Mem. Differential Equations Math. Phys. 21(2000), 138–141

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LOCAL REPRESENTATIONS FOR THE VARIATION OF SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS

(Reported on March 13, 2000)

1. Let $J = [a, b]$ be a finite interval; $O \subset \mathbb{R}^n$ be an open set; E be a space of functions $f : J \times O^s \to \mathbb{R}^n$ satisfying the conditions:

1) for a fixed $t \in J$ the function $f(t, x_1, \dots, x_s)$ is continuously differentiable with respect to $(x_1 \dots, x_s) \in O^s$; 2) for a fixed $(x_1, \dots, x_s) \in O^s$ the functions f, f_{x_i}, $i = 1, \dots, s$ are measurable with respect to $t \in J$; for an arbitrary compact $K \subset O$ there exists a function $m_{f,K}(\cdot) \in L(J, \mathbb{R}_0^+)$, $\mathbb{R}_0^+ = [0, \infty)$, such that

$$|f(t, x_1, \dots, x_s)| + \sum_{i=1}^{s} |f_{x_i}(\cdot)| \le m_{f,K}(t), \quad \forall (t, x_1, \dots, x_s) \in J \times K^s.$$

Let now $\tau_i(t)$, $i = 1, \dots, s$, $t \in J$, be absolutely continuous functions satisfying the conditions: $\tau_i(t) \le t$, $\dot{\tau}_i(t) > 0$; Δ be a space of piecewise continuous functions $\varphi : J_1 = [\tau, b] \to O$, $\tau = \min\{\tau_1(a), \dots, \tau_s(a)\}$, with a finite number of discontinuity points of the first kind, satisfying the conditions: $cl\{\varphi(t) : t \in J_1\}$ is a compact lying in O; $\|\varphi(t)\| = \sup\{|\varphi(t)| : t \in J_1\}$.

To every element $\mu = (t_0, x_0, \varphi, f) \in A = [a, b) \times O \times \Delta \times E$ there corresponds the delay differential equation

$$\dot{x}(t) = f(t, x(\tau_1(t), \dots, x(\tau_s(t))), \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\tau, t_0), \quad x(t_0) = x_0. \tag{2}$$

Definition 1. The function $x(t) = x(t; \mu) \in O, t \in [\tau, t_1], t_1 \in (a, b], t_0 < t_1$ is said to be *a solution corresponding to the element* $\mu \in A$, defined on $[\tau, t_1]$, if the function $x(t)$ on the interval $[\tau, t_0]$ satisfies the condition (2), while on the interval $[t_0, t_1]$ it is absolutely continuous and satisfies the equation (1) almost everywhere.

Introduce the set $V = \{\delta\mu = (\delta t_0, \delta x_0, \delta\varphi, \delta f) \in A - \mu : |\delta t_0| \le c = const, |\delta x_0|$'$ec, \|\delta\varphi\| \le c, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, |\lambda_i| \le c, i = 1, \dots, k\}$, where $\mu = (t_0, x_0, \varphi, f) \in A; \delta f_i \in E - \tilde{f}, i = 1, \dots, k$ are fixed points. By a standard way it is proved that if $x(t)$ is the solution corresponding to the element $\tilde{\mu}$, defined on $[\tau, \tilde{t}_1], \tilde{t}_1 < b$. Then there exist numbers $\varepsilon_0 > 0$, $\delta_0 > 0$ such that for an arbitrary $(\varepsilon, \delta\mu) \in [0, \varepsilon] \times V$ to the element $\tilde{\mu} + \varepsilon\delta\mu \in A$ there corresponds the solution $x(t; \varepsilon\delta\mu)$ defined on $[\tau, \tilde{t}_1 + \delta_0] \subset J_1$. It is obvious that the solution $x(t; 0)$, $t \in [\tau, \tilde{t}_1 + \delta_0]$ is a continuation of the solution $\tilde{x}(t)$ in the sequel assumed to be defined on the whole interval $[\tau, \tilde{t}_1 + \delta_0]$.

The above presented discussion allows us to introduce the function

$$\Delta x(t; \varepsilon\delta\mu) = x(t; \varepsilon\delta\mu) - \tilde{x}(t), \quad (t, \varepsilon, \delta\mu) \in [\tau, \tilde{t}_1 + \delta_0] \times [0, \varepsilon_0] \times V.$$

2000 *Mathematics Subject Classification.* 34K15.

Key words and phrases. Delay differential equations, variation of solution.

The function $\Delta x(t;\varepsilon\delta\mu)$ is called the variation of the solution $\tilde{x}(t)$. In order to formulate the main results, we will need the following notation:

$$\left.\begin{array}{c}
\omega_i^- = (\tilde{t}_0, \underbrace{\tilde{x}_0,\dots,\tilde{x}_0}_{\text{i-times}}, \underbrace{\tilde{\varphi}(\tilde{t}_0-)\dots,\tilde{\varphi}(\tilde{t}_0-)}_{\text{(p-i)-times}}, \tilde{\varphi}(\tau_{p+1}(\tilde{t}_0-)),\dots,\tilde{\varphi}(\tau_s(\tilde{t}_0-))),\\
i=0,\dots,p;\\
\omega_i^- = (\gamma_i, \tilde{x}(\tau_1(\gamma_i)),\dots,\tilde{x}(\tau_{i-1}\gamma_i)), \tilde{x}_0, \tilde{\varphi}(\tau_{i+1}(\gamma_i-)),\dots,\tilde{\varphi}(\tau_s(\gamma_i-))),\\
\overset{\circ}{\omega}{}_i^- = (\gamma_i, \tilde{x}(\tau_1(\gamma_i)),\dots,\tilde{x}(\tau_{i-1}(\gamma_i)), \tilde{\varphi}(\tilde{t}_0-), \tilde{\varphi}(\tau_{i+1}(\gamma_i-)),\dots,\tilde{\varphi}(\tau_s(\gamma_i-))),\\
i=p+1,\dots,s;\ \gamma_i=\gamma_i(\tilde{t}_0),\ \dot{\gamma}_i^- = \dot{\gamma}_i(\tilde{t}_0-),\ i=1,\dots,s;
\end{array}\right\} \tag{3}$$

$\gamma_i(t)$ is the function inverse to $\tau_i(t)$.

$$\left.\begin{array}{c}
\lim_{\omega\to\omega_i^-} \tilde{f}(\omega) = f_i^-,\ \omega=(t,x_1,\dots,x_s)\in R_{\tilde{t}_0}^- \times O^s,\ i=0,\dots,p,\\
R_{\tilde{t}_0}^- = (-\infty,\tilde{t}_0];\quad \lim_{(\omega_1,\omega_2)\to(\omega_i^-,\overset{\circ}{\omega}{}_i^-)} [\tilde{f}(\omega_1)-\tilde{f}(\omega_2)] = f_i^-,\\
\omega_1,\omega_2\in R_{\gamma_i}^-\times O^s,\ i=p+1,\dots,s.
\end{array}\right\} \tag{4}$$

Theorem 1. *Let $\gamma_i=\tilde{t}_0$, $i=1,\dots,p$, $\tilde{t}_0<\gamma_{p+1}<\dots<\gamma_s<\tilde{t}_1$, there exist the finite limits: f_i^-, $i=0,\dots,s$; $\dot{\gamma}_i^-$, $i=1,\dots,s$, there exist a left semi-neighborhood $V^-(\tilde{t}_0)$ of the point $\tilde{t}_0$ such that*

$$t\le\gamma_1(t)\le\dots\le\gamma_(t),\quad \forall t\in V^-(\tilde{t}_0). \tag{5}$$

Then there exist numbers $\varepsilon_1\in(0,\varepsilon_0],\delta_1\in(0,dl_0]$ such that for an arbitrary $(t,\varepsilon,\delta\mu)\in[\tilde{t}_1-\delta_1,\tilde{t}_1+\delta_1]\times V^-;V^-=\delta\mu\in V:\delta t_0\le 0$, the formula

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu), \tag{6}$$

is valid, where

$$\delta x(t;\delta\mu)=$$

$$\{Y(\tilde{t}_0;t)\sum_{i=0}^{p}(\hat{\gamma}_{i+1}^- - \hat{\gamma}_i^-)f_i^- - \sum_{i=p+1}^{s} Y(\gamma_i;t)f_i^+\dot{\gamma}_i^+\}\delta t_0+\alpha(t;\delta\mu),$$

$$\hat{\gamma}_0^-=1, \hat{\gamma}_i^-=\dot{\gamma}_i^-, i=1,\dots,p,\ \dot{\gamma}_{p+1}^-=0,$$

$$\alpha(t;\delta\mu)=Y(\tilde{t}_0;t)\delta x_0+\sum_{i=p+1}^{s}\int_{\tau_i(\tilde{t}_0)}^{tlt_0} Y(\gamma_i(\xi);t)\tilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)\delta\varphi(\xi)d\xi+$$

$$+\int_{\tilde{t}_0}^{t} Y(\xi;t)\delta f[\xi]d\xi,\quad \tilde{f}_{x_i}[\xi]=\tilde{f}_{x_i}(\xi,\tilde{x}(\tau_1(\xi)),\dots,\tilde{x}(\tau_s(\xi))),$$

$$\delta f[\xi]=\delta f(\xi,\tilde{x}(\tau_1(\xi)),\dots,\tilde{x}(\tau_s(\xi))),$$

$\lim_{\varepsilon\to 0}\frac{|o(t;\varepsilon\delta\mu)|}{\varepsilon}=0$, *uniformly with recpect to $(t,\delta\mu)\in[\tilde{t}_1-\delta_1,\tilde{t}_1+\delta_1]\times V^-$ and $Y(\xi;t)$ is a matrix function satisfying the equation*

$$\frac{\partial Y(\xi;t)}{\partial\xi}=-\sum_{i=1}^{s}Y(\gamma_i(\xi);t)\tilde{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi),\quad \xi\in[\tilde{t}_0,t],$$

and the condition

$$Y(\xi;t)=\begin{cases} I, & s=t,\\ \Theta, & s>t,\end{cases}$$

I is the identity matrix, Θ is the zero matrix.

Remark 1. *If $\tilde\varphi(\tilde t_0-)=\tilde x_0$, then $f_0^-=\cdots=f_p^-$, $f_i^-=0$, $i=p+1,\dots,s$. If $\dot\gamma_p^-<\cdots<\dot\gamma_1^-<1$, then the condition (5) is fulfilled.*

Theorem 2. *Let $\gamma_i=\tilde t_0$, $i=1,\dots,p$, $\tilde t_0<\gamma_{p+1}<\cdots<\gamma_s<\tilde t_1$, there exist the finite limits f_i^+, $i=0,\dots,s$; $\dot\gamma_i^+$, $i=1,\dots,s$ (see (3), (4)), and there exist a right-hand semi-neighborhood $V^+(\tilde t_0)$ of the point $\tilde t_0$ such that*

$$t<\gamma_1(t)\le\cdots\le\gamma_p(t),\quad \forall t\in V^+(\tilde t_0). \tag{7}$$

Then there exist numbers $\varepsilon_1\in(0,\varepsilon_0]$, $\delta_1\in(0,\epsilon_0]$, such that for an arbitrary $(t,\varepsilon,\delta\mu)\in[\tilde t_1-\delta_1,\tilde t_1+\delta_1]\times[0,\varepsilon_1]\times V^+=\{\delta\mu\in V:\ \delta t_0\ge 0\}$ the formula (6) is valid, where

$$\delta x(t;\delta\mu)=$$

$$\{Y(\tilde t_0;t)\sum_{i=0}^{p}(\hat\gamma_{i+1}^+-\hat\gamma_i^+)f_i^+-\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i^+\dot\gamma_i^+\}\delta t_0+\alpha(t;\delta\mu),$$

$\hat\gamma_0^+=1$, $\hat\gamma_i^+=\dot\gamma_i^+$, $i=1,\dots,p,\hat\gamma_{p+1}^+=0$.

Remark 2. *If $\tilde\varphi(\tilde t_0+)=\tilde x_0$, then $f_0^+=\cdots=f_p^+$, $f_i^+=o$, $i=p+1,\dots,s$. If $1<\dot\gamma_1^+<\cdots<\dot\gamma_p^+$, then the condition (7) is fulfilled.*

Theorem 3. *Let the assumptions of Theorems 1, 2 are fulfilled and*

$$\sum_{i=0}^{p}(\hat\gamma_{i+1}^--\hat\gamma_i^-)f_i^-=\sum_{i=0}^{p}(\hat\gamma_{i+1}^+-\hat\gamma_i^+)f_i^+=f_0;$$
$$f_i^-\dot\gamma_i^-=f_i^+\dot\gamma_i^+=f_i,\quad i=p+1,\dots,s.$$

Then there exist numbers $\varepsilon_1\in(0,\varepsilon_0]$, $\delta_1\in(0,\delta_0]$, such that for an arbitrary $(t,\varepsilon,\delta\mu)\in[\tilde t-\delta_1,\tilde t+\delta_1]\times[0,\varepsilon_0]\times V$ the formula (6) is valid, where

$$\delta x(t,\delta\mu)=[Y(\tilde t_0;t)f_0-\sum_{i=p+1}^{s}Y(\gamma_i;t)f_i]\delta f_0+\alpha(t;\delta\mu).$$

For the case $s=2$, $\tau_1(t)\equiv t$ analogous theorems are proved in [1].

2. To every element $\zeta=(t_0,\varphi,f)\in A_1=[a.b)\times\Delta\times E$ there eorresponds the delay differential equation (1) with the initial condition $x(t)=\varphi(t)$, $t\in[\tau,t_0]$.

Introduce the set

$$V_1=\{\delta\zeta=(\delta t_0,\delta\varphi,\delta f)\in A_1-\tilde\zeta:$$
$$|\delta t_0|\le c,\delta\varphi=\sum_{i=1}^{k}\lambda_i\delta f_i,\ \ |\lambda_i|\le c,\ \ i=1,\dots,k\},$$

where $\zeta=(t_0,\varphi,f)\in A_1$; $\delta f_i\in E-f$, $\delta\varphi\in\Delta-\tilde\varphi$, $i=1,\dots,k$ are fixed points.

Analogously we set the function (see Section 1)

$$\Delta x(t;\varepsilon\delta\zeta)=x(t;\varepsilon\delta\zeta)-\tilde x(t),\quad (t,\varepsilon,\delta\zeta)\in[\tau,\tilde t_1+\delta_0]\times[0,\varepsilon_0]\times V_1.$$

Theorem 4. *Let $\tilde{\varphi}(t)$ be absolutely continuous in a left semi-neighborhood of the point $\tilde{t}_0$, there exist the finite limits $\dot{\varphi}^- = \dot{\tilde{\varphi}}(\tilde{t}_0-)$ and*

$$\lim_{\omega\to\overset{\circ}{\omega}{}^-} \tilde{f}^-(\omega) = \overset{\circ}{f}{}^-, \quad \omega \in \mathbb{R}^-_{\tilde{t}_0} \times O^s, \quad \overset{\circ}{\omega}{}^- = (\tilde{t}_0, \tilde{\varphi}(\tau_1(\tilde{t}_0-)), \dots, (\tau_s(\tilde{t}_0-))).$$

Then for an arbitrary $(t,\varepsilon,\delta) \in [\tilde{t}_0, \tilde{t}_1 + \delta_0] \times [0, \varepsilon_0] \times V_1^- = \{\delta\zeta \in V_1 : \ \ \delta t_0 \le 0\}$ the formula

$$\Delta\varepsilon\delta\zeta) = \varepsilon\delta x(t;\delta\zeta) + o(t;\varepsilon\delta\zeta) \tag{8}$$

is valid, where

$$\delta x(t;\delta\zeta) = Y(\hat{t}_0;t)[\delta\varphi^- + (\dot{\varphi}^- - \overset{\circ}{f}{}^-)\delta t_0] + \beta(t;\delta\zeta), \quad \delta\varphi^- = \delta\varphi(\tilde{t}_0-),$$
$$\beta(t;\delta\zeta) =$$
$$= \sum_{i=1}^{s} \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma_i(\xi);t)\hat{f}_{x_i}[\gamma_i(\xi)]\dot{\gamma}_i(\xi)d\xi + \int_{\tilde{t}_0}^{t} Y(\xi;t)\delta f[\xi]d\xi.$$

Theorem 5. *Let $\tilde{\varphi}(t)$ be absolutely continuous in a right semi-neighborhood of the point $\tilde{t}_0$, there exist the finite limits $\dot{\varphi}^+ = \dot{\tilde{\varphi}}(\tilde{t}_0+)$ and*

$$\lim_{\omega\to\overset{\circ}{\omega}+} \tilde{f}(\omega) = \overset{\circ}{f}{}^-, \quad \omega \in \mathbb{R}^+_{\tilde{t}_0} \times O^s, \quad \overset{\circ}{\omega}{}^+ = (\tilde{t}_0, \tilde{\varphi}(\tau_1(\tilde{t}_0+)), \dots, (\tau_s(\tilde{t}_0+))).$$

Then for each $\bar{t} \in (\tilde{t}_0, \tilde{t}_1)$ there exists a number $\varepsilon_1 \in (0, \varepsilon_0]$ such that for an arbitrary $(t,\varepsilon,\delta\zeta) \in [\tilde{t}_0, \tilde{t}_1 + \delta_0] \times [0, \varepsilon_1] \times V_1^+ = \{\delta\zeta \in V_1 : \ \ \delta t_0 \ge 0\}$ the formula (8) is valid, where

$$\delta x(t;\delta\zeta) = Y(\hat{t}_0;t)[\delta\varphi^+ + (\dot{\varphi}^+ - \overset{\circ}{f}{}^+)\delta t_0] + \beta(t;\delta\zeta), \quad \delta\varphi^+ = \delta\varphi(\tilde{t}_0+).$$

Finally we note that the formulas (6), (8) play an important role when invstigating delay optimal problems.

References

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