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BOUNDARY VALUE PROBLEMS VIA VECTOR FIELD AN ALTERNATIVE APPROACH

Abstract. Consider the second order nonlinear scalar differential equations

$$x'' \pm f(t,x) = 0, \quad 0 \le t \le 1, \tag{0.1}$$

where $f \in C([0,1] \times [0,\infty), [0,\infty))$, associated to the boundary conditions

$$\begin{cases} \alpha x(0) \pm \beta x'(0) = 0, \\ \gamma x(1) \mp \delta x'(1) = 0, \end{cases} \tag{0.2}$$

with $\alpha, \beta, \gamma, \delta \ge 0$, or the more general nonlinear one

$$g(x(0), x'(0)) = 0 = \ h(x(1), x'(1)). \tag{0.3}$$

Existence of positive solutions of above BVPs are given, under superlinear and/or sublinear growth in f. The approach is based on an analysis of the coresponding vector field on the (x, x') phase plane and Kneser's property of solutions funnel.

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1. Introduction

In [7], Erbe and Tang noticed that, if the boundary value problem (BVP)

$$-\Delta u = F(|x|, u) \quad \text{in} \quad R < |x| < \hat{R} \tag{1.1}$$

$$\begin{aligned} &u = 0 \ \text{ on } \ |x| = R, \quad u = 0 \ \text{ on } \ |x| = \hat{R} \ \text{ or} \\ &u = 0 \ \text{ on } \ |x| = R, \quad \frac{\partial u}{\partial |x|} = 0 \ \text{ on } \ |x| = \hat{R} \ \text{ or} \\ &\frac{\partial u}{\partial |x|} = 0 \ \text{ on } \ |x| = R, \quad u = 0 \ \text{ on } \ |x| = \hat{R} \end{aligned} \tag{1.2}$$

is radially symmetric, then it can be transformed into a scalar Sturm–Liouville one

$$x''(t) = -f(t, x(t)), \quad 0 \le t \le 1, \tag{1.3}$$

$$\begin{cases} \alpha x(0) - \beta x'(0) = 0, \\ \gamma x(1) + \delta x'(1) = 0, \end{cases} \tag{1.4}$$

where the constants $\alpha, \beta, \gamma, \delta \ge 0$.

The literature for the last BVP is voluminous. Suggestively we refer [1], [8], [9], [13], [14] and the references therein.

In [2], Bebernes and Wilhelmsen by using the shooting method, i.e., properties of the solutions funnel, studied a system of the form

$$x' = g(t, x, y), \quad y' = h(t, x, y).$$

In [3], Bebernes and Fraker obtained the existence of

$$\begin{cases} x' = f(t, x, x') \\ (0, x(0), x'(0)) \in S_1 \ \text{ and } \ (1, x(1), x'(1)) \in S_2 \end{cases}$$

for certain boundary sets S_1 and S_2. Also the requirement of nonlinear boundary constraints has been given attention among others, in [11] by Muldowney and Willett or in [9] by Jackson and Palamides. There are common ingredients in the last papers: an (assumed) Nagumo–Bernstein growth condition on the nonlinearity f or\and the presence of upper and lower solutions.

In [6], Erbe and Wang by using Green's function and Krasnoselskiĭ's fixed point theorem in cones proved the existence of a positive solution of (1.3), (1.4), under the following assumptions:

(A.1) f is continuous and positive, i.e., $f \in C([0,1] \times [0,\infty), [0,\infty))$,

(A.2*) $$\begin{cases} f_0 := \lim\limits_{x \to 0+} \max\limits_{0 \le t \le 1} \frac{f(t,x)}{x} = 0 \\ \quad \text{and} \\ f_\infty := \lim\limits_{x \to +\infty} \min\limits_{0 \le t \le 1} \frac{f(t,x)}{x} = +\infty, \end{cases}$$

i.e., f is *supelinear* at both end points $x = 0$ and $x = \infty$ or

$$(\mathrm{A}.2_*)\qquad \begin{cases} f_0 := \lim\limits_{x\to 0+} \min\limits_{0\le t\le 1} \frac{f(t,x)}{x} = +\infty \\ \quad \text{and} \\ f_\infty := \lim\limits_{x\to +\infty} \max\limits_{0\le t\le 1} \frac{f(t,x)}{x} = 0, \end{cases}$$

i.e., f is *sublinear* at both $x = 0$ and $x = \infty$, and

(A.3) $\qquad \rho := \beta\gamma + \alpha\gamma + \alpha\delta > 0.$

Erbe and Tang in [7] and Davis, Erbe and Henderson in [5] established criteria for the existence of multiple positive solutions of (1.1)–(1.2) under certain growth rate assumptions on f.

Restricting our consideration on the linear case, notice as far as the author is aware, that only the conditions (1.4) have been studied, where the constants $\alpha, \beta, \gamma, \delta \ge 0$.

It is the aim of this work to prove the existence of positive solutions for the boundary value problem (1.3), (1.5), where

$$\begin{cases} \alpha x(0) - \beta x'(0) = 0, \\ \gamma x(1) - \delta x'(1) = 0, \end{cases} \tag{1.5}$$

and still $\alpha, \beta, \gamma, \delta \ge 0$, but now under the condition

(A.3*) $\qquad \rho^* := \beta\gamma + \alpha\gamma - \alpha\delta < 0,$

and similarly, we give existence results for the boundary value problems

$$x''(t) = f(t, x(t)), \quad 0 \le t \le 1, \tag{1.6}$$

$$\begin{cases} \alpha x(0) \pm \beta x'(0) = 0, \\ \gamma x(1) \pm \delta x'(1) = 0, \end{cases} \tag{1.7}$$

where the function $f \in C([0,1] \times [0,\infty), [0,\infty))$ is superlinear or sublinear and the constants $\alpha, \beta, \gamma, \delta \ge 0$ are chosen so that

$$\hat{\rho} := \beta\gamma - \alpha\gamma - \alpha\delta < 0.$$

We furthermore investigate nonlinear boundary conditions of the form

$$g(x(0), x'(0)) = 0, \quad h(x(1), x'(1)) = 0, \tag{1.8}$$

where g and h have an "asymptotic behavior" similar to the above linear functions appearing in (1.5) or (1.7). In these cases the above mentioned Green's function seems not to exists or at least it is not always nonnegative. This possibly makes Erbe and Wang's method not applicable to those cases.

Remark 1.1. We notice here that the differential equation (1.3) (or (1.6)) defines a vector field, the properties of which will be crucial for our study.

More specifically, let's look at the (x, x') phase semi-plane $(x > 0)$. By the sign condition on f (see assumption (A.1)) we immediately see that $x'' < 0$. Thus any trajectory $(x(t), x'(t))$, $t \ge 0$, emanating from the semi-line

$$E_0 := \{(x, y) : \alpha x - \beta y = 0,\ x > 0\}$$

"trends" in a natural way, initially (when $x'(t) > 0$) toward the positive $x-$semi-axis and then (when $x'(t) < 0$) turns toward the semi-line

$$E_1 := \{(x, y) : \gamma x - \delta y = 0,\ x > 0\}.$$

Finally, by setting a certain growth rate on f (say superlinearity) we can control the vector field, so that some trajectory reaches E_1 at the time $t = 1$.

These properties will be referred as "*The nature of the vector field*" throughout the rest of the paper.

So the technique presented here is different from that given in the above mentioned papers [6], [5] or [7], but it is closely related with [2], [3] or [11]. Actually, we relay on the above "nature of the vector field" and the Kneser's property (continuum) of the cross-sections of the solutions funnel.

Finally, we cite for completeness the well-known Kneser's theorem (see for example the Copel's text-book [4]).

Theorem 1.2. *Consider a system* $(*)$ $x' = f(t, x)$, $(t, x) \in \Omega := [\alpha, \beta] \times \mathbb{R}^n$, *with f continuous. Let* $\hat{E}_0$ *be a continuum* (*compact and connected*) *in* $\Omega_0 := \{(t, x) \in \Omega : t = \alpha\}$ *and let* $\mathcal{X}(\hat{E}_0)$ *be the family of all solutions of* $(*)$ *emanating from* $\hat{E}_0$. *If any solution* $x \in \mathcal{X}(\hat{E}_0)$ *is defined on the interval* $[\alpha, \tau]$, *then the set* (cross-section)

$$\mathcal{X}(\tau; \hat{E}_0) := \left\{x(\tau) : x \in \mathcal{X}(\hat{E}_0)\right\}$$

is a continuum in $\mathbb{R}^n$.

2. Main Results

For technical reasons and since readers are more familiar with boundary conditions (1.4), we prefer to re-establish first some well-known existence results for the problem (1.3)–(1.4) and then (see Theorems 2.3, 3.1 and 3.3) we exhibit our main results.

Theorem 2.1. *Assume* (A.1) *and* (A.3) *hold. Then the boundary value problem* (1.3), (1.4) *has a positive solution provided that the function f is sublinear* (*see* (A.2$_*$)) *or superlinear* (*see* (A.2*)). *Furthermore there exists* $0 < \eta_0 < H$ *such that*

$$\eta_0 \le x(t) \le H, \quad 0 \le t \le 1,$$

for any such solution.

Proof. To begin with, let's study first the

1) *Superlinear case.* Since $f_\infty = +\infty$, for any $K > \max\left\{\frac{2\alpha}{\beta},\ \frac{2\rho}{\beta(\gamma+2\delta)}\right\}$ there exists $H > 0$ such that

$$\min_{0 \le t \le 1} f(t, x) > Kx, \quad x \ge H. \tag{2.1}$$

Consider an arbitrary point $P := (x_0, y_0) \in E_0$ with $x_0 \geq H$ and let $x \in \mathcal{X}(P)$ be any solution of the differential equation (1.3) starting at the point P. By the assumption (A.1) (i.e., the nature of the vector field, see Remark 1.1) it is obvious that $x(t) \geq x_0$ for all t in a sufficiently small neighborhood of $t = 0$.

Let's suppose that there is $t^* \in (0, 1]$ such that

$$x(t) \geq x_0, \quad 0 \leq t < t^* \quad \text{and} \quad x(t^*) = x_0.$$

Then since $P \in E_0$, by the Taylor formula we get $\bar{t} \in [0, t^*]$ such that

$$x(t^*) \leq x_0\Big[1 + \frac{\alpha}{\beta}\Big] - \frac{1}{2} f(\bar{t}, x(\bar{t})), \tag{2.2}$$

and thus

$$x_0 \frac{2\alpha}{\beta} \geq f(\bar{t}, x(\bar{t})).$$

But since $x(\bar{t}) \geq x_0 \geq H$, by (2.1) we have

$$f(\bar{t}, x(\bar{t})) \geq \min_{0 \leq t \leq 1} f(\bar{t}, x(\bar{t})) \geq K x(\bar{t})) \geq K x_0,$$

and so we obtain

$$x_0 \frac{2\alpha}{\beta} \geq K x_0$$

contrary to the choice $K > \frac{2\alpha}{\beta}$. Furthermore, by (2.2) we get the estimate

$$x_0 \leq x(t) < x_0\Big[1 + \frac{\alpha}{\beta}\Big], \quad 0 \leq t \leq 1. \tag{2.3}$$

Now by the Taylor formula, for some $\hat{t}, t^* \in [0, 1]$ we have

$$\begin{aligned} x(1) &= x_0 + y_0 - \frac{1}{2} f(\hat{t}, x(\hat{t})), \\ x'(1) &= y_0 - f(t^*, x(t^*)), \end{aligned}$$

and since $P = (x_0, y_0) \in E_0$, we get

$$\begin{cases} x(1) = x_0[1 + \frac{\alpha}{\beta}] - \frac{1}{2} f(\hat{t}, x(\hat{t})) \;\text{ and} \\ x'(1) = x_0 \frac{\alpha}{\beta} - f(t^*, x(t^*)). \end{cases}$$

In order to verify that $(x(1), x'(1)) \in E_1$, consider the function

$$G(P) := \gamma x(1) + \delta x'(1).$$

Then we have

$$G(P) = x_0\Big[\frac{\rho}{\beta} - \frac{\gamma}{2} \frac{f(\hat{t}, x(\hat{t}))}{x_0} - \delta\, \frac{f(t^*, x(t^*))}{x_0}\Big], \tag{2.4}$$

where we recall that $\rho := \beta\gamma + \alpha\gamma + \alpha\delta > 0$. By (2.3), we get $H \le x_0 \le \min\{x(\hat{t}), x(t^*)\}$ and thus in view of (2.1),

$$G(P) \le x_0 \Big[\frac{\rho}{\beta} - \frac{\gamma}{2}\,\frac{Kx_0}{x_0} - \delta\,\frac{Kx_0}{x_0}\Big] = x_0 \Big[\frac{\rho}{\beta} - \frac{(\gamma+2\delta)K}{2}\Big].$$

So by the choice of $K > \frac{2\rho}{\beta(\gamma+2\delta)}$, we conclude that

$$G(P_1) < 0, \quad P_1 := (x_1, y_1) \in E_0 \;\text{ with }\; x_1 \ge H. \tag{2.5}$$

Similarly, by the superlinearity of $f(t,x)$ at $x = 0$, for any $\mu > 0$ there is an $\eta > 0$ such that

$$0 < x \le \eta \quad \text{implies} \quad \max_{0\le t\le 1} f(t,x) < \mu x. \tag{2.6}$$

Consider any positive number $\varepsilon < \frac{\beta}{\alpha+\beta}$ and choose

$$\mu < \min\Big\{\frac{2\varepsilon\rho}{\beta(\gamma+2\delta)}, 2\Big[1 - \varepsilon\frac{\alpha+\beta}{\beta}\Big]\Big\}. \tag{2.7}$$

We will show that for any $P = (x_0, y_0) \in E_0$ with $x_0 = \varepsilon\eta$ we have

$$\varepsilon\eta \le x(t) \le \eta, \quad 0 \le t \le 1. \tag{2.8}$$

Indeed, in view of (1.1) let's assume that there exists $t^* \in (0,1]$ such that

$$\varepsilon\eta \le x(t) \le \eta, \;\; 0 \le t < t^*, \;\text{ and }\; x(t^*) = \eta. \tag{2.9}$$

Then by the Taylor formula, assumption (A.1), (2.6) and (2.9), we get $\bar{t} \in (0, t^*)$ such that

$$\begin{aligned} \eta = x(t^*) &\le x_0\Big[1 + \frac{\alpha}{\beta}\Big] - \frac{1}{2}\,f(\bar{t}, x(\bar{t})) \le \\ &\le \varepsilon\eta\Big[1 + \frac{\alpha}{\beta}\Big] + \frac{1}{2}\,\mu x(\bar{t}) \le \varepsilon\eta\Big[1 + \frac{\alpha}{\beta}\Big] + \frac{1}{2}\,\mu\eta. \end{aligned}$$

Consequently we obtain

$$\mu \ge 2\Big[1 - \varepsilon\,\frac{\alpha+\beta}{\beta}\Big]$$

contrary to the choice of μ in (2.7).

Consider now the function $G(P)$ defined in (2.4) and then by (2.6) and (2.8) we get

$$\begin{aligned} G(P) &= x_0\Big[\frac{\rho}{\beta} - \frac{\gamma}{2}\,\frac{f(\hat{t}, x(\hat{t}))}{x_0} - \delta\,\frac{f(t^*, x(t^*))}{x_0}\Big] \ge \\ &\ge \varepsilon\eta\,\frac{\rho}{\beta} - \frac{\gamma}{2}\,\mu x(\hat{t}) - \delta\mu x(t^*) \ge \\ &\ge \varepsilon\eta\,\frac{\rho}{\beta} - \frac{\gamma}{2}\,\mu\eta - \delta\mu\eta = \varepsilon\eta\,\frac{\rho}{\beta} - \mu\eta\,\frac{(\gamma+2\delta)}{2}. \end{aligned}$$

Thus by (2.7) we conclude that there is a point $P_0 = (x_0, y_0) \in E_0$ (with $x_0 = \varepsilon\eta < \frac{\beta}{\alpha+\beta}\eta$) such that

$$G(P_0) > 0. \tag{2.10}$$

Finally consider the segment

$$[P_0, P_1] := \{(x, y) \in E_0 : x_0 \leq x \leq x_1\}$$

and furthermore the cross-section

$$\mathcal{X}(1; [P_0, P_1]) := \{(x(1), x'(1)) : x \in \mathcal{X}(P) \text{ with } P \in [P_0, P_1]\}$$

of the solutions funnel emanating from the segment $[P_0, P_1]$. By the definition of the function G, (2.5) and (2.10), it is clear that

$$E_1 \cap \mathcal{X}(1; [P_0, P_1]) \neq \varnothing$$

and this means that there is a point $P \in [P_0, P_1]$ such that $G(P) = 0$ and so a solution $x \in \mathcal{X}(P)$ which satisfies our boundary value problem (1.3)–(1.4).

Moreover, by the nature of the vector field (1.1), there is $t_P \in (0, 1)$ such that x is strictly increasing on $[0, t_p]$, strictly decreasing on $[t_p, 1]$ and further is strictly positive on $[0, 1]$. So it is clear that

$$||x|| : \max_{0\leq t\leq 1} x(t) = x(t_P) > \varepsilon\eta.$$

Also it holds $x(t) \leq H,\ 0 \leq t \leq 1$, i.e.,

$$\varepsilon\eta \leq ||x|| \leq H.$$

Indeed, let's assume that there exist $t_0, t_1 \in (0, 1)$ such that

$$x(t) \leq H,\ 0 \leq t < t_0,\ x(t_0) = H \text{ and } x(t) > H,\ t_0 \leq t \leq t_1.$$

Then by the nature of vector field, we have $0 < x'(t_0) < \frac{\alpha}{\beta}x(t_0) = \frac{\alpha}{\beta}H$ and further for some $\bar{t} \in (t_0, t_1)$

$$\begin{aligned} H < x(t_1) &= x(t_0) + (t_1 - t_0)x'(t_0) - \frac{1}{2} f(\bar{t}, x(\bar{t})) \leq \\ &\leq H\Big[1 + \frac{\alpha}{\beta}\Big] - \frac{K}{2}\, x(\bar{t}) \leq H\Big[1 + \frac{\alpha}{\beta}\Big] - \frac{K}{2}\, H. \end{aligned}$$

Thus we get the final contradiction $K < \frac{2\alpha}{\beta}$ and this ends the proof for the superlinear case. In the sequel we will study the

2) *Sublinear case.* We choose any $\varepsilon > \frac{\alpha+\beta}{\beta}$. Since $f_\infty = 0$, for any

$$\eta < \min\Big\{\frac{\alpha}{\varepsilon\beta},\ 2\Big[\varepsilon - \frac{\alpha+\beta}{\beta}\Big],\ \frac{2\rho}{\varepsilon\beta(\gamma + 2\delta)}\Big\}$$

there exists $H > 0$ such that

$$\max_{0\leq t\leq 1} f(t, x) < \eta x, \quad x \geq H. \tag{2.11}$$

Let's consider a point $P := (x_0, y_0) \in E_0$ with $x_0 = H$. We will prove first that for any solution $x \in \mathcal{X}(P)$

$$H \le x(t) \le \varepsilon H, \quad 0 \le t \le 1. \tag{2.12}$$

Let's suppose that's not the case. Then by the nature of the vector field, there is $t^* \in (0,1)$ such that either

- $H \le x(t) \le \varepsilon H,\ \ 0 < t < t^*$ and $x(t^*) = \varepsilon H$,
  and then by the Taylor formula we get $\bar{t} \in [0, t^*]$ such that

$$\varepsilon H = x(t^*) \le x_0\Big[1 + \frac{\alpha}{\beta}\Big] - \frac{1}{2}\, f(\bar{t}, x(\bar{t})) <$$
$$< H\Big[1 + \frac{\alpha}{\beta}\Big] + \frac{1}{2}\,\eta x(\bar{t}) \le H\Big[1 + \frac{\alpha}{\beta}\Big] + \frac{1}{2}\eta\varepsilon H$$

  and hence the contradiction

$$\eta > 2\Big[\varepsilon - \frac{\alpha+\beta}{\beta}\Big];$$

  or
- $H \le x(t),\ \ 0 < t < t^*$ and $x(t^*) = H$.
  We assert that $x'(t) > 0,\ 0 \le t \le 1$ and so it can not be true. The last assertion holds, since by (2.11)–(2.12) we get

$$x'(t) = y_0 - f(\hat{t}, x(\hat{t})) \ge \frac{\alpha}{\beta} x_0 - \eta x(\hat{t}) \ge$$
$$\ge \frac{\alpha}{\beta} H - \eta\varepsilon H > 0, \ \ 0 \le t \le 1,$$

  because $\eta < \frac{\alpha}{\varepsilon\beta}$.

Now by (2.11)–(2.12), we have

$$\begin{aligned} G(P) &= x_0\Big[\frac{\rho}{\beta} - \frac{\gamma}{2}\,\frac{f(\hat{t}, x(\hat{t}))}{x_0} - \delta\,\frac{f(t^*, x(t^*))}{x_0}\Big] \ge \\ &\ge H\frac{\rho}{\beta} - \frac{\gamma}{2}\eta x(\hat{t}) - \delta\eta x(t^*) \ge \\ &\ge H\frac{\rho}{\beta} - \eta\varepsilon H\left[\frac{\gamma}{2} + \delta\right]. \end{aligned}$$

Consequently, since $\eta < \frac{2\rho}{\varepsilon\beta(\gamma+2\delta)}$, for every $P_1 = (x_1, y_1) \in E_0$ with $x_1 = H$, we obtain

$$G(P_1) > 0. \tag{2.13}$$

On the other hand, since $f_0 = +\infty$, for any $K > \max\left\{\frac{2(\alpha-\beta)}{\beta},\ \frac{\rho}{\beta(\gamma+2\delta)}\right\}$ there exists $\eta > 0$ such that

$$\min_{0\le t\le 1} f(t,x) > Kx, \quad 0 < x \le \eta. \tag{2.14}$$

As above, by the Taylor formula, (2.14) and the assumption $K > \frac{2(\alpha-\beta)}{\beta}$ we can prove that for $P_0 := (x_0, y_0) \in E_0$ with $x_0 = \frac{\eta}{2}$, we have

$$\frac{\eta}{2} \le x(t) \le \eta, \quad 0 \le t \le 1. \tag{2.15}$$

Further we get

$$\begin{aligned} G(P_0) = x_0\Big[\frac{\rho}{\beta} - \frac{\gamma}{2}\,\frac{Kx(\hat{t})}{x_0} - \delta\frac{Kx(t^*)}{x_0}\Big] \le \\ \le \eta\Big[\frac{\rho}{2\beta} - \frac{\gamma}{2}\,\frac{K}{2} - \delta\,\frac{K}{2}\Big] \end{aligned} \tag{2.16}$$

and so by the choice $K > \frac{\rho}{\beta(\gamma+2\delta)}$, we conclude that

$$G(P_0) < 0, \ \ P_0 := (x_0, y_0) \in E_0 \ \text{ with } \ x_0 = \frac{\eta}{2}.$$

Consequently by this and (2.13), we obtain the desired solution. $\square$

Remark 2.2. By the above given proof of (2.5), this inequality is obviously independent of the sign of the quantity $\rho = \beta\gamma + \alpha\gamma + a\delta$ (and so of the assumption (A.3)).

A similar observation can be done for the inequality (2.16).

Theorem 2.3. *Assume that* (A.1) *and* (A.3*) *hold. Then the boundary value problem* (1.3), (1.5) *has a positive solution provided that the function* f *is sublinear or superlinear.*

Proof. 1) *Superlinear case.* As in the proof of the previous Theorem 2.1 (see Remark 2.2), we can see that for any $K > \max\left\{\frac{2\alpha}{\beta}, \ \frac{2\rho}{\beta(\gamma+2\delta)}\right\}$ there exists $H > 0$ such that for all $P := (x_0, y_0) \in E_0$ with $x_0 \ge H > 0$ and $x \in \mathcal{X}(P)$,

$$x_0 \le x(t) \le x_0\Big[1 + \frac{\alpha}{\beta}\Big], \ \ 0 \le t \le 1$$

and further

$$G(P) = \gamma x(1) + \delta x'(1) < 0.$$

Consequently $-x'(1) > \frac{\gamma}{\delta}x(1)$ and then defining the function

$$G^*(P) := \gamma x(1) - \delta x'(1),$$

we immediately get

$$G^*(P) > 2\gamma x(1) > 0, \ \ P \in E_0 \ \text{ with } \ x_0 \ge H. \tag{2.17}$$

Now, by the assumption $f_0 = 0$, for an $\varepsilon < \frac{\beta}{\alpha+\beta}$ and any small enough μ (i.e., $\mu < \min\left\{-\frac{\varepsilon\rho^*}{2\beta\delta)}, 2\left[1 - \varepsilon\frac{\alpha+\beta}{\beta}\right]\right\}$, see (2.6)–(2.9)), there is an $\eta > 0$ such that for any $P \in E_0$ with $x_0 = \varepsilon\eta$ we have

$$0 < \varepsilon\eta \le x(t) \le \eta, \quad 0 \le t \le 1.$$

Consequently, by (2.6) and assumption (A.1) we get

$$G^*(P) = x_0\Big[\frac{\rho^*}{\beta} - \frac{\gamma}{2}\,\frac{f(\hat{t},x(\hat{t}))}{x_0} + \delta\,\frac{f(t^*,x(t^*))}{x_0}\Big] \le$$
$$\le \varepsilon\eta\,\frac{\rho^*}{\beta} + \delta\,f(t^*,x(t^*)) \le$$
$$\le \varepsilon\eta\,\frac{\rho^*}{\beta} + \delta\mu x(t^*) \le \varepsilon\eta\,\frac{\rho^*}{\beta} + \delta\mu\eta < 0,$$

given that

$$\mu < -\frac{\varepsilon\rho^*}{\beta\delta}.$$

Thus the existence follows once again, by (2.17) and Kneser's property.

2) *Sublinear case.* Assume that $f_\infty = 0$. Then in view of (2.11)–(2.12), for an $\varepsilon > \frac{\alpha+\beta}{\beta}$, any $\eta < \min\left\{\frac{\alpha}{\varepsilon\beta},\ 2\left[\varepsilon - \frac{\alpha+\beta}{\beta}\right], -\frac{\rho^*}{\varepsilon\beta\delta}\right\}$, and $x_0 = H$, we easily get (see (2.13) and Remark 2.2) that

$$G^*(P) = x_0\Big[\frac{\rho^*}{\beta} - \frac{\gamma}{2}\,\frac{f(\hat{t},x(\hat{t}))}{x_0} + \delta\,\frac{f(t^*,x(t^*))}{x_0}\Big] \le$$
$$\le H\,\frac{\rho^*}{\beta} + \delta\eta x(t^*) = H\,\frac{\rho^*}{\beta} + \delta\eta\varepsilon H < 0, \qquad (2.18)$$

given that $\eta < -\frac{\rho^*}{\varepsilon\beta\delta}$.

On the other hand, by the assumption $f_0 = \infty$ and previous results (see (2.15)), for any $K > \max\left\{\frac{2\alpha}{\beta},\ \frac{2\rho}{\beta(\gamma+2\delta)}\right\}$ there exists $\eta > 0$ such that for the point $P_0 = (x_0, y_0) \in E_0$ with $x_0 = \frac{\eta}{2}$, we have

$$0 < \frac{\eta}{2} \le x(t) \le \eta, \quad 0 \le t \le 1,$$

and further $G(P_0) < 0$. Hence, as in (2.17) we readily get

$$G^*(P_0) > 0.$$

Thus the desired result follows from (2.18). □

3. More Results

Consider now the boundary value problem

$$x''(t) = f(t, x(t)), \quad 0 \le t \le 1, \qquad (3.1)$$

$$\begin{cases} \alpha x(0) + \beta x'(0) = 0, \\ \gamma x(1) + \delta x'(1) = 0, \end{cases} \qquad (3.2)$$

where we still assume that the function $f \in C([0,1] \times [0,\infty), [0,\infty))$ is superlinear or sublinear, the constants $\alpha, \beta, \gamma, \delta \ge 0$ are such that

$$\hat{\rho} := \beta\gamma - \alpha\gamma - \alpha\delta < 0. \qquad (3.3)$$

Theorem 3.1. *Assume* (A.1), $(A.2_*)$ (*or* $(A.2^*)$) *and* (3.3) *hold. Then the boundary value problem* (3.1), (3.2) *has a positive solution.*

Proof. We will study only the *superlinear case.* So since $f_\infty = \infty$, for any $K > -\frac{4\hat\rho}{\beta(\gamma+2\delta)}$ there exists $H > 0$ such that for any

$$P_0 := (x_0, y_0) \in E_0^* := \{(x,y) : \alpha x + \beta y = 0,\ x \geq 0\}$$

with $x_0 = 2H$ and any $x \in \mathcal{X}(P_0)$,

$$H \leq x(t) \leq 2H, \quad 0 \leq t \leq 1,$$

and further

$$\begin{aligned} G(P_0) = x_0 \Big[\frac{\hat\rho}{\beta} + \frac{\gamma}{2}\,\frac{f(\hat t, x(\hat t))}{x_0} + \delta\,\frac{f(t^*, x(t^*))}{x_0} \Big] \geq \\ \geq 2H\,\frac{\hat\rho}{\beta} + \frac{\gamma}{2} KH + \delta KH > 0. \end{aligned} \tag{3.4}$$

On the other hand, since $f_0 = 0$, we can easily prove that for any $\mu < \min\left\{\frac{\alpha+\beta}{\beta}, -\frac{\hat\rho}{\beta(\gamma+2\delta)}\right\}$ there exists $\eta > 0$ such that for all $P_1 := (x_1, y_1) \in E_0^*$ with $x_1 = \frac{\eta}{2}$ and any $x \in \mathcal{X}(P_1)$,

$$\frac{\eta}{2} \leq x(t) \leq \eta, \quad 0 \leq t \leq 1,$$

and further

$$\begin{aligned} G(P_1) = x_0 \Big[\frac{\hat\rho}{\beta} + \frac{\gamma}{2}\,\frac{f(\hat t, x(\hat t))}{x_0} + \delta\,\frac{f(t^*, x(t^*))}{x_0} \Big] \leq \\ \leq \eta\,\frac{\hat\rho}{2\beta} + \frac{\gamma}{2}\,\mu\eta + \delta\mu\eta < 0. \end{aligned}$$

Hence the existence result follows, by (3.4). $\square$

Remark 3.2. In the same spirit, one can study the symmetric boundary value problem (3.1)–(3.5), where

$$\begin{cases} \alpha x(0) - \beta x'(0) = 0, \\ \gamma x(1) - \delta x'(1) = 0. \end{cases} \tag{3.5}$$

We end this work by establishing an existence result for the nonlinear BVP

$$x''(t) = -f(t, x(t)), \quad 0 \leq t \leq 1, \tag{3.6}$$

$$g(x(0), x'(0)) = 0, \quad h(x(1), x'(1)) = 0. \tag{3.7}$$

Actually, we only pattern the linear conditions (1.4), although we could study more cases from those given above. So, for the functions g and h, we assume that

$$
\text{(C.1)} \quad \begin{cases} \text{the graph of } g(x,y)=0 \text{ is a (continuous) curve which can} \\ \text{be parametrized } x = p_0(\gamma) > 0,\ y = q_0(\gamma) > 0,\ \gamma \in \mathbb{R}, \\ \text{where } p,\ q \text{ are continuous and} \\ \lim\limits_{\gamma\to-\infty} p_0(\gamma) = 0+,\ \lim\limits_{\gamma\to-\infty} q_0(\gamma) = 0+ \text{ and} \\ \lim\limits_{\gamma\to+\infty} p_0(\gamma) = +\infty,\ \lim\limits_{\gamma\to+\infty} q_0(\gamma) = +\infty \end{cases}
$$

and similarly

$$
\text{(C.2)} \quad \begin{cases} \text{the graph of } h(x,y)=0 \text{ is also a curve:} \\ x = p_1(\gamma) > 0,\ y = q_1(\gamma) < 0,\ \gamma \in \mathbb{R}, \text{ with} \\ \lim\limits_{\gamma\to-\infty} p_1(\gamma) = 0+,\ \lim\limits_{\gamma\to-\infty} q_1(\gamma) = 0- \text{ and} \\ \lim\limits_{\gamma\to+\infty} p_0(\gamma) = +\infty,\ \lim\limits_{\gamma\to+\infty} q_0(\gamma) = -\infty. \end{cases}
$$

Then it is clear that the graphs

$$
G(g) := \{(x,y) : g(x,y) = 0,\ x > 0\}\,.\ G(g) := \{(x,y) : h(x,y) = 0,\ x > 0\}
$$

contain continua E_0 and E_1, respectively, such that

$$
(0,0) \in \bar{E}_0 \cap \bar{E}_1 \tag{3.8}
$$

and for any $K_i > 0$ and $\Lambda_i > 0,\ (i = 0,1)$

$$
\begin{aligned} E_0 \cap \{(x,y) :\ x > K_0,\ y > \Lambda_0\} \neq \varnothing \ \text{ and} \\ E_1 \cap \{(x,y) :\ x > K_1,\ y < -\Lambda_1\} \neq \varnothing. \end{aligned} \tag{3.9}
$$

As in (2.4), define the function

$$
G_h(P) := h(x(1), y(1)),\ \ x \in \mathcal{X}(P) \ \text{ and } \ P \in E_0,
$$

and then following the proof of Theorem 2.1, we can easily show that there exist P_0 and $P_1 \in E_0$ such that

$$
G_h(P_0) G_h(P_1) < 0.
$$

Since E_1 is a continuum, by (3.8) and (3.9) we obtain a point $P \in E_0$ such that $G_h(P) = 0$.

So we have arrived to the next general result

Theorem 3.3. *Assume* (A.1), (C.1), (C.2) *and* ($A.2_*$) (*or* ($A.2^*$)) *hold. Then the boundary value problem* (3.6)–(3.7) *has a positive solution.*

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