1}$$

$$\det(I + \Delta^+ A(t)) \neq 0 \ \text{ on } \ [0,1); \tag{4.3.2}$$

$$M \in \mathbb{R}^{m\times n};\ K \in \mathbb{BV}^{m\times n}; \tag{4.3.3}$$

$$f \in \mathbb{G}_L^n \quad \text{and} \quad r \in \mathbb{R}^m. \tag{4.3.4}$$

Remark 4.3.1. The assumptions (4.3.1) and (4.3.3) ensure that

$$\mathscr{L} : x \in \mathbb{G}_L^n \mapsto x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) \tag{4.3.5}$$

defines a linear bounded operator on $\mathbb{G}_L^n$ (cf. Proposition 2.3.16 and Theorem 2.3.8) and

$$\mathscr{K} : x \in \mathbb{G}_L^n \mapsto M\, x(0) + \int_0^1 K(s)\, \mathrm{d}[x(s)] \tag{4.3.6}$$

defines a linear bounded mapping of $\mathbb{G}_L^n$ into $\mathbb{R}^m$ (cf. Theorem 2.4.8). Hence, by

$$\mathscr{A} : x \in \mathbb{G}_L^n \mapsto \begin{pmatrix} \mathscr{L}\, x \\ \mathscr{K}\, x \end{pmatrix} \in \mathbb{G}_L^n \times \mathbb{R}^m \tag{4.3.7}$$

we define a linear bounded mapping of $\mathbb{G}_L^n$ into $\mathbb{G}_L^n \times \mathbb{R}^m$.

Moreover, notice that according to Theorem 2.4.8, any linear continuous mapping $\mathscr{K}$ of $\mathbb{G}_L^n$ into $\mathbb{R}^m$ can be expressed in the form (4.3.6), where $M \in \mathbb{R}^{m\times n}$ and $K \in \mathbb{BV}^{m\times n}$.

Remark 4.3.2. It is well-known (cf. [45, Theorem III.2.10]) that under the assumptions (4.3.1)-(4.3.2) there exists a unique $n \times n-$matrix valued function $U(t,s)$ such that

$$U(t,s) = \mathrm{I} + \int_0^t \mathrm{d}[A(\tau)]\, U(\tau,s) \quad \text{for} \quad t,s \in [0,1]. \tag{4.3.8}$$

It is called the ***fundamental matrix*** of the homogeneous equation

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) = 0 \ \text{ on } \ [0,1] \tag{4.3.9}$$

and possesses the following properties

$$|U(t,s)| + \mathrm{var}_0^1 U(t,.) + \mathrm{var}_0^1 U(.,s) + \mathrm{v}\,(U) \le M < \infty \quad \text{for} \quad t,s \in [0,1],$$

$$\begin{array}{ll}
U(t,\tau)\, U(\tau,s) = U(t,s) & \text{for } \ t,s,\tau \in [0,1], \\
\det U(t,s) \ne 0 & \text{for } \ t,s \in [0,1], \\
U(t+,s) = [I + \Delta^+ A(t)]\, U(t,s) & \text{for } \ t \in [0,1), s \in [0,1], \\
U(t-,s) = U(t,s) & \text{for } \ t \in (0,1], s \in [0,1], \\
U(t,s+) = U(t,s)\, [I + \Delta^+ A(t)]^{-1} & \text{for } \ t \in [0,1], s \in [0,1), \\
U(t,s-) = U(t,s) & \text{for } \ t \in [0,1], s \in (0,1].
\end{array}$$

For a given $c \in \mathbb{R}^n$, the equation (4.3.9) possesses a unique solution $x\colon [0,1] \mapsto \mathbb{R}^n$ on $[0,1]$ such that $x(0) = c$ and this solution is given by (cf. [45, Theorem III.2.4])

$$x(t) = U(t,0)\, c, \quad t \in [0,1].$$

It is well-known (cf. [45, Theorem III.2.13]) that for any $f : [0,1] \mapsto \mathbb{R}^n$ of bounded variation on $[0,1]$ ($f \in \mathbb{BV}^n$) and any $c \in \mathbb{R}^n$ there exists a unique solution x of (4.1.1) on $[0,1]$ such that $x(0) = c$. This solution has a bounded variation on $[0,1]$ and is given on $[0,1]$ by

$$x(t) = U(t,0)\, c + f(t) - f(0) + \int_0^t \mathrm{d}_s[U(t,s)]\, (f(s) - f(0)).$$

To extend this assertion also to equations (4.1.1) with right-hand sides $f \in \mathbb{G}_L^n$, the following lemma will be helpful.

Lemma 4.3.3. *Assume* (4.3.1) *and* (4.3.2). *Then for any* $f \in \mathbb{G}_L^n$ *the function*

$$\psi(t) = f(t) - f(0) - \int_0^t \mathrm{d}_s[U(t,s)]\,(f(s) - f(0)) \tag{4.3.10}$$

is defined and regulated on $[0,1]$ *and left-continuous on* $(0,1)$. *The operator*

$$\Psi : \, f \in \mathbb{G}_L^n \mapsto \psi \in \mathbb{G}_L^n \tag{4.3.11}$$

is linear and bounded.

Proof. The function ψ is obviously defined on $[0,1]$. Let us put

$$W(t,s) = U(t,s) \text{ if } \; t \ge s \quad \text{and} \quad W(t,s) = U(t,t) \text{ if } \; t < s. \tag{4.3.12}$$

Then

$$\int_0^t \mathrm{d}_s[U(t,s)]\,(f(s) - f(0)) = \int_0^t \mathrm{d}_s[W(t,s)]\,(f(s) - f(0)) \tag{4.3.13}$$

holds for any $t \in [0,1]$ and $f \in \mathbb{G}_L^n$. Since obviously

$$\mathrm{v}\,(W) + \mathrm{var}_0^1 W(0,.) < \infty, \tag{4.3.14}$$

we may use Lemma 4.2.1 to show that $\psi \in \mathbb{G}_L^n$ for any $f \in \mathbb{G}_L^n$. The boundedness of the operator Ψ follows from the inequality

$$|\psi(t)| \le 2\,(\mathrm{var}_0^1 W(t,.))\,\|f\| \le 2\,(\mathrm{v}\,(W) + \mathrm{var}_0^1 W(0,.))\,\|f\|$$

(cf. [45, Lemma I.6.6]). □

Proposition 4.3.4. *Assume* (4.3.1) *and* (4.3.2). *Then for any* $f \in \mathbb{G}_L^n$ *and any* $c \in \mathbb{R}^n$ *the equation* (4.1.1) *possesses on* $[0,1]$ *a unique solution* $x \in \mathbb{G}_L^n$ *such that* $x(0) = c$. *This solution belongs to* $\mathbb{G}_L^n$ *and is given by*

$$x = \Phi\, c + \Psi f, \tag{4.3.15}$$

where Ψ *is the linear bounded operator on* $\mathbb{G}_L^n$ *given by* (4.3.10) *and* (4.3.11) *and* Φ *is the linear bounded mapping of* $\mathbb{R}^n$ *into* $\mathbb{G}_L^n$ *given by*

$$\Phi : \, c \in \mathbb{R}^n \mapsto U(t,0)\, c.$$

Proof. Let $f \in \mathbb{G}_L^n$ and $c \in \mathbb{R}^n$ be given. By Lemma 4.3.3 the function x given by (4.3.15) is defined on $[0,1]$ and belongs to $\mathbb{G}_L^n$. Hence the integral

$$\int_0^t \mathrm{d}[A(s)]\, x(s)$$

is defined for any $t \in [0,1]$. Inserting (4.3.10) into this integral and taking into account (4.3.5) and (4.3.12)-(4.3.14) we obtain by Theorems 2.3.19 (*substitution*) and 2.3.20 (*change of the integration order*)

$$\begin{aligned}\int_0^t \mathrm{d}[A(s)]\,x(s) &= [U(t,0)-I]\,c + \int_0^t \mathrm{d}[A(s)]\,(f(s)-f(0)) \\ &\qquad - \int_0^t \mathrm{d}\Big[\int_0^t \mathrm{d}[A(\tau)]\,W(\tau.s)\Big]\,(f(s)-f(0)) \\ &= [U(t,0)-I]\,c - \int_0^t \mathrm{d}_s[U(t,s)]\,(f(s)-f(0)) \\ &= x(t)-x(0)-f(t)+f(0)\end{aligned}$$

for any $t \in [0,1]$. Hence x is a solution of (4.1.1) on $[0,1]$. Obviously, $x(0) = c$. The uniqueness of this solution follows from the uniqueness of the zero solution to the equation

$$u(t) = \int_0^t \mathrm{d}[A(s)]\,u(s)$$

on $[0,1]$ (cf. [45, Theorem III.1.4]). The boundedness of the operator Φ is evident and the boundedness of Ψ was shown in Lemma 4.3.3. $\square$

Now, by a standard technique due to D.Wexler (cf. [56]) we can prove the normal solvability of the operator $\mathscr{A}$ given by (4.3.7).

Proposition 4.3.5. *Assume* (4.3.1)-(4.3.3). *Then the operator* $\mathscr{A}$ *has a closed range in* $\mathbb{G}^n_L \times \mathbb{R}^m$.

Proof. By (4.3.15) a couple $(f,r) \in \mathbb{G}^n_L \times \mathbb{R}^m$ belongs to the range $\mathscr{R}(\mathscr{A})$ of the operator $\mathscr{A}$ if and only if there exists a $c \in \mathbb{R}^n$ such that

$$(\mathscr{K}\Phi)\,c = r - (\mathscr{K}\,\Psi)\,f,$$

i.e. $\mathscr{R}(\mathscr{A}) = \Theta_{-1}(\mathscr{R}(\mathscr{K}\Phi))$, where

$$\Theta:\ (f,r) \in \mathbb{G}^n_L \times \mathbb{R}^m \mapsto r - (\mathscr{K}\Psi)f \in \mathbb{R}^m$$

is obviously a continuous operator. $\mathscr{R}(\mathscr{K}\Phi)$ being finite dimensional, it is closed and consequently $\mathscr{R}(\mathscr{A})$ is closed as well. $\square$

By Theorem 2.4.8 the dual space to $\mathbb{G}^n$ may be represented by the space $\mathbb{BV}^n \times \mathbb{R}^n$, while for $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^n$ the corresponding linear bounded functional on $\mathbb{G}^n_L$ is given by

$$x \in \mathbb{G}^n_L \mapsto \langle x,(y,\delta)\rangle := \delta^{\mathrm{T}}\,x(0) + \int_0^1 y^{\mathrm{T}}(s)\,\mathrm{d}[x(s)] \in \mathbb{R}. \tag{4.3.16}$$

Thus, the adjoint operator to $\mathscr{A}$ may be represented by the operator

$$\mathscr{A}^* : \mathbb{BV}^n \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{BV}^n \times \mathbb{R}^m$$

defined by the relation

$$\langle \mathscr{A}\, x, (y,\gamma,\delta)\rangle := \langle \mathscr{L}\, x, (y,\gamma)\rangle + \delta^{\mathrm{T}}\,(\mathscr{K}\, x) = \langle x, \mathscr{A}^*(y,\gamma,\delta)\rangle \tag{4.3.17}$$
$$\text{for } \ x \in \mathbb{G}_L^n,\ y \in \mathbb{BV}^n,\ \gamma \in \mathbb{R}^n \text{ and } \ \delta \in \mathbb{R}^m.$$

Definition 4.3.6. The operator $\mathscr{A}^* : \mathbb{BV}^n \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{BV}^n \times \mathbb{R}^m$ fulfilling (4.3.17) is called the ***adjoint operator*** to $\mathscr{A}$.

The next theorem provides an explicit form of the adjoint operator to $\mathscr{A}$.

Theorem 4.3.7. *Assume* (4.3.1)-(4.3.3). *Then the adjoint operator* $\mathscr{A}^*$ *is defined by*

$$\begin{aligned}\mathscr{A}^* : (y^{\scriptscriptstyle T},\gamma^{\scriptscriptstyle T},\delta^{\scriptscriptstyle T}) \in \mathbb{BV}^n \times \mathbb{R}^n \times \mathbb{R}^m \mapsto&\\ \big(y^{\scriptscriptstyle T}(t) + \delta^{\scriptscriptstyle T} K(t) - \int_t^1 y^{\scriptscriptstyle T}(t)\,\mathrm{d}[\widetilde{A}(s+)], \delta^{\scriptscriptstyle T} M&\\ - \int_0^1 y^{\scriptscriptstyle T}(s)\,\mathrm{d}[\widetilde{A}(s)]\big) \in \mathbb{BV}^n \times \mathbb{R}^n,&\end{aligned} \tag{4.3.18}$$

where

$$\widetilde{A}(t) = \begin{cases} A(t+) & \text{if } \ t < 1,\\ A(1) & \text{if } \ t = 1.\end{cases} \tag{4.3.19}$$

Proof. Let $x \in \mathbb{G}_L^n$, $y \in \mathbb{BV}^n$, $\gamma \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^m$. Inserting (4.3.5) and (4.3.6) into the left-hand side of (4.3.17) we obtain

$$\begin{aligned}\langle \mathscr{A} x, (y,\gamma,\delta)\rangle &= \int_0^1 y^{\mathrm{T}}(s)\,\mathrm{d}\big[x(t) - \int_0^t \mathrm{d}[A(s)]\,x(s)\big]\\ &\qquad + \delta^{\mathrm{T}}\,\big(M\,x(0) + \int_0^1 K(t)\,\mathrm{d}[x(t)]\big)\\ &= \int_0^1 \big(y^{\mathrm{T}}(t) + \delta^{\mathrm{T}}\,K(t)\big)\,\mathrm{d}[x(t)] + \delta^{\mathrm{T}}\,M\,x(0)\\ &\qquad + \int_0^1 y^{\mathrm{T}}(t)\,\mathrm{d}\big[\int_0^t \mathrm{d}[A(s)]\,x(s)\big].\end{aligned} \tag{4.3.20}$$

Furthermore, by the Substitution Theorem (cf. Theorem 2.3.19) we have

$$\begin{aligned}&\int_0^1 y^{\mathrm{T}}(t)\,\mathrm{d}\big[\int_0^t \mathrm{d}[A(s)]\,x(s)\big]\\ &\qquad = \int_0^1 y^{\mathrm{T}}(t)\,\mathrm{d}[A(t)]\,x(t) = -\int_0^1 \mathrm{d}\big[\int_t^1 y^{\mathrm{T}}(s)\,\mathrm{d}[A(s)]\big]x(t).\end{aligned}$$

Now, integrating by parts (cf. Theorem 2.3.15) we obtain

$$\begin{aligned}\int_0^1 y^{\mathrm T}(t)\,\mathrm d\Big[\int_0^t \mathrm d[A(s)]\,x(s)\Big]\\ &= \Big(\int_0^1 y^{\mathrm T}(s)\,\mathrm d[A(s)]\Big)\,x(0) + \int_0^1 \Big(\int_t^1 y^{\mathrm T}(s)\,\mathrm d[A(s)]\Big)\,\mathrm d[x(s)]\\ &\quad + \sum_{0\le t<1} \Delta^+ w^{\mathrm T}(t)\,\Delta^+ x(t) - \sum_{0<t\le 1} \Delta^- w^{\mathrm T}(t)\,\Delta^- x(t), \qquad (4.3.21)\end{aligned}$$

where

$$w^{\mathrm T}(t) = \int_t^1 y^{\mathrm T}(s)\,\mathrm d[A(s)] \quad \text{for} \quad t\in[0,1].$$

As

$$\begin{aligned}&\Delta^+ w^{\mathrm T}(0) = -y^{\mathrm T}(0)\,\Delta^+ A(0) = 0,\\ &\Delta^+ w^{\mathrm T}(t) = -y^{\mathrm T}(t)\,\Delta^+ A(t) \quad \text{for} \quad t\in(0,1)\end{aligned}$$

and

$$\Delta^- w^{\mathrm T}(t) = -y^{\mathrm T}(t)\,\Delta^- A(t) = 0 \quad \text{for} \quad t\in(0,1],$$

the relation (4.3.21) reduces to

$$\begin{aligned}\int_0^1 y^{\mathrm T}(t)\mathrm d\Big[\int_0^t \mathrm d[A(s)]\,x(s)\Big]\\ &= \Big(\int_0^1 y^{\mathrm T}(s)\,\mathrm d[A(s)]\Big)\,x(0) + \int_0^1 \Big(\int_t^1 y^{\mathrm T}(s)\,\mathrm d[A(s)]\Big)\,\mathrm d[x(t)]\\ &\quad - \sum_{0<t<1} y^{\mathrm T}(t)\Delta^+ A(t)\Delta^+ x(t).\end{aligned}$$

Let us put $z^{\mathrm T}(t) = y^{\mathrm T}(t)\Delta^+ A(t)$ for $t\in[0,1)$ and $z^{\mathrm T}(1)=0$. Then $z^{\mathrm T}(t+) = z^{\mathrm T}(t-) = 0$ for $t\in(0,1)$, $z^{\mathrm T}(0) = z^{\mathrm T}(0+) = z^{\mathrm T}(1-) = z^{\mathrm T}(1) = 0$ and $z^{\mathrm T}(t) = 0$ if and only if $\Delta^+ A(t) = 0$. Hence, using Proposition 2.3.12 we get

$$\int_0^1 z^{\mathrm T}(t)\,\mathrm d[x(t)] = \sum_{0<t<1} z^{\mathrm T}(t)\,\Delta x(t) = \sum_{0<t<1} y^{\mathrm T}(t)\,\Delta^+ A(t)\,\Delta^+ x(t)$$

and

$$\begin{aligned}\int_0^1 y^{\mathrm T}(t)\,\mathrm d\Big[\int_0^t \mathrm d[A(s)]\,x(s)\Big] &= \Big(\int_0^1 y^{\mathrm T}(s)\,\mathrm d[A(s)]\Big)\,x(0)\\ &\quad + \int_0^1 \Big(\int_t^1 y^{\mathrm T}(s)\,\mathrm d[A(s)]\Big)\,\mathrm d[x(t)] - \int_0^1 z^{\mathrm T}(t)\,\mathrm d[x(t)].\end{aligned}$$

If we define $B(t) = \Delta^+ A(t)$ on $[0,1]$ (i.e. $B(1) = 0$), then $B(t) = 0$ if and only if $\Delta^+ A(t) = 0$ and, moreover, $B(0) = B(0+) = B(t-) = B(t+) = B(1-) = B(1)$ for any $t \in (0,1)$. Consequently,

$$\int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[B(s)] = y^{\mathrm{T}}(t)\, \Delta^+ B(t) = -y^{\mathrm{T}}(t)\, \Delta^+ A(t) = -z^{\mathrm{T}}(t) \quad \text{on} \quad [0,1)$$

(cf. Corollary 2.3.14). Thus,

$$\begin{aligned}
&\int_0^1 y^{\mathrm{T}}(t) \mathrm{d}\Big[\int_0^t \mathrm{d}[A(s)]\, x(s)\Big] \\
&\quad = \Big(\int_0^1 y^{\mathrm{T}}(s)\, \mathrm{d}[A(s)]\Big)\, x(0) + \int_0^1 \Big(\int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[A(s)]\Big)\, \mathrm{d}[x(t)] \\
&\qquad + \int_0^1 \Big(\int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[B(s)]\Big)\, \mathrm{d}[x(t)] \\
&\quad = \Big(\int_0^1 y^{\mathrm{T}}(t)\, \mathrm{d}[A(t)]\Big)\, x(0) \\
&\qquad + \int_0^1 \Big(\int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[A(s+)]\Big)\, \mathrm{d}[x(t)],
\end{aligned} \tag{4.3.22}$$

where the convention $A(1+) = A(1)$ is used. Finally, inserting (4.3.22) into (4.3.20) we obtain

$$\begin{aligned}
\langle \mathscr{A} x, (y,\gamma,\delta)\rangle = \int_0^1 &\big(y^{\mathrm{T}}(t) + \delta^{\mathrm{T}} K(t) - \int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[A(s+)]\big)\, \mathrm{d}[x(t)] \\
&+ \big(\delta^{\mathrm{T}} M - \int_0^1 y^{\mathrm{T}}(s)\, \mathrm{d}[A(s)]\big) x(0).
\end{aligned}$$

□

Theorem 4.3.8. *Assume* (4.3.1)-(4.3.3) *and let* $y \in \mathbb{BV}^n$, $\gamma \in \mathbb{R}^n$ *and* $\delta \in \mathbb{R}^m$. *Then* $(y,\gamma,\delta) \in \mathscr{N}(\mathscr{A}^*)$ *if and only if*

$$y^{\mathrm{T}}(t) = y^{\mathrm{T}}(1) + \int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[\widetilde{A}(s)] - \delta^{\mathrm{T}}\, (K(t) - K(1)) \quad \textit{on} \quad [0,1], \tag{4.3.23}$$

$$y^{\mathrm{T}}(0) + \delta^{\mathrm{T}}\, (K(0) - M) = 0, \quad y^{\mathrm{T}}(1) + \delta^{\mathrm{T}}\, K(1) = 0, \tag{4.3.24}$$

where $\widetilde{A}$ *is defined by* (4.3.19).

Proof. By (4.3.18), $(y,\gamma,\delta) \in \mathscr{N}(\mathscr{A})$ if and only if

$$y^{\mathrm{T}}(t) = \int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[\widetilde{A}(s)] - \delta^{\mathrm{T}}\, K(t) \quad \text{on} \quad [0,1] \tag{4.3.25}$$

and

$$\delta^{\mathrm{T}}\, M = \int_0^1 y^{\mathrm{T}}(s) \mathrm{d}[A(s)]. \tag{4.3.26}$$

For $t = 1$ the relation (4.3.25) yields $y^{\mathrm{T}}(1) - \delta^{\mathrm{T}} K(1) = 0$. Thus, (4.3.25) may be rewritten as (4.3.23). Furthermore, for $t = 0$ we get from (4.3.25)

$$y^{\mathrm{T}}(0) = \int_0^1 y^{\mathrm{T}}(s)\,\mathrm{d}[A(s)] - \delta^{\mathrm{T}} K(0). \tag{4.3.27}$$

Since

$$\int_0^1 y^{\mathrm{T}}(s)\,\mathrm{d}[\widetilde{A}(s) - A(s)] = 0 \quad \text{for all} \quad y \in \mathbb{BV}^n,$$

the relation (4.3.27) reduces by (4.3.26) to $y^{\mathrm{T}}(0) = \delta^{\mathrm{T}}(M - K(0))$. This completes the proof. □

Definition 4.3.9. The problem of determining a function $y : [0,1] \mapsto \mathbb{R}^n$ of bounded variation on $[0,1]$ and $\delta \in \mathbb{R}^m$ such that (4.3.23) and (4.3.24) are true is called the ***adjoint problem*** to the problem (4.1.1), (4.1.2).

By (4.3.16), Proposition 4.3.5 and Theorem 4.3.7 the linear operator equation

$$\mathscr{A}\, x = \begin{pmatrix} h \\ r \end{pmatrix},$$

where $h \in \mathbb{G}_L^n$ is given by $h(t) = f(t) - f(0)$ on $[0,1]$, fulfils the assumptions of the fundamental theorem on the ***Fredholm alternative*** for linear operator equations (cf. e.g. [30, Theorem 4.12]). Thus, we have:

Theorem 4.3.10. *Assume* (4.3.1)-(4.3.4). *Then the problem* (4.1.1), (4.1.2) *possesses a solution if and only if*

$$\int_0^1 y^{T}(t)\,\mathrm{d}[f(t)] + \delta^{T} r = 0$$

holds for any solution (y, δ) *of the adjoint problem* (4.3.23), (4.3.24).

The adjoint problem 4.3.11. For any $\delta \in \mathbb{R}^m$ fixed, the equation (4.3.23) is a generalized linear differential equation which was treated in detail in Section III.4 in [45]. Let us recall here some basic facts. For given $\delta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$, the equation (4.3.23) possesses a unique solution y on $[0,1]$ such that $y(1) = \eta$. This solution is given on $[0,1]$ by

$$\begin{aligned} y^{\mathrm{T}}(t) = \eta^{\mathrm{T}} V(1,t) - \delta^{\mathrm{T}}(K(t) - K(1)) \\ - \delta^{\mathrm{T}} \int_t^1 (K(s) - K(1))\,\mathrm{d}_s[V(s,t)], \end{aligned} \tag{4.3.28}$$

where V is an $n \times n-$matrix valued function uniquely determined on $[0,1] \times [0,1]$ by the relation

$$V(t,s) = \mathrm{I} + \int_s^t V(t,\tau)\,\mathrm{d}[\widetilde{A}(\tau)], \quad t, s \in [0,1].$$

The relationship of the matrix valued functions U and V is given by Theorem III.4.1 of [45]. Under our assumptions (4.3.1) and (4.3.2) we have according to this theorem

$$\begin{aligned} U(t,s) = V(t,s) + V(t,s)\,\Delta^+ A(s) \\ + \Delta^+ A(t)\,U(t,s) \quad \text{for} \quad t,s \in [0,1] \end{aligned} \tag{4.3.29}$$

(where $\Delta^+ A(1) = 0$). It is easy to verify that a couple $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ is a solution to the adjoint problem (4.3.23), (4.3.24) if and only if y is given by (4.3.28), where $\eta^{\mathrm{T}} = -\delta^{\mathrm{T}} K(1)$ and δ satisfies the algebraic equation

$$\delta^{\mathrm{T}} \big(M + \int_0^1 K(s)\,\mathrm{d}_s[V(s,0)]\big) = 0. \tag{4.3.30}$$

Let us put $W(t) = V(t,0) - U(t,0)$. Then by (4.3.29) $W(t) = \Delta^+ A(t)\,U(t,0)$ and consequently

$$W(0) = W(0+) = W(t+) = W(t-) = W(1-) = W(1) = 0$$

holds for any $t \in (0,1)$. This implies that

$$\int_0^1 K(s)\,\mathrm{d}_s[V(s,0)] = \int_0^1 K(s)\,\mathrm{d}_s[U(s,0)]$$

holds, i.e. the equation (4.3.30) may be rewritten as

$$\delta^{\mathrm{T}} \big(M + \int_0^1 K(s)\,\mathrm{d}_s[U(s,0)]\big) = 0. \tag{4.3.31}$$

Inserting $\eta^{\mathrm{T}} = -\delta^{\mathrm{T}} K(1)$ and

$$\int_t^1 K(1)\,\mathrm{d}_s[V(s,t)] = K(1)\,(V(1,t) - I)$$

into (4.3.28) we may now easily complete the proof of the following characterization of the adjoint problem to (4.1.1), (4.1.2).

Proposition 4.3.12. *Assume* (4.3.1)-(4.3.3). *Then a couple* $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ *is a solution to the problem* (4.3.23), (4.3.24) (*i.e.* $(y,\delta) \in \mathscr{N}(\mathscr{A}^*)$) *if and only if*

$$y^{\mathrm{T}}(t) = -\delta^{\mathrm{T}} \big(K(t) + \int_t^1 K(s)\,\mathrm{d}_s[V(s,t)]\big) \quad \textit{for} \quad t \in [0,1]$$

and δ *verifies the equation* (4.3.31). *Moreover, for the dimension* $\dim \mathscr{N}(\mathscr{A}^*)$ *of the null space* $\mathscr{N}(\mathscr{A}^*)$ *of the operator* $\mathscr{A}$ *the relation*

$$\dim \mathscr{N}(\mathscr{A}^*) = m - \operatorname{rank} \big(M + \int_0^1 K(s)\,\mathrm{d}_s[U(s,0)]\big) \tag{4.3.32}$$

is true.

Since, on the other hand, $x \in \mathbb{G}_L^n$ is a solution of the homogeneous boundary value problem (4.3.9),

$$M\,x(0) + \int_0^1 K(s)\,\mathrm{d}[x(s)] = 0$$

(i.e. $x \in \mathscr{N}(\mathscr{A})$) if and only if $x(t) = U(t,0)\,c$ and

$$\Big(\int_0^1 K(s)\,\mathrm{d}_s[U(s,0)]\Big)\,c = 0,$$

the following assertion follows immediately from (4.3.32).

Theorem 4.3.13. *Assume* (4.3.1)-(4.3.3). *Then*

$$\dim\,\mathscr{N}(\mathscr{A}) - \dim\,\mathscr{N}(\mathscr{A}^*) = n - m.$$

Remark 4.3.14. Let us note that the main assertions of this section (Propositions 4.3.4, 4.3.4 and 4.3.12 and Theorems 4.3.7, 4.3.8, 4.3.10 and 4.3.13) remain valid when the assumptions (4.3.1), (4.3.2) and (4.3.4) are respectively replaced by

$$A \in \mathbb{BV}^{n\times n},\ \Delta^+ A(0) = 0,\ \Delta^- A(t) = \Delta^+ A(t) \tag{4.3.1'}$$

$$\text{on}\ \ (0,1), \Delta^- A(1) = 0, \tag{4.3.33}$$

$$\det(I - (\Delta^- A(t))^2) \neq 0\ \ \text{on}\ \ (0,1), \tag{4.3.2'}$$

and

$$f \in \mathbb{G}^n_{\mathrm{reg}} \quad \text{and} \quad r \in \mathbb{R}^m, \tag{4.3.4'}$$

the space $\mathbb{G}_L^n$ is replaced by the space $\mathbb{G}^n_{\mathrm{reg}}$ and $\widetilde{A}(t) \equiv A(t)$ on $[0,1]$ (see [53], where also some more details concerning the periodic problem (4.1.1), $x(0) = x(1)$ can be found). Notice that by virtue of Theorem 2.5.7 the left-hand side of (4.1.2) represents also a general linear bounded mapping of $\mathbb{G}^n_{\mathrm{reg}}$ into $\mathbb{R}^m$.

Finally, let us note that it is known (cf. [28, Proposition 2.3]) that if A and f fulfil the assumptions (4.3.1'), (4.3.2') and (4.3.4'), then the equation (4.1.1) reduces to the distributional differential equation

$$x' - A'\,x = f',$$

where the product $A'\,x$ is the functional on the usual (cf. [13] and [28, Sec. 1.3]) space $\mathscr{D}^n$ of $n-$dimensional test functions given by:

$$A'\,x : \varphi \in \mathscr{D}^n \mapsto \int_0^1 \varphi^{\mathrm{T}}(s)\,\mathrm{d}\Big(\int_0^t \mathrm{d}[A(s)]\,x(s)\Big) \in \mathbb{R}.$$

For related results concerning linear periodic problems or linear differential equations with distributional coefficients, see also [57], [4], [6] or [24].

4.4. Controllability type problem

Let us assume that

$$\mathbb{U} \text{ is a linear space and } \mathscr{B} \in \mathscr{L}(\mathbb{U}, \mathbb{G}_L^n). \tag{4.4.1}$$

In this section we will consider the problem (4.4.2), (4.1.2) of determining $x \in \mathbb{G}_L^n$ and $u \in \mathbb{U}$ such that

$$\begin{aligned} x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) + (\mathscr{B}\, u)(t) - (\mathscr{B}\, u)(0) \\ = f(t) - f(0) \quad \text{on} \quad [0,1] \end{aligned} \tag{4.4.2}$$

and (4.1.2) are satisfied.

Remark 4.4.1. If $m = n$,

$$M = \begin{pmatrix} \mathrm{I} \\ \mathrm{I} \end{pmatrix}, K(t) = \begin{pmatrix} 0 \\ \mathrm{I} \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} x^0 \\ x^1 \end{pmatrix},$$

then the condition (4.1.2) reduces to the couple of conditions

$$x(0) = x^0, \quad x(1) = x^1.$$

Furthermore, if $\mathbb{U} = \mathbb{L}_2^n$ (the space of $n-$vector valued functions square integrable on $[0,1]$), P and q are Lebesgue integrable on $[0,1]$, Q is square integrable on $[0,1]$,

$$A(t) = \int_0^t P(s)\,\mathrm{d}s, \quad f(t) = \int_0^t q(s)\,\mathrm{d}s \quad \text{on} \quad [0,1]$$

and

$$\mathscr{B} \,:\, u \in \mathbb{L}_2^n \mapsto \int_0^t Q(s)\, u(s)\,\mathrm{d}s,$$

then the equation (4.4.2) reduces to the ordinary differential equation

$$x' = P(t)\, x + Q(t)\, u + q(t)$$

on $[0,1]$. Thus, the given problem (4.4.2), (4.1.2) is a generalization of the controllability problem for linear ordinary differential equations. The problem (4.4.2), (4.1.2) could be also viewed as a (possibly infinite dimensional) perturbation of the boundary value problem (4.1.1), (4.1.2).

To obtain necessary and sufficient conditions for the solvability of the problem (4.4.2), (4.1.2) in the form of the Fredholm alternative the following abstract scheme will be applied.

Abstract controllability type problem 4.4.2. Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Y}^+$ and $\mathbb{U}$ be linear spaces and let

$$h \in \mathbb{Y},\ y \in \mathbb{Y}^+ \mapsto \langle h, y\rangle_{\mathbb{Y}} \in \mathbb{R}$$

be a bilinear form on $\mathbb{Y} \times \mathbb{Y}^+$. For $M \subset \mathbb{Y}$ and $N \subset \mathbb{Y}^+$, let us denote

$$^{\perp}M = \{y \in \mathbb{Y}^+ : \langle m, y\rangle_{\mathbb{Y}} = 0 \quad \text{for all} \quad m \in M\}$$

and

$$N^{\perp} = \{h \in \mathbb{Y} : \langle h, y\rangle_{\mathbb{Y}} \quad = 0 \quad \text{for all} \quad y \in N\}.$$

Let $\mathscr{A} \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$, $\mathscr{Q} \in \mathscr{L}(\mathbb{U}, \mathbb{Y})$ and $h \in \mathbb{Y}$ be given and let us consider the operator equation for $(x, u) \in \mathbb{X} \times \mathbb{U}$

$$\mathscr{A}\, x + \mathscr{Q}\, u = h. \tag{4.4.3}$$

Let us denote

$$\mathscr{N}^+_{\mathscr{A}} = {}^{\perp}\mathscr{R}(\mathscr{A}) \quad \text{and} \quad \mathscr{N}^+_{\mathscr{Q}} = {}^{\perp}\mathscr{R}(\mathscr{Q}). \tag{4.4.4}$$

(Obviously $\mathscr{N}^+_{\mathscr{A}}$ and $\mathscr{N}^+_{\mathcal{Q}}$ are linear subspaces of $\mathbb{Y}^+$.)

Let us assume that

$$({}^{\perp}\mathscr{R}(\mathscr{A}))^{\perp} = \mathscr{R}(\mathscr{A}) \quad \text{and} \quad \dim \mathscr{N}^+_{\mathscr{A}} < \infty. \tag{4.4.5}$$

In particular, we have (cf. (4.4.4))

$$\mathscr{R}(\mathscr{A}) = (\mathscr{N}^+_{\mathscr{A}})^{\perp}. \tag{4.4.6}$$

Furthermore, let $k = \dim\, \mathscr{N}^+_{\mathscr{A}}$ and let $\{y^1, y^2, \ldots, y^k\}$ be a basis of $\mathscr{N}^+_{\mathscr{A}}$. In virtue of (4.4.6), the equation (4.4.3) possesses a solution in $\mathbb{X} \times \mathbb{U}$ if and only if there exists a solution $u \in \mathbb{U}$ to the equation

$$\mathscr{C}\, u = b, \tag{4.4.7}$$

where $\mathscr{C} \in \mathscr{L}(\mathbb{U}, \mathbb{R}^k)$ and $b \in \mathbb{R}^k$ are given by

$$\mathscr{C} : u \in \mathbb{U} \mapsto (\langle \mathscr{Q}\, u, y^j\rangle_{\mathbb{Y}})_{j=1,2,\ldots,k} \in \mathbb{R}^k$$

and

$$b = (\langle h, y\rangle_{Y})_{j=1,2,\ldots,k} \in \mathbb{R}^k.$$

Since $\dim \mathscr{R}(\mathscr{C}) \leq k < \infty$, it follows that $({}^{\perp}\mathscr{R}(\mathscr{C}))^{\perp} = \mathscr{R}(\mathscr{C})$ (cf. [30]) or, in other words, the equation (4.4.7) possesses a solution in $\mathbb{U}$ if and only if

$$v^{\mathrm{T}}\, b = 0 \text{ for all } v \in \mathbb{R}^k \text{ such that } v^{\mathrm{T}}\, (Cu) = 0 \text{ for all } u \in \mathbb{U}. \tag{4.4.8}$$

It is easy to verify that the condition (4.4.8) is equivalent to the condition

$$\langle h, y\rangle_{\mathbb{Y}} = 0 \quad \text{for all} \quad y \in \mathscr{N}^+_{\mathscr{A}} \cap \mathscr{N}^+_{\mathscr{Q}}. \tag{4.4.9}$$

Summarizing the above considerations we get the following proposition.

Proposition 4.4.3. *Assume* $\mathscr{A} \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$, $\mathscr{Q} \in \mathscr{L}(\mathbb{U}, \mathbb{Y})$, $h \in \mathbb{Y}$ *and* (4.4.5). *Then the equation* (4.4.3) *possesses a solution in* $\mathbb{X} \times \mathbb{U}$ *if and only if* (4.4.9) *is satisfied.*

Let us notice that up to now no assumptions on topologies in $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Y}^+$ and $\mathbb{U}$ and on the boundedness of the operators $\mathscr{A}$, $\mathscr{B}$ have been needed. Of course, the assumptions of the above proposition are fulfilled if $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces, $\mathbb{Y}^+$ is the dual space of $\mathbb{Y}$, ($\langle ., y\rangle_{\mathbb{Y}}$ for $y \in \mathbb{Y}^+$ are linear bounded functionals on $\mathbb{Y}$), the range $\mathscr{R}(\mathscr{A})$ of $\mathscr{A}$ is closed in $\mathbb{Y}$ and the null space $\mathscr{N}(\mathscr{A}^*)$ of the adjoint operator $\mathscr{A}^*$ to $\mathscr{A}$ has a finite dimension. (In this case $\mathscr{N}^+_{\mathscr{A}} = \mathscr{N}(\mathscr{A}^*)$.)

The problem (4.4.2), (4.1.2) reduces to the operator equation (4.4.3) if we put

$$\mathbb{X} = \mathbb{G}^n_L, \quad \mathbb{Y} = \mathbb{G}^n_L \times \mathbb{R}^n, \quad \mathbb{Y}^+ = \mathbb{BV}^n \times \mathbb{R}^n \times \mathbb{R}^m,$$

$$\langle (f,r),(y,\gamma,\delta)\rangle_{\mathbb{Y}} = \delta^{\mathrm{T}} r + \gamma^{\mathrm{T}} f(0) + \int_0^1 y^{\mathrm{T}}(s)\,\mathrm{d}[f(s)]$$

$$\text{for } f \in \mathbb{G}^n_L,\ r \in \mathbb{R}^m,\ y \in \mathbb{BV}^n,\ \gamma \in \mathbb{R}^n \quad \text{and} \quad \delta \in \mathbb{R}^m,$$

$$\mathscr{Q} : u \in \mathbb{U} \mapsto \begin{pmatrix} (\mathscr{B}u)(t) - \mathscr{B}u)(0) \\ 0 \end{pmatrix} \in \mathbb{G}^n_L \times \mathbb{R}^m \quad \text{and}$$

$$h(t) = \begin{pmatrix} f(t) - f(0) \\ 0 \end{pmatrix} \in \mathbb{G}^n_L \times \mathbb{R}^m$$

and if we make use of (4.3.7) again. By Propositions 4.3.5 and 4.3.12 the assumptions of the above proposition are fulfilled and hence the following assertions are true (cf. Theorem 4.3.8).

Theorem 4.4.4. *Assume* (4.3.1)-(4.3.4) *and* (4.4.1). *Then the problem* (4.4.2), (4.1.2) *possesses a solution in* $\mathbb{G}^n_L \times \mathbb{U}$ *if and only if*

$$\int_0^1 y^{T}(t)\,\mathrm{d}[f(t)] + \delta^{T} r = 0 \tag{4.4.10}$$

holds for any solution (y, δ) *of the system* (4.3.23), (4.3.24) *such that*

$$\int_0^1 y^{T}(t)\,\mathrm{d}[(\mathscr{B}u)(t)] = 0 \quad \textit{for all} \quad u \in \mathbb{U}. \tag{4.4.11}$$

Corollary 4.4.5. *Assume* (4.3.1)-(4.3.3) *and* (4.4.1). *Then the problem* (4.4.2), (4.1.2) *possesses a solution in* $\mathbb{G}^n_L \times \mathbb{U}$ *for any* $f \in \mathbb{G}^n_L$ *and any* $r \in \mathbb{R}^m$ *if and only if the only solution* (y, δ) *of* (4.3.23), (4.3.24) *which fulfils* (4.4.11) *is the zero solution (i.e.* $y(t) \equiv 0$ *on* $[0,1]$, $\delta = 0$*).*

Remark 4.4.6. In accordance with the usual terminology the system (4.4.2), (4.1.2) is called ***completely controllable*** (or more precisely *completely* ($\mathscr{B}$, M, K)*−controllable*) if it possesses a solution in $\mathbb{G}_L^n \times \mathbb{U}$ for any $f \in \mathbb{G}_L^n$ and any $r \in \mathbb{R}^m$ (cf. [12], [25], [23]). The problem (4.3.23), 4.3.24), (4.4.11) adjoint to (4.4.2), (4.1.2) in the sense of Theorem 4.4.4 is a generalization of classical observability problems for linear ordinary differential equations and Corollary 4.4.5 is a generalization of the well known theorem (cf. e.g. [31], [29]) on the duality between controllability and observability problems for linear ordinary differential equations. Controllability is often considered for homogeneous differential equations. In an analogous situation for the problem (4.4.2), (4.1.2) (i.e. $f(t) \equiv f(0)$ on $[0,1]$) we obtain that the system

$$x(t) - x(0) - \int_0^t \mathrm{d}[A(s)]\, x(s) + (\mathscr{B}\, u)(t) - (\mathscr{B}\, u)(0) = 0 \quad \text{on} \quad [0,1], \qquad (4.4.12)$$

(4.1.2) possesses a solution in $\mathbb{G}_L^n \times \mathbb{U}$ for any $r \in \mathbb{R}^m$ if and only if the only couple $(y, \delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ fulfilling (4.3.23), (4.3.24) and (4.4.11) is the zero one. In fact, it follows immediately from (4.4.10) that (4.4.12), (4.1.2) has a solution in $\mathbb{G}_L^n \times \mathbb{U}$ for any $r \in \mathbb{R}^m$ if and only if $\delta = 0$ holds for any couple $(y, \delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ fulfilling (4.3.23), (4.3.24) and (4.4.11). By 4.3.11 this implies that $y(t) \equiv 0$ on $[0,1]$ for any such couple, of course.

Corollary 4.4.7. *Assume* (4.3.1)-(4.3.3) *and let* $\mathbb{U} = \mathbb{G}_L^h$ *and*

$$\mathscr{B} \,:\, u \in \mathbb{G}_L^h \mapsto \int_0^t \mathrm{d}[B(s)]\, u(s), \quad t \in [0,1],$$

where $B(s)$ *is an* $n \times h-$*matrix valued function of bounded variation on* $[0,1]$, *right-continuous at* 0 *and left-continuous on* $(0,1]$. *Then the problem* (4.4.2), (4.1.2) *has a solution if and only if* (4.4.10) *holds for any solution* (y, δ) *of the system* (4.3.23), (4.3.24) *such that*

$$\int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[B(s+)] = 0 \quad \textit{for all} \quad t \in [0,1].$$

Proof follows from Theorem 4.4.4 and from the relation

$$\int_0^1 y^{\mathrm{T}}(t)\, \mathrm{d}\Big[\int_0^t \mathrm{d}[B(s)]\, u(s)\Big] =$$
$$= \Big(\int_0^1 y^{\mathrm{T}}(t)\, \mathrm{d}[B(t)]\Big)\, u(0) + \int_0^1 \Big(\int_t^1 y^{\mathrm{T}}(s)\, \mathrm{d}[B(s+)]\Big)\, \mathrm{d}[u(t)]$$
$$\text{for all} \quad u \in \mathbb{G}_{\mathrm{L}}^h \quad \text{and} \quad y \in \mathbb{BV}^n,$$

which can be verified analogously to the corresponding relation for the $n \times n-$matrix valued function $A(t)$ in the proof of Theorem 4.3.7.

Corollary 4.4.8. *Assume* (4.3.1)-(4.3.3) *and let* $\mathbb{U} = \mathbb{G}_L^h$ *and*

$$\mathscr{B} \,:\, u \in \mathbb{G}_L^h \mapsto \int_0^t B(s)\,\mathrm{d}[u(s)],$$

where B(s) is an $n \times h-$*matrix valued function of bounded variation on* $[0,1]$*. Then the problem* (4.4.2), (4.1.2) *has a solution if and only if* (4.4.10) *holds for any couple* $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ *fulfilling* (4.3.23), (4.3.24) *and* $y^T(t)\,B(t) = 0$ *on* $[0,1]$.

Proof. Since by the Substitution Theorem (cf. Theorem 2.3.19) the relation

$$\begin{aligned}\int_0^1 y^{\mathrm{T}}(t)\,\mathrm{d}[(Bu)(t)] &= \int_0^1 y^{\mathrm{T}}(t)\,\mathrm{d}\left[\int_0^t B(s)\,\mathrm{d}[u(s)]\right]\\ &= \int_0^1 y^{\mathrm{T}}(t)\,B(t)\,\mathrm{d}[u(t)]\end{aligned}$$

holds for all $y \in \mathbb{BV}^n$ and $u \in G_L^h$, the proof follows immediately from Theorem 4.4.4. □

Definition 4.4.9. Let $T = \{t_1, t_2, \ldots, t_\nu\}$ be such that

$$1 > t_1 > t_2 > \cdots > t_\nu > 0. \tag{4.4.13}$$

Then we denote by $\mathbb{U}_T$ the subset of $\mathbb{G}_L^n$ consisting of all functions $u \in \mathbb{G}_L^n$ which are constant on each of the intervals

$$[0,t_\nu],\ (t_1,1],\ (t_{k+1},t_k], \quad k = 1,2,\ldots,\nu-1.$$

Proposition 4.4.10. *Let* $T = \{t_1, t_2, \ldots, t_\nu\}$ *fulfil* (4.4.13) *and let* $\mathbb{U}_T$ *be defined by Definition* 4.4.9*. Then* $\mathbb{U}_T$ *is a linear space. Furthermore, if* $y \in \mathbb{BV}^n$*, then*

$$\int_0^1 y^T(t)\,\mathrm{d}[u(t)] = 0 \quad \textit{for all} \quad u \in \mathbb{U}_T \tag{4.4.14}$$

holds if and only if

$$y^T(\tau) = 0 \quad \textit{for any} \;\; \tau \in \mathbb{U}_T. \tag{4.4.15}$$

Proof. The first part of the proposition is evident. Let us suppose that (4.4.14) holds. Then for a given $\tau \in T$, the function $\chi_{(\tau,1]}$ belongs to $\mathbb{U}_T$ and (cf. Proposition 2.3.3)

$$\int_0^1 y^{\mathrm{T}}(t)\,\mathrm{d}[\chi_{(\tau,1]}(t)] = y^{\mathrm{T}}(\tau) = 0.$$

Analogously, $\chi_{[1]} \in \mathbb{U}_T$, while

$$\int_0^1 y^{\mathrm{T}}(t)\,\mathrm{d}[\chi_{[1]}(t)] = y^{\mathrm{T}}(1) = 0,$$

i.e., (4.4.15) is true.

On the other hand, since obviously $\mathbb{U}_T \subset \mathbb{BV}^n$, it follows from [45, Lemma I.4.23] that (4.4.14) holds for any $y \in \mathbb{BV}^n$ satisfying (4.4.15) and any $u \in \mathbb{U}_T$. □

Corollary 4.4.11. *Assume* (4.3.1)-(4.3.4) *and let* $\mathbb{U} = \mathbb{U}_T$ *and*

$$\mathscr{B} : u \in \mathbb{U} \mapsto u \in \mathbb{G}_L^n,$$

where $T = \{t_k\}_{k=1}^{\nu} \subset (0,1)$ *and* $\mathbb{U}_T$ *satisfy the assumptions of Proposition* 4.4.10. *Then the problem* (4.4.2), (4.1.2) *has a solution if and only if* (4.4.10) *holds for any couple* $(y, \delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ *fulfilling* (4.3.23), (4.3.24) *and such that* $y(\tau) = 0$ *for any* $\tau \in T$.

Proof follows immediately from Theorem 4.4.4 and Proposition 4.4.10.

Example 4.4.12. Let $P \in \mathbb{L}_1^{n\times n}$, $q \in \mathbb{L}_1^n$, $K \in \mathbb{BV}^{m\times n}$, $M_k, N_k \in \mathbb{R}^{m\times n}$ $(k = 0, 1, \ldots, \nu)$, $r \in \mathbb{R}^m$ and let $T = \{t_k\}_{k=1}^{\nu} \subset (0,1)$ satisfy (4.4.13). Consider the problem (P) of determining a function $x \in \mathbb{G}_L^n$ which is absolutely continuous on every interval $(t_{k+1}, t_k]$, $k = 1, 2, \ldots, \nu$, and satisfies

$$x'(t) - P(t)\,x(t) = q(t) \quad \text{a.e. on } [0,1]$$

and

$$\mathscr{K}\,x := M_0\,x(0) + N_0\,x(1) + \sum_{k=1}^{\nu}[M_k\,x(t_k+) + N_k\,x(t_k-)] \\ + \int_0^1 K_0(s)\,\mathrm{d}[x(s)] = r.$$

Such problems are usually called ***interface boundary value problems*** (cf. e.g. [7], [8], [39] or [59]).

Let $\mathbb{U}_T$ and $\mathscr{B}$ have the same meaning as in Proposition 4.4.10 and let us put

$$A(s) = \int_0^t P(\tau)\,\mathrm{d}\tau, \quad f(t) = \int_0^t q(\tau)\,\mathrm{d}\tau \text{ for } t \in [0,1], \quad M = \sum_{k=0}^{\nu}[M_k + N_k]$$

and

$$K(t) = K_0(t) + \sum_{i=1}^{\nu}[M_k\,\chi_{[0,t_k]}(t) + N_k\,\chi_{[0,t_k)}(t)] + N_0 \quad \text{for} \quad t \in [0,1].$$

Then

$$\mathscr{K}\,x = M\,x(0) + \int_0^1 K(s)\,\mathrm{d}[x(s)] \quad \text{for all} \quad x \in \mathbb{G}_L^n$$

and the function $x \in \mathbb{G}_L^n$ is a solution of the interface problem (P) if and only if there is $u \in \mathbb{U}$ such that the couple $(x,u) \in \mathbb{G}_L^n \times \mathbb{U}$ is a solution to the controllability type problem (4.4.2), (4.1.2). Now, Corollary 4.4.11 yields that (P) *has a solution if and only if* (4.4.10) *is true for all couples* $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ *satisfying the equation* (4.3.23) *together with the conditions*

$$y^{\mathrm{T}}(0) + \delta^{\mathrm{T}}\big(K_0(0) - M_0\big) = 0,\ \ y^{\mathrm{T}}(1) + \delta^{\mathrm{T}}\big(K_0(1) + N_0\big) = 0, \tag{4.4.16}$$

$$y^{\mathrm{T}}(t_k) = 0, \quad k = 1,2,\dots,\nu. \tag{4.4.17}$$

Finally, let us notice that for any $t \in [0,1]$ we have

$$K(t) - K(1) = K_0(t) - K_0(1) + \sum_{k=1}^{\nu}[M_k\,\chi_{[0,t_k]}(t) + N_k\,\chi_{[0,t_k)}(t)].$$

This implies that a couple $(y,\delta) \in \mathbb{BV}^n \times \mathbb{R}^m$ solves the system (4.3.23), (4.4.16), (4.4.17) if and only if $y^{\mathrm{T}} + \delta^{\mathrm{T}}\,K_0$ is absolutely continuous on every interval $[\alpha,\beta]$ such that $[\alpha,\beta] \subset [0,1] \setminus T$ and the relations

$$\begin{aligned}
&-(y^{\mathrm{T}} + \delta^{\mathrm{T}}\,K_0)'(t) + y^{\mathrm{T}}\,P(t) = 0 \quad \text{a.e. on} \ \ [0,1],\\
&\Delta^+(y^{\mathrm{T}} + \delta^{\mathrm{T}}\,K_0)(t_k) = \delta^{\mathrm{T}}\,M_k,\\
&\Delta^-(y^{\mathrm{T}} + \delta^{\mathrm{T}}\,K_0)(t_k) = \delta^{\mathrm{T}}\,N_k, \quad i = 1,2,\dots,\nu
\end{aligned}$$

(4.4.16) and (4.4.17) are satisfied.

Chapter 5

Linear Integral Equations in the Space of Regulated Functions

5.1. Introduction

This chapter is devoted to linear operator equations of the form

$$x - \mathscr{L}x = f, \tag{5.1.1}$$

where $\mathscr{L}$ is a linear compact operator on the space $\mathbb{G}_L^n$ and $f \in \mathbb{G}_L^n$. Due to Schwabik (cf. [42, Theorem 5]) it is known that $\mathscr{L}$ is a linear compact operator on $\mathbb{G}_L^n$ if and only if there are functions $A \in \mathbb{G}_{\mathrm{L}}^{n\times n}$ and $B : [0,1] \times [0,1] \mapsto \mathbb{R}^{n\times n}$ such that $B(t,.) \in \mathbb{BV}^{n\times n}$ for any $t \in [0,1]$,

$$(\mathscr{L}x)(t) = A(t)\,x(0) + \int_0^1 B(t,s)\,\mathrm{d}[x(s)] \quad \text{for } \ x \in \mathbb{G}_L^n \ \text{ and } \ t \in [0,1], \tag{5.1.2}$$

and the mapping

$$\mathfrak{M}_B : t \in [0,1] \mapsto \mathfrak{M}_B(t) = B(t,.) \in \mathbb{BV}^{n\times n}$$

is regulated on $[0,1]$ and left-continuous on $(0,1)$ (i.e. $B \in \mathscr{K}_L^{n\times n}$, see Definitions 5.3.2 and 5.3.3).

In Sections 5.4 and 5.5 we prove basic existence and uniqueness results for the equation (5.1.1) and obtain the explicit form of its adjoint equation. An important tool for the proofs of our main results is in particular the theorem on the interchange of the integration order for Stieltjes type integrals (i.e. the ***Bray Theorem***). Its proof for the Perron-Stieltjes integral is given in Sec. 5.3 (cf. Theorem 5.3.13).

Special attention (cf. Sec. 5.6) is paid to the causal case, i.e. to the Volterra-Stieltjes integral equations of the form

$$x(t) - A(t)\,x(0) - \int_0^t B(t,s)\ \mathrm{d}[x(s)] = f(t), \quad t \in [0,1],$$

where $A(0) = 0$.

Similar problems in the space of regulated functions were treated e.g. by Ch. S. Hönig [15], [16], L. Fichmann [9] and L. Barbanti [5], where the interior (Dushnik-Stieltjes) integral was used.

5.2. Auxiliary lemma

By Theorem 2.4.8, $\Phi \in \mathcal{L}(\mathbb{G}^n_L, \mathbb{R}^m)$ (i.e. Φ is a linear bounded mapping of $\mathbb{G}^n_L$ into $\mathbb{R}^m$, cf. 1.3.2) if and only if there exist $M \in \mathbb{R}^{m\times n}$ and $K \in \mathbb{BV}^{m\times n}$ such that

$$\Phi x = M\,x(0) + \int_0^1 K(t)\,\mathrm{d}[x(t)] \quad \text{for all} \quad x \in \mathbb{G}^n_L.$$

Furthermore, for any $M \in \mathbb{R}^{m\times n}$ and any $K \in \mathbb{BV}^{m\times n}$ the relation

$$M\,x(0) + \int_0^1 K(t)\,\mathrm{d}[x(t)] = 0 \quad \text{for all} \quad x \in \mathbb{G}^n_L$$

holds if and only if

$$M = 0 \quad \text{and} \quad K(t) \equiv 0 \text{ on } [0,1].$$

By a slight modification of Corollary 2 from [42] we can obtain an analogous result also for linear bounded mappings of $\mathbb{G}^n_L$ into $\mathbb{G}^n$:

Lemma 5.2.1. *$\mathscr{L} \in \mathcal{L}(\mathbb{G}^n_L, \mathbb{G}^n)$ if and only if there exist $A \in \mathbb{G}^{n\times n}$ and $B : [0,1]\times[0,1] \mapsto \mathbb{R}^{n\times n}$ such that*

$$B(.,s) \in \mathbb{G}^{n\times n} \quad \textit{for all} \quad s \in [0,1], \tag{5.2.1}$$

$$B(t,.) \in \mathbb{BV}^{n\times n} \quad \textit{for all} \quad t \in [0,1], \tag{5.2.2}$$

$$\textit{there is a } \beta < \infty \quad \textit{such that } \operatorname{var}_0^1 B(t,.) \le \beta \quad \textit{for all} \quad t \in [0,1] \tag{5.2.3}$$

and $\mathscr{L}$ is given by (5.1). *Furthermore, for given functions $A \in \mathbb{G}^{n\times n}$ and $B(t,s)$ fulfilling* (5.2.1)-(5.2.3) *the relation*

$$A(t)\,x(0) + \int_0^1 B(t,s)\,\mathrm{d}[x(s)] \equiv 0 \quad \textit{on} \quad [0,1]$$

holds for all $x \in \mathbb{G}^n_L$ if and only if

$$A(t) \equiv 0 \ \textit{on} \ [0,1] \quad \textit{and} \quad B(t,s) \equiv 0 \ \textit{on} \ [0,1]\times[0,1].$$

5.3. Functions of the class $\mathscr{K}^{n\times n}$ and the Bray Theorem

In this section we will study the properties of the class $\mathscr{K}^{n\times n}$ of $n\times n-$matrix valued functions which will play a crucial role in our investigations of equations of the form (5.1.1).

Notation 5.3.1. For $K:\,[0,1]\times[0,1]\mapsto\mathbb{R}^{n\times n}$ such that $K(t,.)\in\mathbb{BV}^{n\times n}$ for any $t\in[0,1]$ we denote by $\mathfrak{M}_K$ the mapping of $[0,1]$ into $\mathbb{BV}^{n\times n}$ defined by

$$\mathfrak{M}_K:\,t\in[0,1]\mapsto\mathfrak{M}_K(t)=K(t,.)\in\mathbb{BV}^{n\times n}. \tag{5.3.1}$$

Definition 5.3.2. We say that a matrix-valued function $K:\,[0,1]\times[0,1]\mapsto\mathbb{R}^{n\times n}$ belongs to the class $\mathscr{K}^{n\times n}$ if it satisfies the following hypotheses:

(H$_1$) $K(t,.)\in\mathbb{BV}^{n\times n}$ for any $t\in[0,1]$;

(H$_2$)(i) for any $t\in[0,1)$ there exists a function $K_t^+=\mathfrak{M}_K(t+)\in\mathbb{BV}^{n\times n}$ such that

$$\lim_{\tau\to t+}\|\mathfrak{M}_K(\tau)-K_t^+\|_{\mathbb{BV}}=0,$$

(H$_2$)(ii) for any $t\in(0,1]$ there exists a function $K_t^-=\mathfrak{M}_K(t-)\in\mathbb{BV}^{n\times n}$ such that

$$\lim_{\tau\to t-}\|\mathfrak{M}_K(\tau)-K_t^-\|_{\mathbb{BV}}=0.$$

Definition 5.3.3. We say that a matrix-valued function $K:\,[0,1]\times[0,1]\mapsto\mathbb{R}^{n\times n}$ belongs to the class $\mathscr{K}_L^{n\times n}$ if $K\in\mathscr{K}^{n\times n}$ and the mapping $\mathfrak{M}_K:[0,1]\mapsto\mathbb{BV}^{n\times n}$ given by (5.3.1) is left-continuous on $(0,1)$, i.e.

$$\lim_{\tau\to t-}\|K(\tau,.)-K(t,.)\|_{\mathbb{BV}}=0$$

holds for any $t\in(0,1)$.

Remark 5.3.4. Let a matrix-valued function $K:\,[0,1]\times[0,1]\mapsto\mathbb{R}^{n\times n}$ be such that $K(t,.)\in\mathbb{BV}^{n\times n}$ for any $t\in[0,1]$ and let the mapping $\mathfrak{M}_K:[0,1]\mapsto\mathbb{BV}^{n\times n}$ be defined by (5.3.1). We say that $\mathfrak{M}_K$ is regulated on $[0,1]$ if the condition (H$_2$) from Definition 5.3.2 is satisfied. Obviously, (H$_2$) is satisfied if and only if the following assertions are true:

(H$_2$)(i$'$) for any $t\in[0,1)$ and any $\varepsilon>0$ there exists $\delta>0$ such that $t+\delta<1$ and

$$\|K(\tau_2,.)-K(\tau_1,.)\|_{\mathbb{BV}}<\varepsilon\quad\text{for all}\quad\tau_1,\tau_2\in(t,t+\delta),$$

$(\mathrm{H}_2)(\mathrm{ii}')$ for any $t \in (0,1]$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $t - \delta > 0$ and
$$\|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbb{BV}} < \varepsilon \quad \text{for all} \quad \tau_1, \tau_2 \in (t-\delta, t).$$

The following assertion due to Schwabik (cf. [42, Theorem 4]) has been already mentioned in the introduction to this chapter.

Theorem 5.3.5. *$\mathscr{L}$ is a linear compact mapping of $\mathbb{G}^n_L$ into $\mathbb{G}^n$ if and only if there exist $n \times n-$matrix valued functions $A \in \mathbb{G}^{n\times n}$ and $B: [0,1]\times[0,1] \mapsto \mathbb{R}^{n\times n}$ such that $B \in \mathscr{K}^{n\times n}$ and $\mathscr{L}$ is given by* (5.1). *Furthermore, $\mathscr{L}$ is a linear compact mapping of $\mathbb{G}^n_L$ into $\mathbb{G}^n_L$ if and only if there exist $n\times n-$matrix valued functions $A \in \mathbb{G}^{n\times n}_L$ and $B: [0,1]\times[0,1] \mapsto \mathbb{R}^{n\times n}$ such that $B \in \mathscr{K}^{n\times n}_L$ and $\mathscr{L}$ is given by* (5.1).

Let us summarize some of the other properties of functions of the class $\mathscr{K}^{n\times n}$.

Lemma 5.3.6. *If $K \in \mathscr{K}^{n\times n}$, then $K(.,s) \in \mathbb{G}^{n\times n}$ for any $s \in [0,1]$.*

Proof. Let $t \in [0,1)$ and $\varepsilon > 0$ be given. By $(\mathrm{H}_2)(\mathrm{i}')$ (cf. Remark 5.3.4) there exists $\delta > 0$ such that $t + \delta < 1$ and
$$\|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbb{BV}} < \varepsilon \quad \text{for all} \quad \tau_1, \tau_2 \in (t, t+\delta).$$
Consequently, if $s \in [0,1]$ and $\tau_1, \tau_2 \in (t, t+\delta)$, then
$$\begin{aligned}
&|K(\tau_2,s) - K(\tau_1,s)|\\
&\qquad \le |K(\tau_2,0) - K(\tau_1,0)| + |K(\tau_2,s) - K(\tau_1,s) - K(\tau_2,0) + |K(\tau_1,0)|\\
&\qquad \le \|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbb{BV}} < \varepsilon.
\end{aligned}$$
This implies that $K(.,s)$ possesses a limit $\lim_{\tau\to t+} K(\tau,s) = K(t+,s) \in \mathbb{R}^n$ for any $t \in [0,1)$ and any $s \in [0,1]$. Analogously, $K(.,s)$ possesses a limit $\lim_{\tau\to t-} K(\tau,s) = K(t-,s) \in \mathbb{R}^n$ for any $t \in (0,1]$ and any $s \in [0,1]$. $\square$

Lemma 5.3.7. *If $K \in \mathscr{K}^{n\times n}$, then*
$$\varkappa := \sup_{t\in[0,1]} \|K(t,.)\|_{\mathbb{BV}} < \infty.$$

Proof follows directly from Definition 5.3.2 by means of the Vitali Covering Theorem (cf. also Remark 5.3.4). $\square$

Lemma 5.3.8. *If $K \in \mathscr{K}^{n\times n}$ and $\mathfrak{M}_K$ is given by* (5.3.1)*, then*
$$\mathfrak{M}_K(t+) = K(t+,.) \in \mathbb{BV}^{n\times n} \quad \textit{for all} \quad t \in [0,1) \tag{5.3.2}$$
and
$$\mathfrak{M}_K(t-) = K(t-,.) \in \mathbb{BV}^{n\times n} \quad \textit{for all} \quad t \in (0,1]. \tag{5.3.3}$$

Proof. Let $t \in [0,1)$ be given. By (H_2)(ii) there exists $H \in \mathbb{BV}^{n\times n}$ such that

$$\lim_{\tau\to t+} \|K(\tau,.) - H\|_{\mathbb{BV}} = 0,$$

i.e. $H = \mathfrak{M}_K(t+)$. In particular, in virtue of Lemma 5.3.6 we have

$$K(t+,s) = \lim_{\tau\to t+} K(\tau,s) = H(s) \quad \text{for all} \quad s \in [0,1]$$

wherefrom the relation (5.3.2) immediately follows. Analogously we can prove that the relation (5.3.3) is true, as well. □

As a direct consequence of Lemma 5.3.8 we have the following

Corollary 5.3.9. *If $K \in \mathscr{K}^{n\times n}$, then the relations*

$$\lim_{\tau\to t+} \|K(\tau,.) - K(t+,.)\|_{\mathbb{BV}} = 0 \quad \textit{for all} \quad t \in [0,1)$$

and

$$\lim_{\tau\to t-} \|K(\tau,.) - K(t-,.)\|_{\mathbb{BV}} = 0 \quad \textit{for all} \quad t \in (0,1]$$

are true.

Lemma 5.3.10. *Let $K \in \mathscr{K}^{n\times n}$. Then for any $x \in \mathbb{G}^n$ the integrals*

$$\int_0^1 K(t,s)\,\mathrm{d}[x(s)], \qquad t \in [0,1], \tag{5.3.4}$$

$$\int_0^1 K(t+,s)\,\mathrm{d}[x(s)], \quad t \in [0,1) \tag{5.3.5}$$

and

$$\int_0^1 K(t-,s)\,\mathrm{d}[x(s)], \quad t \in (0,1] \tag{5.3.6}$$

exist and the relations

$$\lim_{\tau\to t+} \int_0^1 K(\tau,s)\,\mathrm{d}[x(s)] = \int_0^1 K(t+,s)\,\mathrm{d}[x(s)] \quad \textit{for} \quad t \in [0,1) \tag{5.3.7}$$

and

$$\lim_{\tau\to t-} \int_0^1 K(\tau,s)\,\mathrm{d}[x(s)] = \int_0^1 K(t-,s)\,\mathrm{d}[x(s)] \quad \textit{for} \quad t \in (0,1] \tag{5.3.8}$$

are true.

Proof. All the integrals (5.3.4) - (5.3.6) have values in $\mathbb{R}^n$ according to Theorem 2.3.8. The relations (5.3.7) and (5.3.8) follow by Theorem 2.3.7 and by Corollary 5.3.9. □

Corollary 5.3.11. *Let $K \in \mathscr{K}^{n\times n}$. Then the function*

$$h(t) = \int_0^1 K(t,s)\,\mathrm{d}[x(s)]$$

is for any $x \in \mathbb{G}_L^n$ defined and regulated on $[0,1]$.

Moreover, if $K \in \mathscr{K}_L^{n\times n}$, then h is left-continuous on $(0,1)$.

Lemma 5.3.12. *Let $K \in \mathscr{K}^{n\times n}$. Then the function*

$$h^T(t) = \int_0^1 y^T(s)\,\mathrm{d}_s[K(s,t)]$$

is for any $y \in \mathbb{BV}^n$ defined on $[0,1]$ *and has a bounded variation on* $[0,1]$.

Proof. a) The existence of the integrals $h(t)$, $t \in [0,1]$, follows by Theorem 2.3.8.

b) To prove that $h \in \mathbb{BV}^n$, let us first assume that $n = 1$, $k \in \mathscr{K}^{n\times n}$ and

$$d = \{t_0, t_1, \dots, t_m\} \in \mathcal{D}[0,1].$$

Then for all $x_i \in \mathbb{R}$, $i = 1, 2, \dots, m$ such that $|x_i| \le 1$ we have by Theorem 2.3.8 and Lemma 5.3.7

$$\begin{aligned}
\Big|\sum_{i=1}^m [h(t_i) - h(t_{i-1})]\,x_i\Big| &= \Big|\int_0^1 y(s)\,\mathrm{d}_s\Big[\Big(\sum_{i=1}^m (k(s,t_i) - k(s,t_{i-1}))\Big)\Big]\,x_i\Big| \\
&\le 2\,\|y\|_{\mathbb{BV}}\Big(\sup_{\substack{s\in[0,1]\\ |x_i|\le 1}}\Big|\sum_{i=1}^m (k(s,t_i) - k(s,t_{i-1}))\,x_i\Big|\Big) \\
&\le 2\,\|y\|_{\mathbb{BV}}\Big(\sup_{\substack{s\in[0,1]\\ |x_i|\le 1}}\Big(\sum_{i=1}^m |k(s,t_i) - k(s,t_{i-1})|\,|x_i|\Big)\Big) \\
&\le 2\,\|y\|_{\mathbb{BV}}\sup_{s\in[0,1]}\operatorname{var}_0^1 k(s,.) = 2\,\|y\|_{\mathbb{BV}}\,\varkappa < \infty.
\end{aligned}$$

In particular, if we put

$$x_i = \operatorname{sgn}[h(t_i) - h(t_{i-1})]$$

for $i = 1, 2, \dots, m$ we obtain that the inequality

$$S(h,d) = \sum_{i=1}^m |h(t_i) - h(t_{i-1})| \le 2\,\varkappa\,\|y\|_{\mathbb{BV}}$$

holds for any division $d = \{t_0, t_1, \dots, t_m\} \in \mathcal{D}[0,1]$ of the interval $[0,1]$ and any $y \in \mathbb{BV}$, i.e.

$$\operatorname{var}_0^1 h \le 2\,\varkappa\,\|y\|_{\mathbb{BV}} < \infty \quad \text{for any} \quad y \in \mathbb{BV}.$$

c) In the general case of $n \in \mathbb{N}$, $n \geq 1$, we have for any $j = 1, 2, \dots, n$, any $y \in \mathbb{BV}^n$ and any $t \in [0,1]$

$$h_j(t) = \sum_{i=1}^{n} \int_0^1 y_i(s)\, \mathrm{d}_s[k_{i,j}(s,t)].$$

Consequently, by the second part of the proof of this lemma the inequalities

$$\mathrm{var}_0^1 h_j \leq 2 \Big(\sum_{i=1}^{n} \|y_i\|_{\mathbb{BV}} \Big) \varkappa = 2\, \|y\|_{\mathbb{BV}}\, \varkappa$$

are true. It follows easily that $h \in \mathbb{BV}^n$ for any $y \in \mathbb{BV}^n$. $\square$

Theorem 5.3.13. (Bray Theorem) *If $K \in \mathscr{K}^{n\times n}$, then for any $x \in \mathbb{G}^n$ and any $y \in \mathbb{BV}^n$ the relation*

$$\int_0^1 y^{T}(t) \mathrm{d}_t \Big[\int_0^1 K(t,s)\, \mathrm{d}[x(s)] \Big] \\ = \int_0^1 \Big(\int_0^1 y^{T}(t) \mathrm{d}_t[K(t,s)] \Big)\, \mathrm{d}[x(s)] \tag{5.3.9}$$

is true.

Proof. a) Both the iterated integrals occurring in (5.3.9) exist by Corollary 5.3.11, Lemma 5.3.12 and by Theorem 2.3.8.

b) Let us first assume $n = 1$, $k \in \mathscr{K}^{n\times n}$ and $y \in \mathbb{BV}$. Let $f \in \mathbb{G}$ be a finite step function, i.e., there is a division $\{t_0, t_1, \dots, t_m\}$ of the interval $[0,1]$ such that f on $[0,1]$is a linear combination of the functions

$$\big\{ \chi_{[t_r,1]},\ r = 0, 1, \dots, m,\ \chi_{(t_j,1]},\ j = 0, 1, \dots, m-1 \big\}.$$

To show that the relation

$$\int_0^1 y(t)\, \mathrm{d}_t \Big[\int_0^1 k(t,s)\, \mathrm{d}[f(s)] \Big] = \int_0^1 \Big(\int_0^1 y(t)\, \mathrm{d}_t[k(t,s)] \Big)\, \mathrm{d}[f(s)] \tag{5.3.10}$$

is true for any finite step function f on $[0,1]$, it is sufficient to show that (5.3.10) is true for any function from the set

$$\Big\{ \chi_{[\tau,1]},\ \tau \in [0,1] \Big\} \quad \cup \quad \Big\{ \chi_{(\sigma,1]},\ \sigma \in [0,1) \Big\}.$$

If $f = \chi_{[0,1]}$, i.e. $f(t) \equiv 1$ on $[0,1]$, then obviously both sides of (5.3.10) equal 0. Furthermore, let $\tau \in (0,1]$ and $f = \chi_{[\tau,1]}$. Then by Proposition 2.3.3,

$$\int_0^1 k(t,s)\, \mathrm{d}[f(s)] = k(t,\tau),$$

i.e.

$$\int_0^1 y(t)\,\mathrm{d}_t\Big[\int_0^1 k(t,s)\,\mathrm{d}[f(s)]\Big] = \int_0^1 y(t)\,\mathrm{d}_t[k(t,\tau)].$$

On the other hand, by Proposition 2.3.3 we have

$$\int_0^1 \Big(\int_0^1 y(t)\,\mathrm{d}_t[k(t,s)]\Big)\,\mathrm{d}[f(s)] = \int_0^1 y(t)\mathrm{d}_t[k(t,\tau)],$$

as well.

Analogously we would prove that (5.3.10) holds also for $f = \chi_{(\sigma,1]}$, $\sigma \in [0,1)$. Now, if $x \in \mathbb{G}$, let $\{x_r\}_{r=1}^{\infty}$ be a sequence of finite step functions on $[0,1]$ such that x_r tends to x uniformly on $[0,1]$ as $r \to \infty$. By the previous part of the proof, we have

$$\int_0^1 y(t)\,\mathrm{d}_t\Big[\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)]\Big] = \int_0^1 \Big(\int_0^1 y(t)\,\mathrm{d}_t[k(t,s)]\Big)\,\mathrm{d}[x_r(s)]$$

for any $r \in \mathbb{N}$. According to Corollary 2.3.10 it follows that

$$\begin{aligned}\lim_{r\to\infty}&\Big(\int_0^1 \Big(\int_0^1 y(t)\,\mathrm{d}_t[k(t,s)]\Big)\,\mathrm{d}[x_r(s)]\Big)\\ &= \int_0^1 \Big(\int_0^1 y(t)\,\mathrm{d}_t[k(t,s)]\Big)\,\mathrm{d}[x(s)].\end{aligned}$$

On the other hand, by Lemma 5.3.7 and by Theorem 2.3.8 we have for any $r \in \mathbb{N}$ and any $t \in [0,1]$

$$\begin{aligned}&\Big|\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)] - \int_0^1 k(t,s)\,\mathrm{d}[x(s)]\Big| = \Big|\int_0^1 k(t,s)\,\mathrm{d}[x_r(s) - x(s)]\Big|\\ &\leq 2\,\|k(t,.)\|_{\mathbb{BV}}\,\|x_r - x\| \leq 2\,\varkappa\,\|x_r - x\|\end{aligned}$$

and consequently

$$\lim_{r\to\infty}\Big(\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)]\Big) = \int_0^1 k(t,s)\,\mathrm{d}[x(s)]$$

uniformly with respect to $t \in [0,1]$. Thus, making use of Corollary 2.3.10 once more, we obtain that the relation

$$\lim_{r\to\infty}\int_0^1 y(t)\,\mathrm{d}_t\Big[\int_0^1 k(t,s)\,\mathrm{d}[x_r(s)]\Big] = \int_0^1 y(t)\,\mathrm{d}_t\Big[\int_0^1 k(t,s)\,\mathrm{d}[x(s)]\Big]$$

is true. It follows immediately that the relation (5.3.10) is true for any $y \in \mathbb{BV}$ and any $f \in \mathbb{G}$.

c) The proof can be extended to the general case $n \in \mathbb{N}$, $n \geq 1$, similarly as it was done at the end of the proof of Lemma 5.3.12. $\square$

Remark 5.3.14. For the proof of the Bray Theorem in the case of the interior integral see [15, Theorem II.1.1].

Lemma 5.3.15. *Assume $K \in \mathscr{K}^{n\times n}$ and*

$$H(t,s) = \begin{cases} K(t,s+) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s \in [0,1), \\ K(t,1-) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s = 1. \end{cases}$$

Then $H \in \mathscr{K}^{n\times n}$. Moreover, if $K \in \mathscr{K}_L^{n\times n}$, then $H \in \mathscr{K}_L^{n\times n}$, as well.

Proof. Analogously to the proofs of Lemma 5.3.12 and of Theorem 5.3.13 it is sufficient to show that the assertion of the lemma is true in the scalar case $n = 1$.

Let $n = 1$, $k \in \mathscr{K}^{n\times n}$ and

$$h(t,s) = \begin{cases} k(t,s+) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s \in [0,1), \\ k(t,1-) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s = 1. \end{cases}$$

a) Let $d = \{s_0, s_1, \dots, s_m\}$ be an arbitrary division of the interval $[0,1]$ ($d \in \mathcal{D}[0,1]$). Then

$$\begin{aligned} S(h,d) &= \sum_{j=1}^{m} |h(t,s_j) - h(t,s_{j-1})| \\ &= \sum_{j=1}^{m-1} |k(t,s_j+) - k(t,s_{j-1}+)| + |k(t,1-) - k(t,s_{m-1}+)|. \end{aligned}$$

Let $\delta > 0$ be such that $s_{m-1} + \delta < 1 - \delta$ and let us denote

$$\begin{aligned} &\sigma_0 = 0,\ \sigma_j = s_{j-1} + \delta \quad \text{for} \quad j = 1,2,\dots,m, \\ &\sigma_{m+1} = 1 - \delta,\ \ \sigma_{m+2} = 1. \end{aligned} \tag{5.3.11}$$

Then

$$d_\delta = \{\sigma_0, \sigma_1, \dots, \sigma_{m+2}\} \in \mathcal{D}[0,1] \tag{5.3.12}$$

and according to (H_2), for any $\delta > 0$ sufficiently small we have

$$\begin{aligned} S(k,d_\delta) &= |k(t,\delta) - k(t,0)| + \sum_{j=1}^{m-1} |k(t,s_j+\delta) - k(t,s_{j-1}+\delta)| \\ &= |k(t,1-\delta) - k(t,s_{m-1}+\delta)| + |k(t,1) - k(t,1-\delta)| \\ &\le \operatorname{var}_0^1 k(t,.) < \infty. \end{aligned}$$

Thus

$$\infty > \lim_{\delta\to 0+} S(k,d_\delta) = S(h,d) + |\Delta_2^+ k(t,0)| + |\Delta_2^- k(t,1)|$$

and consequently the inequality

$$S(h,d) \le \operatorname{var}_0^1 k(t,.) - |\Delta_2^+ k(t,0)| - |\Delta_2^- k(t,1)|$$

holds for any division $d \in \mathcal{D}[0,1]$. Hence

$$\begin{aligned}\|h(t,.)\|_{\mathbb{BV}} &= |k(t,0+)| + \operatorname{var}_0^1 h(t,.)\\ &\le |k(t,0)| + |\Delta_2^+ k(t,0)| + \operatorname{var}_0^1 k(t,.) - |\Delta_2^+ k(t,0)| - |\Delta_2^- k(t,1)|\\ &\le \|k(t,.)\|_{\mathbb{BV}},\end{aligned}$$

i.e., h fulfils (H_1).

b) Let $t \in [0,1)$ and $\varepsilon > 0$ be given. According to (H_2)(i') there is a $\delta_0 > 0$ such that $t + \delta_0 < 1$ and

$$\|k(\tau_2,.) - k(\tau_1,.)\|_{\mathbb{BV}} < \varepsilon$$

holds for any couple $\tau_1, \tau_2 \in (t, t+\delta_0)$. In particular,

$$S(k(\tau_2,.) - k(\tau_1,.), \Delta) < \varepsilon \tag{5.3.13}$$

for any division $\Delta \in \mathcal{D}[0,1]$ and any couple $\tau_1, \tau_2 \in (t, t+\delta_0)$. Now, let an arbitrary division $d = \{s_0, s_1, \dots, s_m\} \in \mathcal{D}[0,1]$ be given and let $\delta > 0$ be such that $\delta < \delta_0$ and $s_{m-1} + \delta < 1 - \delta$. Let us define a division $d_\delta = \{\sigma_0, \sigma_1, \dots, \sigma_m\} \in \mathcal{D}[0,1]$ as in (5.3.11) and (5.3.12). Making use of (5.3.13) we obtain

$$\begin{aligned}&S(h(\tau_2,.) - h(\tau_1,.), d)\\ &\quad = |k(\tau_2, s_1+) - k(\tau_1, s_1+) - k(\tau_2, 0+) + k(\tau_1, 0+)|\\ &\qquad + \sum_{j=2}^{m-1} |k(\tau_2, s_j+) - k(\tau_1, s_j+) - k(\tau_2, s_{j-1}+) + k(\tau_1, s_{j-1}+)|\\ &\qquad + |k(\tau_2, 1-) - k(\tau_1, 1-) - k(\tau_2, s_{m-1}+) + k(\tau_1, s_{m-1}+)|\\ &\quad = \lim_{\delta\to 0+} \Big(\sum_{j=1}^{m} |k(\tau_2, \sigma_{j+1}) - k(\tau_1, \sigma_{j+1}) - k(\tau_2, \sigma_j) + k(\tau_1, \sigma_j)| \Big)\\ &\quad = \lim_{\delta\to 0+} \big(S(k(\tau_2,.) - k(\tau_1,.), d_\delta) \big)\\ &\qquad - |\Delta_2^+ (k(\tau_2, 0) - k(\tau_1, 0))| - |\Delta_2^- (k(\tau_2, 1) - k(\tau_1, 1))| < \varepsilon.\end{aligned}$$

This means that for any couple $\tau_1, \tau_2 \in (t, t+\delta)$ we have

$$\|h(\tau_2,.) - h(\tau_1,.)\|_{\mathbb{BV}} < \mathrm{e},$$

i.e., h fulfils (H_2)(i'). Similarly we could show that h fulfils also (H_2)(ii). Thus $h \in \mathscr{K}^{1\times 1}$.

c) Let $\mathfrak{M}_k : t \in [0,1] \mapsto k(t,.) \in \mathbb{BV}$ be left-continuous on $(0,1)$ and let $\varepsilon > 0$ be given. Then there is a $\delta_0 > 0$ such that $t - \delta_0 > 0$ and

$$S(k(t,.) - k(\tau,.), \Delta) < \varepsilon \tag{5.3.14}$$

holds for any $\tau \in (t - \delta_0, t)$ and any $\Delta \in \mathcal{D}[0,1]$. Let an arbitrary division

$$d = \{s_0, s_1, \dots, s_m\} \in \mathcal{D}[0,1]$$

be given and let

$$d_\delta = \{\sigma_0, \sigma_1, \dots, \sigma_{m+2}\} \in \mathcal{D}[0,1]$$

be given for $\delta \in (0, \min\{\delta_0, \frac{1-s_{m-1}}{2}\})$ by (5.3.11) and (5.3.12). Then making use of (5.3.14) we obtain similarly as in part b) of this proof

$$\begin{aligned} S(h(t,.)-&h(\tau,.), d) \\ &= \lim_{\delta \to 0+} \Big(\sum_{j=1}^{m} |k(t,\sigma_{j+1}) - k(\tau,\sigma_{j+1}) - k(t,\sigma_j) + k(\tau,\sigma_j)| \Big) \\ &= \lim_{\delta \to 0+} \big(S(k(t,.) - k(\tau,.), d_\delta) \big) \\ &\quad - |\Delta_2^+(k(t,0) - k(\tau,0))| - |\Delta_2^-(k(t,1) - k(\tau,1))| < \varepsilon, \end{aligned}$$

wherefrom the desired relation

$$\lim_{\tau \to t-} \|h(t,.) - h(\tau,.)\|_{\mathbb{BV}} = 0$$

easily follows. $\square$

Remark 5.3.16. Analogously we could show that if $K \in \mathscr{K}^{n\times n}$ and if

$$H(t,s) = \begin{cases} K(t,0+) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s = 0, \\ K(t,s-) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s \in (0,1], \end{cases}$$

then $H \in \mathscr{K}^{n\times n}$. Moreover, if $K \in \mathscr{K}_L^{n\times n}$, then $H \in \mathscr{K}_L^{n\times n}$, as well.

Lemma 5.3.17. *Let $K \in \mathscr{K}^{n\times n}$ and let*

$$H(t,s) = \begin{cases} K(t+,s) & \textit{for} \quad t \in [0,1) \quad \textit{and} \quad s \in [0,1], \\ K(1-,s) & \textit{for} \quad t = 1 \quad \textit{and} \quad s \in [0,1] \end{cases}$$

and

$$G(t,s) = \begin{cases} K(0+,s) & \textit{for} \quad t = 0 \quad \textit{and} \quad s \in [0,1], \\ K(t-,s) & \textit{for} \quad t \in (0,1] \quad \textit{and} \quad s \in [0,1]. \end{cases}$$

Then $H \in \mathscr{K}^{n\times n}$ and $G \in \mathscr{K}_L^{n\times n}$.

Proof. We shall prove that under the assumptions of the lemma, $H \in \mathscr{K}^{n\times n}$. The proof of the latter relation would be quite similar.

Let $t < 1$ and let $d \in \mathcal{D}[0,1]$ be an arbitrary division of $[0,1]$. Then for any $\delta \in (0, 1-t)$ we have by Lemma 5.3.7

$$S(K(t+\delta,.), d) \leq \operatorname{var}_0^1 K(t+\delta,.) \leq \varkappa < \infty.$$

Letting $\delta \to 0+$ we immediately obtain that the inequality

$$S(H(t,.), d) \leq \varkappa < \infty$$

is true for any $d \in \mathcal{D}[0,1]$. It means that

$$\operatorname{var}_0^1 H(t,.) \leq \varkappa < \infty.$$

Now, let an arbitrary $\varepsilon > 0$ be given. By $(\mathrm{H}_2)(\mathrm{i}')$ there is a $\delta > 0$ such that

$$\|K(\tau_2,.) - K(\tau_1,.)\|_{\mathbb{BV}} < \frac{\varepsilon}{2}$$

holds whenever $t < \tau_1 < \tau_2 < t+\delta$. It means that for all $t_1, t_2 \in (t, t+\frac{\delta}{2})$ and any $\tau \in (0, \frac{\delta}{2})$ we have

$$\|K(t_2+\tau,.) - K(t_1+\tau,.)\|_{\mathbb{BV}} < \frac{\varepsilon}{2}.$$

In particular, for any division $d \in \mathcal{D}[0,1]$ we have

$$|K(t_2+\tau,0) - K(t_1+\tau,0)| < \frac{\varepsilon}{2} \quad \text{and}$$
$$S(K(t_2+\tau,.) - K(t_1+\tau,.), d) < \frac{\varepsilon}{2},$$

wherefrom we get that the relation

$$\|H(t_2,.) - H(t_1,.)\|_{\mathbb{BV}} < \varepsilon$$

is true whenever $t < t_1 < t_2 < t + \frac{\delta}{2}$.

Analogously we would prove that if $t > 0$, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|H(t_2,.) - H(t_1,.)\|_{\mathbb{BV}} < \varepsilon$$

is true whenever $t - \frac{\delta}{2} < t_1 < t_2 < t$. □

Lemma 5.3.18. *Let $K \in \mathscr{K}^{n\times n}$, $t_1, s_1 \in [0,1)$ and $t_2, s_2 \in (0,1]$. Then all the limits*

$$K(t_1+, s_1+) = \lim_{\substack{(\tau,\sigma)\to(t_1,s_1)\\ \tau>t_1,\sigma>s_1}} K(\tau,\sigma), \quad K(t_1+, s_2-) = \lim_{\substack{(\tau,\sigma)\to(t_1,s_2)\\ \tau>t_1,\sigma<s_2}} K(\tau,\sigma),$$
$$K(t_2-, s_1+) = \lim_{\substack{(\tau,\sigma)\to(t_2,s_1)\\ \tau<t_2,\sigma>s_1}} K(\tau,\sigma), \quad K(t_2-, s_2-) = \lim_{\substack{(\tau,\sigma)\to(t_2,s_2)\\ \tau<t_2,\sigma<s_2}} K(\tau,\sigma)$$

exist in $\mathbb{R}^{n\times n}$.

Proof. We will restrict ourselves to proving the existence of the limits

$$K(t_1+, s_1+) \in \mathbb{R}^{n\times n} \quad \text{for} \quad t_1, s_1 \in [0, 1).$$

The modifications of the proofs in the other cases are obvious.

Let $t_1 \in [0, 1)$ and $s_1 \in [0, 1)$ be given. By Lemma 5.3.15 there exists $M \in \mathbb{R}^{n\times n}$ such that

$$\lim_{\sigma\to s+} K(t_1+, \sigma) = \lim_{\sigma\to s+} \Big(\lim_{\tau\to t_1+} K(\tau, \sigma)\Big) = M.$$

Furthermore, since in virtue of Corollary 5.3.9 we have $\lim_{\tau\to t_1+} \|K(\tau, .) - K(t_1+, .)\| = 0$, i.e.

$$\lim_{\tau\to t_1+} K(\tau, \sigma) = K(t_1+, \sigma) \quad \text{uniformly with respect to} \quad \sigma \in [0, 1],$$

it follows that

$$\lim_{\substack{(\tau,\sigma)\to(t_1,s_1)\\ \tau>t_1,\sigma>s_1}} K(\tau, \sigma) = M.$$

□

Lemma 5.3.19. *Assume $K \in \mathscr{K}^{n\times n}$, $s \in (0, 1]$ and $t \in [0, 1)$. Then*

$$\begin{aligned}
\lim_{\tau\to t+} K(\tau, \tau-) &= \lim_{\tau\to t+} K(\tau, \tau+) = K(t+, t+),\\
\lim_{\tau\to t+} K(\tau-, \tau) &= \lim_{\tau\to t+} K(\tau+, \tau) = K(t+, t+),\\
\lim_{\tau\to s-} K(\tau, \tau-) &= \lim_{\tau\to s-} K(\tau, \tau+) = K(s-, s-)
\end{aligned}$$

and

$$\lim_{\tau\to s-} K(\tau-, \tau) = \lim_{\tau\to s-} K(\tau+, \tau) = K(s-, s-).$$

Proof. We will restrict ourselves to the proof of the relations

$$\lim_{\tau\to t+} K(\tau, \tau-) = K(t+, t+), \quad t \in [0, 1).$$

The proofs of the other assertions of the lemma would be quite analogous. By Lemma 5.3.18 there exists $\delta \in (0, 1 - t)$ such that

$$|K(\tau, \sigma) - K(t+, t+)| < \frac{\varepsilon}{2}$$

holds whenever $t < \tau < t + \delta$ and $t < \sigma < t + \delta$. Furthermore, for any $\tau \in (t, t + \delta)$ we can choose a $\sigma_\tau \in (t, \tau)$ such that

$$|K(\tau, \tau-) - K(\tau, \sigma_\tau)| < \frac{\varepsilon}{2}$$

is true. Thus for any $\tau \in (t, t + \delta)$ we have

$$\begin{aligned}
|K(\tau, \tau-) - K(t+, t+)| &\le |K(\tau, \tau-) - K(\tau, \sigma_\tau)|\\
&\quad + |K(\tau, \sigma_\tau) - K(t+, t+)| < \varepsilon.
\end{aligned}$$

□

Remark 5.3.20. Notice that by [45, Corollaries I.6.15 and I.6.16] the set $\mathcal{SBV}^{n\times n}$ of $n\times n-$matrix valued functions of strongly bounded variation on $[0,1]\times[0,1]$ (cf. 1.2.5) is a subset of $\mathscr{K}^{n\times n}$.

On the other hand, the functions of the form

$$K(t,s)=F(t)\,G(s),\quad (t,s)\in[0,1]\times[0,1],$$

where $F\in\mathbb{G}^{n\times n}$ and $G\in\mathbb{BV}^{n\times n}$, provide the simplest examples of the kernels which satisfy the assumptions of this paper, but do not belong in general to $\mathcal{SBV}^{n\times n}$.

5.4. Fredholm-Stieltjes integral equations in $\mathbb{G}^n_L$

In this section we will consider linear integral equations of the form

$$x(t)-A(t)\,x(0)-\int_0^1 B(t,s)\,\mathrm{d}[x(s)]=f(t),\quad t\in[0,1],\tag{5.4.1}$$

where

$$A\in\mathbb{G}^{n\times n}_{\mathrm{L}}\quad\text{and}\quad B\in\mathscr{K}^{n\times n}_L.$$

Remark 5.4.1. Let us recall that the operator $\mathscr{L}$ given by (5.1) is the general form of a linear compact operator on the space $\mathbb{G}^n_L$ (cf. Theorem 5.3.5). The equation (5.4.1) may be also written as the operator equation

$$x-\mathscr{L}x=f.\tag{5.4.2}$$

Remark 5.4.2. It is also known (cf. Theorem 2.4.8) that the dual space $(\mathbb{G}^n_L)^*$ to $\mathbb{G}^n_L$ is isomorphic to the space $\mathbb{BV}^n\times\mathbb{R}^n$, while for a given couple $(y,\gamma)\in\mathbb{BV}^n\times\mathbb{R}^n$ the corresponding linear bounded functional on $\mathbb{G}^n_L$ is given by

$$x\in\mathbb{G}^n_L\mapsto\langle x,(y,\gamma)\rangle:=\gamma^{\mathrm{T}}\,x(0)+\int_0^1 y^{\mathrm{T}}(s)\,\mathrm{d}[x(s)]\in\mathbb{R}.\tag{5.4.3}$$

The compactness of the operator $\mathscr{L}$ immediately implies that the following Fredholm alternative type assertions 5.4.3-5.4.5 are true.

Proposition 5.4.3. *Let $A\in\mathbb{G}^{n\times n}_{\mathrm{L}}$ and $B\in\mathscr{K}^{n\times n}_L$. Then the equation* (5.4.1) *possesses a unique solution $x\in\mathbb{G}^n_L$ for any $f\in\mathbb{G}^n_L$ if and only if the corresponding homogeneous equation $x-\mathscr{L}x=0$, i.e.*

$$x(t)-A(t)\,x(0)-\int_0^1 B(t,s)\,\mathrm{d}[x(s)]=0,\quad t\in[0,1],$$

possesses only the trivial solution.

Proposition 5.4.4. *Let $A \in \mathbb{G}_L^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbb{G}_L^n$. Then the equation* (5.4.1) *possesses a solution in $\mathbb{G}_L^n$ if and only if*

$$\gamma^T f(0) + \int_0^1 y^T(s)\,\mathrm{d}[f(s)] = 0 \tag{5.4.4}$$

holds for any solution $(y,\gamma) \in \mathbb{BV}^n \times \mathbb{R}^n$ of the operator equation

$$(y,\gamma) - \mathscr{L}^*(y,\gamma) = 0 \in \mathbb{BV}^n \times \mathbb{R}^n$$

adjoint to (5.4.1).

Proposition 5.4.5. *Let $A \in \mathbb{G}_L^{n\times n}$ and $B \in \mathscr{K}_L^{n\times n}$. Then the relations*

$$\dim \mathscr{N}(\mathrm{I} - \mathscr{L}) = \dim \mathscr{N}(\mathrm{I} - \mathscr{L}^*) < \infty$$

hold for the dimensions of the null spaces $\mathscr{N}(\mathrm{I} - \mathscr{L})$ and $\mathscr{N}(\mathrm{I} - \mathscr{L}^*)$ corresponding to the operator $\mathscr{L}$ and its adjoint $\mathscr{L}^*$, respectively.

Making use of the above mentioned explicit representation (5.4.3) of the dual space to $\mathbb{G}_L^n$ and of the Bray Theorem we can derive the explicit form of the adjoint operator $\mathscr{L}^*$ to $\mathscr{L}$.

Theorem 5.4.6. *Let $A \in \mathbb{G}_L^{n\times n}$ and $B \in \mathscr{K}_L^{n\times n}$. Then the adjoint operator $\mathscr{L}^*$ to the operator $\mathscr{L}$ from* (5.4.2) *is given by*

$$\mathscr{L}^* : (y,\gamma) \in \mathbb{BV}^n \times \mathbb{R}^n \mapsto \big(\mathscr{L}_1^*(y,\gamma), \mathscr{L}_2^*(y,\gamma)\big) \in \mathbb{BV}^n \times \mathbb{R}^n,$$

where

$$\big(\mathscr{L}_1^*(y,\gamma)\big)(t) = B^T(0,t)\,\gamma + \int_0^t \mathrm{d}_s\big[B^T(s,t)\big]\,y(s) \quad \textit{for} \quad t \in [0,1]$$

and

$$\mathscr{L}_2^*(y,\gamma) = A^T(0)\,\gamma + \int_0^1 \mathrm{d}\big[A^T(s)\big]\,y(s).$$

Proof. Given $x \in \mathbb{G}_L^n$, $y \in \mathbb{BV}^n$ and $\gamma \in \mathbb{R}^n$, we have by (5.4.3) and by Theorem 5.3.13

$$\begin{aligned}
\langle \mathscr{L}x, (y,\gamma)\rangle &= \gamma^{\mathrm{T}}\Big(A(0)\,x(0) + \int_0^1 B(0,t)\,\mathrm{d}[x(t)]\Big) \\
&\quad + \int_0^1 y^{\mathrm{T}}(t)\,\mathrm{d}_s\Big[A(t)\,x(0) + \int_0^1 B(t,s)\,\mathrm{d}[x(s)]\Big] \\
&= \Big(\gamma^{\mathrm{T}} A(0) + \int_0^1 y^{\mathrm{T}}(s)\,\mathrm{d}[A(s)]\Big)\,x(0) \\
&\quad + \int_0^1 \Big(\gamma^{\mathrm{T}} B(0,t) + \int_0^1 y^{\mathrm{T}}(s)\,\mathrm{d}_s[B(s,t)]\Big)\,\mathrm{d}[x(t)]
\end{aligned}$$

$$
\begin{aligned}
&= (\mathscr{L}_2^*(y,\gamma))^{\mathrm{T}}\, x(0) + \int_0^1 (\mathscr{L}_1^*(y,\gamma))^{\mathrm{T}}(t)\,\mathrm{d}[x(t)] \\
&= \langle x, (\mathscr{L}_1^*(y,\gamma), \mathscr{L}_2^*(y,\gamma))\rangle,
\end{aligned}
$$

wherefrom the proof of the theorem immediately follows. □

Proposition 5.4.4 and Theorem 5.4.6 immediately yield the following assertion:

Theorem 5.4.7. *Let $A \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbb{G}_L^n$. Then the equation* (5.4.1) *possesses a solution $x \in \mathbb{G}_L^n$ if and only if* (5.4.4) *holds for any solution $(y,\gamma) \in \mathbb{BV}^n \times \mathbb{R}^n$ of the system*

$$
\begin{aligned}
y(t) - B^{\scriptscriptstyle T}(0,t)\,\gamma - \int_0^t \mathrm{d}_s\big[B^{\scriptscriptstyle T}(s,t)\big]\,y(s) &= 0, \quad t \in [0,1],\\
\gamma - A^{\scriptscriptstyle T}(0)\,\gamma \quad - \int_0^1 \mathrm{d}\big[A^{\scriptscriptstyle T}(s)\big]\,y(s) \quad &= 0.
\end{aligned}
$$

Remark 5.4.8. Let us notice that in virtue of Corollary 5.3.9, for any solution $x \in \mathbb{G}^n$ of (5.4.1) on $[0,1]$ we have

$$
\begin{aligned}
x(t+) &= A(t+)\,x(0) + \int_0^1 B(t+,s)\,\mathrm{d}[x(s)] + f(t+) \quad \text{for all} \quad t \in [0,1),\\
x(t-) &= A(t-)\,x(0) + \int_0^1 B(t-,s)\,\mathrm{d}[x(s)] + f(t-) \quad \text{for all} \quad t \in (0,1].
\end{aligned}
$$

In particular, if $A \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbb{G}_L^n$, then any solution x of (5.4.1) on $[0,1]$ is left-continuous on $(0,1)$, i.e. $x \in \mathbb{G}_L^n$.

Example 5.4.9. Let us consider a linear Stieltjes integral equation

$$
x(t) - \int_0^1 \mathrm{d}_s[P(t,s)]\,x(s) = f(t), \quad t \in [0,1] \tag{5.4.5}
$$

with $P \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbb{G}_L^n$. Such equations with kernels P of strongly bounded variation on $[0,1]\times[0,1]$ (cf. Remark 5.3.20) were treated in [45].

Let $t \in [0,1]$ and $x \in \mathbb{G}_L^n$ be given. Let us put

$$
Q(t,s) = \begin{cases} P(t,s+) & \text{for} \quad s < 1,\\ P(t,1-) & \text{for} \quad s = 1 \end{cases}
$$

and

$$
Z(t,s) = P(t,s) - Q(t,s) \quad \text{for} \quad (t,s) \in [0,1]\times[0,1].
$$

Then

$$
Z(t,s) = \begin{cases} -\Delta_2^+ P(t,s) & \text{for} \quad s < 1,\\ \Delta_2^- P(t,1) & \text{for} \quad s = 1. \end{cases}
$$

Since $Q(t,.)$ and $Z(t,.) \in \mathbb{BV}^{n\times n}$,

$$\lim_{\sigma\to s+} P(t,\sigma+) = P(t,s+) \quad \text{for} \quad s \in [0,1)$$

and

$$\lim_{\sigma\to s-} P(t,\sigma+) = P(t,s-) \quad \text{for} \quad s \in (0,1],$$

it is easy to verify that

$$Z(t,s-) = 0 \quad \text{for all} \quad s \in [0,1) \quad \text{and} \quad Z(t,s+) = 0 \quad \text{for all} \quad s \in (0,1].$$

Since $Z(t,.) \in \mathbb{BV}^{n\times n}$, this implies that there is an at most countable set $W \subset [0,1]$ of points in $[0,1]$ such that $Z(t,s) = 0$ holds for any $s \in [0,1]\setminus W$. Making use of Proposition 2.3.13 we obtain

$$\int_0^1 \mathrm{d}_s[Z(t,s)]\,x(s) = Z(t,1)\,x(1) - Z(t,0)\,x(0).$$

This implies that the relation

$$\int_0^1 \mathrm{d}_s[P(t,s)]\,x(s) = \int_0^1 \mathrm{d}_s[Q(t,s)]\,x(s) + \Delta_2^+ P(t,0)\,x(0) + \Delta_2^- P(t,1)\,x(1)$$

is true. Furthermore, according to the integration-by-parts formula (cf. Theorem 2.3.15) we have

$$\begin{aligned}
&\int_0^t \mathrm{d}_s[P(t,s)]\,x(s) \\
&\quad = Q(t,1)\,x(1) - Q(t,0)\,x(0) - \int_0^1 Q(t,s)\,\mathrm{d}[x(s)] \\
&\qquad\quad + [P(t,0+) - P(t,0)]\,x(0) + [P(t,1) - P(t,1-)]\,x(1) \\
&\quad = P(t,1)\,x(1) - P(t,0)\,x(0) - \int_0^1 Q(t,s)\,\mathrm{d}[x(s)] \\
&\quad = [P(t,1) - P(t,0)]\,x(0) + \int_0^1 (P(t,1) - Q(t,s))\,\mathrm{d}[x(s)] \\
&\quad = [P(t,1) - P(t,0)]\,x(0) + \int_0^1 \left\{ \begin{array}{ll} P(t,1) - P(t,s+), & s < 1 \\ P(t,1) - P(t,1-), & s = 1 \end{array} \right\} \mathrm{d}[x(s)].
\end{aligned}$$

Hence

$$\int_0^t \mathrm{d}_s[P(t,s)]\,x(s) = C(t)\,x(0) + \int_0^1 D(t,s)\,\mathrm{d}[x(s)],$$

where

$$C(t) = \mathrm{I} + P(t,1) - P(t,0)$$

and

$$D(t,s) = \begin{cases} P(t,1) - P(t,s+) & \text{for} \quad s \in [0,1), \\ P(t,1) - P(t,1-) & \text{for} \quad s = 1. \end{cases}$$

Obviously, under our assumptions we have $C \in \mathbb{G}_{\mathrm{L}}^{n\times n}$ and $D \in \mathscr{K}_L^{n\times n}$ (cf. Lemma 5.3.15). Thus, if $P \in \mathscr{K}_L^{n\times n}$ and $f \in \mathbb{G}_L^n$, then the given equation (5.4.5) may be transformed to an equation of the form (5.4.1) with coefficients A, B and f fulfilling the assumptions of Theorem 5.4.7.

5.5. The resolvent couple for the Fredholm-Stieltjes integral equation

In this section we consider the special case when the equation (5.4.1) possesses a unique solution $x \in \mathbb{G}_L^n$ for any $f \in \mathbb{G}_L^n$. This means that in addition to $A \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ we assume that

$$\dim \mathscr{N}(\mathrm{I} - \mathscr{L}) = 0 \tag{5.5.1}$$

(cf. Proposition 5.4.3).

Under these assumptions the Bounded Inverse Theorem [33, Section III.4.1] implies that the linear bounded operator $\mathrm{I} - \mathscr{L} : \mathbb{G}_L^n \to \mathbb{G}_L^n$ possesses a bounded inverse $(\mathrm{I} - \mathscr{L})^{-1} : \mathbb{G}_L^n \to \mathbb{G}_L^n$. Furthermore, as

$$(\mathrm{I} - \mathscr{L})^{-1} = \mathrm{I} + (\mathrm{I} - \mathscr{L})^{-1} \mathscr{L},$$

it follows immediately that the inverse operator $(\mathrm{I} - \mathscr{L})^{-1}$ may be expressed in the form

$$(\mathrm{I} - \mathscr{L})^{-1} = \mathrm{I} + \Gamma, \tag{5.5.2}$$

where Γ is a linear compact operator ($\Gamma \in \mathcal{K}(\mathbb{G}_L^n, \mathbb{G}_L^n)$). By Theorem 5.3.5 there exist functions $U \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ such that Γ is given by

$$\Gamma : f \in \mathbb{G}_L^n \to U(t)\, f(0) + \int_0^1 V(t,s)\, \mathrm{d}[f(s)]. \tag{5.5.3}$$

The following assertion now follows from Lemma 5.2.1 and Theorem 5.3.5.

Theorem 5.5.1. *Assume $A \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and* (5.5.1). *Then there exists a unique couple of functions $U \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ such that for any $f \in \mathbb{G}_L^n$ the corresponding solution $x \in \mathbb{G}_L^n$ to* (5.4.1) *is given by*

$$x(t) = f(t) + U(t)\, f(0) + \int_0^1 V(t,s)\, \mathrm{d}[f(s)], \quad t \in [0,1]. \tag{5.5.4}$$

Theorem 5.5.2. *Let the assumptions of Theorem* 5.5.1 *be satisfied. Then the functions* U, V *given by Theorem* 5.5.1 *satisfy the equations*

$$\begin{aligned} &U(t) - A(t)\,U(0) \\ &\qquad - \int_0^1 B(t,\tau)\,\mathrm{d}[U(\tau)] = A(t), \quad t \in [0,1], \end{aligned} \tag{5.5.5}$$

$$\begin{aligned} &V(t,s) - A(t)\,V(0,s) \\ &\qquad - \int_0^1 B(t,\tau)\,\mathrm{d}_\tau[V(\tau,s)] = B(t,s), \quad t,s \in [0,1]. \end{aligned} \tag{5.5.6}$$

Proof. Let Γ be a linear compact operator defined by (5.5.2). Inserting (5.5.2) into (5.4.1) we obtain that under our assumptions Γ has to satisfy the relation

$$\Gamma f - \mathscr{L}(\Gamma f) = \mathscr{L} f \quad \text{for all} \quad f \in \mathbb{G}_L^n. \tag{5.5.7}$$

Inserting (5.5.3) into (5.5.7) and making use of the Bray Theorem (cf. Theorem 5.3.13) we obtain furthermore that

$$\begin{aligned} &\Big(U(t) - A(t)\,U(0) - \int_0^1 B(t,\tau)\,\mathrm{d}[U(\tau)]\Big)\, f(0) \\ &\qquad + \int_0^1 \Big(V(t,s) - A(t)\,V(0,s) - \int_0^1 B(t,\tau)\,\mathrm{d}_\tau[V(\tau,s)]\Big)\,\mathrm{d}[f(s)] \\ &= A(t)\,f(0) + \int_0^1 B(t,s)\,\mathrm{d}[f(s)] \end{aligned}$$

has to be true for any $f \in \mathbb{G}_L^n$, wherefrom by Lemma 5.2.1 the assertion of the theorem follows immediately. □

Definition 5.5.3. We say that a couple of functions $U \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ is a ***resolvent couple*** for the equation (5.4.1) if for any $f \in \mathbb{G}_L^n$ the unique solution $x \in \mathbb{G}_L^n$ of (5.4.1) is given by (5.5.3).

5.6. Volterra-Stieltjes integral equations in $\mathbb{G}_L^n$

It is natural to expect that the linear operator equation (5.4.2) could possess a unique solution for any $f \in \mathbb{G}_L^n$ if the operator $\mathscr{L}$ is ***causal*** .

Definition 5.6.1. An operator $\mathscr{L} \in \mathcal{L}(\mathbb{G}_L^n)$ is said to be causal if

$$(\mathscr{L}x)(0) = 0 \quad \text{for any} \quad x \in \mathbb{G}_L^n, \tag{5.6.1}$$

and for a given $t \in (0,1)$

$$(\mathscr{L}x)(t) = 0 \quad \text{whenever} \quad x \in \mathbb{G}_L^n \ \text{ and } \ x(\tau) = 0 \ \text{ on } \ [0,t]. \tag{5.6.2}$$

Lemma 5.6.2. *Let* $A \in \mathbb{G}_L^{n\times n}$ *and* $B \in \mathscr{K}_L^{n\times n}$*. Then the operator* $\mathscr{L} \in \mathcal{L}(\mathbb{G}_L^n)$ *given by* (5.1) *is causal if and only if*

$$A(0) = 0 \quad \textit{and} \quad B(t,s) = 0 \quad \textit{for all} \quad t \in [0,1) \quad \textit{and} \quad s \in [t,1]. \quad (5.6.3)$$

Proof. a) If (5.6.3) is satisfied, then

$$\int_0^1 B(t,s)\,\mathrm{d}[x(s)] = \int_0^t B(t,s)\,\mathrm{d}[x(s)]$$

holds for any $x \in \mathbb{G}_L^n$ and any $t \in [0,1]$ whence the causality of $\mathscr{L}$ immediately follows.

b) On the other hand, let us assume that $\mathscr{L}$ is causal. Then by (5.6.1) the relation

$$A(0)\,x(0) + \int_0^1 B(0,s)\,\mathrm{d}[x(s)] = 0$$

has to be satisfied for any $x \in \mathbb{G}_L^n$. By Lemma 5.6.2 this means that the relations

$$A(0) = 0 \quad \text{and} \quad B(0,s) = 0 \quad \text{for all} \quad s \in [0,1]$$

have to be satisfied as well. Furthermore, if $t \in (0,1)$, then (5.6.2) is true if and only if

$$\int_t^1 B(t,s)\,\mathrm{d}[x(s)] = 0 \quad \text{for all} \quad x \in \mathbb{G}_L^n.$$

An obvious modification of Lemma 5.6.2 implies that this may hold only if

$$B(t,s) = 0 \quad \text{for all} \quad s \in [t,1],$$

wherefrom the assertion of the lemma immediately follows. □

Remark 5.6.3. Let us notice that the condition (5.6.3) does not necessarily imply that $B(1,1) = 0$. On the other hand, it is easy to verify that the operator $\mathscr{L} \in \mathcal{L}(\mathbb{G}_L^n)$ given by (5.1) fulfils somewhat stronger causality properties (5.6.1) and

$$(\mathscr{L}x)(t) = 0 \quad \text{for all} \ \ t \in (0,1] \text{ and } \ x \in \mathbb{G}_L^n \text{ such that } \ x(\tau) = 0 \text{ on } \ [0,t)$$

if and only if

$$A(0) = 0 \quad \text{and} \quad B(t,s) = 0 \quad \text{whenever} \ \ 0 \le t \le s \le 1.$$

In fact, if $x(\tau) = 0$ on $[0,1)$, then

$$(\mathscr{L}x)(1) = B(1,1)\,x(1) = 0$$

holds for any $x(1) \in \mathbb{R}^n$ if and only if $B(1,1) = 0$.

Remark 5.6.4. As noticed in the proof of Lemma 5.6.2, if the assumptions of Lemma 5.6.2 and the conditions (5.6.3) are satisfied, then the Fredholm-Stieltjes equation (5.4.1) reduces to the Volterra-Stieltjes equation

$$x(t) - A(t)\,x(0) - \int_0^t B(t,s)\,\mathrm{d}[x(s)] = f(t), \quad t \in [0,1]. \tag{5.6.4}$$

To show that the equation (5.6.4) possesses a unique solution $x \in \mathbb{G}_L^n$ for each $f \in \mathbb{G}_L^n$, it is by Proposition 5.4.4 sufficient to show that the corresponding homogeneous equation

$$x(t) = A(t)\,x(0) + \int_0^t B(t,s)\,\mathrm{d}[x(s)], \quad t \in [0,1] \tag{5.6.5}$$

possesses only the trivial solution $x \equiv 0$.

Let $x \in \mathbb{G}_L^n$ be an arbitrary solution of (5.6.5) on $[0,1]$. Then evidently $x(0) = 0$. Furthermore, since by (5.6.3) $B(0+,s) = 0$ whenever $s > 0$, we have by Lemma 5.3.10

$$\begin{aligned} x(0+) &= \lim_{t\to 0+} \int_0^t B(t,s)\,\mathrm{d}[x(s)] = \lim_{t\to 0+} \int_0^1 B(t,s)\,\mathrm{d}[x(s)] \\ &= \int_0^1 B(0+,s)\,\mathrm{d}[x(s)] \qquad = B(0+,0)\,\Delta^+ x(0) = B(0+,0)\,x(0+), \end{aligned}$$

i.e.

$$[\mathrm{I} - B(0+,0)]\,x(0+) = 0.$$

Thus we have $x(0+) = 0$ whenever

$$\det[\mathrm{I} - B(0+,0)] \neq 0.$$

Analogously, if we assume that $x(\tau) \equiv 0$ holds on $[0,t]$ for a given $t \in (0,1)$, then

$$x(t+) = \int_t^1 B(t+,s)\,\mathrm{d}[x(s)] = B(t+,t)\,x(t+),$$

and thus necessarily $x(t+) = 0$ whenever $\det\big(\mathrm{I} - B(t+,t)\big) \neq 0$. Finally, if we assume that $x(\tau) \equiv 0$ on $[0,1)$, then the equation (5.6.5) reduces to

$$[\mathrm{I} - B(1,1)]\,x(1) = x(1).$$

This indicates that it is possible to expect that the equation (5.6.5) will possess only the trivial solution $x \equiv 0$ on $[0,1]$ if the relations

$$\det[\mathrm{I} - B(1,1)] \neq 0 \quad \text{and} \quad \det[\mathrm{I} - B(t+,t)] \neq 0 \quad \text{for all} \quad t \in [0,1) \tag{5.6.6}$$

are satisfied, or in other words, if $\mathrm{I} - B(1,1)$ and $\mathrm{I} - B(t+,t)$ are invertible matrices.

Theorem 5.6.5. *Assume $A \in \mathbb{G}_L^{n\times n}$, $B \in \mathscr{K}_L^{n\times n}$ and* (5.6.3). *Then the equation* (5.6.4) *has a unique solution for any $f \in \mathbb{G}_L^n$ if and only if the relations* (5.6.6) *are satisfied.*

Proof. a) Let us assume that the relations (5.6.6) are satisfied and let $x \in \mathbb{G}_L^n$ be a solution of (5.6.5). We have $x(0+) = x(0) = 0$ and as in Remark 5.6.4 we get

$$\int_0^t B(0+,s)\,\mathrm{d}[x(s)] = B(0+,0)\,\Delta^+ x(0) = 0 \quad \text{for all} \quad t \in [0,1].$$

Consequently, the equation (5.6.5) can be rewritten as

$$x(t) = \int_0^t (B(t,s) - B(0+,s))\,\mathrm{d}[x(s)].$$

In virtue of Theorem 2.3.8, this yields that the inequality

$$|x(t)| \le 2\,\|B(t,.) - B(0+,.)\|_{\mathbb{BV}} \big(\sup_{s\in[0,t]} |x(s)| \big)$$

is true for any $t \in [0,1]$. Furthermore, by Corollary 5.3.9 there is $\delta > 0$ such that

$$\|B(t,.) - B(0+,.)\|_{\mathbb{BV}} < \frac{1}{4} \quad \text{whenever} \quad t \in (0,\delta]$$

and hence also

$$\sup_{t\in[0,\delta]} |x(s)| < \frac{1}{2} \sup_{t\in[0,\delta]} |x(s)|,$$

which yields

$$x(t) \equiv 0 \quad \text{for} \quad t \in [0,\delta].$$

Now, let us put

$$t^* = \sup\big\{\delta \in [0,1] :\, x(t) = 0 \text{ on } \ [0,\delta]\big\}.$$

We know that $t^* \in (0,1]$ and $x(t) = 0$ on $[0,t^*)$. Since x is left-continuous on $(0,1)$ (cf. Remark 5.4.8), it follows that if $t^* < 1$, then $x(t^*) = x(t^*-) = 0$, as well.

Now, if we had $t^* < 1$, then taking into account the hypothesis (5.6.3) and Lemma 5.3.10 we would obtain

$$\begin{aligned} x(t^*+) &= \lim_{t\to t^*+} \int_0^t B(t,s)\,\mathrm{d}[x(s)] = \int_0^1 B(t^*+,s)\,\mathrm{d}[x(s)] \\ &= B(t^*+,t^*)\,x(t^*+) \end{aligned}$$

and consequently

$$[\mathrm{I} - B(t^*+, t^*)]\, x(t^*+) = 0.$$

Hence, according to (5.6.6) we would have $x(t^*+) = 0$. By an argument analogous to that used above for 0 in the place of t^*, we can get that there exists $\delta > 0$ such that $x(t) = 0$ on $[0, t^* + \delta]$, which contradicts the definition of t^*. Moreover, as $x(t) \equiv 0$ on $[0,1)$, we have $x(1-) = 0$ and the equation (5.6.5) reduces to

$$[\mathrm{I} - B(1,1)]\, x(1) = 0$$

and, in virtue of (5.6.6), we have $x(1) = 0$, i.e. $x(t) \equiv 0$ on $[0,1]$. By Proposition 5.4.3 this implies that (5.6.4) has a unique solution for any $f \in \mathbb{G}_L^n$.
b) Let us assume that the set

$$\mathcal{S}_B := \Big\{ t \in [0,1) : \det \big[\mathrm{I} - B(t+, t) \big] = 0 \Big\}$$

is nonempty. Let us denote

$$t^\star = \inf \mathcal{S}_B.$$

Then t^* is not a point of accumulation of $\mathcal{S}_B$. In fact, if this were not the case, then there would exist a sequence $\{t_k\}_{k=1}^\infty$ of points in $\mathcal{S}_B$ such that $t_k > t^*$ for any $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = t^*$. Since in virtue of (5.6.3) we have for any $\sigma > t^*$

$$\lim_{\tau \to t^*+} B(\tau, \sigma) = 0,$$

it follows by Lemma 5.3.18 that

$$B(t^*+, t^*+) = \lim_{\substack{(\tau,\sigma)\to(t^*,t^*)\\ \tau>t^*, \sigma>t^*}} B(\tau,\sigma) = \lim_{\sigma\to t^*+} \big(\lim_{\tau \to t^*+} B(\tau,\sigma) \big) = 0$$

and consequently

$$0 = \lim_{k\to\infty} \det(\mathrm{I} - B(t_k+, t_k)) = \det(\mathrm{I} - B(t^*+, t^*+)) = \det(\mathrm{I}) = 1,$$

a contradiction.

In particular, $t^* \in \mathcal{S}_B$ and $\det(\mathrm{I} - B(t^*+, t^*)) = 0$. Hence there is a $d \in \mathbb{R}^n$ such that there is no $c \in \mathbb{R}^n$ such that

$$[\mathrm{I} - B(t^*+, t^*)]\, c = d.$$

Now, let us put

$$f(t) = \begin{cases} 0 & \text{for} \quad t \le t^*, \\ d & \text{for} \quad t > t^*. \end{cases}$$

By the first part of the proof, for any possible solution $x \in \mathbb{G}_L^n$ of the equation (5.6.4) on $[0,1]$ we have $x(t) = 0$ on $[0, t^*)$ and thus

$$x(t^*) = \lim_{t \to t^*-} x(t) = 0.$$

By an argument analogous to that used above we can further deduce that the limit $x(t^*+)$ of any possible solution x of (5.6.4) has to verify the relation

$$[\mathrm{I} - B(t^*+, t^*+)]\, x(t^*+) = f(t^*+) = d,$$

which contradicts the definition of d. Thus, $\mathcal{S}_B = \emptyset$ and this completes the proof of the theorem. $\square$

Corollary 5.6.6. *Under the assumptions of Theorem* 5.6.5*, the homogeneous equation* (5.6.5) *possesses only the trivial solution* $x \equiv 0$ *if and only if the relations* (5.6.6) *are satisfied.*

Proof. The sufficiency of (5.6.6) was proved in part a) of the proof of Theorem 5.6.5. The necessity follows from Proposition 5.4.3 and Theorem 5.6.5. $\square$

Similarly, the proof of the following assertion is an easy consequence of Theorems 5.5.1 and 5.5.2 and Corollary 5.6.6.

Corollary 5.6.7. *Let the assumptions of Theorem* 5.6.5 *together with* (5.6.6) *be satisfied. Then there exists a resolvent couple* $U \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ *for the equation* (5.6.4)*. The functions* U *and* V *satisfy in addition the relations*

$$U(0) = 0 \quad \textit{and} \quad V(t,s) = 0 \qquad \textit{for} \quad s \in [0,1), t \in [0,s], \tag{5.6.7}$$

$$U(t) - \int_0^t B(t,\tau)\, \mathrm{d}_\tau[U(\tau)] = A(t) \qquad \textit{for} \quad t \in [0,1], \tag{5.6.8}$$

and

$$V(t,s) - \int_0^t B(t,\tau)\, \mathrm{d}_\tau[V(\tau,s)] = B(t,s) \quad \textit{for} \quad t,s \in [0,1]. \tag{5.6.9}$$

Proof. By Theorems 5.5.1 and 5.5.2 and Corollary 5.6.6 there exists a resolvent couple $U \in \mathbb{G}_{\mathrm{L}}^{n\times n}$, $V \in \mathscr{K}_L^{n\times n}$ for the equation (5.6.4) and the functions U, V satisfy (5.5.5) and (5.5.6). Furthermore, as in virtue of (5.6.3) we have $A(0) = 0$, it follows easily from (5.5.5) that $U(0) = 0$ holds. Consequently, the relation (5.5.5) reduces to (5.6.8).

Furthermore, let $s \in (0,1)$. Since by (5.6.3) we have $B(t,s) = 0$ whenever $t \le s$, it follows that the function $V(.,s)$ fulfils the relation

$$V(t,s) = A(t)\, V(0,s) + \int_0^1 B(t,\tau)\, \mathrm{d}_\tau[V(\tau,s)] \quad \text{for all} \quad t \in [0,s].$$

By an argument analogous to that used in the first part of the proof of Theorem 5.6.5 we can deduce that $V(t,s) = 0$ for any $t \in [0,s]$. Finally, as by (5.6.3) we have $B(0,s) = 0$ for any $s \in [0,1]$, it follows from (5.5.6) that $V(0,s) = 0$ on $[0,1]$, as well. Consequently, (5.6.7) holds. Hence the relation (5.5.6) reduces to (5.6.9). $\square$

Remark 5.6.8. It is easy to verify that under the assumption of Corollary 5.6.7 the resolvent couple (U,V) of (5.6.4) satisfies in addition to the relations (5.6.7)-(5.6.9) also the following relations

$$V(t,1) \equiv 0 \text{ on } \ [0,1) \quad \text{and} \quad V(1,1) = [\mathrm{I} - B(1,1)]^{-1} B(1,1).$$

To show that the results of this section cover also the Volterra analogue of the equation mentioned in Example 5.4.9 the following three lemmas are essential.

Lemma 5.6.9. *Let $K \in \mathscr{K}^{n\times n}$ and $t \in [0,1)$. Then*

$$\begin{aligned} &\textit{for any} \ \ \varepsilon > 0 \ \ \textit{there exists} \ \ \delta \in (0, 1-t) \\ &\textit{such that} \ \ \mathrm{var}_{t_1}^{t_2} K(t_2,.) < \varepsilon \\ &\textit{holds whenever} \ \ 0 < t < t_1 < t_2 < t + \delta \le 1. \end{aligned} \tag{5.6.10}$$

Proof (due to I. Vrkoč). Let $t \in [0,1)$ be given and let us assume that there is $\gamma > 0$ and sequences $\{t_k^1\}$ and $\{t_k^2\}$ of points in $(t,1]$ such that

$$t < t_{k+1}^1 < t_{k+1}^2 < t_k^1 < t_k^2 < 1 \quad \text{holds for any} \ \ k \in \mathbb{N},$$
$$\lim_{k\to\infty} t_k^1 = \lim_{k\to\infty} t_k^2 = t \quad \text{and} \quad \mathrm{var}_{t_k^1}^{t_k^2} K(t_2^k,.) > 2\,\gamma.$$

On the other hand, by (H_2)(ii) there is $k_0 \in \mathbb{N}$ such that

$$\mathrm{var}_0^1 \left(K(t_2^k,.) - K(t_2^{k_0},.)\right) < \gamma.$$

This means that in the case that (5.6.10) does not hold we obtain

$$\begin{aligned} \mathrm{var}_0^1 K(t_2^{k_0},.) &\ge \sum_{k\ge k_0} \mathrm{var}_{t_1^k}^{t_2^k} K(t_2^{k_0},.) \\ &\ge \sum_{k\ge k_0} \left[\mathrm{var}_{t_1^k}^{t_2^k} K(t_2^k,.) - \mathrm{var}_{t_1^k}^{t_2^k} \left(K(t_2^k,.) - K(t_2^{k_0},.)\right)\right] \\ &\ge \sum_{k\ge k_0} \gamma = \infty. \end{aligned}$$

This being impossible in virtue of the assumption (H_1), it follows that the assertion (5.6.10) is true and this completes the proof of the lemma. $\square$

Analogously we could prove the following assertion.

Lemma 5.6.10. *Let $K \in \mathscr{K}^{n\times n}$ and $t \in (0,1]$. Then for any $\varepsilon > 0$ there exists a $\delta \in (0,t)$ such that $\operatorname{var}_{t_1}^{t_2} K(t_2,.) < \varepsilon$ holds whenever $0 \le t-\delta < t_1 < t_2 < t$.*

Lemma 5.6.11. *Let $K \in \mathscr{K}^{n\times n}$ and let K^Δ be given by*

$$K^\Delta(t,s) = \begin{cases} K(t,s) & \text{for } \ t\in[0,1] \ \text{ and } \ s\in[0,t], \\ K(t,t) & \text{for } \ t\in[0,1] \ \text{ and } \ s\in[t,1]. \end{cases} \tag{5.6.11}$$

Then $K^\Delta \in \mathscr{K}^{n\times n}$. Moreover, if $K \in \mathscr{K}_L^{n\times n}$ and

$$K(t,t-) = K(t,t) \quad \text{for all} \quad t \in (0,1), \tag{5.6.12}$$

then $K^\Delta \in \mathscr{K}_L^{n\times n}$, as well.

Proof. Let $t \in (0,1]$ and $\varepsilon > 0$. By our assumption and by Lemma 5.6.9 there is $\delta \in (0,t)$ such that

$$\|K(t_2,.)-K(t_1,.)\|_{\mathbb{BV}} < \frac{\varepsilon}{2} \quad \text{and} \quad \operatorname{var}_{t_1}^{t_2} K(t_2,.) < \frac{\varepsilon}{2}$$

whenever $0 \le t-\delta < t_1 \le t_2 < t$. Now, let $t_1, t_2 \in [0,1]$ be such that $t-\delta < t_1 \le t_2 < t$. Then by (5.6.11) we have

$$K^\Delta(t_2,s) - K^\Delta(t_1,s) = \begin{cases} K(t_2,s) - K(t_1,s) & \text{for } \ 0 \le s \le t_1, \\ K(t_2,s) - K(t_1,t_1) & \text{for } \ t_1 \le s \le t_2, \\ K(t_2,t_2) - K(t_1,t_1) & \text{for } \ t_2 \le s \end{cases}$$

and it is easy to see that this implies that

$$\begin{aligned} &\|K^\Delta(t_2,.) - K^\Delta(t_1,.)\|_{\mathbb{BV}} \\ &\quad \le |K(t_2,0)-K(t_1,0)| + \operatorname{var}_0^{t_1}(K(t_2,.)-K(t_1,.)) \\ &\qquad\quad + \operatorname{var}_{t_1}^{t_2}(K(t_2,.)-K(t_1,t_1)) \\ &\quad \le \|K(t_2,.)-K(t_1,.)\|_{\mathbb{BV}} + \operatorname{var}_{t_1}^{t_2} K(t_2,.) < \varepsilon \end{aligned}$$

holds for any couple $t_1, t_2 \in [0,1]$ such that $t-\delta < t_1 \le t_2 < t$. Analogously we would show that for any $\varepsilon > 0$ there exists a $\delta \in (0,t)$ such that

$$\|K^\Delta(t_2,.) - K^\Delta(t_1,.)\|_{\mathbb{BV}} < \varepsilon$$

holds for any couple $t_1, t_2 \in [0,1]$ such that $t < t_1 \le t_2 < t+\delta$, wherefrom the relation $K^\Delta \in \mathscr{K}^{n\times n}$ follows.

Furthermore, if $K^\Delta \in \mathscr{K}_L^{n\times n}$ and (5.6.12) holds, then we have

$$\begin{aligned} \lim_{\tau\to t-} \|K^\Delta(t,.) - K^\Delta(\tau,.)\|_{\mathbb{BV}} &\le \lim_{\tau\to t-} \|K(t,.)-K(\tau,.)\|_{\mathbb{BV}} \\ &\quad + \lim_{\tau\to t-} \operatorname{var}_\tau^t K(t,.) = 0 \end{aligned}$$

for any $t \in [0,1]$. $\square$

Remark 5.6.12. It follows from Lemmas 5.3.18 and 5.3.19 that, for any $K \in \mathscr{K}_L^{n\times n}$ and any $x \in \mathbb{G}_L^n$, the function

$$z(t) = \int_0^t \mathrm{d}_s[K(t,s)]\,x(s), \quad t \in [0,1],$$

is left-continuous on $(0,1)$ if and only if (5.6.12) holds.

Example 5.6.13. Let us consider the linear Volterra-Stieltjes integral equation

$$x(t) - \int_0^t \mathrm{d}_s[K(t,s)]\,x(s) = f(t), \quad t \in [0,1] \tag{5.6.13}$$

with $K \in \mathscr{K}_L^{n\times n}$ fulfilling the relation (5.6.12) and $f \in \mathbb{G}_L^n$.

Let us define the function $K^\Delta : [0,1]\times[0,1] \to \mathbb{R}^{n\times n}$ again by (5.6.11). Then by Lemma 5.6.11 we have $K^\Delta \in \mathscr{K}_L^{n\times n}$. Obviously,

$$\int_0^t \mathrm{d}_s[K(t,s)]\,x(s) = \int_0^t \mathrm{d}_s[K^\Delta(t,s)]\,x(s)$$

holds for any $x \in \mathbb{G}^n$. Let $t \in [0,1]$ and $x \in \mathbb{G}_L^n$ be given. Analogously to Example 5.4.9 we could show that then

$$\int_0^1 \mathrm{d}_s[K^\Delta(t,s)]\,x(s) = A(t)\,x(0) + \int_0^1 B(t,s)\,\mathrm{d}[x(s)],$$

where

$$A(t) = \mathrm{I} + K^\Delta(t,1) - K^\Delta(t,0) \quad \text{for} \quad t \in [0,1]$$

and

$$B(t,s) = \begin{cases} K^\Delta(t,1) - K^\Delta(t,s+) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s \in [0,1), \\ K^\Delta(t,1) - K^\Delta(t,1-) & \text{for} \quad t \in [0,1] \quad \text{and} \quad s = 1. \end{cases}$$

It is easy to verify that $A \in \mathbb{G}_{\mathrm{L}}^{n\times n}$ and $B \in \mathscr{K}_L^{n\times n}$ (cf. Lemma 5.3.15 and Lemma 5.6.11) and

$$A(t) = \mathrm{I} + K(t,t) - K(t,0) \quad \text{for} \quad t \in [0,1]$$

and

$$B(t,s) = \begin{cases} K(t,t) - K(t,s+) & \text{if } \ 0 \le s < t \le 1, \\ K(t,t) - K(t,t) & \text{if } \ 0 \le t \le s < 1, \\ K(t,t) - K(t,t) & \text{if } \ 0 \le t < s = 1, \\ K(1,1) - K(1,1-) & \text{if } \ t = s = 1. \end{cases}$$

In particular, we have

$$A(0) = 0 \quad \text{and} \quad B(t,s) = 0 \quad \text{whenever} \quad 0 \le t \le s \le 1 \quad \text{and} \quad t < 1.$$

Furthermore, for an arbitrary $t \in [0,1)$ we have

$$B(t+,t) = \lim_{\tau \to t+} (K(\tau,\tau) - K(\tau,t+)) = K(t+,t+) - K(t+,t+) = 0$$

(cf. Lemma 5.3.18). It means that under the above assumptions the Volterra-Stieltjes integral equation (5.6.13) can be converted to the causal integral equation of the form (5.6.4) whose coefficients A and B satisfy the assumptions of Corollary 5.6.7 if we assume in addition that the relation

$$\det(\mathrm{I} - (K(1,1) - K(1,1-)) \neq 0$$

is satisfied.

Acknowledgement

This work was supported by the grants No. 201/97/0218 and 201/01/1199 of the Grant Agency of the Czech Republic.

References

[1] M. Ashordia, On the correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Proc. Georgian Acad. Sci. Math.* **1**(1993), 385–394.

[2] M. Ashordia, Criteria of correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Czechoslovak Math. J.* **46(121)**(1996), 385–403.

[3] G. Aumann, Reelle Funktionen. *Springer-Verlag, Berlin,* 1969.

[4] D. Bainov and P. Simeonov, Impulsive differential equations: periodic solutions and applications. *Pitman Monographs and Surveys in Pure and Applied Mathematics* 66, *Longman Scientific, Harlow,* 1993.

[5] L. Barbanti, Linear Volterra-Stieltjes integral equations and control. *Equadiff* 82. *Proceedings, Würzburg* 1982, *Lecture Notes in Math.* 1017, 67–72, *Springer-Verlag, Berlin,* 1983.

[6] A. A. Boichuk, N. A. Perestyuk, and A. M. Samoilenko, Periodic solutions of impulsive differential equations in critical cases. (Russian) *Differentsial'nye Uravneniya* **27**(1991), 1516–1521.

[7] R. N. Bryan, A nonhomogeneous linear differential system with interface conditions and their adjoints. *Proc. Amer. Math. Soc.* **22**(1969), 270–276.

[8] R. Conti, On ordinary differential equations with interface conditions. *J. Differential Equations* **4**(1968), 4–11.

[9] L. Fichmann, Volterra-Stieltjes integral equations and equations of the neutral type. (Portuguese) *Thesis, University of Sao Paulo*, 1984.

[10] D. Fraňková, Continuous dependence on a parameter of solutions of generalized differential equations. *Časopis pěst. Mat.* **114**(1989), 230–261.

[11] D. Fraňková, Regulated functions. *Math. Bohem.* **116**(1991), 20–59.

[12] A. Halanay, Optimal control of periodic solutions. *Rev. Roumaine Math. Pures Appl.* **14**(1974), 3–16.

[13] I. Halperin, Introduction to the theory of distributions. *University of Toronto Press, Toronto,* 1952.

[14] T. H. Hildebrandt, Introduction to the theory of integration. *Academic Press, New York-London,* 1963.

[15] Ch. S. Hönig, Volterra-Stieltjes integral equations. *North Holland and American Elsevier, Mathematics Studies* 16, *Amsterdam and New York,* 1975.

[16] Ch. S. Hönig, Volterra-Stieltjes integral equations. *Functional Differential Equations and Bifurcation. Proceedings of the Sao Carlos Conference* 1979, 173–216, *Lecture Notes in Math.* 799, *Springer-Verlag, Berlin,* 1980.

[17] Ch. S. Hönig, The adjoint equation of a linear Volterra-Stieltjes integral equation with a linear constraint. *Differential Equations, Proceedings,* 1981, *Lecture Notes in Math.* 957, 118–125, *Springer-Verlag, Berlin,* 1982.

[18] H. S. Kaltenborn, Linear functional operations on functions having discontinuities of the first kind. *Bull. Amer. Math. Soc.* **40**(1934), 702–708.

[19] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. *Czechoslovak Math. J.* **7(82)**(1957), 418–449.

[20] J. Kurzweil, Generalized ordinary differential equations. *Czechoslovak Math. J.* **24(83)**(1958), 360–387.

[21] J. Kurzweil, Nichtabsolut konvergente Integrale. *BSB B. G. Teubner Verlagsgesselschaft, Leipzig,* 1980.

[22] J. Kurzweil and Z. Vorel, Continuous dependence of solutions of differential equations on a parameter. *Czechoslovak Math. J.* **23(82)**(1957), 568–583.

[23] Yu. K. Lando, Controllable integro-differential operators. (Russian) *Differentsial'nya Uravneniya* **9**(1973), 2227–2230.

[24] J. Ligęza, Weak solutions of ordinary differential equations. *Uniwersytet Śląski, Katowice,* 1986.

[25] C. Marchiò, (M,N,F)-controllabilità completa. *Questioni di controllabilità, Istituto U. Dini, Firenze* **1973/2**, 14–26.

[26] S. G. Pandit and S. G. Deo, Differential equations involving impulses. *Lecture Notes in Math.* 954, *Springer-Verlag, Berlin,* 1982.

[27] M. Pelant, On approximations of solutions of generalized differential equations. (Czech) *Dissertation, Charles University, Praha,* 1997.

[28] M. Pelant and M. Tvrdý, Linear distributional differential equations in the space of regulated functions. *Math. Bohem.* **118**(1993), 379–400.

[29] S. Rolewicz, Functional analysis and control analysis (linear systems). *Polish Scientific Publishers (PWN) and D. Reidel, Warszawa and Dordrecht,* 1987.

[30] W. Rudin, Functional analysis. *McGraw-Hill, New York,* 1973.

[31] D. L. Russell, Mathematics of finite-dimensional control systems (theory and design). *M. Dekker, New York and Basel,* 1979.

[32] S. Saks, Theory of the integral. —em Monografie Matematyczne, Warszawa, 1937.

[33] M. Schechter, Principles of functional analysis. *Academic Press, New York,* 1973.

[34] Š. Schwabik, Verallgemeinerte lineare Differentialgleichungssysteme. *Časopis Pěst. Mat.* **96**(1971), 183–211.

[35] Š. Schwabik, On an integral operator in the space of functions of bounded variation. *Časopis Pěst. Mat.* **97**(1972), 297–330.

[36] Š. Schwabik, On the relation between Young's and Kurzweil's concept of Stieltjes integral. *Časopis Pěst. Mat.* **98**(1973) 237–251.

[37] Š. Schwabik, On modified sum integral of Stieltjes type. *Časopis Pěst. Mat.* **98**(1973), 274–277.

[38] Š. Schwabik, On Volterra-Stieltjes integral equations. *Časopis Pěst. Mat.* **99**(1974), 255–278.

[39] Š. Schwabik, Differential equations with interface conditions. *Časopis Pěst. Mat.* **105**(1980) 391–408.

[40] Š. Schwabik, Generalized differential equations (fundamental results). *Rozpravy ČSAV, Řada MPV,* 95 (6). *Academia, Praha,* 1985.

[41] Š. Schwabik, Generalized ordinary differential equations. *World Scientific, Singapore,* 1992.

[42] Š. Schwabik, Linear operators in the space of regulated functions. *Math. Bohem.* **117**(1992), 79–92.

[43] Š. Schwabik and M. Tvrdý, Boundary value problems for generalized linear differential equations. *Czechoslovak Math. J.* **29(104)**(1974) 451–477.

[44] Š. Schwabik and M. Tvrdý, On linear problems in the space BV. *Equadiff IV* (*Proceedings, Prague, August* 22-26*,* 1977) (*ed. by Jiří Fábera*), *Lecture Notes in Math.* 366–373, *Springer-Verlag, Berlin-Heidelberg-New York,* 1979.

[45] Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations: boundary value problems and adjoints. *Academia and D. Reidel, Praha and Dordrecht,* 1979.

[46] M. Tvrdý, Boundary value problems for linear generalized differential equations and their adjoints. *Czechoslovak Math. J.* **23(98)**(1973) 183–217.

[47] M. Tvrdý, Boundary value problems for generalized linear integrodifferential equations with left-continuous solutions. *Časopis Pěst. Mat.* **99**(1974) 147–157.

[48] M. Tvrdý, Regulated functions and the Perron-Stieltjes integral. *Časopis Pěst. Mat.* **114**(1989) 187–209.

[49] M. Tvrdý, Generalized differential equations in the space of regulated functions (Boundary value problems and controllability). *Math. Bohem.* **116**(1991), 225–244.

[50] M. Tvrdý, Linear bounded functionals on the space of regular regulated functions. *Tatra Mt. Math. Publ.* **8**(1996), 203–210.

[51] M. Tvrdý, Linear integral equations in the space of regulated functions. *Mathem. Bohem.* **123**(1998), 177–212.

[52] M. Tvrdý, On the continuous dependence on a parameter of solutions of initial value problems for linear generalized differential equations. *Funct. Differ. Equ.* **5**(1999), 483–498.

[53] M. Tvrdý, Linear boundary value problems for generalized differential equations. *Ann. Math. Sil.* **14**(2000), 51–80.

[54] O. Vejvoda and M. Tvrdý, Existence of solutions to a linear integro-boundary-differential equation with additional conditions. *Ann. Mat. Pura Appl.* **89**(1971), 169–216.

[55] A. J. Ward, The Perron-Stieltjes integral. *Math. Z.* **41**(1936), 578–604.

[56] D. Wexler, On boundary value problems for an ordinary linear differential system. *Ann. Mat. Pura Appl.* **80**(1968) 123–134.

[57] Z. Wyderka, Periodic solutions of linear differential equations with measures as coefficients. *Acta Univ. Palack. Olomuc., Fac. Rerum Natur. Math.* **35** (1996), 199–214.

[58] S. G. Zavalishchin and A. N. Sesekin, Impulse processes, models and applications. (Russian) *Nauka, Moscow,* 1991.

[59] A. Zettl, Adjoint and self-adjoint boundary value problems with interface conditions. *SIAM J. Appl. Math.* **16**(1968), 851–859.

(Received 6.06.2001)

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