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Lothar Jentsch and David Natroshvili

THREE-DIMENSIONAL MATHEMATICAL PROBLEMS OF THERMOELASTICITY OF ANISOTROPIC BODIES, I

Dedicated to V. D. Kupradze
on the occasion of his 95th birthday

Abstract. A wide class of basic, mixed, and crack type boundary value and interface problems for the steady state and pseudo-oscillation equations of the thermoelasticity theory of anisotropic bodies are considered. The generalized Sommerfield–Kupradze type thermo-radiation conditions are formulated and uniqueness and existence theorems are proved by the potential method and the theory of pseudodifferential equations on manifold. The almost best regularity properties of solutions are established.

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Introduction

Boundary value problems (BVPs) of the theory of thermoelasticity have a long history. They encounter in many physical, mechanical, and engineering applications where the thermal stresses appear. Therefore, the mathematical model of thermoelasticity have received considerable attention in the scientific literature (for exhaustive historical and bibliographical material see [45], [63]).

Without trying to discuss the history in detail we note that three-dimensional *regular* problems of statics, pseudo-oscillations, steady state oscillations, and general dynamics of the thermoelasticity theory of *homogeneous isotropic* elastic bodies are completely investigated by many authors (see, for example, [45], [8], [24], [63], [66], [29]–[31] and references therein). The main mathematical tools applied for the investigation of various aspects of the above problems are variational and functional methods ([14], [63]), the potential methods and the direct and indirect boundary integral equations (BIE) methods ([45], [29]–[31], [28]), different versions of the Bubnov-Galerkin method and the method of generalized Fourier series (method of regular sources) ([45]).

To the best of the authors' knowledge the problems of thermoelastic pseudo-oscillations and steady state oscillations for *anisotropic* bodies have not been treated systematically in the scientific literature (cf. [33]).

In the present memoir we undertake to examine a wide class of the basic regular, mixed, and crack type boundary value and interface problems for the systems of differential equations of pseudo-oscillations and steady state oscillations of the thermoelasticity theory of *homogeneous anisotropic* bodies. We develop the potential method to prove the existence and uniqueness theorems in various functional spaces and to establish the almost best regularity properties of solutions. We note that many problems considered in this memoir have not been treated even in the isotropic thermoelasticity.

It should be mentioned that the methods, developed for the isotropic case in the above cited references, unfortunately, are not always applicable in the case of general anisotropy. It concerns, especially, the steady state oscillation problems where quite new ideas are required. In particular, the exterior BVPs of steady state thermoelastic oscillations in the isotropic case have been studied on the basis of the classical Sommerfeld-Kupradze thermo-radiation conditions and the uniqueness theorems were proved with the help of the well-known Rellich's lemma, since components of the displacement vector and the temperature in the isotropic case can be represented as a sum of metaharmonic functions (for details see [45]).

In the anisotropic case we need a nontrivial generalization of the thermo-radiation conditions at infinity. We notice that the basic difficulties in dealing with the steady state oscillation problems are connected with a very complicated geometrical form of the corresponding characteristic surfaces which play a significant role in the study of the far field behaviour of solutions (cf. [80], [55]).

The monograph consists of six chapters and is organized as follows.

In the first chapter there are constructed the matrices of fundamental solutions to the systems of pseudo-oscillation and steady state oscillation equations of thermoelasticity theory by Fourier transform and limiting absorption principle, and their asymptotic properties at infinity and in a vicinity of the origin are studied.

On the basis of the results obtained the *generalized Sommerfeld-Kupradze type thermo-radiation conditions* are formulated and the Somigliana type integral representation formulae for bounded and unbounded domains (with compact boundaries) are derived.

We emphasize that the above mentioned fundamental matrices are not represented explicitly in terms of elementary functions. This essentially complicates the investigation of corresponding integral operators.

The second chapter deals with the detail formulation of boundary value and interface problems for homogeneous and piecewise homogeneous (composed) anisotropic bodies. Besides the usual classical setting in $\mathrm{C}^{k,\alpha}$-continuous Hölder functional spaces here is given a weak formulation of the problems in the Sobolev W_p^1 ($W_{p,\mathrm{loc}}^1$) spaces with $1 < p < \infty$. The weak setting relies upon the definition of generalized boundary trace functionals in the Besov $B_{p,q}^s$ spaces which are introduced and broadly discussed in Section 4. Note that crack type and mixed problems, in general, do not admit C^α-continuous solutions (with $\alpha > 1/2$) in closed domains even for C^∞-regular boundary data. Therefore, these problems are formulated only in the natural weak setting.

In the third chapter there are proved uniqueness theorems of solutions to the regular and mixed homogeneous boundary value and interface problems in the appropriate functional spaces. Here the crucial moment is selection of the functional classes where the homogeneous steady state oscillation problems in unbounded domain admit only the trivial solution. This is done with the help of the above mentioned generalized Sommerfeld-Kupradze type thermo-radiation conditions.

Chapter IV is entirely devoted to the study of single and double layer potential type operators and boundary integral (pseudodifferential) operators generated by them. These results are the main tools used in the subsequent chapters.

The existence theorems of solutions to the regular nonhomogeneous boundary value and interface problems are proved in the fifth chapter. By the potential method these problems are reduced to the *equivalent* systems of pseudodifferential equations (ΨDE) on the boundary of the elastic body (or on the interface of the composed body) under consideration. It is established that these BIEs are elliptic systems (in general, in the sense of Douglis-Nirenberg) with trivial null-spaces and zero indices. The general theory of pseudodifferential equations on closed smooth manifold and corresponding embedding theorems then immediately lead to the existence results for the above indicated nonhomogeneous problems in $C^{k,\alpha}$ functional spaces with integer $k \geq 1$ and $0 < \alpha < 1$ in the case of classical setting or in W_p^1 ($W_{p,\mathrm{loc}}^1$) spaces with $1 < p < \infty$ in the case of weak setting (provided the boundary data belong to appropriate natural spaces).

Finally, in the last sixth chapter the existence theorems of solutions to the nonhomogeneous mixed and crack type boundary value problems and to the mixed interface problems are proved again by the potential method. These problems are reduced to the equivalent pseudodifferential equations on some proper subset of the boundary (or of the interface). The investigation of these equations is carried out with the help of the theory of ΨDEs on manifold with boundary. The BIEs are again elliptic systems of ΨDEs (in general, in the sense of Douglis-Nirenberg) with positive definite principal homogeneous symbol matricies, trivial null-spaces and indices equal to zero. Making use of these results the existence of solutions to the problems indicated above are proved in the Sobolev W_p^1 ($W_{p,\mathrm{loc}}^1$) spaces with $4/3 < p < 4$. Applying the corresponding embedding theorems it is shown that the solutions possess C^α-smoothness (with arbitrary $\alpha < 1/2$) at the crack edges (in crack problems) and at the collision curves of changing boundary conditions (in mixed problems) provided again that the boundary data belong to appropriate natural spaces.

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CHAPTER I
BASIC EQUATIONS. FUNDAMENTAL MATRICES. THERMO-RADIATION CONDITIONS

In this chapter first we construct exponentially decreasing fundamental solution to the system of pseudo-oscillation equations of the thermoelasticity theory of anisotropic bodies and then by the limiting absorption principle we obtain two fundamental matrices for the system of steady state oscillation equations. Further, we derive the asymptotic formulae for the entries of these matrices and formulate *the generalized Sommerfeld-Kupradze type radiation conditions* in anisotropic thermoelasticity.

1. Basic Differential Equations of Thermoelasticity

In this section we collect an auxiliary material concerning the governing equations and the basic mechanical and physical concepts of the thermoelasticity theory (for details we refer to [63], [45]).

1.1. The system of equations of coupled linear thermoelastodynamics of homogeneous anisotropic elastic medium reads (see [63], Ch. V)

$$\begin{aligned} &c_{kjpq}\,D_jD_q\,u_p(x,t)\,+\,X_k(x,t)\,=\,\varrho D_t^2u_k(x,t)\,+\,\beta_{kj}D_ju_4(x,t),\\ &\lambda_{pq}D_pD_qu_4(x,t)\,-\,c_0D_tu_4(x,t)\,-\,T_0\beta_{pq}D_tD_pu_q(x,t)\,=\,-Q(x,t), \end{aligned} \tag{1.1}$$

where $c_{kjpq}=c_{pqkj}=c_{jkpq}$ are elastic constants, $\lambda_{pq}=\lambda_{qp}$ are heat conductivity coefficients, $c_0>0$ is the thermal capacity, $T_0>0$ is the temperature of the medium in the natural state, $\beta_{pq}=\beta_{qp}$ are expressed in terms of the thermal and elastic constants, ϱ =const>0 is the density of the medium; $u=(u_1,u_2,u_3)^\top$ is the displacement vector, u_4 is the temperature, $X=(X_1,X_2,X_3)^\top$ is the bulk force, Q is the heat source; $x=(x_1,x_2,x_3)$ denotes the spatial variable, while t is the time variable; here and in what follows the summation over repeated indices is meant from 1 to 3, unless otherwise stated; the superscript $\top$ denotes transposition and $D_p=D_{x_p}:=\partial/\partial x_p$, $D_t:=\partial/\partial t$.

In the sequel, we usually consider the homogeneous version of equations (1.1), i.e., we assume $X=0$, $Q=0$. In addition, without any restriction of generality $\varrho=1$ is assumed as well.

In (1.1) the term $-T_0\beta_{pq}D_tD_pu_q(x,t)$ describes the coupling between the temperature and strain fields. It vanishes only for a stationary heat flow. In that case or if this term is neglected, we have the so-called decoupled thermoelasticity theory.

In the thermoelasticity theory the stress tensor $\{\sigma_{kj}\}$, the strain tensor $\{\varepsilon_{kj}\}$ and the temperature field u_4 are related by Duhamel–Neumann law $\sigma_{kj}=c_{kjpq}\varepsilon_{pq}-\beta_{kj}u_4$, $\varepsilon_{kj}=2^{-1}(D_ku_j+D_ju_k)$, $k,j=1,2,3$; the k-th component of the vector of thermostresses, acting on a surface element with the unit normal vector $n=(n_1,n_2,n_3)$, is calculated by the formula

$$\sigma_{kj}n_j=c_{kjpq}\varepsilon_{pq}n_j-\beta_{kj}n_ju_4=c_{kjpq}n_j\,D_qu_p-\beta_{kj}n_ju_4,\ \ k=1,2,3. \tag{1.2}$$

The formal Laplace transform of the equations (1.1) (with respect to t) leads to the so-called *pseudo-oscillation equations* of the thermoelasticity theory

$$
\begin{aligned}
&c_{kjpq}\, D_j D_q u_p(x) = \tau^2 u_k(x) + \beta_{kj} D_j u_4(x),\\
&\lambda_{pq} D_p D_q u_4(x) - \tau c_0 u_4(x) - \tau T_0 \beta_{pq} D_p u_q(x) = 0;
\end{aligned}
\tag{1.3}
$$

here $\tau = \sigma - i\omega$ is a complex parameter with $\omega \in \mathbb{R}$ and $\sigma \in \mathbb{R} \setminus \{0\}$.

If all data involved in (1.1) are harmonic time dependent, i.e., $u_k(x,t) = \overset{1}{u}_k(x)\cos\omega t + \overset{2}{u}_k(x)\sin\omega t$, $k = 1,2,3,4$, $\omega \in \mathbb{R}$, then we get the so-called *steady state oscillation equations* of the theory of thermoelasticity

$$
\begin{aligned}
&c_{kjpq}\, D_j D_q u_p(x) = -\omega^2 u_k(x) + \beta_{kj} D_j u_4(x),\\
&\lambda_{pq} D_p D_q u_4(x) + i\omega c_0 u_4(x) + i\omega T_0 \beta_{pq} D_p u_q(x) = 0,
\end{aligned}
\tag{1.4}
$$

where the following notation $u_k(x) = \overset{1}{u}_k(x) + i\overset{2}{u}_k(x)$, $k = 1,2,3,4$, is employed.

It is evident that system (1.4) formally can be obtained from (1.3) provided $\sigma = 0$, but this similarity is a very formal one and it will become apparent later on.

Finally, let us note that, if the displacement vector and the temperature do not depend on the time variable t, then from (1.1) we obtain equations of the so-called *decoupled thermoelastostatics*

$$
c_{kjpq}\, D_j D_q u_p(x) = \beta_{kj} D_j u_4(x), \quad k = 1,2,3, \tag{1.5}
$$

$$
\lambda_{pq} D_p D_q u_4(x) = 0. \tag{1.6}
$$

In this monograph we shall not systematically treat the equations of decoupled thermoelastostatics (1.5)-(1.6), since in this case all the boundary value and interface problems, we intend to consider, are also completely decoupled into two independent problems for the temperature field and the dicplacement field. The corresponding problems of elastostatics of anisotropic bodies for the system (1.5) have been studied in [8], [56], while the problems for the stationary distribution of the temperature field which, in fact, are BVPs for the second order scalar elliptic differential equation (1.6) can be found, for example, in [52].

1.2. In order to rewrite the above equations in the matrix form, let us set

$$
U = (u_1, u_2, u_3, u_4)^\top = (u, u_4)^\top, \quad u = (u_1, u_2, u_3)^\top,
$$

$$
C(D) = [C_{kp}(D)]_{3\times 3}, \quad C_{kp}(D) = c_{kjpq}\, D_j D_q, \tag{1.7}
$$

$$
\Lambda(D) = \lambda_{pq}\, D_p D_q, \quad D = \nabla = (D_1, D_2, D_3). \tag{1.8}
$$

For the sake of simplicity we shall use also the notation either $[A]_{m\times n}$ or $[A_{kp}]_{m\times n}$ for the $m \times n$ matrix A.

Now we can represent equations (1.3) and (1.4) in the following form, respectively,

$$A(D,\tau)U(x)=0, \tag{1.9}$$

$$A(D,-i\omega)U(x)=0, \tag{1.10}$$

where

$$A(D,\varkappa)=\left[\begin{array}{cc}[C(D)-\varkappa^2\,I_3]_{3\times 3} & [-\beta_{kj}D_j]_{3\times 1}\\ [-\varkappa T_0\beta_{kj}D_j]_{1\times 3} & \Lambda(D)-\varkappa c_0\end{array}\right]_{4\times 4}, \tag{1.11}$$

$I_m=[\delta_{kj}]_{m\times m}$ stands for the identity $m\times m$ matrix, δ_{kj} is Kronecker's symbol.

Clearly, $\varkappa=\tau=\sigma-i\,\omega$ corresponds to the pseudo–oscillations, while $\varkappa=-i\omega$ corresponds to the steady state oscillations, and $\varkappa=0$ to the decoupled thermoelastostatics.

Further we introduce the classical stress operator

$$T(D,n)=[T_{kp}(D,n)]_{3\times 3}=[c_{kjpq}\,n_j\,D_q]_{3\times 3}, \tag{1.12}$$

and the thermoelastic stress operator

$$P(D,n)=[\,[T(D,n)]_{3\times 3},\ [-\beta_{kj}n_j]_{3\times 1}\,]_{3\times 4}\,. \tag{1.13}$$

Due to (1.2) we have

$$[P(D,n)U]_k=\sigma_{kj}n_j=[T(D,n)u]_k-\beta_{kj}n_ju_4,\ \ k=1,2,3.$$

1.3. From the physical considerations it follow that (see [22], [63]):

a) the matrix $[\lambda_{pq}]_{3\times 3}$ is positive definite, i.e.,

$$\Lambda(\xi)=\lambda_{pq}\xi_p\xi_q\geq\delta_0\,|\xi|^2,\ \ \xi\in\mathbb{R}^3,\ \ \delta_0=\text{const}>0; \tag{1.14}$$

b) the quadratic form $c_{kjpq}e_{kj}e_{pq}$ is positive definite in the real symmetric variables $e_{kj}=e_{jk}$,

$$c_{kjpq}e_{kj}e_{pq}\geq\delta' e_{kj}e_{kj},\ \ \delta'=\text{const}>0, \tag{1.15}$$

which implies positive definiteness of the matrix $C(\xi),\ \xi\in\mathbb{R}^3\setminus\{0\}$, defined by (1.7), i.e.,

$$C_{kj}(\xi)\eta_j\eta_k\geq\delta_1|\xi|^2|\eta|^2,\ \ \xi,\eta\in\mathbb{R}^3,\ \ \delta_1=\text{const}>0. \tag{1.16}$$

Inequalities (1.14) and (1.16) together with the symmetry properties of the matrices $[\lambda_{pq}]$ and $C(\xi)$ yield

$$C(\xi)\eta\cdot\eta=C_{kj}(\xi)\eta_j\overline{\eta_k}\geq\delta_1|\xi|^2\,|\eta|^2,\ \ \xi\in\mathbb{R}^3, \tag{1.17}$$

$$\lambda_{pq}\eta_p\overline{\eta_q}\geq\delta_0|\eta|^2, \tag{1.18}$$

for an arbitrary complex vector $\eta\in\mathbb{C}^3$. Here $a\cdot b=\sum_{k=1}^m a_k\overline{b_k}$ denotes the usual scalar product of the two complex vectors $a=(a_1,\cdots,a_m)$ and $b=(b_1,\cdots,b_m)$ in $\mathbb{C}^m$, while upper bar denotes complex conjugate. We shall also employ the following notation ("real" scalar product of complex vectors)

$$\langle a\,,\,b\rangle=\sum_{k=1}^{m}a_kb_k,\quad a,b\in\mathbb{C}^m. \tag{1.19}$$

1.4. We emphasize that the differential operator $A(D,\varkappa)$ defined by (1.11) is not formally self-adjoint. Denote by $A^*(D,\varkappa)$ the operator formally adjoint to $A(D,\varkappa)$

$$A^*(D,\varkappa)=\overline{A^\top(-D,\varkappa)}=A^\top(-D,\overline{\varkappa})=$$
$$=\begin{bmatrix}[C(D)-\overline{\varkappa}^2I_3]_{3\times3} & [\overline{\varkappa}T_0\beta_{kj}D_j]_{3\times1}\\ [\beta_{kj}D_j]_{1\times3} & \Lambda(D)-\overline{\varkappa}c_0\end{bmatrix}_{4\times4}. \tag{1.20}$$

Let us note here that throughout this memoir we shall use the following notations (when no confusion can be caused by this):

a) if all elements of a vector $v=(v_1,...,v_m)$ (matrix $a=[a_{kj}]_{m\times n}$) belong to one and the same space X, we shall write $v\in X$ $(a\in X)$ instead of $v\in[X]^m$ $(a\in[X]_{m\times n})$;

b) if $K\,:\,X_1\times\cdots\times X_m\to Y_1\times\cdots\times Y_n$ and $X_1=\cdots=X_m=X,\ Y_1=\cdots=Y_n=Y$, we shall write $K\,:\,X\to Y$ rather than $K\,:\,[X]^m\to[Y]^n$.

Let $\Omega^+\subset\mathbb{R}^3$ be a bounded domain with a C^2-smooth connected boundary $S=\partial\Omega^+$, $\overline{\Omega^+}=\Omega^+\cup S$ and $\Omega^-=\mathbb{R}^3\setminus\overline{\Omega^+}$. We assume that $\overline{\Omega^+}$ $(\overline{\Omega^-})$ is filled by a homogeneous anisotropic medium with the elastic and thermal characteristics described above.

Now we present the so-called Green formulae for the operator $A(D,\varkappa)$ which will be used many times in the sequel.

Let $U=(u_1,u_2,u_3,u_4)^\top,\ V=(v_1,v_2,v_3,v_4)^\top\in\mathrm{C}^2(\Omega^+)\cap\mathrm{C}^1(\overline{\Omega^+})$ (i.e., U and V are regular vectors in Ω^+) and $A(D,\varkappa)U,\ A^*(D,\varkappa)V\in L_1(\Omega^+)$. Then the following equations hold for arbitrary $\varkappa\in\mathbb{C}$ (cf. [57], [55], [16]):

$$\int_{\Omega^+}A(D,\varkappa)U\cdot V\,dx=\int_S[B(D,n)U]^+\cdot[V]^+\,dS-\int_{\Omega^+}E(U,V)\,dx, \tag{1.21}$$

$$\int_{\Omega^+}\{A(D,\varkappa)U\cdot V-U\cdot A^*(D,\varkappa)V\}\,dx=\int_S\Big\{[B(D,n)U]^+\cdot[V]^+-$$
$$-[U]^+\cdot[\overline{Q(D,n,\varkappa)}V]^+\Big\}\,dS, \tag{1.22}$$

$$\int_{\Omega^+}\Big\{[A(D,\varkappa)U]_k\,\overline{u}_k+\tfrac{1}{\overline{\varkappa}T_0}[\overline{A(D,\varkappa)U}]_4\,u_4\Big\}\,dx=$$
$$=-\int_{\Omega^+}\Big\{c_{kjpq}D_pu_qD_k\overline{u}_j+\varkappa^2|u|^2+\tfrac{1}{\overline{\varkappa}T_0}\lambda_{kj}D_ku_4D_j\overline{u}_4+\tfrac{c_0}{T_0}|u_4|^2\Big\}\,dx+$$
$$+\int_S\Big\{[B(D,n)U]_k^+[\overline{u}_k]^++\tfrac{1}{\overline{\varkappa}T_0}[u_4]^+[\partial_n\overline{u}_4]^+\Big\}\,dS, \tag{1.23}$$

where

$$\partial_n=\lambda(D,n):=\lambda_{pq}n_pD_q, \tag{1.24}$$

$$B(D,n)=\begin{bmatrix}[T(D,n)]_{3\times3} & [-\beta_{kj}n_j]_{3\times1}\\ [0]_{1\times3} & \lambda(D,n)\end{bmatrix}_{4\times4}, \tag{1.25}$$

$$Q(D,n,\varkappa)=\begin{bmatrix} [T(D,n)]_{3\times 3} & [\varkappa T_0\beta_{kj}n_j]_{3\times 1} \\ [0]_{1\times 3} & \lambda(D,n) \end{bmatrix}_{4\times 4}, \tag{1.26}$$

$$\begin{aligned} E(U,V) &= c_{kjpq}\,D_p u_q\,D_k\overline{v}_j+\varkappa^2 u_k\overline{v}_k-\beta_{kj}u_4\,D_j\overline{v}_k+ \\ &\quad +\lambda_{pq}\,D_q u_4\,D_p\overline{v}_4+c_0\varkappa u_4\overline{v}_4+\varkappa T_0\overline{v}_4\beta_{pq}\,D_p u_q. \end{aligned} \tag{1.27}$$

Here and in what follows $n(x)$ denotes the exterior unit normal vector of S at the point $x\in S$. The symbols $[\,\cdot\,]^{\pm}$ denote limits on S from $\Omega^{\pm}$.

Note that, if we consider the first three components of the U as the displacement vector and the fourth one as the temperature, then the vector $B(D,n)U$ has the following thermo-mechanical sense: the first three components of the $B(D,n)U$ represent the corresponding vector of thermal stresses (see (1.13)), while the fourth component describes the heat flux through the surface S.

The similar formulae hold valid also for the domain Ω^-, when $\varkappa=0$ or $\mathrm{Re}\varkappa>0$, with the following changes (related to the choice of direction of the normal vector): the superscript "+" must be replaced everywhere by the superscript "−" and in front of the surface integrals the sign "−" is to be put.

In this case the vectors U and V have to satisfy the conditions

$$U,\,V\in \mathrm{C}^2(\Omega^-)\cap \mathrm{C}^1(\overline{\Omega^-}),\;\; A(D,\varkappa)U,\;A^*(D,\varkappa)V\in L_1(\Omega^-), \tag{1.28}$$

$A(D,\varkappa)U$ and $A^*(D,\varkappa)V$ have compact supports and, in addition, U and V have the following asymptotic behaviour at infinity

$$u_k(x),\,v_k(x)=\begin{cases} o(1) & \text{for} \quad \varkappa=0, \\ O(|x|^N) & \text{for} \quad \mathrm{Re}\varkappa=\sigma>0,\;\; k=1,2,3,4, \end{cases} \tag{1.29}$$

with an arbitrary fixed positive number N. In fact, it can be proved that, if U and V are solutions of the corresponding homogeneous equations , then the conditions (1.29) imply

$$D^\beta u_k(x),D^\beta v_k(x)=\begin{cases} O(|x|^{-1-|\beta|}) & \text{for} \;\; \varkappa=0, \\ O(|x|^{-\nu}) & \text{for} \;\; \mathrm{Re}\varkappa=\sigma>0,\;\; k=1,2,3,4, \end{cases} \tag{1.30}$$

where ν is an arbitrary positive number, $\beta=(\beta_1,\beta_2,\beta_3)$ is an arbitrary multi-index and $|\beta|=\beta_1+\beta_2+\beta_3$ (see, for example, [7], [44], [56].

The principal remark here is that for solutions U and V of the steady state oscillation equation (1.10) (i.e., when $\varkappa=-i\omega$) the Green formulae, similar to (1.21)-(1.23), are not valid any more for the unbounded domain Ω^-.

1.5. In this subsection, before starting the construction of the fundamental matrices, we shall analyse the so-called characteristic matrices corresponding to the above differential operators of the thermoelasticity theory. They will play a fundamental role in the sequel.

Let us introduce the characteristic polynomial of the operator $A(D,\varkappa)$

$$M(\xi,\varkappa)=\det A(-i\xi,\varkappa). \tag{1.31}$$

Denote by $N(-i\xi,\varkappa)$ the matrix adjoint to $A(-i\xi,\varkappa)$, i.e.,

$$A(-i\xi,\varkappa)N(-i\xi,\varkappa)=N(-i\xi,\varkappa)A(-i\xi,\varkappa)=M(\xi,\varkappa)I_4. \tag{1.32}$$

Clearly, we have

$$[A(-i\xi,\varkappa)]^{-1}=[M(\xi,\varkappa)]^{-1}N(-i\xi,\varkappa), \tag{1.33}$$

where $[A(-i\xi,\varkappa)]^{-1}$ is the matrix inverse to $A(-i\xi,\varkappa)$. Equations (1.31), (1.11), and (1.7) yield

$$\begin{aligned} M(\xi,\varkappa)=\det\begin{bmatrix} [-C(\xi)-\varkappa^2\, I_3]_{3\times3} & [i\beta_{kj}\xi_j]_{3\times1}\\ [i\varkappa T_0\beta_{kj}\xi_j]_{1\times3} & -\varkappa c_0\end{bmatrix}_{4\times4}+\\ +\det\begin{bmatrix} [-C(\xi)-\varkappa^2\, I_3]_{3\times3} & [i\beta_{kj}\xi_j]_{3\times1}\\ [0]_{1\times3} & -\Lambda(\xi)\end{bmatrix}_{4\times4}=\Lambda(\xi)\,\det[C(\xi)+\varkappa^2\, I_3]-\\ -\varkappa T_0\,\det\begin{bmatrix} [-C(\xi)-\varkappa^2\, I_3]_{3\times3} & [\beta_{kj}\xi_j]_{3\times1}\\ [\beta_{kj}\xi_j]_{1\times3} & c_0T_0^{-1}\end{bmatrix}_{4\times4}=\\ =\Lambda(\xi)\,\det[C(\xi)+\varkappa^2\, I_3]-\\ -\varkappa T_0\det\begin{bmatrix} [-C(\xi)-\varkappa^2\, I_3]_{3\times3}-[c_0^{-1}T_0\beta_{kj}\xi_j\beta_{pq}\xi_q]_{3\times3} & [\beta_{kj}\xi_j]_{3\times1}\\ [0]_{1\times3} & c_0T_0^{-1}\end{bmatrix}_{4\times4}=\\ =\Lambda(\xi)\,\det[C(\xi)+\varkappa^2\, I_3]+\varkappa c_0\,\det[\widetilde{C}(\xi)+\varkappa^2\, I_3], \end{aligned} \tag{1.34}$$

where $C(\xi)$ and $\Lambda(\xi)$ are defined by (1.7) and (1.8), respectively, and

$$\begin{aligned} &\widetilde{C}(\xi)=[\widetilde{C}_{kp}(\xi)]_{3\times3}=C(\xi)+[c_0^{-1}T_0\beta_{kj}\beta_{pq}\xi_j\xi_q]_{3\times3}, \\ &\widetilde{C}_{kp}(\xi)=(c_{kjpq}+c_0^{-1}T_0\beta_{kj}\beta_{pq})\xi_j\xi_q,\ \ k,p=1,2,3. \end{aligned} \tag{1.35}$$

Next, we set

$$\Psi(\xi,\varkappa)=\det[C(\xi)+\varkappa^2\, I_3], \tag{1.36}$$

$$\widetilde{\Psi}(\xi,\varkappa)=\det[\widetilde{C}(\xi)+\varkappa^2\, I_3]. \tag{1.37}$$

The relations (1.35) and (1.17) imply that the matrix $\widetilde{C}(\xi)$ for any $\xi\in\mathbb{R}^3\setminus\{0\}$ is positive definite and, therefore,

$$\widetilde{C}(\xi)\eta\cdot\eta=C(\xi)\eta\cdot\eta+c_0^{-1}T_0|\beta_{kj}\xi_j\eta_k|^2\geq\delta_1|\xi|^2|\eta|^2 \tag{1.38}$$

for an arbitrary $\eta\in\mathbb{C}^3$ and the same δ_1 as in (1.17).

Thus, we have

$$M(\xi,\varkappa)=\Lambda(\xi)\Psi(\xi,\varkappa)+\varkappa c_0\widetilde{\Psi}(\xi,\varkappa). \tag{1.39}$$

It is evident that, if $|\varkappa|<\varkappa_0$ with some positive $\varkappa_0$, then there exists a positive number ϱ_0 such that

$$|\Psi(\xi,\varkappa)|\geq1,\ \ |\widetilde{\Psi}(\xi,\varkappa)|\geq1,\ \ |M(\xi,\varkappa)|\geq1, \tag{1.40}$$

for $|\xi|\geq\varrho_0$; here ϱ_0 depends on $\varkappa_0$ and the thermoelastic constants.

Lemma 1.1. *Let* $\tau=\sigma-i\omega$, $\operatorname{Re}\tau=\sigma>0$ *and* $\xi\in\mathbb{R}^3$. *Then* $M(\xi,\tau)\neq0$ *for any* $\omega\in\mathbb{R}$. *Moreover,* $[A(-i\xi,\tau)]^{-1}\in L_2(\mathbb{R}^3)$.

Proof. Let us suppose that the assertion of the lemma is false, i.e., $M(\xi,\tau)=0$. Then the homogeneous system of linear algebraic equations

$$A(-i\xi,\tau)\ a=0 \tag{1.41}$$

has some nontrivial solution $a=(a_1,\cdots,a_4)^\top\in\mathbb{C}^4\setminus\{0\}$.

Multiplying the $k-$th equation of (1.41) by $\overline{a}_k$ and summing the first three equations we get

$$\begin{aligned}-c_{kjpq}\xi_j\xi_q a_p\overline{a}_k-\tau^2\delta_{kp}a_p\overline{a}_k+i\beta_{kj}\xi_j a_4\overline{a}_k&=0,\\ i\tau T_0\beta_{kj}\xi_j a_k\overline{a}_4-\lambda_{pq}\xi_p\xi_q|a_4|^2-\tau c_0|a_4|^2&=0.\end{aligned}$$

Deviding the latter equation by τT_0, taking the complex conjugate and adding to the first one, we obtain

$$c_{kjpq}\xi_j\xi_q a_p\overline{a}_k+\tau^2 a_k\overline{a}_k+\tau[|\tau|^2T_0]^{-1}\lambda_{pq}\xi_p\xi_q|a_4|^2+c_0T_0^{-1}|a_4|^2=0.$$

Due to (1.17) we deduce by separating the real and imaginary parts

$$\left\{\begin{array}{l}C(\xi)\,\widetilde{a}\cdot\widetilde{a}+(\sigma^2-\omega^2)|\widetilde{a}|^2+\sigma[|\tau|^2T_0]^{-1}\Lambda(\xi)|a_4|^2+c_0T_0^{-1}|a_4|^2=0,\\ \omega\{2\sigma|\widetilde{a}|^2+[|\tau|^2T_0]^{-1}\Lambda(\xi)|a_4|^2\}=0,\end{array}\right.$$

where $\widetilde{a}=(a_1,a_2,a_3)^\top$.

From this system and the inequality (1.14) it follows that $a_1=\cdots=a_4=0$, for any $\xi\in\mathbb{R}^3$, $\omega\in\mathbb{R}$, and $\sigma>0$. This contradiction proves the first part of the lemma.

The second part of the lemma is a consequence of the inequality

$$[A(-i\xi,\tau)]^{-1}_{kj}\leq\frac{c(\sigma)}{1+|\xi|^2}\ \text{ for }\ \xi\in\mathbb{R}^3,$$

where the positive constant $c(\sigma)$ does not depend on ξ (it depends on σ and on the thermoelastic constants of the medium in question). $\square$

1.6. Now we shall analyse the characteristic polynomial $M(\xi,-i\omega)$ of the operator $A(D,-i\omega)$. It can be easily shown that (see (1.36), (1.37), (1.39))

$$M(\xi,-i\omega)=\Lambda(\xi)\Phi(\xi,\omega)-i\omega c_0\widetilde{\Phi}(\xi,\omega), \tag{1.42}$$

where

$$\Phi(\xi,\omega)=\det[C(\xi)-\omega^2\,I_3]=\Psi(\xi,-i\omega), \tag{1.43}$$

$$\widetilde{\Phi}(\xi,\omega)=\det[\widetilde{C}(\xi)-\omega^2\,I_3]=\widetilde{\Psi}(\xi,-i\omega). \tag{1.44}$$

Characteristic surfaces of the operator $A(D,-i\omega)$ are defined by the equation

$$M(\xi,-i\omega)=0,\ \ \xi\in\mathbb{R}^3, \tag{1.45}$$

which, in turn, due to (1.42), is equivalent to the following system

$$\left\{\begin{array}{l}\Phi(\xi,\omega)=0,\\ \widetilde{\Phi}(\xi,\omega)=0,\quad\xi\in\mathbb{R}^3.\end{array}\right. \tag{1.46}$$

Passing on the spherical co-ordinates

$$\xi_1 = \varrho \cos\varphi \sin\theta,\;\; \xi_2 = \varrho \sin\varphi \sin\theta,\;\; \xi_3 = \varrho \cos\theta,$$
$$0 \le \varrho < +\infty,\;\; 0 \le \varphi < 2\pi,\;\; 0 \le \theta \le \pi,$$

and, taking into account formulae (1.43), (1.44), (1.17) and (1.38), we conclude that each equation of the system (1.46) has three positive roots with respect to ϱ^2. These roots are proportional to ω^2, and polynomials $\Phi(\xi,\omega)$ and $\widetilde{\Phi}(\xi,\omega)$ can be represented in the form:

$$\begin{aligned}\Phi(\xi,\omega) &= \Phi(\eta,0)\,[\varrho^2-\omega^2\varrho_1^2(\theta,\varphi)][\varrho^2-\omega^2\varrho_2^2(\theta,\varphi)][\varrho^2-\omega^2\varrho_3^2(\theta,\varphi)],\\ \widetilde{\Phi}(\xi,\omega) &= \widetilde{\Phi}(\eta,0)\,[\varrho^2-\omega^2\widetilde{\varrho}_1^2(\theta,\varphi)][\varrho^2-\omega^2\widetilde{\varrho}_2^2(\theta,\varphi)][\varrho^2-\omega^2\widetilde{\varrho}_3^2(\theta,\varphi)],\end{aligned} \tag{1.47}$$

where $\eta = \xi/\varrho$, $\varrho = |\xi|$, $\Phi(\eta,0) = \det C(\eta) > 0$, $\widetilde{\Phi}(\eta,0) = \det\widetilde{C}(\eta) > 0$; here $\{\varrho_k^2(\theta,\varphi)\}_{k=1}^3$ and $\{\widetilde{\varrho}_k^2(\theta,\varphi)\}_{k=1}^3$ do not depend on ω and are solutions of the following equations (with respect to ϱ^2):

$$\Phi(\xi,1) = \Phi(\eta,0)\varrho^6 + \Phi^{(2)}(\eta)\varrho^4 + \Phi^{(1)}(\eta)\varrho^2 - 1 = 0, \tag{1.48}$$
$$\widetilde{\Phi}(\xi,1) = \widetilde{\Phi}(\eta,0)\varrho^6 + \widetilde{\Phi}^{(2)}(\eta)\varrho^4 + \widetilde{\Phi}^{(1)}(\eta)\varrho^2 - 1 = 0, \tag{1.49}$$

where $\Phi^{(j)}(\eta)$ and $\widetilde{\Phi}^{(j)}(\eta)$ are even, homogeneous functions of order $2j$ in η (see (1.43), (1.44)).

In what follows we consider the so-called regular case, i.e., we assume the following conditions to be fulfilled (cf. [55], [80]):

I^0. $\nabla_\xi \Phi(\xi,\omega) \ne 0$ *at real zeros of the polynomial* $\Phi(\xi,\omega)$;

II^0. *Gaussian curvature of the manifold, defined by the real zeros of the polynomial* $\Phi(\xi,\omega)$, *does not vanish anywhere.*

From the above conditions I^0–II^0 it follows that the real zeros of the polynomial $\Phi(\xi,\omega)$ form nonselfintersecting, closed, convex two-dimensional surfaces S_j^0, $j=1,2,3$, enveloping the origin of co-ordinates. For an arbitrary vector $x \in \mathbb{R}^3 \setminus \{0\}$ there exist exactly two points on each S_j^0, namely $\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j)$ and $\xi_*^j = -\xi^j$, at which the exterior unit normal is parallel to the vector x. We provide that at ξ^j the normal vector $n(\xi^j)$ and x have the same direction, while at ξ_*^j they are opposite directed. Note that, if $\xi^j \in S_j^0$ and $\xi^k \in S_k^0$ correspond to the same vector x, then (due to the convexity property of the above surfaces) $(\xi^j \cdot x) \ne (\xi^k \cdot x)$ for $k \ne j$.

In the sequel, the $\xi^j \in S_j^0$ will be referred to as the point which corresponds to the vector x (i.e., to the direction $x/|x|$).

Clearly, $\varrho = |\omega|\;\varrho_k(\theta,\varphi) > 0$, $k=1,2,3$, represent the equations of the surfaces S_k^0 in the spherical co-ordinates.

The set of points in $\mathbb{R}^3$ defined by the system of equations (1.46) may have a very complicated geometric form. Among these forms we single out and study the following *regular case*: The system (1.46) is either inconsistent in $\mathbb{R}^3$ (i.e., it defines the empty set) or it defines a two-dimensional manifold, i.e., equations (1.48) and (1.49) have m $(1 \le m \le 3)$ common roots and, if $1 \leq m < 3$, the remaining two groups of the roots form disjoint sets for arbitrary values of θ and φ. We denote these common roots by $\nu_1(\theta,\varphi), \cdots, \nu_m(\theta,\varphi)$ $(1 \leq m \leq 3)$ and without loss of generality assume that

$$0<\varrho_1(\theta,\varphi)<\varrho_2(\theta,\varphi)<\varrho_3(\theta,\varphi), \;\; 0<\nu_1(\theta,\varphi)<\cdots<\nu_m(\theta,\varphi). \tag{1.50}$$

Thus, in this case the characteristic equation (1.45) (i.e., the system (1.46)) defines analytic (characteristic) surfaces $S_1^c, \cdots, S_m^c$, whose equations in the spherical co-ordinates read as $\varrho = |\omega|\;\nu_k(\theta,\varphi) > 0$, $k = 1, \cdots, m$.

The BVPs corresponding to the case $m = 0$ turned out to be very similar to those of the pseudo-oscillation ones (see Remark 2.7) and therefore in what follows we shall mainly consider the case $1 \leq m \leq 3$.

1.7. From the above arguments it follows that

$$\Psi(\xi,\varkappa)=\Phi(\eta,0)[\varrho^2+\varkappa^2\varrho_1^2(\theta,\varphi)][\varrho^2+\varkappa^2\varrho_2^2(\theta,\varphi)][\varrho^2+\varkappa^2\varrho_3^2(\theta,\varphi)], \tag{1.51}$$

$$\widetilde{\Psi}(\xi,\varkappa)=\widetilde{\Phi}(\eta,0)[\varrho^2+\varkappa^2\widetilde{\varrho}_1^2(\theta,\varphi)][\varrho^2+\varkappa^2\widetilde{\varrho}_2^2(\theta,\varphi)][\varrho^2+\varkappa^2\widetilde{\varrho}_3^2(\theta,\varphi)], \tag{1.52}$$

for any $\xi \in \mathbb{R}^3$ and $\varkappa \in \mathbb{C}$.

Consequently, according to (1.39) we have

$$\begin{aligned} M(\xi,\varkappa)=\Phi(\eta,0)\,\Lambda(\xi)\,[\varrho^2 + \varkappa^2\varrho_1^2(\theta,\varphi)][\varrho^2 + \varkappa^2\varrho_2^2(\theta,\varphi)][\varrho^2 + \varkappa^2\varrho_3^2(\theta,\varphi)]+ \\ +\varkappa c_0\,\widetilde{\Phi}(\eta,0)\,[\varrho^2 + \varkappa^2\widetilde{\varrho}_1^2(\theta,\varphi)][\varrho^2 + \varkappa^2\widetilde{\varrho}_2^2(\theta,\varphi)][\varrho^2 + \varkappa^2\widetilde{\varrho}_3^2(\theta,\varphi)] = \\ = \Phi_m(\varrho,\theta,\varphi;\varkappa)\,\Psi_m(\varrho,\theta,\varphi;\varkappa), \end{aligned} \tag{1.53}$$

where

$$\begin{aligned} \Phi_m(\varrho,\theta,\varphi;\varkappa) = \Phi_m(\xi,\varkappa) = \Phi_m(-\xi,\varkappa) = \Phi_m(\xi,-\varkappa) = \\ = (-1)^m\,[\varrho^2 + \varkappa^2\nu_1^2(\theta,\varphi)]\cdots[\varrho^2 + \varkappa^2\nu_m^2(\theta,\varphi)], \end{aligned} \tag{1.54}$$

$$\begin{aligned} \Psi_m(\varrho,\theta,\varphi;\varkappa) = \Psi_m(\xi,\varkappa) = \Psi_m(-\xi,\varkappa) = \\ = (-1)^m\,\{\Phi(\eta,0)\,\Lambda(\xi)\;[\varrho^2 + \varkappa^2\lambda_1^2(\theta,\varphi)]\cdots[\varrho^2 + \varkappa^2\lambda_{3-m}^2(\theta,\varphi)]+ \\ +\varkappa c_0\,\widetilde{\Phi}(\eta,0)\,[\varrho^2 + \varkappa^2\widetilde{\lambda}_1^2(\theta,\varphi)]\cdots[\varrho^2 + \varkappa^2\widetilde{\lambda}_{3-m}^2(\theta,\varphi)]; \end{aligned} \tag{1.55}$$

here $\lambda_j^2(\theta,\varphi)$ and $\widetilde{\lambda}_j^2(\theta,\varphi)$ denote the different (non-common) roots of the equations (1.48) and (1.49), respectively. Note that formulae (1.51)-(1) are valid for arbitrary $\xi \in \mathbb{R}^3$ and $\varkappa \in \mathbb{C}$.

The multiplier $(-1)^m$ in (1) ensures the inequality

$$\Phi_m(0,-i\omega) > 0 \tag{1.56}$$

which will be employed later on.

Remark 1.2. Note that the polynomial $\Phi_m(\varrho,\theta,\varphi;-i\omega)$ in ϱ vanishes on S_j^c, $j = 1, \cdots, m$ (i.e., when $\varrho = |\omega|\nu_j(\theta,\varrho)$) while $\Psi_m(\varrho,\theta,\varphi;-i\omega)$ is different from zero for any real ϱ and ω. Therefore, for any fixed ω and ϱ_0 there exists a positive number ε_0 such that $|\Psi_m(\varrho,\theta,\varphi;\varkappa)| > 0$ for $|\mathrm{Im}\,\varrho| \leq \varepsilon_0$, $|\mathrm{Re}\,\varkappa| \leq \varepsilon_0$ and $|\varrho| \leq 2\varrho_0$, where $\varrho = \varrho' + i\varrho''$, $\varkappa = \sigma - i\omega$.

Now from equations (1) and (1) it follows that, if $|\mathrm{Re}\,\varkappa| = |\sigma| < \varepsilon_0$ and $|\sigma\,\nu_j(\theta,\varphi)| < \varepsilon_0$, then the complex numbers $\pm(\omega + i\sigma)\nu_j(\theta,\varphi) = \pm i\varkappa\nu_j(\theta,\varphi)$, $j = 1,\cdots,m$, are the only zeros of the polynomial (1) with respect to ϱ in the strip $|\mathrm{Im}\,\varrho| = |\varrho''| < \varepsilon_0$. As a consequence we have that $M(\xi,\varkappa) \neq 0$ for $\xi \in \mathbb{R}^3$ and $0 < |\sigma| = |\mathrm{Re}\,\varkappa| < \varepsilon_0$. □

2. Fundamental Matrices

In this section with the help of the fundamental matrix of the pseudo-oscillation equations we will construct maximally decreasing fundamental matrices of the steady state oscillation operator by limiting absorption principle (cf. [55]).

Denote by $\Gamma(x,\tau)$ a fundamental matrix of the operator $A(D,\tau)$: $A(D,\tau)\Gamma(x,\tau) = I_4\,\delta(x)$, $\tau = \sigma - i\omega$, $\sigma \neq 0$, where $\delta(x)$ is Dirac's distribution.

Let $0 < |\mathrm{Re}\tau| = |\sigma| < \varepsilon_0$ with $\varepsilon_0 > 0$ from Remark 1.2 or $\sigma > 0$. Then due to the representation (1), Remark 1.2, equation (1.33) and Lemma 1.1 we have

$$M(\xi,\tau) \neq 0,\ \ \xi \in \mathbb{R}^3, \qquad [A(-i\xi,\tau)]^{-1} \in L_2(\mathbb{R}^3). \tag{2.1}$$

Therefore, we can represent $\Gamma(x,\tau)$ by the Fourier integral [57]

$$\Gamma(x,\tau) = \mathcal{F}^{-1}_{\xi\to x}\left([A(-i\xi,\tau)]^{-1}\right) = (2\pi)^{-3}\lim_{R\to\infty}\int\limits_{|\xi|<R}[A(-i\xi,\tau)]^{-1}e^{-ix\xi}d\xi. \tag{2.2}$$

By $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ we denote the generalized Fourier and inverse Fourier transforms which for summable functions are defined as follows (see, e.g., [20])

$$\mathcal{F}_{x\to\xi}[f] = \int_{\mathbb{R}^n} f(x)\,e^{ix\xi}\,dx, \qquad \mathcal{F}^{-1}_{\xi\to x}[g] = (2\pi)^{-n}\int_{\mathbb{R}^n} g(\xi)\,e^{-ix\xi}\,d\xi.$$

From the conditions $\sigma \neq 0$ and (2.1), and properties of the Fourier transform it easily follows that the entries of the matrix $\Gamma(x,\tau)$ together with all derivatives decrease more rapidly than any negative power of $|x|$ as $|x| \to +\infty$. The behaviour of this matrix in a neighbourhood of the origin will be established below (see Lemma 2.1) (cf. [23]).

Let h be a cut off function with properties

$$\begin{aligned} &h(\xi) = h(-\xi),\ \ h \in \mathrm{C}^\infty(\mathbb{R}^3),\ \ h(\xi) = 1 \quad \text{for} \quad |\xi| < \varrho_0,\\ &h(\xi) = 0 \quad \text{for} \quad |\xi| > 2\varrho_0 \end{aligned} \tag{2.3}$$

with ϱ_0 from (1.40).

Now we decompose (2.2) into the two parts

$$\Gamma(x,\tau) = \Gamma^{(1)}(x,\tau) + \Gamma^{(2)}(x,\tau),$$

where

$$\begin{aligned} \Gamma^{(1)}(x,\tau) &= \mathcal{F}^{-1}_{\xi\to x}\left([1-h(\xi)]\,[A(-i\xi,\tau)]^{-1}\right), \qquad (2.4)\\ \Gamma^{(2)}(x,\tau) &= \mathcal{F}^{-1}_{\xi\to x}\left(h(\xi)\,[A(-i\xi,\tau)]^{-1}\right) = \end{aligned}$$

$$= (2\pi)^{-3} \int_{|\xi|<2\varrho_0} h(\xi)\, [A(-i\xi,\tau)]^{-1}\, e^{-ix\xi}\, d\xi. \tag{2.5}$$

The main result of this section will follow from two the lemmata which we now present.

Let $\Gamma^{(0)}(x)$ be the homogeneous (of order -1) fundamental matrix of the operator $C(D)$ (see [55], [56])

$$\Gamma^{(0)}(x) = \mathcal{F}^{-1}_{\xi\to x}\ \big([C(-i\xi)]^{-1}\big) = (-8\pi^2|x|)^{-1} \int\limits_0^{2\pi} [C(a\eta)]^{-1}\, d\varphi, \tag{2.6}$$

where $x \in \mathbb{R}^3 \setminus \{0\}$, $a = [a_{kj}]_{3\times 3}$ is an orthogonal matrix with property $a^\top x^\top = (0,0,|x|)^\top$, $\eta = (\cos\varphi, \sin\varphi, 0)^\top$. Further, let $\gamma^{(0)}(x)$ be the homogeneous (of order -1) fundamental function of the operator $\Lambda(D)$ (see [52])

$$\gamma^{(0)}(x) = \mathcal{F}^{-1}_{\xi\to x}\ \big([\Lambda(-i\xi)]^{-1}\big) = -[4\pi\, |L|^{1/2} (L^{-1}x\cdot x)^{1/2}]^{-1} \tag{2.7}$$

with $L = [\lambda_{pq}]_{3\times 3}$, $|L| = \det L$.

Lemma 2.1. *The entries of the matrix $\Gamma^{(1)}(x,\tau)$ belong to $C^\infty(\mathbb{R}^3 \setminus \{0\})$ and for an arbitrary $\sigma \in [-\varepsilon_0, \varepsilon_0]$ together with all derivatives decrease more rapidly than any negative power of $|x|$ as $|x| \to +\infty$.*

The limit

$$\lim_{\sigma\to 0} D^\beta_x \Gamma^{(1)}(x, \sigma - i\omega) = D^\beta_x \Gamma^{(1)}(x, -i\omega)$$

exists uniformly for $|x| > \delta$ with an arbitrary $\delta > 0$ and in a neigbourhood of the origin (say $|x| < 1/2$) the following inequalities

$$|D^\beta_x \Gamma^{(1)}_{kj}(x, \sigma - i\omega) - D^\beta_x \Gamma^{(1)}_{kj}(x, -i\omega)| \le |\sigma|\, c\, \varphi^{(kj)}_{|\beta|}(x),$$
$$|D^\beta_x \Gamma^{(1)}_{kj}(x, \sigma - i\omega) - D^\beta_x \Gamma_{kj}(x)| \le c\, \varphi^{(kj)}_{|\beta|}(x)$$

hold, where $c = const > 0$ does not depend on σ,

$$\Gamma(x) = \begin{bmatrix} [\Gamma^{(0)}(x)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \gamma^{(0)}(x) \end{bmatrix}_{4\times 4}, \tag{2.8}$$

$$\varphi^{(kj)}_0(x) = 1,\ \ \varphi^{(kj)}_1(x) = -\ln|x|,\ \ \varphi^{(kj)}_l(x) = |x|^{1-l},\ \ l \ge 2,$$

for $1 \le k, j \le 3$ and $k = j = 4$;

$$\varphi^{(k4)}_0(x) = \varphi^{(4k)}_0(x) = -\ln|x|,\ \ \varphi^{(k4)}_m(x) = \varphi^{(4k)}_m(x) = |x|^{-m},\ \ m \ge 1,$$

for $k = 1,2,3$; β is an arbitrary multi-index.

Proof. Note that the relations $D^\beta [A(-i\xi,\tau)]^{-1}_{kj} = O([1+|\xi|]^{-2-|\beta|})$ and

$$[A(-i\xi,\tau)]^{-1} = \begin{bmatrix} [(C(-i\xi))^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & [\Lambda(-i\xi)]^{-1} \end{bmatrix} +$$
$$+ \begin{bmatrix} [O(|\xi|^{-4})]_{3\times 3} & [O(|\xi|^{-3})]_{3\times 1} \\ [O(|\xi|^{-3})]_{1\times 3} & O(|\xi|^{-4}) \end{bmatrix},$$

hold for sufficiently large $|\xi|$.

Now the proof follows from Lemma 1.1, equations (2.6), (2.7), and properties of the Fourier transform of homogeneous functions (see, for example, [20], [54], Lemma 2.17, [55], Lemma 3.1). $\square$

Now we analyse properties of the matrix $\Gamma^{(2)}(x,\tau)$.

Going to the spherical co-ordinates in the integral (2) we get

$$\Gamma^{(2)}(x,\tau) = (2\pi)^{-3} \int_{\Sigma_1} d\Sigma_1 \Big\{ \int\limits_0^{\varrho_0} + \int\limits_{\varrho_0}^{2\varrho_0} \Big\} h(\xi)[A(-i\xi,\tau)]^{-1}\, e^{-ix\xi} \varrho^2\, d\varrho, \qquad (2.9)$$

where Σ_1 is the unit sphere in $\mathbb{R}^3$ centered at the origin.

Taking into account Remark 1.2, the analyticity of the integrand with respect to ϱ, and introducing the complex $\varrho = \varrho' + i\varrho''$ plane we can rewrite (2.9) by Cauchy theorem as follows

$$\Gamma^{(2)}(x,\tau) = (2\pi)^{-3} \int_{\Sigma_1} d\Sigma_1 \Big\{ \int\limits_{l^{\pm}} [A(-i\xi,\tau)]^{-1}\, e^{-ix\xi} \varrho^2\, d\varrho +$$

$$+ \int\limits_{\varrho_0}^{2\varrho_0} h(\xi)\,[A(-i\xi,\tau)]^{-1}\, e^{-ix\xi} \varrho^2\, d\varrho \Big\}, \qquad (2.10)$$

where $l^{\pm} = [0, |\omega|\nu_1 - \delta] \cup l^{\pm}_{1,\delta} \cup [|\omega|\nu_1 + \delta, |\omega|\nu_2 - \delta] \cup l^{\pm}_{2,\delta} \cup \cdots \cup l^{\pm}_{m,\delta} \cup [|\omega|\nu_m + \delta, \varrho_0]$, $\delta > 0$ is a sufficiently small number, $l^{+}_{j,\delta}\,[l^{-}_{j,\delta}]$ is the semicircle in the upper [lower] half-plane centered at $|\omega|\nu_j$ and radius δ oriented clockwise [counter-clockwise]; in (2.10) the contour $l^{+}\,[l^{-}]$ corresponds to the case $\sigma\omega < 0$ $[\sigma\omega > 0]$.

Now passing to the limit in (2.10) as $\sigma \to 0\pm$ we get

$$\lim_{\sigma\to 0} \Gamma^{(2)}(x, \sigma - i\omega) =$$

$$= (2\pi)^{-3} \int\limits_{\Sigma_1} d\Sigma_1 \Big\{ \int\limits_{l^-} [A(-i\xi,-i\omega)]^{-1}\, e^{-ix\xi} \varrho^2\, d\varrho +$$

$$+ \int\limits_{\varrho_0}^{2\varrho_0} h(\xi)\,[A(-i\xi,-i\omega)]^{-1}\, e^{-ix\xi} \varrho^2\, d\varrho \Big\} =: \Gamma^{(2)}_{+}(x,-i\omega), \quad \sigma\omega > 0, \quad (2.11)$$

$$\lim_{\sigma\to 0} \Gamma^{(2)}(x, \sigma - i\omega) =$$

$$= (2\pi)^{-3} \int\limits_{\Sigma_1} d\Sigma_1 \Big\{ \int\limits_{l^+} [A(-i\xi,-i\omega)]^{-1}\, e^{-ix\xi} \varrho^2\, d\varrho +$$

$$+ \int\limits_{\varrho_0}^{2\varrho_0} h(\xi)\,[A(-i\xi,-i\omega)]^{-1}\, e^{-ix\xi} \varrho^2\, d\varrho \Big\} =: \Gamma^{(2)}_{-}(x,-i\omega), \quad \sigma\omega < 0. \quad (2.12)$$

These limits exist uniformly for $|x| < R_0$ with an arbitrary R_0.

Such type of integrals have been studied in [55]. Applying the arguments quite similar to that of [55] we arrive at the formulae

$$\Gamma^{(2)}_{\pm}(x,-i\omega) = (2\pi)^{-3} \Big[\lim_{\delta\to 0} \int\limits_{|\Phi_m|>\delta} h(\xi)\,[A(-i\xi,-i\omega)]^{-1}\, e^{-ix\xi}\, d\xi \,\pm$$

$$\pm i\pi \sum_{j=1}^{m} \int\limits_{\Sigma_1} \left\{ \frac{N(-i\xi, -i\omega)e^{-ix\xi}\varrho^2}{[\partial/\partial\varrho\, \Phi_m(\varrho,\theta,\varphi;-i\omega)]\Psi_m(\varrho,\theta,\varphi;-i\omega)} \right\}_{\varrho=|\omega|\nu_j} d\Sigma_1 \Big], \quad (2.13)$$

where Φ_m and Ψ_m are defined by (1) and (1), respectively.

We need to go over to the integrals over S_j^c in the last summand of (2). To this end let us note that the exterior unit normal of S_j^c is defined by the equation

$$n(\xi) = (-1)^j \frac{\nabla_\xi \Phi_m(\xi, -i\omega)}{|\nabla_\xi \Phi_m(\xi, -i\omega)|}, \ \xi \in S_j^c, \ j = 1, \dots, m,$$

since due to (1), (1.50) and(1.56)

$$(-1)^j [\partial/\partial\varrho \Phi_m(\xi, -i\omega)]_{\varrho=|\omega|\nu_j} > 0, \ j = 1, \dots, m. \quad (2.14)$$

Further,

$$d\Sigma_1 = \left[\frac{\xi/|\xi| \cdot n(\xi)}{\varrho^2} \right]_{\varrho=|\omega|\nu_j} dS_j^c = (-1)^j \left[\frac{\partial/\partial\varrho \Phi_m(\xi, -i\omega)}{\varrho^2 |\nabla \Phi_m(\xi, -i\omega)|} \right]_{\varrho=|\omega|\nu_j} dS_j^c.$$

Therefore, (2) implies

$$\Gamma_\pm^{(2)}(x, -i\omega) = (2\pi)^{-3} \Big[\text{V.P.} \int_{\mathbb{R}^3} h(\xi)\, [A(-i\xi, -i\omega)]^{-1}\, e^{-ix\xi}\, d\xi \pm$$

$$\pm i\pi \sum_{j=1}^{m} (-1)^j \int\limits_{S_j^c} \frac{N(-i\xi, -i\omega)e^{-ix\xi}}{|\nabla \Phi_m(\xi, -i\omega)| \Psi_m(\xi, -i\omega)}\, dS_j^c \Big], \quad (2.15)$$

where

$$\text{V.P.} \int\limits_{\mathbb{R}^3} h(\xi)\, [A(-i\xi, -i\omega)]^{-1}\, e^{-ix\xi}\, d\xi =$$

$$= \lim_{\delta \to 0} \int\limits_{|\Phi_m(\xi, -i\omega)| > \delta} h(\xi)\, [A(-i\xi, -i\omega)]^{-1}\, e^{-ix\xi}\, d\xi.$$

Existence and asymptotic behaviour of integrals similar to the above ones are investigated in [21], [81], [82]. Namely, in [81] there are analysed the following functions (n-dimensional version of the case in question)

$$I_j(x) = \int\limits_{S_j^c} \frac{f(\xi)e^{ix\xi}}{|\nabla \Phi_m(\xi)|}\, dS_j^c, \ \ j = 1, \dots, m, \quad (2.16)$$

$$J(x) = \text{V.P.} \int_{\mathbb{R}^n} \frac{f(\xi)e^{ix\xi}}{\Phi_m(\xi)}\, d\xi, \ \ n \geq 2, \quad (2.17)$$

where

i) $\operatorname{diam}(\operatorname{supp} f) < \infty$; $f, \Phi_m \in C^\infty(\mathbb{R}^n)$,

ii) the equation $\Phi_m(\xi) = 0$, $\xi \in \mathbb{R}^n$, defines $(n-1)$-dimensional closed nonselfintersecting surfaces S_j^c, $j = 1, \dots, m$, with the Gaussian curvature different from zero everywhere; moreover, $\nabla \Phi_m(\xi) \neq 0$ for $\xi \in S_j^c$;

iii) for an arbitrary unit vector η the system

$$\begin{cases} \Phi_m(\xi) = 0, \\ \nabla \Phi_m(\xi) |\nabla \Phi_m(\xi)|^{-1} = \pm \eta, \end{cases} \quad (2.18)$$

has only a finite number of solutions with respect to ξ.

Clearly, in the case under consideration the above conditions for the functions occured in (2) are fulfilled due to (2.3) and I^0-II^0. Moreover, $\Phi_m(\xi,-i\omega)=\Phi_m(\xi,i\omega)=\Phi_m(-\xi,i\omega)$, and the corresponding system of type (2.18) defines $2m$ points $\pm\xi^j\in S_j^c$ $j=1,\dots,m$ (the so–called stationary points); we emphasize also that the unit exterior normal vector $n(\xi^j)$ has the same direction as η, while $n(-\xi^j)$ is opposite directed.

We assume the function $\Phi_m(\xi)$ in (2.16) and (2.17) to possess the analogous symmetry property with respect to ξ.

Now let $|x|$ be sufficiently large, $\eta=x/|x|$, and let $\pm\xi^j\in S_j^c$, $j=1,\dots,m$, be the stationary points corresponding to η, i.e., $n(\xi^j)=\eta$, $n(-\xi^j)=-n(\xi^j)=-\eta$.

According to the results in references [21], [81], we have then the following asymptotic formulae for the functions I_j and J:

$$\begin{aligned} I_j(x)&=[a_je^{ix\xi^j}+\widetilde{a}_je^{-ix\xi^j}]\,|x|^{-(n-1)/2}+O(|x|^{-(n+1)/2}),\\ J(x)&=\sum_{j=1}^m[b_je^{ix\xi^j}+\widetilde{b}_je^{-ix\xi^j}]\,|x|^{-(n-1)/2}+O(|x|^{-(n+1)/2}), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} a_j&=a_j(\xi^j)=(2\pi)^{(n-1)/2}\frac{1}{[\kappa(\xi^j)]^{1/2}}\,\frac{f(\xi^j)}{|\nabla\Phi_m(\xi^j)|}\,e^{-i(n-1)\pi/4},\\ \widetilde{a}_j&=\widetilde{a}_j(-\xi^j)=(2\pi)^{(n-1)/2}\frac{1}{[\kappa(-\xi^j)]^{1/2}}\,\frac{f(-\xi^j)}{|\nabla\Phi_m(-\xi^j)|}\,e^{i(n-1)\pi/4},\\ b_j&=i\pi a_j\operatorname{sgn}(\eta\cdot\nabla\Phi_m(\xi^j))=i\pi(-1)^ja_j,\\ \widetilde{b}_j&=i\pi\widetilde{a}_j\operatorname{sgn}(\eta\cdot\nabla\Phi_m(-\xi^j))=-i\pi(-1)^j\widetilde{a}_j, \end{aligned} \tag{2.20}$$

$\kappa(\xi)$ is the Gaussian curvature at the point $\xi\in S_j^c$.

The asymptotic formulae (2.19) can be differentiated any times with respect to x.

It is easy to see that the symmetry properties of S_j^c imply

$$\kappa(\xi)=\kappa(-\xi),\quad \nabla\Phi_m(-\xi)=-\nabla\Phi_m(\xi) \tag{2.21}$$

for any $\xi\in S_j^c$, $j=1,\dots,m$.

By virtue of (2.16), (2.17), and (2.19) we derive

$$\begin{aligned} J(x)+\lambda\sum_{j=1}^m i\pi(-1)^jI_j(x)&=\sum_{j=1}^m i\pi(-1)^j[(1+\lambda)a_je^{ix\xi^j}-\\ &-(1-\lambda)\widetilde{a}_je^{-ix\xi^j}]\,|x|^{-(n-1)/2}+O(|x|^{-(n+1)/2}) \end{aligned} \tag{2.22}$$

with a_j and $\widetilde{a}_j$ defined by (2.20) and an arbitrary λ.

Lemma 2.2. *Entries of matrices* (2) *belong to* $C^\infty(\mathbb{R}^3)$ *and for sufficiently large* $|x|$ *the asymptotic formulae*

$$\Gamma^{(2)}_{\pm}(x,-i\omega)=\sum_{j=1}^{m} c^{(j)}_{\pm}(\xi^j,-i\omega)\,e^{\pm ix\xi^j}|x|^{-1}+O(|x|^{-2}) \tag{2.23}$$

hold, where the point $\xi^j\in S^c_j$ *corresponds to* x (*i.e.*, $n(\xi^j)=x/|x|$) *and*

$$\begin{aligned} c^{(j)}_{+}=c^{(j)}_{1}(\xi^j,-i\omega)&:=(-1)^j\frac{1}{2\pi[\kappa(\xi^j)]^{1/2}}\,\frac{N(i\xi^j,-i\omega)}{|\nabla\Phi_m(\xi^j,-i\omega)|\Psi_m(\xi^j,-i\omega)},\\ c^{(j)}_{-}=c^{(j)}_{2}(\xi^j,-i\omega)&:=(-1)^j\frac{1}{2\pi[\kappa(\xi^j)]^{1/2}}\,\frac{N(-i\xi^j,-i\omega)}{|\nabla\Phi_m(\xi^j,-i\omega)|\Psi_m(\xi^j,-i\omega)}; \end{aligned} \tag{2.24}$$

moreover, (2.23) *can be differentiated any times with respect to* x.

Proof. The first part of the lemma is evident due to (2.3) and I^0–II^0. To prove the asymptotic formulae (2.23), we first perform the change of variable ξ by $-\xi$ in (2) and afterwards rewrite it as follows

$$\Gamma^{(2)}_{\pm}(x,-i\omega)=(2\pi)^{-3}[J(x)\pm\sum_{j=1}^{m} i\pi(-1)^j I_j(x)], \tag{2.25}$$

where $I_j(x)$ and $J(x)$ are given by (2.16) and (2.17), respectively, with $n=3$; moreover,

$$f(\xi)=\frac{h(\xi)N(i\xi,-i\omega)}{\Psi_m(\xi,-i\omega)}, \tag{2.26}$$

$h(\xi)$, $\Phi_m(\xi,-i\omega)$, and $\Psi_m(\xi,-i\omega)$ are defined by (2.3), (1), and (1), respectively; here we have used the fact that h, Φ_m, and Ψ_m are even functions in ξ.

Now (2.23) follows from (2.25), (2), (2.21), (2.26), and (2.20). □

Thus, we have proved that there exist one sided limits of the matrix (2.2) as $\mathrm{Re}\tau=\sigma\to 0\pm$.

Let us set

$$\sigma\omega>0:\ \lim_{\sigma\to 0}\Gamma(x,\sigma-i\omega)=\Gamma^{(1)}(x,-i\omega)+\Gamma^{(2)}_{+}(x,-i\omega)=:\Gamma(x,\omega,1), \tag{2.27}$$

$$\sigma\omega<0:\ \lim_{\sigma\to 0}\Gamma(x,\sigma-i\omega)=\Gamma^{(1)}(x,-i\omega)+\Gamma^{(2)}_{-}(x,-i\omega)=:\Gamma(x,\omega,2), \tag{2.28}$$

where $\Gamma^{(1)}$, $\Gamma^{(2)}_{+}$ and $\Gamma^{(2)}_{-}$ are given by (2.4), (2.11) and (2.12), respectively.

Combining the two latter formulae we have

$$\begin{aligned} \Gamma(x,\omega,r)&=\mathcal{F}^{-1}_{\xi\to x}[(1-h(\xi))\{A(-i\xi,-i\omega)\}^{-1}]+\\ &+(2\pi)^{-3}\,\mathrm{V.P.}\int_{\mathbb{R}^3} h(\xi)\{A(-i\xi,-i\omega)\}^{-1}e^{-ix\xi}\,d\xi+\\ &+(-1)^{r+1}\frac{i\pi}{(2\pi)^3}\sum_{j=1}^{m}(-1)^j\int_{S^c_j}\frac{N(-i\xi,-i\omega)e^{-ix\xi}}{|\nabla\Phi_m(\xi,-i\omega)|\Psi_m(\xi,-i\omega)}dS^c_j,\ r=1,2. \end{aligned} \tag{2.29}$$

Now we formulate the main result of this section.

Theorem 2.3. *The matrix–functions* $\Gamma(x,\omega,r)$, $r=1,2$, *defined by* (2), *are fundamental matrices of the operator* $A(D,-i\omega)$ *and satisfy the following conditions:*

i) $\Gamma(\cdot,\omega,r)\in C^{\infty}(\mathbb{R}^3\setminus\{0\})$ *and in a neighbourhood of the origin* ($|x|<1/2$)

$$|D_x^{\beta}\Gamma_{kj}(x,\omega,r)-D_x^{\beta}\Gamma_{kj}(x)|\le c\,\varphi_{|\beta|}^{(kj)}(x),\ c=const>0,\ k,j=1,\dots,4,$$

where $\Gamma_{kj}(x)$, $\varphi_{|\beta|}^{(kj)}$, $c=const>0$ *and* β *are the same as in Lemma* 2.1;

ii) for sufficiently large $|x|$

$$\Gamma(x-y,\omega,r)=\sum_{j=1}^{m}c_r^{(j)}(\xi^j,-i\omega)\ e^{(-1)^{r+1}i(x-y)\xi^j}\ |x|^{-1}+O(|x|^{-2}),\quad (2.30)$$

where $c_r^{(j)}$ *are defined by* (2.24), $\xi^j\in S_j^c$ *corresponds to the vector* x *and the range of the variable* y *is a bounded subset of* $\mathbb{R}^3$; *the equation* (2.30) *can be differentiated any times with respect to* x *and* y.

Proof. It follows immediately from Lemmata 2.1 and 2.2. □

Remark 2.4. Note that, if in (2.30) the vector $(x-y)$ is replaced by $-(x-y)$, then the point ξ^j is to be changed by $-\xi^j$, simultaneously, since to the vector $-x$ there corresponds the point $-\xi^j\in S_j^c$ $(-x/|x|=n(-\xi^j))$. As a result the exponential factor in (2.30) will not be changed. □

Remark 2.5. The fundamental matrix of the adjoint operator $A^*(D,\tau)$, clearly, has the form

$$\begin{aligned}\Gamma^*(x,\tau)&=\mathcal{F}^{-1}_{\xi\to x}[\{A^*(-i\xi,\tau)\}^{-1}]=\mathcal{F}^{-1}_{\xi\to x}[\{A^{\top}(i\xi,\overline{\tau})\}^{-1}]=\\&=\mathcal{F}^{-1}_{\xi\to x}[\{\overline{A^{\top}(-i\xi,\tau)}\}^{-1}]=(2\pi)^{-3}\,\overline{\int_{\mathbb{R}^3}[A^{\top}(-i\xi,\tau)]^{-1}\,e^{ix\xi}\,d\xi}=\\&=\overline{\Gamma^{\top}(-x,\tau)},\ \ \tau=\sigma-i\omega,\ \ \sigma\neq 0,\end{aligned}\quad (2.31)$$

where $\Gamma(x,\tau)$ is given by (2.2).

Therefore, there exist limits similar to (2.27) and (2.28)

$$\Gamma^*(x,\omega,r)=\lim_{\sigma\to 0}\Gamma^*(x,\tau)=\lim_{\sigma\to 0}\overline{\Gamma^{\top}(-x,\tau)}=\overline{\Gamma^{\top}(-x,\omega,r)},\ \ r=1,2,\quad (2.32)$$

where $(-1)^{r+1}\sigma\omega>0$ is assumed.

The entries of matrix (2.5) and their derivatives decrease more rapidly then any negative power of $|x|$ as $|x|\to+\infty$ if $0<|\sigma|<\varepsilon_0$ (see Remark 1.2). The asymptotic formulae for $\Gamma^*(x,\omega,r)$ follow from (2.32) and Theorem 2.3

$$\Gamma^*(x,\omega,r)=\sum_{j=1}^{m}\widetilde{c}_r^{(j)}e^{(-1)^r ix\xi^j}\ |x|^{-1}+O(|x|^{-2}),$$

where $|x|$ is sufficiently large, $\widetilde{c}_r^{(j)}=\overline{[c_r^{(j)}(-\xi^j,-i\omega)]}^{\top}$ with $c_r^{(j)}$ defined by (2.24), and $\xi^j\in S_j^c$ corresponds to x.

From Lemmata 2.1, 2.2, and Theorem 2.3 together with the equations (2.5), (2.32), and $\Gamma(x)=\overline{\Gamma(x)}=\Gamma^{\top}(x)=\Gamma(-x)$, $\Gamma(tx)=t^{-1}\Gamma(x)$, $t>0$, we infer that the matrices $\Gamma(x,\tau)$, $\Gamma(x,\omega,r)$, $\Gamma^*(x,\tau)$, and $\Gamma^*(x,\omega,r)$ have the matrix $\Gamma(x)$ as the dominant singular part in a neighbourhood of the origin. $\square$

Remark 2.6. Equation (2.30) implies the following representation

$$\Gamma(x-y,\omega,r)=\sum_{j=1}^{m}\overset{(j)}{\Gamma}(x-y,\omega,r),$$

where for sufficiently large $|x|$

$$\overset{(j)}{\Gamma}(x-y,\omega,r)=c_r^{(j)}\,e^{(-1)^{r+1}i(x-y)\xi^j}\,|x|^{-1}+O(|x|^{-2}),$$
$$D_{x_p}\overset{(j)}{\Gamma}(x-y,\omega,r)+i(-1)^r\,\xi_p^j\,\overset{(j)}{\Gamma}(x-y,\omega,r)=O(|x|^{-2}),$$
$$j=1,\dots,m,\ \ p=1,2,3,\ \ r=1,2,$$

$\xi^j\in S_j^c$ corresponds to x and the range of y is again a bounded subset of $\mathbb{R}^3$; here the matrices $c_r^{(j)}$ are given by (2.24). $\square$

Remark 2.7. If the system of equations (1.46) is inconsistent in $\mathbb{R}^3$ for some $\omega>0$, then $M(\xi,-i\omega)=\det A(-i\xi,-i\omega)\neq 0$ for arbitrary $\xi\in\mathbb{R}^3$ and $\omega\in\mathbb{R}$, and

$$\Gamma(x,-i\omega)=\mathcal{F}^{-1}_{\xi\to x}\left([A(-i\xi,-i\omega)]^{-1}\right)\in \mathrm{C}^\infty(\mathbb{R}^3\setminus\{0\}) \tag{2.33}$$

is a fundamental matrix of the operator $A(D,-i\omega)$ whose entries together with all derivatives decrease more rapidly than any negative power of $|x|$ as $|x|\to+\infty$. The main singular part of (2.33) in a neighbourhood of the origin is again the matrix $\Gamma(x)$. Therefore this case is very similar to the pseudo-oscillation one [57]. $\square$

3. Thermo-Radiation Conditions. Somigliana Type Integral Representations

In this section we formulate the generalized Sommerfeld-Kupradze type radiation conditions in the thermoelasticity theory of anisotropic bodies and derive Somigliana type integral representation formulae.

3.1. Let us introduce the classes $\mathrm{SK}_r^m(\Omega^-)$ of vector-functions defined on an unbounded domain of type Ω^- (which is the complement to a compact region $\overline{\Omega^+}$ in $\mathbb{R}^3$).

A vector-function $U=(u_1,u_2,u_3,u_4)^\top$ belongs to the class $\mathrm{SK}_r^m(\Omega^-)$, $r=1,2$, if it is C^1-smooth in Ω^-, and for sufficiently large $|x|$ the following relations hold (no summation over the repeated index j in the last equation)

$$U(x)=\sum_{j=1}^{m}\overset{(j)}{U}(x),\qquad \overset{(j)}{U}(x)=(\overset{(j)}{u}_1,\cdots,\overset{(j)}{u}_4)^\top=O(|x|^{-1}),$$

$$D_p\overset{(j)}{U}(x)+i(-1)^r\,\xi_p^j\,\overset{(j)}{U}(x)=O(|x|^{-2}),\ p=1,2,3,\ j=1,\dots,m, \tag{3.1}$$

where $\xi^j\in S_j^c$ corresponds to the vector x.

Clearly, this definition is essentially related to the operator $A(D,-i\omega)$ and its characteristic equation (1.45). The conditions (3.1) will be referred to as *generalized Sommerfeld-Kupradze type radiation conditions* in the thermoelasticity theory of anisotropic bodies (cf. [45]).

A four-dimensional vector $U=(u_1,\cdots,u_4)^\top$, satisfying conditions (3.1), will also be referred to as $(m,r)-$*thermo-radiating vector*. We say that a 4×4 matrix belongs to the class $\mathrm{SK}_r^m(\Omega^-)$ if each column of the matrix is a $(m,r)-$*thermo-radiating vector*.

Remark 2.6 implies that $\Gamma(\cdot,\omega,r)\in \mathrm{SK}_r^m(\mathbb{R}^3\setminus\{0\})$.

In the isotropic case $m=1$ and S_1^c is defined by the equation $\varrho^2=k_1^2$ with $k_1^2=\omega^2\mu^{-1}$ (μ is the Lamé constant and ω is the oscillation parameter). Therefore the point $\xi^1\in S_1^c$, which corresponds to the given direction (vector) x, is given by $\xi^1=k_1\eta$, $\eta=x/|x|$, and conditions (3.1) are equivalent to the well-known thermoelastic radiation conditions (see, e.g., [45], Ch. III).

3.2. Let $U=(u_1,\cdots,u_4)^\top$ be a regular vector-function in $\Omega^\pm$, i.e., $U\in \mathrm{C}^2(\Omega^\pm)\cap \mathrm{C}^1(\overline{\Omega^\pm})$.

In addition, let $A(D,\tau)U\in L_1(\Omega^\pm)$ and conditions (1.30) be satisfied (in the case of the domain Ω^-). If we assume that either $0<|\mathrm{Re}\tau|=|\sigma|<\varepsilon_0$ or $\sigma>0$, and use the identity (1.22), by standard arguments we obtain the following integral representation formulae (see, for example, [56], [16])

$$\int\limits_{\Omega^\pm}\Gamma(x-y,\tau)A(D_y,\tau)U(y)dy \pm \int\limits_{S}\big\{[Q(D_y,n(y),\tau)\Gamma^\top(x-y,\tau)]^\top[U(y)]^\pm -$$
$$-\Gamma(x-y,\tau)[B(D_y,n(y))U(y)]^\pm\big\}\,dS_y=\begin{cases} U(x), & x\in\Omega^\pm,\\ 0, & x\in\Omega^\mp,\end{cases} \qquad (3.2)$$

where boundary operators B and Q are given by (1.25) and (1.26), respectively, and the fundamental matrix $\Gamma(x,\tau)$ is defined by (2.2); $n(y)$ is the outward unit normal vector of S at the point $y\in S$ and S is a C^2-smooth surface.

From the representation formula (3) it follows that any solution of equation (1.9) for $\sigma>0$, satisfying the condition (1.29), actually, is a C^∞-regular in $\Omega^\pm$ vector-function which decrease, together with all derivatives, more rapidly than any negative power of $|x|$ as $|x|\to+\infty$.

Due to Theorem 2.3 and equalities (2.27), (2.28) analogous representation formulae can be written by means of the fundamental matrices $\Gamma(x,\omega,r)$ in the case of the domain Ω^+. One needs only to replace $A(D,\tau)$ and $\Gamma(x,\tau)$ in (3) by $A(D,-i\omega)$ and $\Gamma(x,\omega,r)$, respectively. Concerning the domain Ω^- we will prove the following proposition.

Theorem 3.1. *Let $\partial\Omega^-=S$ be a C^2-smooth surface and U be a regular $(m,r)-$thermo-radiating vector in Ω^-, i.e., $U\in \mathrm{C}^2(\Omega^-)\cap \mathrm{C}^1(\overline{\Omega^-})\cap \mathrm{SK}_r^m(\Omega^-)$. Let, in addition, $A(D,-i\omega)U$ have a compact support and belong to the space $\mathrm{L}_1(\Omega^-)$. Then*

$$U(x)=\int\limits_{\Omega^-}\Gamma(x-y,\omega,r)\,A(D_y,-i\omega)U(y)\,dy+$$

$$+\int_S \{\Gamma(x-y,\omega,r)[B(D_y,n(y))U(y)]^- -$$

$$-[Q(D_y,n(y),-i\omega)\Gamma^\top(x-y,\omega,r)]^\top[U(y)]^-\}\;dS_y,\; x\in\Omega^-; \qquad (3.3)$$

here B, Q and n are the same as in (3).

Proof. Let R be a sufficiently large positive number and $\overline{\Omega^+}\subset B_R := \{x\in\mathbb{R}^3 : |x|<R\}$. We assume also that $\operatorname{supp} A(D,-i\omega)U\subset B_R$. Denote $\Omega_R^- = \Omega^-\cap B_R$ and $\partial B_R = \Sigma_R$. Then the vector-function U is regular in Ω_R^-. Therefore, we can write the following integral representation (cf. (3))

$$U(x) = \int_{\Omega_R^-}\Gamma(x-y,\omega,r)\,A(D_y,-i\omega)U(y)\,dy+$$

$$+\Big\{\int_{\Sigma_R}-\int_S\Big\}\{[Q(D_y,n(y),-i\omega)\Gamma^\top(x-y,\omega,r)]^\top[U(y)]-$$

$$-\Gamma(x-y,\omega,r)[B(D_y,n(y))U(y)]\}\,dS_y,\; x\in\Omega_R^-, \qquad (3.4)$$

where $n(y)$ is the exterior normal on the both surfaces S and Σ_R; clearly, $n(y)=y/R$ for $y\in\Sigma_R$. Note that in the first integral the domain Ω_R^- can be replaced by Ω^-, since $A(D_y,-i\omega)U$ has a compact support.

Our goal is to show that the integral over Σ_R tends to zero as $R\to+\infty$.

To this end, denote the right-hand side expression in (3.1) by $\mathcal{T}[U]$. Then by integrating of (3) from ν to 2ν with respect to R and deviding the result by ν, we get $U(x)=\mathcal{T}[U](x)+X(\nu)$, where

$$X(\nu)=\frac{1}{\nu}\int_\nu^{2\nu}dR\int_{\Sigma_R}\{[Q(D_y,\eta,-i\omega)\Gamma^\top(x-y,\omega,r)]^\top[U(y)]-$$

$$-\,\Gamma(x-y,\omega,r)[B(D_y,\eta)U(y)]\}\,d\Sigma_R,\quad \eta=n(y)=y/R.$$

Next we prove that $X(\nu)\to 0$ as $\nu\to+\infty$.

It can be done by applying the arguments similar to that of [80]. In fact, for definiteness, let $r=1$. Then due to the thermo-radiation conditions (3.1)

$$B(D_y,\eta)U(y)=\sum_{j=1}^m B(i\xi^j,\eta)\overset{(j)}{U}(y)+O(R^{-2}),$$

where $\xi^j\in S_j^c$ corresponds to the vector η.

According to Remarks 2.4, 2.6, and Theorem 2.3 analogous formulae hold also for $[Q(D_y,\eta,-i\omega)\Gamma^\top(x-y,\omega,1)]^\top$ and $\Gamma(x-y,\omega,1)$ (note that x is some fixed point in Ω_R^-). The terms corresponding to $O(R^{-3})$ in the expression of $X(\nu)$ decay as $O(\nu^{-1})$, while all other summands have the following structure

$$v_{st}(\nu)=\frac{1}{\nu}\int_\nu^{2\nu}dR\int_{\Sigma_1}\psi(\eta)\,g_s(R\eta)\,h_t(R\eta)\,R^2\,d\Sigma_1,$$

where $\psi \in \mathrm{C}^\infty(\Sigma_1)$, $\eta \in \Sigma_1$, g_s and h_t $(s,t = 1,\cdots,m)$ are smooth functions satisfying the following inequalities

$$\begin{aligned} |g_s(R\eta)| &< c\,R^{-1}, \qquad |\partial/\partial R\, g_s(R\eta) - i\mu_s(\eta) g_s(R\eta)| < c\,R^{-2}, \\ |h_t(R\eta)| &< c\,R^{-1}, \qquad |\partial/\partial R\, h_t(R\eta) - i\mu_t(\eta) h_t(R\eta)| < c\,R^{-2}, \\ &\mu_j(\eta) = (\eta \cdot \xi^j) > 0, \quad c = const > 0, \end{aligned}$$

due to (3.1).

The last inequality is a consequence of (2.14), since

$$(\eta \cdot \xi^j) = (n(\xi^j)\cdot \xi^j) = (-1)^j \left(\frac{\nabla \Phi_m(\xi^j, -i\omega)}{|\nabla \Phi_m(\xi^j, -i\omega)|} \cdot \xi^j \right) =$$
$$= (-1)^j \frac{|\xi^j|}{|\nabla \Phi_m(\xi^j, -i\omega)|} \left[\frac{\partial}{\partial |\xi|} \Phi_m(\xi^j, -i\omega) \right]_{\xi = \xi^j} > 0.$$

Now we proceed as follows

$$v_{st}(\nu) = \frac{1}{i\nu} \int_\nu^{2\nu} dR \int_{\Sigma_1} \frac{\psi(\eta)}{\mu_s(\eta) + \mu_t(\eta)} [i\mu_s(\eta)\, g_s(R\eta)\, h_t(R\eta) +$$
$$+ g_s(R\eta)\, i\mu_t(\eta)\, h_t(R\eta)]\, R^2\, d\Sigma_1 =$$
$$= \frac{1}{i\nu} \int_{\Sigma_1} d\Sigma_1 \int_\nu^{2\nu} \left\{ \frac{\psi(\eta)}{\mu_s(\eta) + \mu_t(\eta)} \frac{\partial}{\partial R} [g_s(R\eta)\, h_t(R\eta)] + O(R^{-3}) \right\} R^2\, dR =$$
$$= \frac{1}{i\nu} \int_{\Sigma_1} \frac{\psi(\eta)}{\mu_s(\eta) + \mu_t(\eta)} \{ (2\nu)^2 g_s(2\nu\eta)\, h_t(2\nu\eta) - \nu^2 g_s(\nu\eta)\, h_t(\nu\eta) -$$
$$- \int_\nu^{2\nu} g_s(R\eta)\, h_t(R\eta)\, 2R\, dR \}\, d\Sigma_1 + O(\nu^{-1}) = O(\nu^{-1}).$$

Thus, $X(\nu) \to 0$ as $\nu \to +\infty$ which completes the proof. □

Remark 3.2. From the above proof it follows that, if U satisfies the assumptions of Theorem 3.1 and R is a sufficiently large positive number such that supp $A(D, -i\omega)U \subset B_R$, then

$$\int_{\Sigma_R} \{ [Q(D_y, n(y), -i\omega) \Gamma^\top (x-y, \omega, r)]^\top [U(y)] -$$
$$- \Gamma(x-y, \omega, r)[B(D_y, n(y))U(y)] \}\, d\Sigma_R = 0$$

for an arbitrary $x \in B_R \cap \Omega^-$. □

Corollary 3.3. *Let U be the same as in Theorem* 3.1. *Then the derivatives $D^\beta U$ are again $(m,r)-$thermo-radiating vectors for an arbitrary multi-index β and the asymptotic representation of $D^\beta U$ at infinity can be obtained by the direct differentiation from the corresponding asymptotic formula of U.*

Corollary 3.4. *Let $A(D, -i\omega)U(x) = 0$ in $\mathbb{R}^3$ and $U \in \mathrm{SK}^m_r(\mathbb{R}^3)$. Then $U = 0$ in $\mathbb{R}^3$.*

Corollary 3.5. *Let* $F = (F_1, \dots, F_4)^\top \in \mathrm{C}^1(\mathbb{R}^3)$ *and* $\operatorname{diam}\operatorname{supp} F < +\infty$. *Then the equation* $A(D, -i\omega)U(x) = F(x)$, $x \in \mathbb{R}^3$ *is uniquely solvable in the class* $\mathrm{C}^2(\mathbb{R}^3) \cap \mathrm{SK}_r^m(\mathbb{R}^3)$ *and the solution is representable by the following convolution type integral*

$$U(x) = \int_{\mathbb{R}^3} \Gamma(x - y, \omega, r)\, F(y)\, dy, \quad x \in \mathbb{R}^3 .$$

CHAPTER II
FORMULATION OF BOUNDARY VALUE AND INTERFACE PROBLEMS

Here we present the classical and weak formulations of the boundary value and interface problems of the thermoelasticity theory which will be investigated in the subsequent chapters.

4. Functional Spaces

In this section we introduce some functional spaces which will be needed in the formulation of boundary value and interface problems. We recall here some properties of these spaces and for details refer to, for example, [78], [79], [49], [47], [1].

Let Ω^+, Ω^-, and S be the same as in Subsection 1.5.

By $\mathrm{C}^k(\Omega^\pm)$, $\mathrm{C}^k(\overline{\Omega^\pm})$, $\mathrm{C}^k(S)$, and $\mathrm{C}^{k,\alpha}(\Omega^\pm)$, $\mathrm{C}^{k,\alpha}(\overline{\Omega^\pm})$, $\mathrm{C}^{k,\alpha}(S)$, with integer $k \geq 0$ and $0 < \alpha \leq 1$, we denote the usual k-smooth and Hölder (k,α)-smooth function spaces. Note that here we assume S to be a $\mathrm{C}^{k,\alpha}$-smooth manifold. Further, $\mathrm{C}^\infty_{\mathrm{comp}}(\Omega^-)$ stands for the class of C^∞-regular functions with compact supports in Ω^-, $\mathrm{C}(\Omega^\pm)$ and $\mathrm{C}(S)$ denote the spaces of continuous functions in $\Omega^\pm$ and S, respectively, and $\mathrm{C}^\alpha := \mathrm{C}^{0,\alpha}$ for $0 < \alpha < 1$.

By $W^1_p(\Omega^\pm)$, $W^1_{p,\mathrm{loc}}(\Omega^\pm)$, and $W^1_{p,\mathrm{comp}}(\Omega^\pm)$ we denote the usual Sobolev spaces, i.e., spaces of measurable, in general, complex-valued functions that together with their first order generalized derivatives are p-integrable, locally p-integrable, and compactly supported p-integrable functions, respectively, in corresponding domains. Further, $L_p(\Omega^\pm)$, $L_{p,\mathrm{loc}}(\Omega^\pm)$, $L_{p,\mathrm{comp}}(\Omega^\pm)$, and $L_p(S)$ denote the usual (Lebesgue) measurable function spaces.

Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, and $S \in \mathrm{C}^\infty$. Then $B^s_{p,q}(\Omega^\pm)$, $B^s_{p,q,\mathrm{loc}}(\Omega^\pm)$, $B^s_{p,q}(S)$, and $H^s_p(\Omega^\pm)$, $H^s_{p,\mathrm{loc}}(\Omega^\pm)$, $H^s_p(S)$, stand for the Besov and the Bessel-potential spaces, respectively.

Next, let S_1 be a submanifold of S with a C^∞-smooth boundary ∂S_1. We introduce the following spaces on S_1:

$$B^s_{p,q}(S_1) = \{f|_{S_1} : f \in B^s_{p,q}(S)\}, \quad H^s_p(S_1) = \{f|_{S_1} : f \in H^s_p(S)\},$$
$$\widetilde{B}^s_{p,q}(S_1) = \{f \in B^s_{p,q}(S) : \operatorname{supp} f \subset \overline{S}_1\},$$
$$\widetilde{H}^s_p(S_1) = \{f \in H^s_p(S) : \operatorname{supp} f \subset \overline{S}_1\},$$

where $f|_{S_1}$ denotes the restriction of f to S_1, and s, p, and q are as above. The appearance of the Besov and Bessel-potential spaces with $p \neq 2$ and $q \neq 2$ is not only of mathematical interest. The case is that for particular mixed and crack type boundary value and interface problems with specific geometry studied in mathematical physics and mechanics it is well known that, in general, solutions or their derivatives have singularities at the collision curves of changing boundary conditions or edge points of cracks and they do not belong to the class of C^1-regular functions in closed domains (see,e.g.,[74], [84]).

Because of this fact and in order to allow a wide class of boundary data, on one side, and to establish optimal regularity properties of the solutions, on the other hand, we state the basic and mixed interface (transmission) problems in Sobolev spaces with $p > 1$. If we invoke that $u \in W_p^1(\Omega^+)$ $[W^1_{p,\mathrm{loc}}(\Omega^-)]$ implies $u|_{\partial\Omega^\pm} \in B^{1-1/p}_{p,p}(\partial\Omega^\pm)$, then the need of Besov spaces in formulation of our BVPs and interface problems becomes transparent. Clearly, here $u|_S$ is defined in the trace sense.

We recall that $H_2^s = W_2^s = B^s_{2,2}$, $W_p^t = B^t_{p,p}$, and $H_p^k = W_p^k$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

It is evident that first order derivatives of functions from $W_p^1(\Omega^+)$ and $W^1_{p,\mathrm{loc}}(\Omega^-)$ belong to $L_p(\Omega^+)$ and $L_{p,\mathrm{loc}}(\Omega^-)$, respectively, and, in general, they have no traces on S. However, for vector-functions $U \in W_p^1(\Omega^+)$ $[W^1_{p,\mathrm{loc}}(\Omega^-)]$, satisfying, in addition, $A(D,\varkappa)U \in L_p(\Omega^+)$ $[L_{p,\mathrm{loc}}(\Omega^-)]$ the functionals $[P(D,n)U]^\pm_S \in [B^{-1/p}_{p,p}(S)]^3$ and $[\lambda(D,n)U_4]^\pm_S \in B^{-1/p}_{p,p}(S)$, i.e., the functional $[B(D,n)U]^\pm_S \in [B^{-1/p}_{p,p}(S)]^4$ (see (1.25)), can be defined correctly by means of the Green formulae (1.21).

To this end, let us set

$$\langle\, [B(D,n)U]^+_S \,,\, [V]^+_S \,\rangle_S := \int\limits_{\Omega^+} E(U,\overline{V})\, dx + \int\limits_{\Omega^+} A(D,\varkappa)U \cdot \overline{V}\, dx \tag{4.1}$$

$$\Big[\langle [B(D,n)U]^-_S , [V]^-_S \rangle_S := -\int\limits_{\Omega^-} E(U,\overline{V}) dx - \int\limits_{\Omega^-} A(D,\varkappa)U \cdot \overline{V} dx\Big], \tag{4.2}$$

where $E(U,V)$ is given by (1.27), and $V \in W^1_{p'}(\Omega^+)$ $[\,V \in W^1_{p',\mathrm{comp}}(\Omega^-)\,]$, $1/p + 1/p' = 1$. Clearly, by the trace theorem $[V]^\pm_S \in B^{1-1/p'}_{p',p'}(S)$.

It is easy to see that the right-hand side expression in (4.1) [(4.2)] gives the same value for arbitrary vector-functions $V \in W^1_{p'}(\Omega^+)$ $[V \in W^1_{p',\mathrm{comp}}(\Omega^-)]$ having the same traces on S (provided U is fixed). This in turn shows, that the functionals defined by the above equations are, actually, supported on S. We also note that, if $U \in \mathrm{C}^1(\overline{\Omega^+})$ $[\,U \in \mathrm{C}^1(\overline{\Omega^-})\,]$ and $A(D,\varkappa)U \in L_1(\Omega^+)$ $[\,L_{1,\mathrm{loc}}(\Omega^-)\,]$, then the above introduced functionals correspond to the usual boundary values $[B(D,n)U]^+$ and $[B(D,n)U]^-$, respectively. Therefore, we can consider $\langle \,\cdot\, , \,\cdot\, \rangle_S$ in (4.1) and (4.2) as dualities between the spaces $B^{-1/p}_{p,p}(S)$ and $B^{1/p}_{p',p'}(S)$. Note that

$$\langle\, f \,,\, g \,\rangle_S = \int\limits_S \langle\, f \,,\, g \,\rangle\, dS = \int\limits_S \sum_{j=1}^4 f_j\, g_j\, dS$$

for the smooth vector functions $f = (f_1, \cdots, f_4)^\top$ and $g = (g_1, \cdots, g_4)^\top$, i.e., the above duality extends the usual "real" L_2-scalar product.

Throuhgout this monograph all boundary and interface conditions for the displacement vector and temperature always are understood in the trace sense, while for the stress vector and heat flux they are to be concidered in the above duality sense, i.e., in the sense of continuous linear functionals.

Remark 4.1. Let us note the following two simple things. Firstly, the condition $[B(D,n)U]^+ = F$ on S, where $U \in [W_p^1(\Omega^+)]^4$, $A(D,\varkappa)U \in [L_p(\Omega^+)]^4$, and $F \in [B_{p,p}^{-1/p}(S)]^4$, means in the above functional sense that

$$\int_{\Omega^+} E(U,\overline{V})\,dx + \int_{\Omega^+} A(D,\varkappa)U\cdot\overline{V}\,dx = \langle\, F\,,\,[V]_S^+\,\rangle_S \tag{4.3}$$

for arbitrary $V \in [W_{p'}^1(\Omega^+)]^4$.

Secondly, let $U \in [W_p^1(\Omega^+)]^4$, $A(D,\varkappa)U \in [L_p(\Omega^+)]^4$, $F \in [B_{p,p}^{-1/p}(S_1)]^4$, where S_1 is a submanifold of the surface S as described above. Then the condition $[B(D,n)U]^+ = F$ on S_1, is understood as follows

$$\int\limits_{\Omega^+} E(U,\overline{V})\,dx + \int\limits_{\Omega^+} A(D,\varkappa)U\cdot\overline{V}\,dx = \langle\, F\,,\,[V]_S^+\,\rangle_S =: \langle\, F\,,\,[V]_{S_1}^+\,\rangle_{S_1} \tag{4.4}$$

for arbitrary $V \in [W_{p'}^1(\Omega^+)]^4$ whose trace $[V]_S^+$ is supported on S_1, i.e., $[V]_{S\setminus S_1}^+ = 0$. Evidently, $[V]_{S_1}^+ \in [\widetilde{B}_{p',p'}^{1/p}(S_1)]^4$. Here $\langle\,\cdot\,,\,\cdot\,\rangle_{S_1}$ is the duality between the spaces $[B_{p,p}^{-1/p}(S_1)]^4$ and $[\widetilde{B}_{p',p'}^{1/p}(S_1)]^4$. Boundary conditions for the exterior domain Ω^- are understood quite analogously. We have only to change the sign "+" by the sign "−" in front of the volume integrals in the left-hand sides of (4.3) and (4.4), and the superscript "+" is to be replaced by the superscript "−" in the right-hand sides. Moreover, a test function V is to be taken from the same type of Sobolev spaces as above but now with a compact support in Ω^-. □

5. Formulation of the Basic and Mixed BVPs

In this section and in what follows boundary value and interface problems for the pseudo-oscillation and steady state oscillation equations will be marked by the subscripts τ and ω, respectively (unless otherwise stated). We note that in the pseudo-oscillation problems $\tau = \sigma - i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$.

We start by the formulation of the so-called basic and mixed boundary value problems for the bounded domain Ω^+ and its unbounded complement Ω^-. As above, we assume that $S = \partial\Omega^\pm$ is a C^2-smooth manifold. Moreover, $U = (u, u_4)^\top$ is again a four-dimensional vector-function whose first three components correspond to the displacement vector, while the fourth component describes the temperature field.

We consider the following BVPs.

Find a solution U to the system of differential equations (1.9) [(1.10)] in $\Omega^\pm$ satisfying one of the boundary conditions on S:

Problem $(\mathcal{P}_1)_\tau^\pm$ $[(\mathcal{P}_1)_\omega^\pm]$:

$$[u]^\pm = \widetilde{f}, \quad \widetilde{f} = (f_1, f_2, f_3)^\top, \tag{5.1}$$

$$[u_4]^\pm = f_4, \tag{5.2}$$

i.e., the dicplacement vector and the temperature are prescribed on S.

Problem $(\mathcal{P}_2)^{\pm}_{\tau}$ $[(\mathcal{P}_2)^{\pm}_{\omega}]$:

$$[u]^{\pm} = \widetilde{f}, \tag{5.3}$$

$$[\lambda(D,n)\, u_4]^{\pm} = F_4, \tag{5.4}$$

i.e., the dicplacement vector and the heat flux through the surface S are given on S. Here $\lambda(D,n) = \partial_n$ is given by (1.24). The case $[\partial_n u_4]^{\pm} = 0$ describes a thermal insulation over the surface bounding the body.

Problem $(\mathcal{P}_3)^{\pm}_{\tau}$ $[(\mathcal{P}_3)^{\pm}_{\omega}]$:

$$[P(D,n)U]^{\pm} = \widetilde{F}, \quad \widetilde{F} = (F_1, F_2, F_3)^{\top}, \tag{5.5}$$

$$[u_4]^{\pm} = f_4, \tag{5.6}$$

i.e., the vector of thermal stresses and the temperature are given on S. Here $P(D,n)$ is defined by (1.13).

Problem $(\mathcal{P}_4)^{\pm}_{\tau}$ $[(\mathcal{P}_4)^{\pm}_{\omega}]$:

$$[P(D,n)U]^{\pm} = \widetilde{F}, \tag{5.7}$$

$$[\lambda(D,n)\, u_4]^{\pm} = F_4, \tag{5.8}$$

i.e., the vector of thermal stresses and the heat flux are prescribed on S.

Problem $(\mathcal{P}_{mix})^{\pm}_{\tau}$ $[(\mathcal{P}_{mix})^{\pm}_{\omega}]$:

$$[u]^{\pm} = \widetilde{f}^{(1)} \text{ and } [u_4]^{\pm} = f_4^{(1)} \text{ on } S_1, \;\; \widetilde{f}^{(1)} = (f_1^{(1)}, f_2^{(1)}, f_3^{(1)})^{\top}, \tag{5.9}$$

$$\begin{gathered} [P(D,n)U]^{\pm} = \widetilde{F}^{(2)} \text{ and } [\lambda(D,n)u_4]^{\pm} = F_4^{(2)} \text{ on } S_2, \\ \widetilde{F}^{(2)} = (F_1^{(2)}, F_2^{(2)}, F_3^{(2)})^{\top}, \end{gathered} \tag{5.10}$$

where $\overline{S}_1 \cup \overline{S}_2 = S$, $S_1 \cap S_2 = \varnothing$, $S_j \neq \varnothing$, $j = 1,2$; we assume here that the common boundary of $\partial S_1 = \partial S_2$ is also a smooth curve.

The functions f_k, F_k, $f_k^{(1)}$ and $F_k^{(2)}$ are given functions and in the sequel they will be referred as *boundary data* of the BVPs.

Let us introduce the matrix boundary operators

$$\begin{gathered} B_{(1)}(D,n) := I_4 = [\delta_{kj}]_{4\times 4}, \;\; B_{(2)}(D,n) := \begin{bmatrix} I_3 & [0]_{3\times 1} \\ [0]_{1\times 3} & \lambda(D,n) \end{bmatrix}_{4\times 4}, \\ B_{(3)}(D,n) := \begin{bmatrix} [T(D,n)]_{3\times 3} & [-\beta_{kj} n_j]_{3\times 1} \\ [0]_{1\times 3} & 1 \end{bmatrix}_{4\times 4}, \;\; B_{(4)}(D,n) := B(D,n), \end{gathered} \tag{5.11}$$

where $T(D,n)$ and $B(D,n)$ are given by formulae (1.12) and (1.25), respectively. The boundary conditions corresponding to the above problems $(\mathcal{P}_k)^{\pm}_{\tau}$ $[(\mathcal{P}_k)^{\pm}_{\omega}]$ can be then written as follows

$$[B_{(k)}(D,n)U]^{\pm} = g, \;\; k = 1,2,3,4, \tag{5.12}$$

where the four-dimensional vector g is constructed by the boundary data of the corresponding problem.

By a solution of the interior BVPs $(\mathcal{P}_k)^{+}_{\tau}$ and $(\mathcal{P}_k)^{+}_{\omega}$ we understand a vector U from the space either $\mathrm{C}^1(\overline{\Omega^+}) \cap \mathrm{C}^2(\Omega^+)$ or $W^1_p(\Omega^+)$ with $p > 1$.

The mixed BVPs $(\mathcal{P}_{mix})^+_\tau$ and $(\mathcal{P}_{mix})^+_\omega$ will be considered only in the space $W^1_p(\Omega^+)$ since, in general, they have no solutions in the space of smooth functions $\mathrm{C}^1(\overline{\Omega^+})$.

Clearly, in the case of the Sobolev spaces $W^1_p(\Omega^+)$ the differential equations (1.9) and (1.10) are to be considered in the distributional (weak) sense, while the boundary conditions are to be understood in the functional-trace sense described in the previous section.

Moreover, in the exterior BVPs for the domain Ω^- we provide that a solution to the pseudo-oscillation equations (1.9) has to satisfy the conditions (1.29) at infinity (i.e., (1.30)), while a solution to the steady state oscillation equations (1.10) has to meet the generalized Sommerfeld-Kupradze type $(m,r)-$thermo-radiation conditions (3.1). It is also evident that in the exterior problems for the homogeneous pseudo-oscillation equations we may assume $U \in W^1_p(\Omega^-)$ (due to the required asymptotic behaviour at infinity), while in the exterior problems for the homogeneous steady state oscillation equations we have to look for solution in the space $W^1_{p,\mathrm{loc}}(\Omega^-)$.

We remark that every solution to the homogeneous elliptic equations with constant coefficients (1.9) and (1.10) is C^∞-regular in Ω^+ and Ω^-. Therefore, we have to control the smoothness of the solutions only near the boundary S.

Concerning the boundary data in the above formulated problems we note that the precised functional spaces for them will be given below when we start the systematic study of the existence of solutions to the nonhomogeneous BVPs (see Chapter V).

However, we mention here only some necessary (compatibility) conditions. Namely, when we look for a solution $U \in \mathrm{C}^1(\overline{\Omega^\pm})$, then the boundary functions f_k and F_k $(k = 1, \cdots, 4)$ have to belong to some subspaces of $\mathrm{C}^1(S)$ and $\mathrm{C}^0(S)$, respectively, while the following natural conditions $f_k \in B^{1-1/p}_{p,p}(S)$ and $F_k \in B^{-1/p}_{p,p}(S)$ must be satisfied when we seek a solution U in the space $W^1_p(\Omega^\pm)$ $[W^1_{p,\mathrm{loc}}(\Omega^\pm)]$. Analogously, in the mixed BVPs we have to require the natural restrictions $f^{(1)}_k \in B^{1-1/p}_{p,p}(S_1)$ and $F^{(2)}_k \in B^{-1/p}_{p,p}(S_2)$.

We note here that in the elasticity theory of isotropic bodies the basic BVPs in the classical setting by potential methods have been exaustively investigated in [45], while the mixed BVPs have been studied in [50], [13], [75], [76] (L_2-setting) (see also references therein). The same problems of the elasticity theory of anisotropic bodies are considered in [56], [8], [59] (classical and L_p-setting).

6. Formulation of Crack Type Problems

This type of problems appear when the elastic body under consideration has interior cracks of the form of two-dimensional open manifolds. We consider the case when these crack surfaces are disjoint and do not hit the boundary of the body.

We deal with the following model problems.

Let S_1 be an open, two-dimensional, C^∞-regular, two-sided, connected manifold with C^∞-regular boundary ∂S_1. Moreover, we assume S_1 to be a subset of some closed C^∞-regular surface S surrounding a bounded domain, say Ω^+. Further, let $\mathbb{R}^3_{S_1} = \mathbb{R}^3 \setminus \overline{S}_1$, $\overline{S}_1 = S_1 \cup \partial S_1$, and as usual, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. We choose that direction of the unit normal vector on S_1 which corresponds to the outward normal vector on S (with respect to Ω^+). Due to this choice, the symbols $[\,\cdot\,]^\pm$ denote again limits on S_1 from $\Omega^\pm$ either in the usual classical-trace sense or in the functional-trace sense described in Section 5.

Let the whole unbounded domain $\mathbb{R}^3_{S_1}$ be filled up by an anisotropic elastic material with thermoelastic characteristics introduced in Section 1.

The crack type problems in the thermoelasticity theory are formulated as follows (cf. [16], [38]).

Find a solution $U = (u, u_4)^\top \in W^1_{p,\mathrm{loc}}(\mathbb{R}^3_{S_1})$, $p > 1$, to the system of steady state oscillation equation (1.10) in $\mathbb{R}^3_{S_1}$ satisfying the generalized Sommerfeld-Kupradze type $(m, r)-$thermo-radiation conditions at infinity (3.1) and one of the following boundary conditions on S_1:

Problem $(\mathcal{CR.D})_\omega$:

$$\begin{cases} [u]^+ = \widetilde{f}^{(+)}, \\ [u_4]^+ = f_4^{(+)}, \end{cases} \qquad \text{and} \qquad \begin{cases} [u]^- = \widetilde{f}^{(-)}, \\ [u_4]^- = f_4^{(-)}, \end{cases} \tag{6.1}$$

where $\widetilde{f}^\pm = (f_1^\pm, f_2^\pm, f_3^\pm)^\top$, $f^\pm = (f_1^\pm, \cdots, f_4^\pm)^\top$;

Problem $(\mathcal{CR.N})_\omega$:

$$\begin{cases} [P(D,n)U]^+ = \widetilde{F}^{(+)}, \\ [\lambda(D,n)u_4]^+ = F_4^{(+)}, \end{cases} \qquad \text{and} \qquad \begin{cases} [P(D,n)U]^- = \widetilde{F}^{(-)}, \\ [\lambda(D,n)u_4]^- = F_4^{(-)}, \end{cases} \tag{6.2}$$

where $\widetilde{F}^\pm = (F_1^\pm, F_2^\pm, F_3^\pm)^\top$, $F^\pm = (F_1^\pm, \cdots, F_4^\pm)^\top$.

The boundary data $f_k^\pm$ and $F_1^\pm$ belong again to the natural spaces

$$f_k^\pm \in B^{1-1/p}_{p,p}(S_1), \quad F_k^\pm \in B^{-1/p}_{p,p}(S_1), \;\; k = 1, \cdots, 4. \tag{6.3}$$

Moreover, we assume

$$f_k^+ - f_k^- \in \widetilde{B}^{1-1/p}_{p,p}(S_1), \quad F_k^+ - F_k^- \in \widetilde{B}^{-1/p}_{p,p}(S_1), \;\; k = 1, \cdots, 4, \tag{6.4}$$

which is stipulated by the fact that an arbitrary solution U to the equation (1.10) is C^∞-regular in $\mathbb{R}^3_{S_1}$ and, obviously,

$$[U]^+ - [U]^- = 0 \;\text{ and }\; [B(D,n)U]^+ - [B(D,n)U]^- = 0, \;\text{ on } S \setminus \overline{S_1}. \tag{6.5}$$

The formulation of crack type BVPs for the pseudo-oscillation equations are similar to the above ones.

In this case we look for a solution $U = (u, u_4)^\top \in W^1_p(\mathbb{R}^3_{S_1})$, $p > 1$, to the system of equations (1.9) in $\mathbb{R}^3_{S_1}$ satisfying the decay conditions (1.30) at infinity, and either the boundary conditions (6.1) (in **Problem** $(\mathcal{CR.D})_\tau$) or the boundary conditions (6.2) (in **Problem** $(\mathcal{CR.N})_\tau$) on S_1. The boundary data $f_k^\pm$ and $F_1^\pm$ are supposed again to meet embeddings (6.3) and (6.4).

If one considers the crack type problems for the domains $\Omega^\pm$ with the interior cut S_1, then to the above boundary conditions (6.1) and (6.2) on S_1, clearly, one has to add one of the basic boundary conditions on S corresponding to the BVPs $(\mathcal{P}_k)_\tau^\pm$ $[(\mathcal{P}_k)_\omega^\pm]$. As it becomes transparent later on, these type of BVPs can be investigated by slight and evident modifications of our analysis developed in the next chapters. Therefore, we confine ourselves to deal with only the above formulated model problems.

We remark that analogous problems of elastostatics of isotropic and anisotropic bodies have been investigated in [13], [17], [18] (see also references therein). The above formulated crack problems for the pseudo-oscillation equations of the thermoelasticity theory in the general anisotropic case have been treated in [16].

7. Basic and Mixed Interface Problems

In this section we formulate the basic and mixed interface problems of the thermoelasticity theory for piecewise homogeneous anisotropic bodies. In the scientific literature the mixed interface problems are called also as *interface crack problems*.

The most general case of the structure of a piecewise homogeneous elastic body under consideration can be mathematicaly described as follows. In three-dimensional Euclidean space $\mathbb{R}^3$ we have some closed, smooth, connected, nonselfintersecting surfaces $\widetilde{S}_1, \widetilde{S}_2, \dots, \widetilde{S}_n$ $(\widetilde{S}_j \cap \widetilde{S}_k = \varnothing,\ j \neq k)$. By these surfaces the whole space $\mathbb{R}^3$ is devided into several connected domains $\Omega_1, \dots, \Omega_l$. Each domain is supposed to be filled up by an anisotropic material with corresponding, in general, different thermoelastic coefficients.

Common boundaries of the two distinct materials are called *interfaces* or *contact surfaces* of the piecewise homogeneous elastic body. If some domains represent empty inclusions, then corresponding to them surrounding surfaces are called *boundary surfaces* of the composed elastic body in question. Such type of piecewise homogeneous structures encounter in many physical, mechanical and engineering applications. Therefore, besides the theoretical importance of the transmission problems we intend to study, this interest is also motivated by their fundamental applications to many areas of science and technology.

7.1. For illustration of the method suggested we consider the following model problems. We assume that the piecewise homogeneous composed anisotropic body consists of two elastic components occupying bounded domain $\Omega^1 = \Omega^+$ and its unbounded complement $\Omega^2 = \Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$; $\partial\Omega^\pm = S$, $\overline{\Omega^\mu} = \Omega^\mu \cup S$, $\mu = 1, 2$. Thus, the whole space $\mathbb{R}^3$ can be considered as a piecewise homogeneous anisotropic body with the single contact (interface) surface S.

Let a smooth, connected, nonselfintersecting curve $l \subset S$ devide the contact surface S into two open parts S_1 and S_2: $S = S_1 \cup S_2 \cup l$, $S = S_1 \cap S_2 = \varnothing$, $\overline{S}_j = S_j \cup l$, $j = 1, 2$.

We treat the two groups of interface conditions:

I. **Basic interface problems**. On the whole contact surface S there are given

a) jumps of the displacement vector, the temperature, the vector of thermal stresses, and the heat flux (**Problem** $(\mathcal{C})$) or

b) jumps of the temperature, the heat flux, and the normal components of the displacement and the stress vectors; in addition to these conditions, the limits of either the tangent components of the stress vectors (**Problem** $(\mathcal{G})$) or the tangent components of the displacement vectors (**Problem** $(\mathcal{H})$) are given from both sides of the interface (cf. [45], [29], [32], [34]).

II. **Mixed interface problems**. On the submanifold S_1 the conditions of Problem $(\mathcal{C})$ are prescribed, while on S_2 there are given:

a) the conditions of Problem $(\mathcal{G})$ (**Problem** $(\mathcal{C} - \mathcal{G})$) or

b) the conditions of Problem $(\mathcal{H})$ (**Problem** $(\mathcal{C} - \mathcal{H})$) or

c) the displacement vector and the temperature (on the both sides of S_2) (**Problem** $(\mathcal{C} - \mathcal{DD})$) or

d) the thermal stresses and the heat flux (on the both sides of S_2) (**Problem** $(\mathcal{C} - \mathcal{NN})$) or

e) the displacement [stress] vector (on the both sides of S_2) and the jumps of the temperature and the heat flux (**Problem** $(\mathcal{C} - \mathcal{DC})$ [**Problem** $(\mathcal{C} - \mathcal{NC})$]) (cf. [58], [33], [35], [41], [40]).

The analogous basic interface problems in the classical elasticity and thermoelasticity of isotropic bodies have been studied by the potential and variational methods in [45], [32], [67], [84] (see also [75], [61], [62]). In anisotropic elasticity the basic interface problems have been considered in [34], [41], [22], while the mixed interface problems have been investigated in [35], [58], [67], [41], [9].

7.2. Before we start the mathematical formulation of the above interface problems let us introduce some notations.

We assume that the domain Ω^μ $(\mu = 1, 2)$ is filled up by elastic material whose thermoelastic constants are $c^{(\mu)}_{kjpq}$, $\lambda^{(\mu)}_{pq}$, $\beta^{(\mu)}_{pq}$, $c^{(\mu)}_0$, with the same properties as in Section 1. The displacement vector and the temperature in Ω^μ are denoted by $u^{(\mu)}$ and $u^{(\mu)}_4$, respectively. All operators and thermomechanical characteristics corresponding to the elastic material occupying the domain Ω^μ we mark with the superscript μ. For example, the basic equations of pseudo-oscillations and steady state oscillations now read as (see (1.7)–refn1.12)

$$A^{(\mu)}(D, \tau) U^{(\mu)}(x) = 0 \quad \text{in} \quad \Omega^\mu, \tag{7.1}$$

$$A^{(\mu)}(D, -i\omega) U^{(\mu)}(x) = 0 \quad \text{in} \quad \Omega^\mu. \tag{7.2}$$

The symbols $T^{(\mu)}(D,n)$, $P^{(\mu)}(D,n)$, and $\lambda^{(\mu)}(D,n)$ stand now for the corresponding classical stress operator, thermo-stress operator, and heat flux operator, respectively (see (1.11), (1.13), (1.24)).

First we formulate the basic interface problems for the steady state oscillation equations of thermoelasticity.

Find vector functions $U^{(\mu)}$ ($\mu = 1,2$) that solve the equations (7.2) in Ω^μ and that satisfy the following interface (transmission) conditions on S:

Problem $(\mathcal{C})_\omega$:

$$[u^{(1)}]^+ - [u^{(2)}]^- = \widetilde{f}, \quad [u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4, \tag{7.3}$$

$$\left.\begin{aligned} &[P^{(1)}(D,n)U^{(1)}]^+ - [P^{(2)}(D,n)U^{(2)}]^- = \widetilde{F}, \\ &[\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4, \end{aligned}\right\} \tag{7.4}$$

where $f = (\widetilde{f}, f_4)^\top$, $\widetilde{f} = (f_1, f_2, f_3)^\top$, $F = (\widetilde{F}, F_4)^\top$, $\widetilde{F} = (F_1, F_2, F_3)^\top$.

Problem $(\mathcal{G})_\omega$:

$$[P^{(1)}(D,n)U^{(1)} \cdot l]^+ = \widetilde{F}_l^{(+)}, \quad [P^{(1)}(D,n)U^{(1)} \cdot m]^+ = \widetilde{F}_m^{(+)}, \tag{7.5}$$

$$[P^{(2)}(D,n)U^{(2)} \cdot l]^- = \widetilde{F}_l^{(-)}, \quad [P^{(2)}(D,n)U^{(2)} \cdot m]^- = \widetilde{F}_m^{(-)}, \tag{7.6}$$

$$\begin{aligned}&[u^{(1)} \cdot n]^+ - [u^{(2)} \cdot n]^- = \widetilde{f}_n, \quad [P^{(1)}(D,n)U^{(1)} \cdot n]^+ - \\ &\qquad -[P^{(2)}(D,n)U^{(2)} \cdot n]^- = \widetilde{F}_n, \end{aligned}\tag{7.7}$$

$$[u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4, \quad [\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4. \tag{7.8}$$

Problem $(\mathcal{H})_\omega$: conditions (7.7), (7.8), and

$$[u^{(1)} \cdot l]^+ = \widetilde{f}_l^{(+)}, \quad [u^{(1)} \cdot m]^+ = \widetilde{f}_m^{(+)}, \tag{7.9}$$

$$[u^{(2)} \cdot l]^- = \widetilde{f}_l^{(-)}, \quad [u^{(2)} \cdot m]^- = \widetilde{f}_m^{(-)}. \tag{7.10}$$

Here and in what follows we denote by $n(x)$ again the outward (to Ω^+) unit normal vector at the point $x \in S$, and by $l(x)$ and $m(x)$ orthogonal unit vectors in the tangent plane. The orthogonal local co-ordinate system n, l, and m at $x \in S$ is orientated as follows: $l \times m = n$, where $\cdot \times \cdot$ denotes the vector product of two vectors.

The conditions (7.5)–(7.6) and (7.9)–(7.10), in fact, represent limits on S of the tangent components of the thermo-stress vector and the displacement vector, respectively, while the second equation in (7.4) represents the jump of the heat flux on S.

The conditions (7.3) and (7.4) can be written then as follows:

$$[U^{(1)}]^+ - [U^{(2)}]^- = f \ \text{ on } \ S, \tag{7.11}$$

$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F \ \text{ on } \ S, \tag{7.12}$$

where $B^{(\mu)}(D,n)$ is defined by (1.25).

Next, we recall that S_1 and S_2 are the two disjoint submanifolds of S such that $\overline{S}_1 \cup \overline{S}_2 = S$, and formulate the mixed interface problems.

Find vector functions $U^{(\mu)}$ $(\mu = 1,2)$ that solve the equations (7.2) in Ω^μ and that satisfy one of the following mixed interface conditions on S:

Problem $(\mathcal{C}-\mathcal{D}\mathcal{D})_\omega$:

$$\left.\begin{array}{l} [U^{(1)}]^+ - [U^{(2)}]^- = f^{(1)} \\ [B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F^{(1)} \end{array}\right\} \quad \text{on} \quad S_1, \tag{7.13}$$

$$[U^{(1)}]^+ = \varphi^{(+)}, \quad [U^{(2)}]^- = \varphi^{(-)} \quad \text{on} \quad S_2, \tag{7.14}$$

where

$$f^{(1)} = (\widetilde{f}^{(1)}, f_4^{(1)})^\top, \quad \widetilde{f}^{(1)} = (f_1^{(1)}, f_2^{(1)}, f_3^{(1)})^\top, \quad F^{(1)} = (\widetilde{F}^{(1)}, F_4^{(1)})^\top,$$
$$\widetilde{F}^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})^\top, \ \varphi^{(\pm)} = (\widetilde{\varphi}^{(\pm)}, \varphi_4^{(\pm)})^\top, \ \widetilde{\varphi}^{(\pm)} = (\varphi_1^{(\pm)}, \varphi_2^{(\pm)}\ \varphi_3^{(\pm)})^\top.$$

Problem $(\mathcal{C}-\mathcal{N}\mathcal{N})_\omega$: conditions (7.13) on S_1 and

$$[B^{(1)}(D,n)U^{(1)}]^+ = \Phi^{(+)}, \quad [B^{(2)}(D,n)U^{(2)}]^- = \Phi^{(-)} \quad \text{on} \quad S_2, \tag{7.15}$$
$$\Phi^{(\pm)} = (\widetilde{\Phi}^{(\pm)}, \Phi_4^{(\pm)})^\top, \quad \widetilde{\Phi}^{(\pm)} = (\Phi_1^{(\pm)}, \Phi_2^{(\pm)}, \Phi_3^{(\pm)})^\top.$$

Problem $(\mathcal{C}-\mathcal{D}\mathcal{C})_\omega$: condition (7.8) on S and

$$\begin{array}{r} [u^{(1)}]^+ - [u^{(2)}]^- = \widetilde{f}^{(1)}, \ [P^{(1)}(D,n)U^{(1)}]^+ - \\ -[P^{(2)}(D,n)U^{(2)}]^- = \widetilde{F}^{(1)} \quad \text{on} \quad S_1, \end{array} \tag{7.16}$$

$$[u^{(1)}]^+ = \widetilde{\varphi}^{(+)}, \quad [u^{(2)}]^- = \widetilde{\varphi}^{(-)} \quad \text{on} \quad S_2. \tag{7.17}$$

Problem $(\mathcal{C}-\mathcal{N}\mathcal{C})_\omega$: conditions (7.8) on S, (7.16) on S_1, and

$$[P^{(1)}(D,n)U^{(1)}]^+ = \widetilde{\Phi}^{(+)}, \ [P^{(2)}(D,n)U^{(2)}]^- = \widetilde{\Phi}^{(-)} \quad \text{on} \quad S_2. \tag{7.18}$$

Problem $(\mathcal{C}-\mathcal{G})_\omega$: conditions (7.8) on S, (7.16) on S_1, and

$$\left.\begin{array}{l} [u^{(1)}\cdot n]^+ - [u^{(2)}\cdot n]^- = \widetilde{f}_n^{(2)} \\ [P^{(1)}(D,n)U^{(1)}\cdot n]^+ - [P^{(2)}(D,n)U^{(2)}\cdot n]^- = \widetilde{F}_n^{(2)} \end{array}\right\} \quad \text{on} \quad S_2, \tag{7.19}$$

$$[P^{(1)}(D,n)U^{(1)}\cdot l]^+ = \widetilde{\Phi}_l^{(+)}, \quad [P^{(1)}(D,n)U^{(1)}\cdot m]^+ = \widetilde{\Phi}_m^{(+)} \quad \text{on} \quad S_2,$$
$$[P^{(2)}(D,n)U^{(2)}\cdot l]^- = \widetilde{\Phi}_l^{(-)}, \quad [P^{(2)}(D,n)U^{(2)}\cdot m]^- = \widetilde{\Phi}_m^{(-)} \quad \text{on} \quad S_2.$$

Problem $(\mathcal{C}-\mathcal{H})_\omega$: conditions (7.8) on S, (7.16) on S_1, (7.19) on S_2, and

$$[u^{(1)}\cdot l]^+ = \widetilde{\varphi}_l^{(+)}, \quad [u^{(1)}\cdot m]^+ = \widetilde{\varphi}_m^{(+)} \quad \text{on} \quad S_2,$$
$$[u^{(2)}\cdot l]^- = \widetilde{\varphi}_l^{(-)}, \quad [u^{(2)}\cdot m]^- = \widetilde{\varphi}_m^{(-)} \quad \text{on} \quad S_2.$$

In the all above steady state oscillation problems we require that the vector function $U^{(2)}$ satisfies the $(m,r)-$thermo-radiation conditions at infinity.

Moreover, by a solution to the above interface problems we understand a pair of vector-functions $(U^{(1)}, U^{(2)})$ satisfying the conditions of the corresponding problem.

We note that the basic interface problems formulated above will be studied in both the regular $(\mathrm{C}^1(\overline{\Omega^1}), \mathrm{C}^1(\overline{\Omega^2}))$ and the Sobolev $(W^1_p(\Omega^1), W^1_{p,\mathrm{loc}}(\Omega^2))$ spaces.

Therefore, the given data of the interface problems belong to the corresponding natural functional spaces, and the transmission conditions are to be understood in the classical sense and in the functional-trace sense, respectively.

Particularly, in the regular case, all data corresponding to the displacement vector and the temperature are embedded in $\mathrm{C}^1(S)$ space, while the data corresponding to the thermo-stress vector and the heat flux are embedded in $\mathrm{C}^0(S)$ space. In the case of weak setting (in Sobolev spaces), these data are in $B^{1-1/p}_{p,p}(S)$ and $B^{-1/p}_{p,p}(S)$ spaces, respectively.

The above mixed type interface problems will be treated only in the weak setting, i.e., in this case we look for the unknown vector functions $U^{(1)}$ and $U^{(2)}$ in the Sobolev spaces

$$U^{(1)} \in W^1_p(\Omega^1) \quad \text{and} \quad U^{(2)} \in W^1_{p,\mathrm{loc}}(\Omega^2) \cap \mathrm{SK}^m_r(\Omega^2), \;\; 1<p<\infty. \tag{7.20}$$

This implies that the data of the mixed interface problems have to meet the following natural restrictions caused by (7.20):

$$\begin{gathered} f_4 \in B^{1-1/p}_{p,p}(S), \quad F_4 \in B^{-1/p}_{p,p}(S), \\ f^{(1)}_k \in B^{1-1/p}_{p,p}(S_1), \;\; F^{(1)}_k \in B^{-1/p}_{p,p}(S_1), \;\; \varphi^{(\pm)}_k, \widetilde{f}^{(2)}_n, \widetilde{\varphi}^{(\pm)}_l, \widetilde{\varphi}^{(\pm)}_m \in B^{1-1/p}_{p,p}(S_2), \\ \Phi^{(\pm)}_k, \widetilde{F}^{(2)}_n, \widetilde{\Phi}^{(\pm)}_l, \widetilde{\Phi}^{(\pm)}_m \in B^{-1/p}_{p,p}(S_2), \quad k=\overline{1,4}. \end{gathered} \tag{7.21}$$

Moreover, the inclusions (7.20) lead also to the following necessary (compatibility) conditions:

a) in the problem $(\mathcal{C}-\mathcal{DD})_\omega$:

$$f = \begin{cases} f^{(1)} & \text{on} \;\; S_1, \\ \varphi^{(+)} - \varphi^{(-)} & \text{on} \;\; S_2, \end{cases} \quad f \in [B^{1-1/p}_{p,p}(S)]^4; \tag{7.22}$$

b) in the problem $(\mathcal{C}-\mathcal{NN})_\omega$:

$$F = \begin{cases} F^{(1)} & \text{on} \;\; S_1, \\ \Phi^{(+)} - \Phi^{(-)} & \text{on} \;\; S_2, \end{cases} \quad F \in [B^{-1/p}_{p,p}(S)]^4; \tag{7.23}$$

c) in the problem $(\mathcal{C}-\mathcal{DC})_\omega$:

$$\widetilde{f} = \begin{cases} \widetilde{f}^{(1)} & \text{on} \;\; S_1, \\ \widetilde{\varphi}^{(+)} - \widetilde{\varphi}^{(-)} & \text{on} \;\; S_2, \end{cases} \quad \widetilde{f} \in [B^{1-1/p}_{p,p}(S)]^3; \tag{7.24}$$

d) in the problem $(\mathcal{C}-\mathcal{NC})_\omega$:

$$\widetilde{F} = \begin{cases} \widetilde{F}^{(1)} & \text{on} \;\; S_1, \\ \widetilde{\Phi}^{(+)} - \widetilde{\Phi}^{(-)} & \text{on} \;\; S_2, \end{cases} \quad \widetilde{F} \in [B^{-1/p}_{p,p}(S)]^3; \tag{7.25}$$

e) in the problem $(\mathcal{C}-\mathcal{G})_\omega$:

$$\widetilde{f}_n = \begin{cases} \widetilde{f}^{(1)}\cdot n & \text{on} \quad S_1, \\ \widetilde{f}_n^{(2)} & \text{on} \quad S_2, \end{cases} \qquad \widetilde{f}_n \in B_{p,p}^{1-1/p}(S), \tag{7.26}$$

$$\widetilde{F} = \begin{cases} \widetilde{F}^{(1)} & \text{on } S_1, \\ [\widetilde{\Phi}_l^{(+)} - \widetilde{\Phi}_l^{(-)}]l + [\widetilde{\Phi}_m^{(+)} - \widetilde{\Phi}_m^{(-)}]m + \widetilde{F}_n^{(2)}n & \text{on } S_2, \end{cases} \quad \widetilde{F} \in [B_{p,p}^{-1/p}(S)]^3; \tag{7.27}$$

f) in the problem $(\mathcal{C}-\mathcal{H})_\omega$:

$$\widetilde{f} = \begin{cases} \widetilde{f}^{(1)} & \text{on } S_1, \\ [\widetilde{\varphi}_l^{(+)} - \widetilde{\varphi}_l^{(-)}]l + [\widetilde{\varphi}_m^{(+)} - \widetilde{\varphi}_m^{(-)}]m + \widetilde{f}_n^{(2)}n & \text{on } S_2, \end{cases} \quad \widetilde{f} \in [B_{p,p}^{1-1/p}(S)]^3, \tag{7.28}$$

$$\widetilde{F}_n = \begin{cases} \widetilde{F}^{(1)}\cdot n & \text{on} \quad S_1, \\ \widetilde{F}_n^{(2)} & \text{on} \quad S_2, \end{cases} \qquad \widetilde{F}_n \in B_{p,p}^{-1/p}(S). \tag{7.29}$$

In the sequel all these conditions are supposed to be fulfilled. Note that the conditions (7.22), (7.24), (7.26), (7.28), and (7.23), (7.25), (7.27), (7.29), hold for arbitrary functions satisfying (7) with $1 < p < 2$ and $2 < p < \infty$, respectively. This follows from the multiplication properties of Besov spaces (see [79], Ch. 3, Section 3.3.2).

Finally, we note that for the domains of general structure, described in the beginning of the section, the basic and mixed transmission problems mathematically could be formulated quite similarly: on the contact surfaces the conditions one of the interface problems stated above are assigned, while on the boundary of the composed body the conditions of the basic (or mixed) boundary value problemes are given. We observe that the all principal difficulties arising in the study of problems for the composed bodies of general structure are presented in the above model problems as well.

7.3 The basic and mixed interface problems for the pseudo-oscillation case are formulated in the same way. The only difference is that a solution $U^{(2)}$ to the equation (7.1) in Ω^2 has to satisfy the natural decay condition (1.30) at infinity. Therefore, in the weak setting, we look for solutions in the spaces

$$U^{(1)} \in W_p^1(\Omega^1) \quad \text{and} \quad U^{(2)} \in W_p^1(\Omega^2), \;\; 1 < p < \infty. \tag{7.30}$$

These problems, due to the above agreement, we denote by symbols $(\mathcal{C})_\tau$, $(\mathcal{G})_\tau$, $(\mathcal{H})_\tau$, $(\mathcal{C}-\mathcal{D}\mathcal{D})_\tau$, $(\mathcal{C}-\mathcal{N}\mathcal{N})_\tau$, $(\mathcal{C}-\mathcal{D}\mathcal{C})_\tau$, $(\mathcal{C}-\mathcal{N}\mathcal{C})_\tau$, $(\mathcal{C}-\mathcal{G})_\tau$, $(\mathcal{C}-\mathcal{H})_\tau$, respectively.

The interface conditions on S in the regular and weak setting of these problems read again as in the steady state oscillation case.

CHAPTER III
UNIQUENESS THEOREMS

In this chapter we study the homogeneous versions of the above problems and prove the corresponding uniqueness theorems. The problems in the classical formulation will be analysed completely, while the problems in the weak setting will be treated only partially. Namely, we consider here the case $p = 2$. The general case ($p > 1$) will be considered later together with the existence questions.

8. Uniqueness Theorems for Pseudo-Oscillation Problems

8.1. Let us begin with the consideration of the basic BVPs of pseudo-oscillations.

Theorem 8.1. *The homogeneous versions of the problems* $(\mathcal{P}_k)^+_\tau$, $k = 1, 2, 3, 4$, *have only the trivial solutions in the class of regular vector functions* $\mathrm{C}^1(\overline{\Omega^+})$.

Proof. Let $U = (u, u_4)^\top \in \mathrm{C}^1(\overline{\Omega^+}) \cap \mathrm{C}^\infty(\Omega^+)$ be a solution to one of the homogeneous BVPs indicated in the theorem. Making use of the identity (1.23) with $\varkappa = \tau = \sigma - i\omega$, where $\sigma > 0$ and $\omega \in \mathbb{R}$, we get

$$\int_{\Omega^+} \Big\{ {}_{kjpq} D_p u_q \overline{D_k u_j} + \tau^2 |u|^2 + \tfrac{1}{\overline{\tau} T_0} \lambda_{pq} D_q u_4 \overline{D_p u_4} + \tfrac{c_0}{T_0} |u_4|^2 \Big\} \, dx = 0, \quad (8.1)$$

since the two other integrals in (1.23) vanish due to the homogeneity of the differential equation (1.9) and the boundary conditions (see (5.1)-(5.8)). Separating the real and imaginary parts leads to the system of equations

$$\begin{aligned} &\int_{\Omega^+} \Big\{ c_{kjpq} D_p u_q \overline{D_k u_j} + (\sigma^2 - \omega^2)|u|^2 + \\ &+ \frac{\sigma}{|\tau|^2 T_0} \lambda_{pq} D_q u_4 \overline{D_p u_4} + \frac{c_0}{T_0} |u_4|^2 \Big\} \, dx = 0, \end{aligned} \quad (8.2)$$

$$\omega \int_{\Omega^+} \Big\{ 2\sigma |u|^2 + \frac{1}{|\tau|^2 T_0} \lambda_{pq} D_q u_4 \overline{D_p u_4} \Big\} \, dx = 0. \quad (8.3)$$

Hence, by (1.14) and (1.15), we infer that $u = 0$ and $u_4 = 0$ in Ω^+. $\square$

Theorem 8.2. *Let* $U = (u, u_4)^\top \in W^1_2(\Omega^+)$ *be a solution to one of the homogeneous BVPs* $(\mathcal{P}_k)^+_\tau$, $k = 1, 2, 3, 4$. *Then* $U = 0$ *in* Ω^+.

Proof. We prove the theorem for the problem $(\mathcal{P}_4)^+_\tau$. The other problems can be treated analogously.

In the case under consideration the homogeneous boundary conditions (5.7) and (5.8) (with $F = 0$) are understood in the functional-trace sense described in Section 4. Invoking the definition (4.1) with $\varkappa = \tau$, and noting that $A(D, \tau)U(x) = 0$ in Ω^+, we conclude

$$\langle \, [B(D, n)U]^+_S \, , \, [\overline{V}]^+_S \, \rangle_S = \int_{\Omega^+} E(U, V) \, dx, \quad (8.4)$$

where $V=(v,v_4)^\top$, with $v=(v_1,v_2,v_3)^\top$, is an arbitrary vector function of the space $[W_2^1(\Omega^+)]^4$, and $E(U,V)$ is given by (1.27). Clearly, (8.4) implies

$$\begin{gathered}\langle\, [P(D,n)U]_S^+\,,\, [\overline{v}]_S^+\,\rangle_S = \\ = \int_{\Omega^+} \left\{c_{kjpq}D_p u_q \overline{D_k v_j} + \tau^2 u_p\overline{v_p} - \beta_{pq}u_4\overline{D_p v_q}\right\}\, dx,\end{gathered} \tag{8.5}$$

$$\begin{gathered}\langle\, [\lambda(D,n)u_4]_S^+\,,\, [\overline{v_4}]_S^+\,\rangle_S = \\ = \int_{\Omega^+} \left\{\lambda_{pq}D_q u_4\overline{D_p v_4} + c_0\tau u_4\overline{v_4} + \tau T_0\overline{v_4}\beta_{pq}D_p u_q\right\}\, dx,\end{gathered} \tag{8.6}$$

where $v=(v_1,v_2,v_3)^\top$ and v_4 are arbitrary elements of the spaces $[W_2^1(\Omega^+)]^3$ and $W_2^1(\Omega^+)$, respectively.

Multiplying (8) by $(\tau T_0)^{-1}$, taking its complex conjugate, and adding the result termwise to the (8) lead then us to the equation

$$\begin{gathered}\langle\, [P(D,n)U]_S^+\,,\, [\overline{v}]_S^+\,\rangle_S + \tfrac{1}{\overline{\tau}T_0}\overline{\langle\, [\lambda(D,n)u_4]_S^+\,,\, [\overline{v_4}]_S^+\,\rangle_S} = \\ = \int_{\Omega^+}\Big\{c_{kjpq}D_p u_q\overline{D_k v_j} + \tau^2 u_p\overline{v_p} - \beta_{pq}[u_4\overline{D_p v_q} - v_4\overline{D_p u_q}] + \\ + \tfrac{1}{\overline{\tau}T_0}\lambda_{pq}\overline{D_q u_4}D_p v_4 + \tfrac{c_0}{T_0}\overline{u_4}v_4\Big\}\, dx.\end{gathered} \tag{8.7}$$

It is evident that, if U is a solution to the homogeneous BVP $(\mathcal{P}_4)_\tau^+$, then the left-hand side expression in (8.7) vanishes. Whence

$$\begin{gathered}\int_{\Omega^+}\Big\{c_{kjpq}D_p u_q\overline{D_k v_j} + \tau^2 u_p\overline{v_p} - \beta_{pq}[u_4\overline{D_p v_q} - v_4\overline{D_p u_q}] + \\ + \tfrac{1}{\overline{\tau}T_0}\lambda_{pq}\overline{D_q u_4}D_p v_4 + \tfrac{c_0}{T_0}\overline{u_4}v_4\Big\}\, dx = 0\end{gathered} \tag{8.8}$$

for arbitrary $v_j \in W_2^1(\Omega^+)$, $j=\overline{1,4}$. Since we are allowed to put here $v_j=u_j$ and apply the arguments of the proof of Theorem 8.1, we get $u_j=0$ $(j=\overline{1,4})$ in Ω^+.

Now we make some remarks concerning the other homogeneous boundary value problems. First of all we note that the starting point to prove the uniqueness of solutions in Sobolev spaces always is the formula (8.4). For example, let us consider the homogeneous problem $(\mathcal{P}_1)_\tau^+$, and let some vector-function $U\in W_2^1(\Omega^+)$ be its solution. Due to the homogeneity of the problem, obviousely, $[U]^+=0$ on S in the usual trace sense. Next, let us calculate the corresponding thermo-stress vector and the heat flux on S, i.e., the vector $[B(D,n)U]_S^+$ which is understood in the functional sense. To this end we have to apply the definition (4.1) which in the case in question reads as (8.4). Surely, we may substitute the solution $U\in W_2^1(\Omega^+)$ in the place of the vector-function $V\in W_2^1(\Omega^+)$ in the equations (8.4)–(8.8). Since the trace $[U]_S^+$ vanishes on S, we again arrive at the equations (8) and (8.3). Whence $U=0$ in Ω^+ follows. □

Theorem 8.3. *The homogeneous mixed BVP* $(\mathcal{P}_{mix})_\tau^+$ *in the class* $W_2^1(\Omega^+)$ *has only the trivial solution.*

Proof. Denote by $U=(u,u_4)^\top\in W_2^1(\Omega^+)$ an arbitrary solution of the homogeneous mixed problem $(\mathcal{P}_{mix})_\tau^+$. Clearly, $[U]_{S_1}^+=0$ in the usual trace sense and, therefore, $[U]_{S_2}^+\in\widetilde{B}_{2,2}^{1/2}(S_2)$, since $U\in B_{2,2}^{1/2}(S)$. Further, let us note that the homogeneous boundary conditions for the vector U on S_2, due to Remark 4.1, imply

$$\int_{\Omega^+} E(U,V)\,dx = \langle\, [B(D,n)U]^+_{S_2}\,,\, [\overline{V}]^+_{S_2}\,\rangle_{S_2} = 0 \tag{8.9}$$

for arbitrary $V \in W^1_2(\Omega^+)$ with the property $[V]^+_{S_2} \in \widetilde{B}^{1/2}_{2,2}(S_2)$. Clearly, the equation (8.9) is equivalent to (8.8), where we may again substitute the vector-function U in the place of V, since the U satisfies the restrictions required above for V in (8.9). Therefore, with the help of the arguments in the proof of Theorems 8.1 and 8.2 we easily conclude that $u_j = 0$ $(j = \overline{1,4})$ in Ω^+. □

The uniqueness theorems for the exterior basic BVPs for the pseudo-oscillation equations can be proved quite analogously.

Theorem 8.4. *The homogeneous BVPs $(\mathcal{P}_k)^-_\tau$, $k = 1,2,3,4$, and $(\mathcal{P}_{mix})^-_\tau$ have only the trivial solutions in the space $W^1_2(\Omega^-)$.*

Proof. We will prove the theorem only for the problem $(\mathcal{P}_{mix})^-_\tau$, since for the other problems it is verbatim.

Let $U = (u,u_4)^\top \in W^1_2(\Omega^-) \cap C^\infty(\Omega^-)$ be an arbitrary solution to the mixed homogeneous BVP for the pseudo-oscillation equations. Then, in addition, the U satisfies the decay condition (1.30) at infinity. Due to Remark 4.1 and the homogeneity of the boundary conditions for stresses on S_2 the following equation

$$\langle\, [B(D,n)U]^-_{S_2}\,,\, [\overline{V}]^-_{S_2}\,\rangle_S = -\int_{\Omega^-} E(U,V)\,dx = 0 \tag{8.10}$$

holds for arbitrary $V \in W^1_{2,\mathrm{comp}}(\Omega^-)$ with $[V]^-_{S_2} \in \widetilde{B}^{1/2}_{2,2}(S_2)$, i.e., $[V]^-_{S_1} = 0$.

As in the proof of Theorem 8.2 we can easily show that (8.10) yields

$$\begin{aligned}\int_{\Omega^-} \Big\{ c_{kjpq} D_p u_q \overline{D_k v_j} + \tau^2 u_p \overline{v_p} - \beta_{pq}[u_4 \overline{D_p v_q} - v_4 \overline{D_p u_q}] +\\ + \tfrac{1}{\overline{\tau} T_0} \lambda_{pq} \overline{D_q u_4} D_p v_4 + \tfrac{c_0}{T_0} \overline{u_4} v_4 \Big\}\,dx = 0.\end{aligned} \tag{8.11}$$

Note that C^∞-regular vector functions having compact supports in $\overline{\Omega^-}$ and zero traces on S_1 are densely embedded in the space $X = \{V \in W^1_2(\Omega^-) \ :\ [V]^-_{S_1} = 0\}$. Thus, for $V \in X$ we can choose a sequence $\{V^{(n)} \in C^\infty_{\mathrm{comp}}(\overline{\Omega^-}) \ :\ [V^{(n)}]^-_{S_1} = 0\}$ which converges to the vector function V in the $W^1_2(\Omega^-)$-norm. Therefore, simple limiting arguments yield that (8.11) is valid for $V \in X$. Now, we may substitute u_k in the place of v_k in (8.11). As a result we finally obtain

$$\int_{\Omega^-} \{ c_{kjpq} D_p u_q \overline{D_k u_j} + \tau^2 |u|^2 + \tfrac{1}{\overline{\tau} T_0} \lambda_{pq} D_q u_4 \overline{D_p u_4} + \tfrac{c_0}{T_0} |u_4|^2 \}\,dx = 0, \tag{8.12}$$

which completes the proof (see the proof of Theorem 8.1). □

8.2. Now we consider the crack type problems.

Theorem 8.5. *The homogeneous problems $(\mathcal{CR.D})_\tau$ and $(\mathcal{CR.N})_\tau$ have only the trivial solutions in the space $W^1_2(\mathbb{R}^3_{S_1})$.*

Proof. Let $U \in W_2^1(\mathbb{R}^3_{S_1})$ be some solution to the homogeneous problem $(\mathcal{CR.D})_\tau$. Clearly, $[U]^+_{S_1} = 0$ and $[U]^-_{S_1} = 0$ in the usual trace sense. Recall that $S_1 \subset S$, where $S = \partial\Omega^+$ for some bounded domain Ω^+. Next, let us calculate the functional traces $[B(D,n)U]^\pm_S$. Note that $[B(D,n)U]^\pm_{S\setminus S_1}$ exist in the usual trace sense and $[B(D,n)U]^+_{S\setminus\overline{S_1}} = [B(D,n)U]^-_{S\setminus\overline{S_1}}$ since $U \in \mathrm{C}^\infty(\mathbb{R}^3_{\overline{S_1}})$. We apply again the definitions (4.1) and (4.2) to write the equations

$$\langle\, [B(D,n)U]^+_S \,,\, [\overline{V}]^+_S \,\rangle_S = \int_{\Omega^+} E(U,V)\,dx, \tag{8.13}$$

$$\langle\, [B(D,n)U]^-_S \,,\, [\overline{V'}]^-_S \,\rangle_S = -\int_{\Omega^-} E(U,V')\,dx, \tag{8.14}$$

where $V = (v,v_4)^\top \in W_2^1(\Omega^+)$, $V' = (v',v_4')^\top \in W^1_{2,\mathrm{comp}}(\Omega^-)$, $v = (v_1,v_2,v_3)^\top$, $v' = (v_1',v_2',v_3')^\top$. Making again use of the limiting arguments from the proof of Theorem 8.4, we easily conclude by virtue of (8.13) and (8.14)

$$\begin{aligned}\int_{\Omega^+} E(U,V)dx + \int_{\Omega^-} E(U,V')dx =\\ = \langle [B(D,n)U]^+_S \,,\, [\overline{V}]^+_S\rangle_S - \langle [B(D,n)U]^-_S \,,\, [\overline{V'}]^-_S\rangle_S\end{aligned} \tag{8.15}$$

for arbitrary $V \in [W_2^1(\Omega^+)]^4$ and arbitrary $V' \in [W^1_{2,\mathrm{comp}}(\Omega^-)]^4$.

By the same manipulations as in the proof of Theorem 8.2, we derive from (8.15)

$$\begin{aligned}&\int_{\Omega^+}\big\{c_{kjpq}D_pu_q\overline{D_kv_j} + \tau^2u_p\overline{v_p} - \beta_{pq}[u_4\overline{D_pv_q} - v_4\overline{D_pu_q}] +\\ &\qquad + \tfrac{1}{\tau T_0}\lambda_{pq}\overline{D_qu_4}D_pv_4 + \tfrac{c_0}{T_0}\overline{u_4}v_4\big\}\,dx +\\ &+\int_{\Omega^-}\big\{c_{kjpq}D_pu_q\overline{D_kv_j'} + \tau^2u_p\overline{v_p'} - \beta_{pq}[u_4\overline{D_pv_q'} - v_4'\overline{D_pu_q}] +\\ &\qquad + \tfrac{1}{\tau T_0}\lambda_{pq}\overline{D_qu_4}D_pv_4' + \tfrac{c_0}{T_0}\overline{u_4}v_4'\big\}\,dx =\\ &= \langle\, [P(D,n)U]^+_S \,,\, [\overline{v}]^+_S \,\rangle_S + \tfrac{1}{\tau T_0}\overline{\langle\, [\lambda(D,n)u_4]^+_S \,,\, [\overline{v_4}]^+_S \,\rangle_S} -\\ &-\langle\, [P(D,n)U]^-_S \,,\, [\overline{v'}]^-_S \,\rangle_S - \tfrac{1}{\tau T_0}\overline{\langle\, [\lambda(D,n)u_4]^-_S \,,\, [\overline{v_4'}]^-_S \,\rangle_S}.\end{aligned} \tag{8.16}$$

We may substitute in this equation $V = U|_{\Omega^+}$ and $V' = U|_{\Omega^-}$, where $U|_{\Omega^\pm}$ denotes the restriction of U onto $\Omega^\pm$. Taking into account the equalities $[U]^\pm_{S_1} = 0$, $[B(D,n)U]^+_{S\setminus\overline{S_1}} = [B(D,n)U]^-_{S\setminus\overline{S_1}}$, and $[U]^+_{S\setminus\overline{S_1}} = [U]^-_{S\setminus\overline{S_1}}$, we easily see that (see also Remark 4.1)

$$\begin{aligned}&\langle\, [P(D,n)U]^+_S \,,\, [\overline{u}]^+_S \,\rangle_S + \tfrac{1}{\tau T_0}\overline{\langle\, [\lambda(D,n)u_4]^+_S \,,\, [\overline{u_4}]^+_S \,\rangle_S} -\\ &-\langle\, [P(D,n)U]^-_S \,,\, [\overline{u}]^-_S \,\rangle_S - \tfrac{1}{\tau T_0}\overline{\langle\, [\lambda(D,n)u_4]^-_S \,,\, [\overline{u_4}]^-_S \,\rangle_S} =\\ &= \langle\, [P(D,n)U]^+_{S\setminus\overline{S_1}}, [\overline{u}]^+_{S\setminus\overline{S_1}} \,\rangle_{S\setminus\overline{S_1}} + \tfrac{1}{\tau T_0}\overline{\langle\, [\lambda(D,n)u_4]^+_{S\setminus\overline{S_1}}, [\overline{u_4}]^+_{S\setminus\overline{S_1}} \,\rangle_{S\setminus\overline{S_1}}} -\\ &-\langle\, [P(D,n)U]^-_{S\setminus\overline{S_1}}, [\overline{u}]^-_{S\setminus\overline{S_1}} \,\rangle_{S\setminus\overline{S_1}} - \tfrac{1}{\tau T_0}\overline{\langle [\lambda(D,n)u_4]^-_{S\setminus\overline{S_1}}, [\overline{u_4}]^-_{S\setminus\overline{S_1}} \rangle_{S\setminus\overline{S_1}}} = 0.\end{aligned}$$

Therefore, (8.16) implies

$$\int_{\mathbb{R}^3_{S_1}}\Big\{c_{kjpq}D_pu_q\overline{D_ku_j} + \tau^2|u|^2 + \tfrac{1}{\tau T_0}\lambda_{pq}D_qu_4\overline{D_pu_4} + \tfrac{c_0}{T_0}|u_4|^2\Big\}\,dx = 0.$$

Whence $U=0$ in $\mathbb{R}^3_{S_1}$ follows. □

8.3. To prove the uniqueness theorems for the basic and mixed homogeneous interface problems, one has to apply the arguments quite similar to the above ones to derive the following basic equation for solutions of the indicated homogeneous problems

$$\sum_{\mu=1}^{2}\int_{\Omega^\mu}\Big\{c^{(\mu)}_{kjpq}D_p u^{(\mu)}_q\overline{D_k u^{(\mu)}_j}+\tau^2|u^{(\mu)}|^2+\tfrac{1}{\overline{\tau}T_0}\lambda^{(\mu)}_{pq}D_q u^{(\mu)}_4\,\overline{D_p u^{(\mu)}_4}+$$
$$.+\tfrac{c^{(\mu)}_0}{T_0}|u^{(\mu)}|^2\Big\}dx=0. \tag{8.17}$$

For regular solutions this formula can be obtained from the following Green identities for Ω^μ $(\mu=1,2)$

$$\int_{\Omega^\mu}\Big\{[A^{(\mu)}(D,\tau)U^{(\mu)}]_k\,\overline{u^{(\mu)}}_k+\tfrac{1}{\overline{\tau}T_0}\overline{[A^{(\mu)}(D,\tau)U^{(\mu)}]_4\,u^{(\mu)}_4}\Big\}\,dx=$$
$$=(-1)^{\mu+1}\int_S\Big\{[B^{(\mu)}(D,n)U^{(\mu)}]^{(\mu)}_k[\overline{u^{(\mu)}}_k]^{(\mu)}+$$
$$+\tfrac{1}{\overline{\tau}T_0}[u^{(\mu)}_4]^{(\mu)}\overline{[\lambda^{(\mu)}(D,n)\overline{u^{(\mu)}}_4]^{(\mu)}}\Big\}\,dS-\int_{\Omega^\mu}\Big\{c^{(\mu)}_{kjpq}\,D_p u^{(\mu)}_q\,D_k\overline{u^{(\mu)}}_j+$$
$$\tau^2|u^{(\mu)}|^2++\tfrac{1}{\overline{\tau}T_0}\lambda^{(\mu)}_{kj}\,D_k u^{(\mu)}_4\,D_j\overline{u^{(\mu)}}_4+\tfrac{c^{(\mu)}_0}{T_0}|u^{(\mu)}_4|^2\Big\}\,dx, \tag{8.18}$$

where $[\cdot]^{(1)}:=[\cdot]^+_S$ and $[\cdot]^{(2)}:=[\cdot]^-_S$.

For solutions of the homogeneous problems in the Sobolev spaces $W^1_2(\Omega^\mu)$ formula (8.17) follows from the definitions of functional traces given in Section 4.

Now we formulate the uniqueness results for the interface problems of thermoelastic pseudo-oscillations.

Theorem 8.6. *The homogeneous basic and mixed interface problems* $(\mathcal{C})_\tau$, $(\mathcal{G})_\tau$, $(\mathcal{H})_\tau$, $(\mathcal{C}-\mathcal{DD})_\tau$, $(\mathcal{C}-\mathcal{NN})_\tau$, $(\mathcal{C}-\mathcal{DC})_\tau$, $(\mathcal{C}-\mathcal{NC})_\tau$, $(\mathcal{C}-\mathcal{G})_\tau$, $(\mathcal{C}-\mathcal{H})_\tau$, *have only the trivial solutions in the corresponding Sobolev spaces, i.e., if* $(U^1,U^2)\in(W^1_2(\Omega^1),W^1_2(\Omega^2))$ *solves one of the above homogeneous problems, then* $U^{(\mu)}=0$ *in* Ω^μ, $\mu=1,2$.

Proof. By the reasonings similar to the already applied ones in the previous subsection, we can easily conclude that for the pair of vector functions $(U^1,U^2)\in(W^1_2(\Omega^1),W^1_2(\Omega^2))$, which is solution to one of the homogeneous problems indicated in the theorem, the formula (8.17) holds. Whence the proof follows. □

We remark that the regular case (i.e., when $(U^{(1)},U^{(2)})\in(\mathrm{C}^1(\overline{\Omega^1}),\mathrm{C}^1(\overline{\Omega^2}))$) is covered by this theorem.

9. Uniqueness Theorems for the Steady State Oscillation Problems

9.1. First we shall establish some auxiliary results concerning the coefficients of asymptotic formulae (2.30) and ascertain the structure of the matrix functions (2.24).

We recall that

$$N(-i\xi,-i\omega)=[N_{kj}(-i\xi,-i\omega)]_{4\times 4} \tag{9.1}$$

is the adjoint matrix to

$$A(-i\xi,-i\omega)=\begin{bmatrix} [\omega^2 I_3 - C(\xi)]_{3\times 3} & [i\beta_{kj}\xi_j]_{3\times 1} \\ [\omega T_0\beta_{kj}\xi_j]_{1\times 3} & -\Lambda(\xi)+i\omega c_0 \end{bmatrix}_{4\times 4}, \tag{9.2}$$

where $C(\xi)$ and $\Lambda(\xi)$ are defined by (1.7) and (1.8), respectively, while $N_{kj}(-i\xi,-i\omega)$ denotes the cofactor of the element $A_{jk}(-i\xi,-i\omega)$ of the matrix (9.2) (cf. (1.32), (1.33)).

Let us set

$$C(\xi,\omega)=\omega^2 I_3 - C(\xi),\ \ \tilde{C}(\xi,\omega)=\omega^2 I_3-\tilde{C}(\xi), \tag{9.3}$$

where $\tilde{C}(\xi)$ is given by (1.35). Denote by $C^*(\xi,\omega)$ and $\tilde{C}^*(\xi,\omega)$ the corresponding adjoint matrices.

Due to (1.43) and (1.44) we have

$$C(\xi,\omega)\,C^*(\xi,\omega)=-\Phi(\xi,\omega)\,I_3,\quad \tilde{C}(\xi,\omega)\,\tilde{C}^*(\xi,\omega)=-\tilde{\Phi}(\xi,\omega)\,I_3. \tag{9.4}$$

From the condition I^0 (see Subsection 1.6) it follows that $\mathrm{rank}C(\xi,\omega)=2$ and, consequently, $\mathrm{rank}C^*(\xi,\omega)=1$ for an arbitrary $\xi\in S_l^0$. Moreover (for the same $\xi\in S_l^0$) there exists an orthogonal real matrix $G(\xi,\omega)$ such that

$$G^\top(\xi,\omega)\,C^*(\xi,\omega)G(\xi,\omega)=\lambda_1\mathcal{I}_0,\ \ \mathcal{I}_0=\begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{9.5}$$

where $\lambda_1=\lambda_1(\xi,\omega)\neq 0$ is a real eigenvalue of the matrix $C^*(\xi,\omega)$ (two other eigenvalues are equal to zero; for details see [55]).

Further, let $d(\xi,\omega)=-\omega c_0[\Lambda(\xi)]^{-1}$ and

$$d(\xi,\omega)G^\top(\xi,\omega)\,\tilde{C}^*(\xi,\omega)G(\xi,\omega)=[b_{kj}(\xi,\omega)]_{3\times 3}. \tag{9.6}$$

Lemma 9.1. *Let $\xi\in S_j^c$, $j=1,\dots,m$, where S_j^c are the characteristic surfaces defined in Subsection* 1.6. *Then the matrix N has the following structure*

$$N(\pm i\xi,-i\omega)=\begin{bmatrix} [\mathcal{N}(\xi,\omega)]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & 0\end{bmatrix}_{4\times 4},$$

where $\mathcal{N}(\xi,\omega)=-\Lambda(\xi)[1+ib_{11}(\xi,\omega)\lambda_1^{-1}(\xi,\omega)]C^*(\xi,\omega)$.

Proof. Let $\xi\in S_j^c$ be an arbitrary point ($1\le j\le m$). Clearly, ξ belongs to some surface S_l^0, $1\le l\le 3$, as well (see Subsection 1.6). Therefore,

$$N_{44}(\pm i\xi,-i\omega)=-\Phi(\xi,\omega)=0, \tag{9.7}$$

due to (1.46).

By direct calculations we get

$$N_{4k}(-i\xi,-i\omega)=-i\omega T_0 N_{k4}(-i\xi,-i\omega),\ \ k=1,2,3, \tag{9.8}$$

$$\begin{aligned} N_{pq}(-i\xi,-i\omega)&=-\Lambda(\xi)C^*_{pq}(\xi,\omega)+i\omega c_0\tilde{C}^*_{pq}(\xi,\omega)=\\ &=N_{pq}(i\xi,-i\omega),\ \ 1\le p,q\le 3. \end{aligned} \tag{9.9}$$

The condition I^0 of Subsection 1.6 implies (see (1.42)) $\nabla M(\xi,-i\omega)=\Lambda(\xi)\nabla\Phi(\xi,\omega)-i\omega c_0\nabla\tilde{\Phi}(\xi,\omega)\neq 0$, since $\Lambda(\xi)\neq 0$ on S_j^c.

This relation together with the equations (1.31), (1.32), (1), and

$$\det A(-i\xi',-i\omega)=\det A(i\xi',-i\omega)=M(-\xi',-i\omega)=M(\xi',-i\omega),\ \xi'\in\mathbb{R}^3,$$

yields

$$\operatorname{rank} A(i\xi,-i\omega)=3,\quad \operatorname{rank} N(i\xi,-i\omega)=1, \tag{9.10}$$

i.e., any two columns (rows) of the matrix (9.1) are linearly dependent.

Taking into account the equations (9.8) and (9.7) it can be easily proved that $N_{k4}(-i\xi,-i\omega)=0$, $N_{4k}(-i\xi,-i\omega)=0$, $k=1,2,3$.

Thus, we have obtained the following representation

$$N(\pm i\xi,-i\omega)=\begin{bmatrix}[N^{(0)}(\xi,\omega)]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & 0\end{bmatrix}_{4\times 4}$$

with

$$N^{(0)}(\xi,\omega)=[N_{pq}(i\xi,-i\omega)]_{3\times 3}, \tag{9.11}$$

where $N_{pq}(i\xi,-i\omega)=N_{qp}(i\xi,-i\omega)$ are given by (9.9).

Now from (9.9) and (9.11) together with (9.5) and (9.6) it follows

$$\begin{gathered}N^{(0)}(\xi,\omega)=-\Lambda(\xi)C^*(\xi,\omega)+i\omega c_0\tilde{C}^*(\xi,\omega)\,,\\ G^\top(\xi,\omega)N^{(0)}(\xi,\omega)G(\xi,\omega)=-\Lambda(\xi)\lambda_1(\xi,\omega)\mathcal{I}_0+\\ +i\omega c_0G^\top(\xi,\omega)\tilde{C}^*(\xi,\omega)G(\xi,\omega)=-\Lambda(\xi)\begin{bmatrix}\lambda_1(\xi,\omega)+ib_{11} & ib_{12} & ib_{13}\\ ib_{12} & ib_{22} & ib_{23}\\ ib_{13} & ib_{23} & ib_{33}\end{bmatrix},\end{gathered} \tag{9.12}$$

where b_{pq} are real functions defined by (9.6).

By virtue of (9.10) we have $\operatorname{rank} N^{(0)}(\xi,-i\omega)=1$, and, consequently,

$$\operatorname{rank}\,[G^\top(\xi,\omega)N^{(0)}(\xi,\omega)G(\xi,\omega)]=1,$$

since G is an orthogonal matrix. This, in turn, implies that the matrix (9.12) has only one linearly independent column (row). Inasmuch as $\lambda_1\neq 0$, there exist complex numbers $\alpha=\alpha_1+i\alpha_2$ and $\beta=\beta_1+i\beta_2$ such that

$$\begin{pmatrix}ib_{12}\\ ib_{22}\\ ib_{23}\end{pmatrix}=\alpha\begin{pmatrix}\lambda_1+ib_{11}\\ ib_{12}\\ ib_{13}\end{pmatrix},\quad\begin{pmatrix}ib_{13}\\ ib_{23}\\ ib_{33}\end{pmatrix}=\beta\begin{pmatrix}\lambda_1+ib_{11}\\ ib_{12}\\ ib_{13}\end{pmatrix}. \tag{9.13}$$

Equating the corresponding elements and separating the real and imaginary parts lead to the equations $(\alpha_1^2+\alpha_2^2)\lambda_1=0$, $(\beta_1^2+\beta_2^2)\lambda_1=0$, i.e., $\alpha=\beta=0$. But then from (9.13), (9.12), and (9.5) we derive

$$\begin{gathered}N^{(0)}(\xi,\omega)=-\Lambda(\xi)\{\lambda_1(\xi,\omega)G(\xi,\omega)\mathcal{I}_0G^\top(\xi,\omega)+\\ +ib_{11}(\xi,\omega)G(\xi,\omega)\mathcal{I}_0G^\top(\xi,\omega)\}=-\Lambda(\xi)[\lambda_1(\xi,\omega)+\\ +ib_{11}(\xi,\omega)]G(\xi,\omega)\mathcal{I}_0G^\top(\xi,\omega)=-\Lambda(\xi)[1+i\lambda_1^{-1}(\xi,\omega)b_{11}(\xi,\omega)]C^*(\xi,\omega),\end{gathered}$$

which completes the proof. $\square$

Remark 9.2. Due to equation (2.24) and Lemma 4.1 we get (for arbitrary $\xi \in S_j^c$, $j = 1, \ldots, m$, and $r = 1, 2$)

$$c_r^{(j)}(\xi, -i\omega) = d_j(\xi, -i\omega) \begin{bmatrix} [C^*(\xi,\omega)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 0 \end{bmatrix}_{4\times 4} \tag{9.14}$$

with

$$d_j(\xi, -i\omega) = (-1)^{j+1} \frac{\Lambda(\xi)[1 + i\lambda_1^{-1}(\xi,\omega) b_{11}(\xi,\omega)]}{[2\pi(\kappa(\xi))^{1/2} |\nabla\Phi_m(\xi, -i\omega)| \Psi_m(\xi, -i\omega)]} \,.$$

Lemma 9.3. *Let $U = (u, u_4)^\top$ be a regular vector in Ω^- of the class $\mathrm{SK}_r^m(\Omega^-)$, and let $A(D, -i\omega)U$ have a compact support.*

Then for sufficiently large $|x|$

$$u(x) = \sum_{j=1}^m |x|^{-1} d_j(\xi^j, -i\omega) e^{(-1)^{r+1} i x \xi^j} C^*(\xi^j, \omega) \widetilde{b}(\xi^j) + O(|x|^{-2}), \tag{9.15}$$

$$u_4(x) = O(|x|^{-2}), \tag{9.16}$$

with the same d_j as in Remark 9.2; *here $C^*(\xi,\omega)$ is the adjoint matrix to $C(\xi,\omega)$, $\widetilde{b} = (b_1, b_2, b_3)^\top$ is uniquely determined by the vector U (see below* (9.18)), *and the point $\xi^j \in S_j^c$ corresponds to the vector $x/|x|$.*

Proof. Denote by Ω the support of $A(D, -i\omega)U$. Then by Theorems 2.3, 3.1 and Remark 2.6 we have (for sufficiently large $|x|$)

$$\begin{aligned} U(x) = \sum_{j=1}^m \Big\{ &\int_\Omega |x|^{-1} e^{(-1)^{r+1} i(x-y)\xi^j} c_r^{(j)}(\xi^j, -i\omega)[A(D_y, -i\omega)U(y)]\, dy + \\ &+ \int_S |x|^{-1} e^{(-1)^{r+1} i(x-y)\xi^j} c_r^{(j)}(\xi^j, -i\omega)[B(D_y, n(y))U(y)]^-\, dS_y - \\ &- \int_S |x|^{-1} e^{(-1)^{r+1} i(x-y)\xi^j} \{Q((-1)^r i\xi^j, n(y), -i\omega) \times \\ &\times [c_r^{(j)}(\xi^j, -i\omega)]^\top\}^\top [U(y)]^-\, dS_y \Big\} + \\ + O(|x|^{-2}) = \sum_{j=1}^m &|x|^{-1} e^{(-1)^{r+1} i x\xi^j} c_r^{(j)}(\xi^j, -i\omega)\, b(\xi^j) + O(|x|^{-2}), \end{aligned} \tag{9.17}$$

where

$$\begin{aligned} b(\xi^j) = (\widetilde{b}(\xi^j), b_4(\xi^j))^\top = &\int_\Omega e^{(-1)^r i y\xi^j} [A(D_y, -i\omega)U(y)]\, dy + \\ &+ \int_S e^{(-1)^r i y\xi^j} [B(D_y, n(y))U(y)]^-\, dS_y - \\ &- \int_S e^{(-1)^r i y\xi^j} Q^\top((-1)^r i\xi^j, n(y), -i\omega)[U(y)]^-\, dS_y; \end{aligned} \tag{9.18}$$

here ξ^j corresponds to the vector $x/|x|$.

Now (9.15) and (9.16) follow immediately from (9.17) and (9.14). Note that the vector $b(\xi^j)$ is represented explicitly by (9.18). $\square$

Remark 9.4. From (9.15) with the help of equation (9.5) we get the following equivalent asymptotic formula for u

$$u(x) = \sum_{j=1}^{m} |x|^{-1} e^{(-1)^{r+1} i x \xi^j} \lambda_1(\xi^j,\omega) G(\xi^j,\omega) \mathcal{I}_0 G^\top(\xi^j!,\omega) a^{(j)}(\xi^j,\omega) + \\ + O(|x|^{-2}), \tag{9.19}$$

where

$$a^{(j)}(\xi^j,\omega) = d_j(\xi^j,-i\omega)\,\widetilde{b}(\xi^j), \tag{9.20}$$

d_j and $\widetilde{b}$ are the same as in Lemma 9.3. Note that due to (9.5)

$$\mathcal{I}_0\, G^\top\, a^{(j)} = ([G^\top\, a^{(j)}]_1, 0, 0)^\top. \tag{9.21}$$

9.2. In this subsection we assume $S = \partial\Omega^-$ to be a connected C^1-regular surface and prove the following uniqueness theorem.

Theorem 9.5. *Let U be a regular solution to the homogeneous exterior problem $(\mathcal{P}_k)^-_\omega$ $(k = 1,\dots,4)$ and $U \in \mathrm{SK}^m_r(\Omega^-)$ with $r = 1$ for $\omega > 0$ and $r = 2$ for $\omega < 0$.*

Then $U = 0$ in Ω^-.

Proof. Let $R,\ B_R,\ \Sigma_R$ and Ω^-_R be the same as in the proof of Theorem 3.1. Since U satisfies the homogeneous conditions of the problem $(\mathcal{P}_k)^-_\omega$, from (1.23) (with $\Omega^+ = \Omega^-_R$ and $\mu = -i\omega$) it follows that

$$\int_{\Omega^-_R} \{c_{kjpq}\, D_p u_q\, D_k \overline{u}_j - \omega^2\,|u|^2 - i(\omega T_0)^{-1} \lambda_{kj}\, D_k u_4\, D_j \overline{u}_4 + \\ + c_0(T_0)^{-1}|u_4|^2\}\ dx = \int\limits_{\Sigma_R} \Big\{[B(D,n)U]_k\,[\overline{u}_k] - \tfrac{i}{\omega T_0}[u_4]\,[\partial_n \overline{u}_4]\Big\}\ d\Sigma_R,$$

where $B(D,n)$ and ∂_n are defined by (1.25) and (1.24), respectively.

Owing the fact that $c_{kjpq}\, D_p u_q\, D_k \overline{u}_j$ and $\lambda_{kj}\, D_k u_4\, D_j \overline{u}_4$ are non-negative real quantities, from the last equation by separating the imaginary part we get

$$\mathrm{Im}\Big\{ \int\limits_{\Sigma_R} \Big\{[B(D_x,\eta)U(x)]_k\,[\overline{u}_k(x)] - \tfrac{i}{\omega T_0}[u_4(x)]\,[\partial_\eta \overline{u}_4(x)]\Big\}\ d\Sigma_R \Big\} + \\ + \tfrac{1}{\omega T_0} \int_{\Omega^-_R} \lambda_{kj}\, D_k u_4(x)\, D_j \overline{u}_4(x)\, dx = 0, \tag{9.22}$$

where $\eta = x/|x|$ is the unit outward normal at the point $x \in \Sigma_R$.

Due to Lemma 9.3 it is easily seen that

$$\int_{\Omega^-_R} \lambda_{kj}\, D_k u_4(x)\, D_j \overline{u}_4(x)\, dx = \int_{\Omega^-} \lambda_{kj}\, D_k u_4(x)\, D_j \overline{u}_4(x)\, dx + O(R^{-1}),$$

$$\int_{\Sigma_R} |u_4(x)\, \partial_\eta \overline{u}_4(x)|\, d\Sigma_R = O(R^{-2}), \quad \int_{\Sigma_R} |u_4(x)\, \overline{u}_k(x)|\, d\Sigma_R = O(R^{-1}),$$

as $R \to +\infty$ $(k = 1,2,3)$. Clearly, $\partial_\eta = \partial_n$ on Σ_R.

Taking into account (1.25) and applying the above relations to (9.22) we obtain

$$\mathrm{Im}\Big\{ \int\limits_{\Sigma_R} [T(D_x,\eta)u]_k\,[\overline{u}_k]\, d\Sigma_R \Big\} + \tfrac{1}{\omega T_0} \int\limits_{\Omega^-} \lambda_{kj} D_k u_4 D_j \overline{u}_4 dx = O(R^{-1}), \tag{9.23}$$

where $T(D,\eta)$ is the stress operator of elastostatics defined by (1.12).

In the same way as in the proof of Theorem 3.1 (by integrating with respect to R from ν to 2ν and deviding the result by ν) from (9.23) we derive

$$\operatorname{Im}\Big\{\frac{1}{\nu}\int\limits_{\nu}^{2\nu}\int\limits_{\Sigma_R}[T(D_x,\eta)u]_k[\overline{u}_k]d\Sigma_R dR\Big\}+\frac{1}{\omega T_0}\int\limits_{\Omega^-}\lambda_{kj}D_k u_4 D_j\overline{u}_4\,dx=O(\nu^{-1}),\tag{9.24}$$

where ν is large enough.

Further, by Lemma 9.3 the first summand in the left-hand side of (9.24) can be transformed as follows

$$\begin{aligned}F(\nu):=\operatorname{Im}\left\{\frac{1}{\nu}\int\limits_{\nu}^{2\nu}\int\limits_{\Sigma_R}[T(D,\eta)u]_k\,[\overline{u}_k]\,d\Sigma_R\,dR\right\}=\\
=\operatorname{Im}\left\{\frac{1}{\nu}\int\limits_{\nu}^{2\nu}\int\limits_{\Sigma_R}\sum_{j=1}^{m}[i(-1)^{r+1}R^{-1}d_j(\xi^j,-i\omega)\times\right.\\
\times e^{(-1)^{r+1}ix\xi^j}T(\xi^j,\eta)C^*(\xi^j,\omega)\widetilde{b}(\xi^j)]_k\times\\
\left.\times\sum_{l=1}^{m}[R^{-1}\overline{d_l(\xi^l,-i\omega)}e^{(-1)^r ix\xi^l}C^*(\xi^l,\omega)\overline{\widetilde{b}(\xi^l)}]_k\,d\Sigma_R\,dR+O(\nu^{-1})\right\}=\\
=\operatorname{Re}\left\{\frac{(-1)^{r+1}}{\nu}\int\limits_{\Sigma_1}\sum_{j,l=1}^{m}d_j(\xi^j,-i\omega)\overline{d_l(\xi^l,-i\omega)}[T(\xi^j,\eta)C^*(\xi^j,\omega)\widetilde{b}(\xi^j)]_k\times\right.\\
\left.\times[C^*(\xi^l,\omega)\overline{\widetilde{b}(\xi^l)}]_k\left(\int\limits_{\nu}^{2\nu}e^{(-1)^{r+1}iR[\mu_j(\eta)-\mu_l(\eta)]}\,dR\right)\,d\Sigma_1\right\}+O(\nu^{-1}),\end{aligned}\tag{9.25}$$

where $\mu_j(\eta)=(\eta\cdot\xi^j)$ and ξ^j corresponds to the vector $x/|x|$.

It can be easily proved that $\mu_j(\eta)\neq\mu_l(\eta)$ if $j\neq l$ (see Subsection 1.6). Therefore, if $j\neq l$, clearly,

$$\int\limits_{\nu}^{2\nu}e^{\pm iR[\mu_j(\eta)-\mu_l(\eta)]}\,dR=O(1),$$

and (9.25) implies

$$\begin{aligned}F(\nu)=\operatorname{Re}\Big\{(-1)^{r+1}\sum_{j=1}^{m}\int\limits_{\Sigma_1}T(\xi^j,\eta)C^*(\xi^j,\omega)a^{(j)}\cdot C^*(\xi^j,\omega)a^{(j)}\,d\Sigma_1\Big\}+\\
+O(\nu^{-1})\end{aligned}\tag{9.26}$$

with $a^{(j)}$ defined by (9.20).

In view of the symmetry property of $C^*(\xi,\eta)$ and equality $T^\top(\xi,\eta)=T(\eta,\xi)$ we have from (9.26)

$$\begin{aligned}F(\nu)=\frac{(-1)^{r+1}}{2}\sum_{j=1}^{m}\int\limits_{\Sigma_1}C^*(\xi^j,\omega)[T(\xi^j,\eta)+\\
+T(\eta,\xi^j)]C^*(\xi^j,\omega)a^{(j)}\cdot a^{(j)}\,d\Sigma_1+O(\nu^{-1}).\end{aligned}\tag{9.27}$$

Now passing to the limit in (9.24) as $\nu \to +\infty$ and bearing in mind (9.25) and (9.27) we arrive at the equation

$$\tfrac{1}{\omega T_0} \int\limits_{\Omega^-} \lambda_{kj}\, D_k u_4\, D_j \overline{u}_4\, dx + \tfrac{(-1)^{r+1}}{2} \sum_{j=1}^{m} \int\limits_{\Sigma_1} E_j(\xi^j, \omega)\, d\Sigma_1 = 0 \quad (9.28)$$

with

$$E_j(\xi^j, \omega) = C^*(\xi^j, \omega)[T(\xi^j, \eta) + T(\eta, \xi^j)]C^*(\xi^j, \omega)a^{(j)} \cdot a^{(j)}, \qquad (9.29)$$

where $\xi^j \in S_j^c$ corresponds again to η, i.e., $n(\xi^j) = \eta$.

In what follows we claim that the integral in the second term of (9.28) is a non-negative function for all $\xi^j \in S_j^c$.

To see this, let us note that

$$T(\xi, \eta) + T(\eta, \xi) = \tfrac{\partial}{\partial n(\xi)} C(\xi) = -\tfrac{\partial}{\partial n(\xi)} C(\xi, \omega),$$

where $\eta = n(\xi)$, $\partial/\partial n(\xi) = n_k(\xi)\, D_k$ is a directional derivative, $C(\xi)$ and $C(\xi, \omega)$ are defined by (1.7) and (9.3), respectively.

We recall that in Subsection 1.6 we introduced the two sets of surfaces $\{S_j^c\}_{j=1}^m$ and $\{S_p^0\}_{p=1}^3$ defined by equations (1.46) and by the first equation of the same system, respectively. Therefore, each S_j^c coincides with some S_p^0 for some $p = p(j)$. Let us fix this correspondence, i.e., $S_j^c = S_{p(j)}^0$.

Further, we proceed as follows. Note that

$$-\left[C^*(\xi, \omega) \left(\tfrac{\partial}{\partial n(\xi^j)} C(\xi, \omega)\right) C^*(\xi, \omega)\right] =$$

$$= -\tfrac{\partial}{\partial n(\xi^j)}[C^*(\xi, \omega)C(\xi, \omega)C^*(\xi, \omega)] = \left[\tfrac{\partial}{\partial n(\xi^j)} \Phi(\xi, \omega)\right] C^*(\xi, \omega) \quad (9.30)$$

for all $\xi = \xi^j \in S_j^c$ (see (9.4)).

With the help of (9.5), (9.30), and (9.29) we deduce

$$E_j(\xi^j, \omega) = \left\{ \left[\tfrac{\partial}{\partial n(\xi)} \Phi(\xi, \omega)\right] C^*(\xi, \omega) a^{(j)} \cdot a^{(j)} \right\}_{\xi=\xi^j} =$$

$$= \left\{ \left[\tfrac{\partial}{\partial n(\xi)} \Phi(\xi, \omega)\right] \lambda_1(\xi, \omega) \mathcal{I}_0 G^\top(\xi, \omega) a^{(j)} \cdot G^\top(\xi, \omega) a^{(j)} \right\}_{\xi=\xi^j} =$$

$$= \left\{ \left[\tfrac{\partial}{\partial n(\xi)} \Phi(\xi, \omega)\right] \lambda_1(\xi, \omega)\ \left|\left[G^\top(\xi, \omega) a^{(j)}\right]_1\right|^2 \right\}_{\xi=\xi^j}. \qquad (9.31)$$

Now we show that the function

$$\psi(\xi) = \left[\tfrac{\partial}{\partial n(\xi)} \Phi(\xi, \omega)\right] \lambda_1(\xi, \omega), \ \ \xi \in S_j^c, \qquad (9.32)$$

is strictly positive.

Since $\lambda_1(\xi, \omega)$ is the only nonzero eigenvalue of the matrix $C^*(\xi, \omega)$ for $\xi \in S_j^c = S_p^0$, we have

$$\{\lambda_1(\xi, \omega)\}_{\xi \in S_j^c} = \{\mathrm{Sp} C^*(\xi, \omega)\}_{\xi \in S_j^c} = \{C^*_{11}(\xi, \omega) + C^*_{22}(\xi, \omega) + C^*_{33}(\xi, \omega)\}_{\xi \in S_j^c} =$$

$$= \frac{1}{2\omega} \left\{ \frac{\partial}{\partial \omega} \begin{vmatrix} \omega^2 - C_{11}(\xi) & -C_{12}(\xi) & -C_{13}(\xi) \\ -C_{12}(\xi) & \omega^2 - C_{22}(\xi) & -C_{23}(\xi) \\ -C_{13}(\xi) & -C_{23}(\xi) & \omega^2 - C_{33}(\xi) \end{vmatrix} \right\}_{\xi \in S_j^c} =$$

$$= -\frac{1}{2\omega}\left\{\frac{\partial}{\partial\omega}\Phi(\xi,\omega)\right\}_{\xi\in S_j^c} = -\frac{1}{2\omega}\left\{\frac{\partial}{\partial\omega}\Phi(\xi,\omega)\right\}_{\xi\in S_p^0} = \Phi(\zeta,0)\omega^4\{\varrho_1^2(\varrho_p^2 -$$

$$-\varrho_2^2)(\varrho_p^2-\varrho_3^2)+\varrho_2^2(\varrho_p^2-\varrho_1^2)(\varrho_p^2-\varrho_3^2)+\varrho_3^2(\varrho_p^2-\varrho_1^2)(\varrho_p^2-\varrho_2^2)\} =$$

$$= (-1)^{p+1}\left|\left\{\tfrac{\varrho}{2\omega^2}\tfrac{\partial}{\partial\varrho}\Phi(\xi,\omega)\right\}_{\varrho=|\omega|\varrho_p}\right| = (-1)^{p+1}\left|\{\lambda_1(\xi,\omega)\}_{\xi\in S_p^0}\right|, \tag{9.33}$$

where $\zeta = \xi/|\xi|$, $\Phi(\zeta,0) > 0$; here we employed the representation (1.47).

It is easy to check that the exterior unit normal vector of S_p^0 is calculated by the following formula

$$n(\xi) = (-1)^{p+1}\tfrac{\nabla\Phi(\xi,\omega)}{|\nabla\Phi(\xi,\omega)|},\ \ \xi\in S_p^0.$$

Therefore,

$$\left\{\tfrac{\partial}{\partial n(\xi)}\Phi(\xi,\omega)\right\}_{\xi\in S_j^c} = \left\{(-1)^{p+1}\tfrac{\nabla\Phi(\xi,\omega)}{|\nabla\Phi(\xi,\omega)|}\cdot\nabla\Phi(\xi,\omega)\right\}_{\xi\in S_p^0} =$$

$$= \left\{(-1)^{p+1}|\nabla\Phi(\xi,\omega)|\right\}_{\xi\in S_p^0}, \tag{9.34}$$

which together with (9) yields

$$\psi(\xi) = |\nabla\Phi(\xi,\omega)|\ |\lambda_1(\xi,\omega)| > 0 \ \text{ for } \ \xi\in S_p^0 = S_j^c. \tag{9.35}$$

Hence by virtue of (9.31)-(9.35) we get

$$E_j(\xi^j,\omega) = \left\{|\nabla\Phi(\xi,\omega)|\ |\lambda_1(\xi,\omega)|\ \left|\left[G^\top(\xi,\omega)a^{(j)}\right]_1\right|^2\right\}_{\xi=\xi^j} \geq 0. \tag{9.36}$$

Now from (9.28) it follows that $\lambda_{kj}\, D_k u_4(x)\, D_j\overline{u_4(x)} = 0$, $x\in\Omega^-$, $E_j(\xi^j,\omega) = 0$, $\xi\in S_j^c$, if $(-1)^{r+1}\omega > 0$.

Applying (1.18), (9.35), (9.36), and (9.19)–(9.21) we conclude that $u_4(x) = 0$ in Ω^- and $[G^\top(\xi^j,\omega)a^{(j)}(\xi^j,\omega)]_1 = 0$, i.e.,

$$D^\beta u(x) = O(|x|^{-2}) \ \text{ as } \ |x|\to+\infty \tag{9.37}$$

for an arbitrary multi-index β.

Thus, we have obtained that u is a solution to the steady state oscillation equations of elasticity theory $C(D)u(x)+\omega^2 u(x) = 0$, $x\in\Omega^-$, satisfying the homogeneous boundary condition either $[u]^- = 0$ or $[Tu]^- = 0$ on S (see (5.1)-(5.8)) and the decay condition (9.37) at infinity.

Due to Lemma 3.4 in [41] (see also [55], Section 4) we then have $u(x) = 0$ in Ω^-, which completes the proof. □

9.3 In this subsection we consider the same basic BVPs $(\mathcal{P}_k)_\omega^-$ $(k = \overline{1,4})$ together with the mixed BVP $(\mathcal{P}_{mix})_\omega^-$ in the weak setting in the Sobolev space $W^1_{2,\mathrm{loc}}(\Omega^-)$. Here the principal difference in comparison with the pseudo-oscillation case is that the steady state oscillation equations do not admit nontrivial square integrable in Ω^- solutions, as it can be seen from the previous subsection (see the corresponding results for the Helmholtz equation and for the elastic oscillation equations, for example, in [10], [11], [80], [83], [45]).

As it is evident from the proof of Theorem 8.5, one of the central moments to establish the uniqueness of solutions to the homogeneous steady state oscillation problems is the derivation of formula (9.22) which follows from the corresponding Green identities for regular functions. In the sequel we shall show that the same type formula can be derived for weak solutions as well.

Theorem 9.6. *The homogeneous exterior BVPs* $(\mathcal{P}_k)^-_\omega$ $(k=1,\dots,4)$ *and* $(\mathcal{P}_{mix})^-_\omega$ *have only the trivial solutions in the class* $W^1_{2,\mathrm{loc}}(\Omega^-)\cap \mathrm{SK}^m_r(\Omega^-)$ *with* $r=1$ *for* $\omega>0$ *and* $r=2$ *for* $\omega<0$.

Proof. For definiteness, let $U\in W^1_{2,\mathrm{loc}}(\Omega^-)\cap \mathrm{SK}^m_r(\Omega^-)$ be a solution of the homogeneous problem $(\mathcal{P}_4)^-_\omega$.

Due to the definition (4.2) the homogeneous boundary condition $[B(D,n)U]^-=0$, which is understood in the functional sense, is equivalent to the equation

$$\langle\,[B(D,n)U]^-_S\,,\,[\overline{V}]^-_S\,\rangle_S=-\int_{\Omega^-}E(U,V)\,dx=0, \tag{9.38}$$

where $V\in W^1_{2,\mathrm{comp}}(\Omega^-)$ is an arbitrary vector function and $E(U,V)$ is defined by (1.27) with $\varkappa=-i\omega$.

In the same way as in the proof of Theorem 8.2 we easily derive from (9.38)

$$\begin{aligned}\langle\,[P(D,n)U]^-_S\,,\,[\overline{v}]^-_S\,\rangle_S-\tfrac{i}{\omega T_0}\overline{\langle\,[\lambda(D,n)u_4]^-_S\,,\,[\overline{v_4}]^-_S\,\rangle_S}=\\ =-\int_{\Omega^-}\big\{c_{kjpq}\,D_pu_q\,D_k\overline{v}_j-\omega^2u_k\overline{v}_k-\beta_{pq}[u_4\,D_p\overline{v}_q-v_4D_p\overline{u}_q]-\\ -\tfrac{i}{\omega T_0}\lambda_{pq}\,D_q\overline{u}_4\,D_pv_4+\tfrac{c_0}{T_0}\overline{u}_4v_4\big\}\,dx=0.\end{aligned} \tag{9.39}$$

Further, let $h_R(x)$ be a real cut off function with the following properties:

$$h_R\in \mathrm{C}^\infty(\mathbb{R}^3),\ \ h_R(x)=1\ \text{ for }\ |x|\le R,\ \ h_R(x)=0\ \text{ for }\ |x|\ge 2R, \tag{9.40}$$

where $R>0$ is an arbitrary real number such that the open ball $B_R=\{x\in\mathbb{R}^3\ :\ |x|<R\}$ contains the closed domain $\overline{\Omega^+}$ as a proper subset. Recall that $\partial B_R=:\Sigma_R$.

Next, we set $V_R(x):=h_R(x)U(x)$. Clearly, $V_R(x)\in W^1_{2,\mathrm{comp}}(\Omega^-)\cap \mathrm{C}^\infty(\Omega^-)$. Substitution of this vector function in (9.39) in the place of V implies

$$\mathcal{E}_1+\mathcal{E}_2=0, \tag{9.41}$$

where

$$\mathcal{E}_1=\int\limits_{\Omega^-_R}\Big\{c_{kjpq}D_pu_qD_k\overline{u}_j-\omega^2|u|^2-\tfrac{i}{\omega T_0}\lambda_{pq}D_q\overline{u}_4D_pu_4+\tfrac{c_0}{T_0}|u_4|^2\Big\}\,dx, \tag{9.42}$$

$$\begin{aligned}\mathcal{E}_2=\int\limits_{B_{2R}\setminus B_R}\big\{c_{kjpq}\,D_pu_q\,D_k(h_R\overline{u}_j)-\omega^2h_R|u|^2-\beta_{pq}[u_4\,D_p(h_R\overline{u}_q)-\\ -h_Ru_4D_p\overline{u}_q]-\,-\tfrac{i}{\omega T_0}\lambda_{pq}\,D_q\overline{u}_4\,D_p(h_Ru_4)+\tfrac{c_0}{T_0}h_R|u_4|^2\Big\}\ dx;\end{aligned} \tag{9.43}$$

here $\Omega^-_R=\Omega^-\cap B_R$.

The integration by parts in (9.43) leads to the equation

$$\mathcal{E}_2=-\int\limits_{B_{2R}\setminus B_R}[A(D,-i\omega)U]_k\overline{u}_k h_R dx+\frac{i}{\omega T_0}\int\limits_{B_{2R}\setminus B_R}\overline{[A(D,-i\omega)U]_4}u_4 h_R dx-$$

$$-\int_{\Sigma_R}[P(D,n)U]_k\overline{u}_k\, d\Sigma_R+\frac{i}{\omega T_0}\int_{\Sigma_R}\overline{[\lambda(D,n)u_4]}u_4\, d\Sigma_R=$$

$$=-\int_{\Sigma_R}\Big\{[P(D,n)U]_k\overline{u}_k-\frac{i}{\omega T_0}\overline{[\lambda(D,\eta)u_4]}u_4\Big\}\, d\Sigma_R, \qquad (9.44)$$

since $A(D,-i\omega)U=0$ in Ω^- and $n=\eta$ on Σ_R.

Therefore, (9.41), (9.42), and (9.43), due to the formulae (1.13) and (1.25), yield

$$\mathrm{Im}\Big\{\int_{\Sigma_R}\Big\{[B(D,n)U]_k\overline{u}_k-\frac{i}{\omega T_0}\overline{[\lambda(D,\eta)u_4]}u_4\Big\}\, d\Sigma_R+$$

$$+\frac{1}{\omega T_0}\int_{\Omega_R^-}\lambda_{pq}D_q\overline{u}_4 D_p u_4 dx\Big\}=0 \qquad (9.45)$$

for arbitrary solution $U\in W^1_{2,\mathrm{loc}}(\Omega^-)$ to the homogeneous problem $(\mathcal{P}_4)^-_\omega$.

Thus, we have obtained again the relation (9.22). This formula can be derived in the same way for weak solutions of the other basic and mixed BVPs indicated in the theorem. Now applying the same analysis as in the proof of Theorem 9.5 we can show that $U=0$ in Ω^-. □

9.4. The uniqueness theorems for the homogeneous crack type problems of thermoelastic oscillations can be proved by quite the same approach as above. To avoid the repetition of the arguments outlined in the previous subsections, we only note here that with the help of the identity (9.45) these problems by the analysis given in the proof of Theorem 9.5 are again reduced to the corresponding homogeneous BVPs of steady state oscillations of the elasticity theory with the displacement vector which behaves like $O(|x|^{-2})$ at infinity. Therefore, due to the results in [55], [56], [17], [41], such a displacement vector identically vanishes in the domain of analyticity. This finally leads to the corresponding uniqueness results for the above mentioned homogeneous crack type problems of the steady state thermoelastic oscillations. As a consequence we have the following uniqueness theorem.

Theorem 9.7. *The homogeneous crack type BVPs* $(\mathcal{CR.D})_\omega$ *and* $(\mathcal{CR.N})_\omega$ *have only the trivial solutions in the class* $W^1_{2,\mathrm{loc}}(\mathbb{R}^3_{S_1})\cap\mathrm{SK}^m_r(\mathbb{R}^3_{S_1})$ *with* $r=1$ *for* $\omega>0$ *and* $r=2$ *for* $\omega<0$.

9.5. For the homogeneous basic and mixed interface problems of the steady state thermoelastic oscillations we have a different situation since not all of them have only the trivial solution.

Let us first consider the basic homogeneous problem $(\mathcal{C})_\omega$ (see (7.3), (7.4)).

Theorem 9.8. *The homogeneous problem* $(\mathcal{C})_\omega$ *has only the trivial solution in the class* $(\mathrm{C}^1(\overline{\Omega^1}),\, \mathrm{C}^1(\overline{\Omega^2})\cap\mathrm{SK}^m_r(\Omega^2))$ *with* $r=1$ *for* $\omega>0$ *and* $r=2$ *for* $\omega<0$.

Proof. Let $(U^{(1)},U^{(2)})$ be a solution of the homogeneous problem $(\mathcal{C})_\omega$ from the class indicated in the theorem. Further, let R, B_R, Σ_R, and $\Omega_R^- =: \Omega_R^2$

be the same as in the proof of Theorem 9.6. By the Green formula (1.23) then we have

$$\int\limits_{\Omega^1}\left\{c^{(1)}_{kjpq}D_pu^{(1)}_q\overline{D_ku^{(1)}_j}-\omega^2|u^{(1)}|^2-\frac{i}{\omega T_0}\lambda^{(1)}_{pq}D_qu^{(1)}_4\overline{D_pu^{(1)}_4}+\frac{c^{(1)}_0}{T_0}|u^{(1)}|^2\right\}dx=$$

$$=\int_S\left\{[B^{(1)}(D,n)U^{(1)}]^+_k[\overline{u^{(1)}}_k]^+-\frac{i}{\omega T_0}[u^{(1)}_4]^+[\lambda^{(1)}(D,n)\overline{u^{(1)}}_4]^+\right\}dS, \qquad (9.46)$$

$$\int\limits_{\Omega^2_R}\left\{c^{(2)}_{kjpq}D_pu^{(2)}_q\overline{D_ku^{(2)}_j}-\omega^2|u^{(2)}|^2-\frac{i}{\omega T_0}\lambda^{(2)}_{pq}D_qu^{(2)}_4\overline{D_pu^{(2)}_4}+\frac{c^{(2)}_0}{T_0}|u^{(2)}|^2\right\}dx=$$

$$=-\int\limits_S\left\{[B^{(2)}(D,n)U^{(2)}]^-_k[\overline{u^{(2)}}_k]^- - \frac{i}{\omega T_0}[u^{(2)}_4]^-[\lambda^{(2)}(D,n)\overline{u^{(2)}}_4]^-\right\}dS+$$

$$+\int\limits_{\Sigma_R}\left\{[B^{(2)}(D,n)U^{(2)}]_k[\overline{u^{(2)}}_k] - \frac{i}{\omega T_0}[u^{(2)}_4][\lambda^{(2)}(D,n)\overline{u^{(2)}}_4]\right\}d\Sigma_R. \qquad (9.47)$$

Whence

$$\int_{\Omega^1}\{c^{(1)}_{kjpq}D_pu^{(1)}_q\overline{D_ku^{(1)}_j}-\omega^2|u^{(1)}|^2-\frac{i}{\omega T_0}\lambda^{(1)}_{pq}D_qu^{(1)}_4\ \overline{D_pu^{(1)}_4}+$$
$$+\frac{c^{(1)}_0}{T_0}|u^{(1)}|^2\}dx+\int_{\Omega^2_R}\left\{c^{(2)}_{kjpq}D_pu^{(2)}_q\overline{D_ku^{(2)}_j}-\omega^2|u^{(2)}|^2-\right.$$
$$\left.-\frac{i}{\omega T_0}\lambda^{(2)}_{pq}D_qu^{(2)}_4\ \overline{D_pu^{(2)}_4}+\frac{c^{(2)}_0}{T_0}|u^{(2)}|^2\right\}dx=$$
$$=\int_{\Sigma_R}\left\{[B^{(2)}(D,n)U^{(2)}]_k[\overline{u^{(2)}}_k]-\frac{i}{\omega T_0}[u^{(2)}_4][\lambda^{(2)}(D,n)\overline{u^{(2)}}_4]\right\}d\Sigma_R, (9.48)$$

due to the homogeneity of the transmission conditions.

In turn (9.48) implies (if we look at the imaginary part)

$$\mathrm{Im}\left\{\int_{\Sigma_R}\left\{[B^{(2)}(D,n)U^{(2)}]_k[\overline{u^{(2)}}_k]-\frac{i}{\omega T_0}[u^{(2)}_4][\lambda^{(2)}(D,n)\overline{u^{(2)}}_4]\right\}d\Sigma_R\right\}+$$
$$+\frac{1}{\omega T_0}\int_{\Omega^1}\lambda^{(1)}_{pq}D_qu^{(1)}_4\ \overline{D_pu^{(1)}_4}dx+\frac{1}{\omega T_0}\int_{\Omega^2_R}\lambda^{(2)}_{pq}D_qu^{(2)}_4\ \overline{D_pu^{(2)}_4}dx=0. (9.49)$$

From this equation, as in the proof of Theorem 9.5, we can show that $u^{(1)}_4=0$ in Ω^1, $u^{(2)}_4=0$ in Ω^2, and $u^{(2)}=0$ in Ω^2 with $r=1$ for $\omega>0$ and $r=2$ for $\omega<0$.

Next, the homogeneous interface conditions (7.3) and (7.4) imply that $[U^{(1)}]^+=0$ and $[B^{(1)}(D,n)U^{(1)}]^+=0$ on S, which together with the following general integral representation formula of the solution $U^{(1)}$ in Ω^1

$$U^{(1)}(x)=\int_S\{[Q^{(1)}(D,n,-i\omega)[\Gamma^{(1)}(x-y),\omega,r)]^\top]^\top\,[U^{(1)}]^+-$$
$$-\Gamma^{(1)}(x-y,\omega,r)\,[B^{(1)}(D,n)U^{(1)}]^+\}\,dS,\quad x\in\Omega^1, \qquad (9.50)$$

completes the proof. □

It is evident that in the case of the homogeneous problems $(\mathcal{G})_\omega$ and $(\mathcal{H})_\omega$ we again obtain the equation (9.49). Therefore,

$$U^{(2)}(x)=0 \quad \text{in} \quad \Omega^2, \qquad (9.51)$$

$$u^{(1)}_4(x)=0 \quad \text{in} \quad \Omega^1. \qquad (9.52)$$

From these equations and the corresponding homogeneous transmission conditions we conclude:

i) In the case of the homogeneous problem $(\mathcal{G})_\omega$ the displacement vector $u^{(1)}$ solves the following BVP

$$\left.\begin{aligned} &C^{(1)}(D)u^{(1)}(x)+\omega^2\, u^{(1)}(x)=0 \\ &\beta^{(1)}_{kj}D_j u^{(1)}_k(x)=0 \end{aligned}\right\} \quad \text{in} \quad \Omega^1, \tag{9.53}$$

$$[T^{(1)}(D,n)u^{(1)}]^+=0 \quad \text{and} \quad [u^{(1)}\cdot n]^+=0 \quad \text{on} \quad S. \tag{9.54}$$

ii) In the case of the homogeneous problem $(\mathcal{H})_\omega$ the displacement vector $u^{(1)}$ solves the following BVP

$$\left.\begin{aligned} &C^{(1)}(D)u^{(1)}(x)+\omega^2\, u^{(1)}(x)=0 \\ &\beta^{(1)}_{kj}D_j u^{(1)}_k(x)=0 \end{aligned}\right\} \quad \text{in} \quad \Omega^1, \tag{9.55}$$

$$[u^{(1)}]^+=0 \quad \text{and} \quad [T^{(1)}(D,n)u^{(1)}\cdot n]^+=0 \quad \text{on} \quad S. \tag{9.56}$$

These homogeneous problems for the elastic field have not, in general, the only trivial solutions.

Denote by $J_{\mathcal{G}}(\Omega^1)$ and $J_{\mathcal{H}}(\Omega^1)$, respectively, the set of values of the frequency parameter ω for which the above problems (9.53)-(9.54) and (9.55)-(9.56) admit nontrivial solutions. Obviously, $J_{\mathcal{G}}(\Omega^1)$ is the intersection of the spectral sets of the so-called second and third interior BVPs of the theory of steady state elastic oscillations (in terms of the monograph [45]), while $J_{\mathcal{H}}(\Omega^1)$ is the intersection of the spectral sets of the first and fourth interior BVPs.

Such frequencies are called also *Jones eigenfrequencies*, while the corresponding nontrivial solutions are referred to as *Jones modes*. Spectral problems similar to (9.53)-(9.54) encounter also in the fluid-structure interaction problems (see, e.g., [26], [27], [48], [36], [39], and references therein).

Clearly, $J_{\mathcal{G}}(\Omega^1)$ and $J_{\mathcal{H}}(\Omega^1)$ are at most countable and to each Jones eigenfrequency there correspond only finitely many linearly independent Jones modes (cf. [56]). In general, $J_{\mathcal{G}}(\Omega^1)$ and $J_{\mathcal{H}}(\Omega^1)$ are not empty (see [45], [42]), hoewer there exist domains for which they are empty sets (for details see [45], [25], [37]).

The above arguments easily lead to the following proposition.

Theorem 9.9. *The homogeneous problems $(\mathcal{G})_\omega$ and $(\mathcal{H})_\omega$ have only the trivial solutions in the class $(\mathrm{C}^1(\overline{\Omega^1})\,,\, \mathrm{C}^1(\overline{\Omega^2})\cap \mathrm{SK}^m_r(\Omega^2))$ with $r=1$ for $\omega>0$ and $r=2$ for $\omega<0$, provided that ω is not a corresponding Jones eigenfrequency.*

Analogous uniqueness theorems hold valid also in the case of the weak formulation of the basic steady state oscillation interface problems.

Theorem 9.10. *The homogeneous interface problem $(\mathcal{C})_\omega$ has only the trivial solution in the class $(W^1_2(\Omega^1)\,,\, W^1_{2,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2))$ with $r=1$ for $\omega>0$ and $r=2$ for $\omega<0$.*

Theorem 9.11. *The homogeneous interface problems $(\mathcal{G})_\omega$ and $(\mathcal{H})_\omega$ have only the trivial solutions in the class $(W_2^1(\Omega^1)\,,\, W^1_{2,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2))$ with $r=1$ for $\omega>0$ and $r=2$ for $\omega<0$, provided that ω is not a corresponding Jones eigenfrequency.*

The proofs of these assertions are quite similar to the proof of Theorem 9.6.

The uiqueness theorems for the homogeneous mixed interface problems requires some new ideas which will be presented below.

Theorem 9.12. *The homogeneous mixed interface problems $(\mathcal{C}-\mathcal{DD})_\omega$, $(\mathcal{C}-\mathcal{NN})_\omega$, $(\mathcal{C}-\mathcal{DC})_\omega$, $(\mathcal{C}-\mathcal{NC})_\omega$, $(\mathcal{C}-\mathcal{G})_\omega$, $(\mathcal{C}-\mathcal{H})_\omega$, have only the trivial solutions in the class $(W_2^1(\Omega^1)\,,\, W^1_{2,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2))$ with $r=1$ for $\omega>0$ and $r=2$ for $\omega<0$.*

Proof. We demonstrate the proof for the problem $(\mathcal{C}-\mathcal{DD})_\omega$ since it is verbatim for the other problems.

Let $(U^{(1)},U^{(2)})$ be an arbitrary solution of the homogeneous interface problem $(\mathcal{C}-\mathcal{DD})_\omega$ of the class indicated in the theorem. By the same analysis as in the proof of Theorems 9.6 and 9.8 we again arrive at the equations (9.51) and (9.52). To see this, one has to apply the identities (9) and (9) where the surface integrals over S should be replaced by the appropriate duality relations, in accordance with the definitions of functional traces, and afterwards to take into account the homogeneity of the corresponding transmission and boundary conditions of the problem in question (see (7.13), (7.14)).

As a result we obtain that the vector function $U^{(1)}=(u^{(1)},0)^\top\in W_2^1(\Omega^1)$ has to satisfy the conditions:

$$A^{(1)}(D,-i\omega)U^{(1)}(x)=0 \quad \text{in} \quad \Omega^1, \tag{9.57}$$

$$[U^{(1)}]^+=0 \quad \text{on} \quad S=\overline{S}_1\cup\overline{S}_2, \tag{9.58}$$

$$[B^{(1)}(D,n)U^{(1)}]^+=0 \quad \text{on} \quad S_1. \tag{9.59}$$

Note that we may apply the representation (9.50) for the vector-function $U^{(1)}$ under consideration (see Theorem 10.8, item ii) in Section 10). Therefore, we have

$$U^{(1)}(x)=\int_{S_2}\Gamma^{(1)}(x-y,\omega,r)\,[B^{(1)}(D,n)U^{(1)}]^+dS,\ \ x\in\Omega^1, \tag{9.60}$$

where $[B^{(1)}(D,n)U^{(1)}]^+\in \widetilde{B}^{-1/2}_{2,2}(S_2)$ due to the condition (9.59).

It is evident that we can extend the vector function $U^{(1)}$ from Ω^1 onto the whole $\mathbb{R}^3_{S_2}$ by the same formula (9.60) since the right-hand side integral is defined in $\mathbb{R}^3_{S_2}$. Denote this extension by the symbol $\widetilde{U}^{(1)}$

From the above representation it follows that (cf. Theorem 10.8)

$$\widetilde{U}^{(1)}\in W^1_{2,\mathrm{loc}}(\mathbb{R}^3_{S_2})\cap \mathrm{SK}^m_r(\mathbb{R}^3_{S_2}), \tag{9.61}$$

$$[\widetilde{U}^{(1)}]^+=0 \quad \text{and} \quad [\widetilde{U}^{(1)}]^-=0 \quad \text{on} \quad S_2, \tag{9.62}$$

$$A^{(1)}(D,-i\omega)\widetilde{U}^{(1)}(x)=0 \quad \text{in} \quad \mathbb{R}^3_{S_2}. \tag{9.63}$$

The second equation in (9.62) is a consequence of the "continuity" property of the so-called single layer integral operator (9.60) (see below Theorem 10.8).

Thus, we have established that the vector function $\widetilde{U}^{(1)}$ given by the integral (9.60) solves the homogeneous crack type problem (9.61)-(9.63) in the sapce $W^1_{2,\mathrm{loc}}(\mathbb{R}^3_{S_2}) \cap \mathrm{SK}^m_r(\mathbb{R}^3_{S_2})$ where r and ω are as in Theorem 9.12. Due to Theorem 9.7 we then conclude that $\widetilde{U}^{(1)}$ vanishes in $\mathbb{R}^3_{S_2}$, which completes the proof. $\square$

We note that properties of surface potentials similar to (9.60) and boundary integral operators corresponding to them will be studied in detail in various functional spaces in the next chapter.

CHAPTER IV
POTENTIALS AND BOUNDARY INTEGRAL OPERATORS

In this chapter we introduce and study the generalized single and double layer potentials of the thermoelastisity theory of anisotropic bodies. We investigate their smoothness properties in the closed domains, asymtotic behaviour at infinity and establish jump relations on the surface of integration. We analyse also boundary integral (pseudodifferential) operators generated by these potentials and consider their mapping properties in various functional spaces. Note that the analogous questions for the potential type operators in the elasticity theory of isotropic and anisotropic bodies have been exaustively studied in [45], [8], [34], [35], [59], [17], [41], [13], [56], [32].

In Section 10 we examine in detail properties of the thermoelastic steady state oscillation potentials and afterwards, in Section 11, we briefly treat the same topics for the pseudo-oscillation potentials.

10. Thermoelastic Steady State Oscillation Potentials

10.1. Let us introduce the following generalized single and double layer steady state oscillation potentials constructed by the fundamental solution (2.29)

$$V(g)(x) := \int\limits_S \Gamma(x-y,\omega,r)\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{10.1}$$

$$W(g)(x) := \int\limits_S [Q(D_y, n(y), -i\omega)\Gamma^\top(x-y,\omega,r)]^\top g(y) dS_y, \ x \in \mathbb{R}^3 \setminus S, \tag{10.2}$$

where $S = \partial\Omega^\pm$, $g = (g_1, \dots, g_4)^\top = (\widetilde{g}, g_4)^\top$, $\widetilde{g} = (g_1, g_2, g_3)^\top$; the operator $Q(D, n, -i\omega)$ is defined by (1.26) with $\varkappa = -i\omega$.

Note that here and in what follows, for simplicity of the notations, we do not mark with the subscript ω the steady state oscillation potentials and the integral operators corresponding to them.

To investigate the existence of solutions to the nonhomogeneous BVPs posed in Chapter II we need special mapping properties of the above potentials and the boundary integral (pseudodifferential) operators generated by them.

Let

$$\mathcal{H}\, g(z) = \int\limits_S \Gamma(z-y,\omega,r)\, g(y)\, dS_y, \quad z \in S, \tag{10.3}$$

$$\mathcal{K}_1\, g(z) = \int\limits_S [B(D_z, n(z))\Gamma(z-y,\omega,r)]\, g(y)\, dS_y, \quad z \in S, \tag{10.4}$$

$$\mathcal{K}_2\, g(z) = \int\limits_S [Q(D_y, n(y), -i\omega)\Gamma^\top(z-y,\omega,r)]^\top\, g(y)\, dS_y, \quad z \in S, \tag{10.5}$$

$$\mathcal{L}^\pm\, g(z) = \lim_{\Omega^\pm \ni x \to z \in S} B(D_x, n(z)) W(g)(x), \quad z \in S, \tag{10.6}$$

where the boundary differential operator $B(D,n)$ is given by (1.25). Here the integrals (10.4) and (10.5) are understood in the Cauchy principal value sense.

In the sequel everywhere the two positive numbers α and α' are subjected to the inequalities $0<\alpha<\alpha'\le 1$.

Lemma 10.1. *Let $k\ge 0$ be an integer and $S\in \mathrm{C}^{k+1,\alpha'}$. Then for an arbitrary summable g the potentials $V(g)$ and $W(g)$ are C^∞-smooth solutions to the equation* (1.10) *in $\Omega^\pm$ and belong to the class $\mathrm{SK}^m_r(\Omega^-)$.*

The following formulae

$$[V(g)(z)]^+=[V(g)(z)]^-=\mathcal{H}\,g(z),\ \ g\in \mathrm{C}(S), \tag{10.7}$$

$$[B(D,n)V(g)(z)]^\pm=(\mp 2^{-1}I_4+\mathcal{K}_1)\,g(z),\ \ g\in \mathrm{C}^\alpha(S), \tag{10.8}$$

$$[W(g)(z)]^\pm=(\pm 2^{-1}I_4+\mathcal{K}_2)\,g(z),\ \ g\in \mathrm{C}^\alpha(S), \tag{10.9}$$

hold and the operators

$$\mathcal{H}\ :\ \mathrm{C}^{l,\alpha}(S)\to \mathrm{C}^{l+1,\alpha}(S), \tag{10.10}$$

$$\mathcal{K}_1,\ \mathcal{K}_2\ :\ \mathrm{C}^{l,\alpha}(S)\to \mathrm{C}^{l,\alpha}(S), \tag{10.11}$$

$$V\ :\ \mathrm{C}^{l,\alpha}(S)\to \mathrm{C}^{l+1,\alpha}(\overline{\Omega^\pm}), \tag{10.12}$$

$$W\ :\ \mathrm{C}^{l,\alpha}(S)\to \mathrm{C}^{l,\alpha}(\overline{\Omega^\pm}), \tag{10.13}$$

where $0\le l\le k$, are bounded.

Proof. The first part of the lemma follows immediately from the properties of the fundamental matrix $\Gamma(x-y,\omega,r)$ and is trivial, since the columns of $\Gamma(x-y,\omega,r)$ are solutions of the homogeneous equation (1.10) for $x\ne y$.

To prove the second part, we proceed as follows.

From equations (1.25), (1.26), and Theorem 2.3 we have

$$\Gamma(x-y,\omega,r)-\Gamma(x-y)=:\widetilde{\Gamma}(x-y,\omega,r), \tag{10.14}$$

$$B(D,n)=B_0(D,n)-\widetilde{B}(n), \tag{10.15}$$

$$Q(D,n,-i\omega)=B_0(D,n)-i\omega T_0\widetilde{B}(n), \tag{10.16}$$

where $|D^\beta\widetilde{\Gamma}_{kj}(x,\omega,r)|<c\,\varphi^{(kj)}_{|\beta|}(x)$, $k,j=1,\ldots,4$, in a vicinity of the origin,

$$B_0(D,n)=\begin{bmatrix}[T(D,n)]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & \partial_n\end{bmatrix}_{4\times 4},\quad \widetilde{B}(n)=\begin{bmatrix}[0]_{3\times 3} & [\beta_{kj}n_j]_{3\times 1}\\ [0]_{1\times 3} & 0\end{bmatrix}_{4\times 4};$$

here $\Gamma(x)$, β, c and $\varphi^{(kj)}_{|\beta|}$ are as in Lemma 2.1.

Therefore, we can single out the dominant singular terms in the above potentials and represent them in the form

$$V(g)(x)=V_0(g)(x)+\widetilde{V}(g)(x), \tag{10.17}$$

$$W(g)(x)=W_0(g)(x)+\widetilde{W}(g)(x), \tag{10.18}$$

$$B(D,n)V(g)(x)-B_0(D,n)V_0(g)(x)=:R(g)(x),$$

where

$$V_0(g)(x) = \int\limits_S \Gamma(x-y)\, g(y)\, dS_y,$$

$$W_0(g)(x) = \int\limits_S [B_0(D_y, n(y))\Gamma(x-y)]^\top g(y)\, dS_y.$$

The kernels of the potentials $\widetilde{V}(g)$, $\widetilde{W}(g)$ and $R(g)$ have singularities of type $O(|x-y|^{-1})$ as $|x-y| \to 0$. Therefore, $\widetilde{V}$, $\widetilde{W}$, and R are continuous vectors in $\mathbb{R}^3$ provided $g \in \mathrm{C}(S)$.

It is easy to see that

$$V_0(g) = (v^{(0)}(\widetilde{g}), v_4^{(0)}(g_4))^\top, \quad W_0(g) = (w^{(0)}(\widetilde{g}), w_4^{(0)}(g_4))^\top,$$

$$B_0(D,n)V_0(g) = (T(D,n)v^{(0)}(\widetilde{g}), \partial_n v_4^{(0)}(g_4))^\top,$$

where $v^{(0)}(\widetilde{g})$ and $w^{(0)}(\widetilde{g})$ are single and double layer potentials of elastostatics (corresponding to the operator $C(D)$) constructed by the fundamental matrix $\Gamma^{(0)}(x)$:

$$v^{(0)}(\widetilde{g})(x) := \int\limits_S \Gamma^{(0)}(x-y)\, \widetilde{g}(y)\, dS_y, \tag{10.19}$$

$$w^{(0)}(\widetilde{g})(x) := \int\limits_S [T(D_y, n(y))\Gamma^{(0)}(y-x)]^\top\, \widetilde{g}(y)\, dS_y, \tag{10.20}$$

while $v_4^{(0)}(g_4)$ and $w_4^{(0)}(g_4)$ are potentials of the same type (corresponding to the homogeneous operator $\Lambda(D)$) constructed by the fundamental function $\gamma^{(0)}(x)$:

$$v_4^{(0)}(g_4)(x) := \int\limits_S \gamma^{(0)}(x-y)\, g_4(y)\, dS_y, \tag{10.21}$$

$$w_4^{(0)}(g_4)(x) := \int\limits_S \partial_{n(y)}\gamma^{(0)}(y-x)\, g_4(y)\, dS_y \tag{10.22}$$

(see Lemma 2.1).

The properties of the latter potentials and boundary integral operators on S, generated by them, are studied in detail for regular function spaces in [8], [52], [56], [57], [59]. The results in the above mentioned references together with the representation formulae (10.17)-(10.18) yield equations (10.7)-(10.9) and mapping properties (10.10)- (10.13). $\square$

For a pseudodifferential operator (ΨDO) $\mathcal{K}$ on S we denote by $(\mathcal{K})_0$ and $\sigma(\mathcal{K})(x,\widetilde{\xi})$ $(x \in S, \widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\})$ the dominant singular part and the principal homogeneous symbol, respectively. As usual, if no confusion arises, in the sequel the arguments x and $\widetilde{\xi}$ will be omitted.

Lemma 10.2. *The operators $\mathcal{H}$, $\pm 2^{-1}I_4 + \mathcal{K}_1$, and $\pm 2^{-1}I_4 + \mathcal{K}_2$ are elliptic ΨDOs of order -1, 0, and 0, respectively, with index equal to zero.*

Proof. From equations (10.14)-(10.16) and (10.3)-(10.5) it follows that

$$(\mathcal{H})_0 = \begin{bmatrix} [\mathcal{H}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{H}_4^{(0)} \end{bmatrix}_{4\times 4}, \tag{10.23}$$

$$(\pm 2^{-1}I_4+\mathcal{K}_1)_0=\begin{bmatrix}[\pm 2^{-1}I_3+\mathcal{K}^{(0)}]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & \pm 2^{-1}I_1+\mathcal{K}_4^{(0)}\end{bmatrix}_{4\times 4}, \tag{10.24}$$

$$(\pm 2^{-1}I_4+\mathcal{K}_2)_0=\begin{bmatrix}[\pm 2^{-1}I_3+\overset{*}{\mathcal{K}}{}^{(0)}]_{3\times 3} & [0]_{3\times 1}\\ [0]_{1\times 3} & \pm 2^{-1}I_1+\overset{*}{\mathcal{K}}{}_4^{(0)}\end{bmatrix}_{4\times 4}, \tag{10.25}$$

where

$$\begin{gathered}\mathcal{H}^{(0)}\,\widetilde{g}(z)=\int\limits_S\Gamma^{(0)}(z-y)\,\widetilde{g}(y)\,dS_y,\quad \mathcal{H}_4^{(0)}\,g_4(z)=\int\limits_S\gamma^{(0)}(z-y)\,g_4(y)\,dS_y,\\ \mathcal{K}^{(0)}\,\widetilde{g}(z)=\int\limits_S[T(D_z,n(z))\Gamma^{(0)}(z-y)]\,\widetilde{g}(y)\,dS_y,\\ \overset{*}{\mathcal{K}}{}^{(0)}\,\widetilde{g}(z)=\int\limits_S[T(D_y,n(y))\Gamma^{(0)}(y-z)]^\top\,\widetilde{g}(y)\,dS_y,\\ \mathcal{K}_4^{(0)}\,g_4(z)=\int\limits_S\partial_{n(z)}\gamma^{(0)}(z-y)\,g_4(y)\,dS_y,\\ \overset{*}{\mathcal{K}}{}_4^{(0)}\,g_4(z)=\int\limits_S\partial_{n(y)}\gamma^{(0)}(y-z)\,g_4(y)\,dS_y.\end{gathered} \tag{10.26}$$

Due to the general theory of ΨDOs (see, e.g., [77], [20]) we have to show that the principal symbol matrices of the operators (10.23), (10.24), and (10.25) are nonsingular and that the indices of these operators are equal to zero.

It is evident that $\mathcal{K}^{(0)}$ $[\mathcal{K}_4^{(0)}]$ and $\overset{*}{\mathcal{K}}{}^{(0)}$ $[\overset{*}{\mathcal{K}}{}_4^{(0)}]$ are mutually adjoint singular integral operators while $\mathcal{H}^{(0)}$ $[\mathcal{H}_4^{(0)}]$ is a formally self-adjoint integral operator with a weakly singular kernel of the type $O(|x-y|^{-1})$.

For the principal symbols we have (see [56], [59], [39])

$$\sigma(\mathcal{H}^{(0)})=-\frac{1}{2\pi}\int\limits_{l^\mp}[C(a\xi)]^{-1}\,d\xi_3=-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}[C(a\xi)]^{-1}\,d\xi_3, \tag{10.27}$$

$$\begin{aligned}\sigma(\pm 2^{-1}I_3+\mathcal{K}^{(0)})&=\frac{i}{2\pi}\int\limits_{l^\mp}T(a\xi,n)\,[C(a\xi)]^{-1}\,d\xi_3=\\ &=\overline{[\sigma(\pm 2^{-1}I_3+\overset{*}{\mathcal{K}}{}^{(0)})]^\top},\end{aligned} \tag{10.28}$$

$$\sigma(\mathcal{H}_4^{(0)})=-\frac{1}{2\pi}\int\limits_{l^\mp}[\Lambda(a\xi)]^{-1}\,d\xi_3=-\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}[\Lambda(a\xi)]^{-1}\,d\xi_3<0, \tag{10.29}$$

$$\begin{aligned}\sigma(\pm 2^{-1}I_1+\mathcal{K}_4^{(0)})&=\frac{i}{2\pi}\int\limits_{l^\mp}\lambda(a\xi,n)\,[\Lambda(a\xi)]^{-1}\,d\xi_3=\\ &=\overline{\sigma(\pm 2^{-1}I_1+\overset{*}{\mathcal{K}}{}_4^{(0)})}=\pm 2^{-1},\end{aligned} \tag{10.30}$$

where $\xi=(\widetilde{\xi},\xi_3)$, $\widetilde{\xi}=(\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}$, $\lambda(\xi,n)$ is defined by (1.24),

$$a(x)=\begin{bmatrix}l_1(x) & m_1(x) & n_1(x)\\ l_2(x) & m_2(x) & n_2(x)\\ l_3(x) & m_3(x) & n_3(x)\end{bmatrix}$$

is an orthogonal matrix with $\det a(x) = +1$, $l = (l_1, l_2, l_3)^\top$, $m = (m_1, m_2, m_3)^\top$ and $n = (n_1, n_2, n_3)^\top$ is a triple of orthogonal vectors at $x \in S$ (l and m lie in the tangent plane at $x \in S$ and n is again the exterior unit normal), l^- (l^+) is a closed clockwise (counter-clockwise) oriented contour in the lower (upper) complex half-plane $\xi_3 = \xi_3' + i\xi_3''$ enclosing all roots of the equations $\det C(a\xi) = 0$, $\Lambda(a\xi) = 0$, with respect to ξ_3 with negative (positive) imaginary parts. The last equation in (10.30) follows due to the fact that the kernel-function of the integral operators $\mathcal{K}_4^{(0)}$ and $\overset{*}{\mathcal{K}}{}_4^{(0)}$ have weak singularities of type $O(|x-y|^{-2+\alpha'})$ on a $\mathrm{C}^{1,\alpha'}$-smooth manifold.

The entries of the matrices (10.28) are homogeneous functions of order 0, while (10.27) and (10.29) are homogeneous functions of order -1 in $\widetilde{\xi}$. Moreover, all the above principal homogeneous symbols are nonsingular for $|\widetilde{\xi}| = 1$, the corresponding integral operators are elliptic ΨDOs of order 0 and -1, respectively, and their indices are equal to zero (for details see [56], [59], [41], [16]).

Now (10.23), (10.24), and (10.25) imply

$$\sigma(\mathcal{H}) = \begin{bmatrix} [\sigma(\mathcal{H}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{H}_4^{(0)}) \end{bmatrix}_{4\times 4}, \tag{10.31}$$

$$\sigma(\pm 2^{-1}I_4 + \mathcal{K}_1) = [\overline{\sigma(\pm 2^{-1}I_4 + \mathcal{K}_2)}]^\top =$$
$$= \begin{bmatrix} [\sigma(\pm 2^{-1}I_3 + \mathcal{K}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\pm 2^{-1}I_1 + \mathcal{K}_4^{(0)}) \end{bmatrix}_{4\times 4}, \tag{10.32}$$

which together with equations (10.23)–(10.25) completes the proof. □

Remark 10.3. More subtle analysis of the fundamental solution $\Gamma(x, \omega, r)$ shows that in a vicinity of the origin the following representation

$$\Gamma(x, \omega, r) = \Gamma(x) + i\widetilde{\Gamma}'(x) - \omega T_0[\widetilde{\Gamma}'(x)]^\top + \widetilde{\Gamma}''(x, \omega, r), \tag{10.33}$$
$$\widetilde{\Gamma}'(x) = \begin{bmatrix} [0]_{3\times 3} & [\widetilde{\Gamma}'_{k4}(x)]_{3\times 1} \\ [0]_{1\times 3} & 0 \end{bmatrix}_{4\times 4},$$

holds, where $\Gamma(x)$ is the same as in Lemma 2.1 and $\widetilde{\Gamma}'_{k4}(x)$ is independent of ω; first order derivatives of $\widetilde{\Gamma}'_{k4}(x)$ are homogeneous functions of order -1 and $|D^\beta \widetilde{\Gamma}'_{k4}(x)| < c\,\varphi_{|\beta|}^{(k4)}(x)$ with the same $\varphi_{|\beta|}^{(k4)}(x)$ as in Lemma 2.1; the second order derivatives of the entries of the matrix $\widetilde{\Gamma}''(x, \omega, r)$ have singularities of the type $O(|x|^{-1})$.

Remark 10.4. Note that the operator $-\mathcal{H}^{(0)}$ $[-\mathcal{H}_4^{(0)}]$ is a positive operator which implies that the corresponding principal homogeneous symbol is a positive definite matrix [is a positive function] (see [56]). Therefore, the principal homogeneous symbol matrix $\sigma(-\mathcal{H})$ is also positive definite due to the equation (10.31) and the inequality (10.29).

10.2. Now we turn our attention to the equation (10.6). To prove the existence of limits (10.6) and to study properties of the operators $\mathcal{L}^{\pm}$ we need some auxiliary results which are now presented.

Lemma 10.5. *Let $U = (u, u_4)^\top$ be a regular solution of the homogeneous interior problem $(\mathcal{P}_1)^+_\omega$. Then $u_4(x) = 0$ in Ω^+ and u is a solution to the following interior homogeneous BVP of steady state oscillations of the elasticity theory*

$$C(D)u(x) + \omega^2 u(x) = 0 \quad \text{in} \quad \Omega^+, \tag{10.34}$$

$$[u(z)]^+ = 0 \quad \text{on} \quad S, \tag{10.35}$$

satisfying, in addition, the equation $\beta_{kj} D_j u_k = 0$ in Ω^+.

Proof. The equation $u_4(x) = 0$ in Ω^+ follows from the identity (1.23), if we look at the imaginary part. Then we obtain the BVP (10.34)-(10.35) for the displacement vector u with the additional equation indicated in the lemma due to the homogeneous conditions of the problem $(\mathcal{P}_1)^+_\omega$. □

By $\Sigma[(\mathcal{P}_1)^+_\omega]$ we denote the spectral set corresponding to the problem $(\mathcal{P}_1)^+_\omega$ (i.e., the set of values of the parameter ω for which the homogeneous problem $(\mathcal{P}_1)^+_\omega$ possesses a nontrivial solution). Note that the spectral set corresponding to the problem (10.34)–(10.35) is at most countable. Therefore, Lemma 10.5 implies the following proposition (cf. [56]).

Corollary 10.6. *The set $\Sigma[(\mathcal{P}_1)^+_\omega]$ is either finite or countable* (*with the only possible accumulation point at infinity*).

Now we are ready to examine the properties of the hypersingular operators $\mathcal{L}^{\pm}$.

Lemma 10.7. *Let $S \in \mathrm{C}^{2,\alpha'}$ and $g \in \mathrm{C}^{1,\alpha}(S)$. Then limits* (10.6) *exist and*

$$\mathcal{L}^+ g(z) = \mathcal{L}^- g(z) =: \mathcal{L}\, g(z), \quad z \in S. \tag{10.36}$$

Moreover, the operator

$$\mathcal{L} \,:\, \mathrm{C}^{l+1,\alpha}(S) \to \mathrm{C}^{l,\alpha}(S), \quad S \in \mathrm{C}^{k+2,\alpha'}, \;\; k \geq 0, \;\; 0 \leq l \leq k, \tag{10.37}$$

is a bounded singular integro-differential operator with nonsingular positive definite principal homogeneous symbol matrix and with index equal to zero.

Proof. First we prove the existence of limits (10.6). With the help of equations (10.15), (10.16), and (10.33) we deduce

$$\begin{aligned} B(D_x, n(x))[Q(D_y, n(y), -i\omega)\Gamma^\top(x-y,\omega,r)]^\top &= \widetilde{K}_3(x,y,x-y) + \\ +[\widetilde{K}_2'(x,y,x-y) + \omega T_0 \widetilde{K}_2''(x,y,x-y)] &+ \widetilde{K}_1(x,y,x-y;\omega), \end{aligned} \tag{10.38}$$

where

$$\widetilde{K}_3(x,y,x-y) = B_0(D_x,n(x))[B_0(D_y,n(y))\Gamma(y-x)]^\top =$$

$$= \begin{bmatrix} [T(D_x,n(x))[T(D_y,n(y))\Gamma(y-x)]^\top]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \partial_{n(x)}\partial_{n(y)}\gamma^{(0)}(y-x) \end{bmatrix}_{4\times 4}$$

is a hypersingular kernel with the entries of the type $O(|x-y|^{-3})$ as $|x-y| \to 0$, while

$$\widetilde{K}_2'(x,y,x-y) = iB_0(D_x, n(x))\{B_0(D_y, n(y))[\widetilde{\Gamma}'(x-y)]^\top\}^\top - \\ -\widetilde{B}(n(x))[B_0(D_y, n(y))\Gamma(x-y)]^\top$$

and

$$\widetilde{K}_2''(x,y,x-y) = -B_0(D_x, n(x))[B_0(D_y, n(y))\widetilde{\Gamma}'(x-y)]^\top - \\ -i[B_0(D_x, n(x))\Gamma(x-y)]\widetilde{B}^\top(n(y))$$

are singular kernels on S with the entries of the type $O(|x-y|^{-2})$ as $|x-y| \to 0$, and the entries of the matrix $\widetilde{K}_1(x,y,x-y;\omega)$ have singularities of the type $O(|x-y|^{-1})$. Note that here either $x \in \Omega^+$ or $x \in \Omega^-$.

In turn, (10.38) implies

$$B(D_x, n(x))W(g)(x) = (T(D_x, n(x))w^{(0)}(\widetilde{g})(x), \partial_{n(x)} w_4^{(0)}(g_4)(x))^\top + \\ + \int_S [\widetilde{K}_2'(x,y,x-y) + \omega T_0 \widetilde{K}_2''(x,y,x-y)]\, g(y)\, dS_y + \\ + \int_S \widetilde{K}_1(x,y,x-y;\omega)\, g(y)\, dS_y, \tag{10.39}$$

where $w^{(0)}(\widetilde{g})$ and $w_4^{(0)}(g_4)$ are defined by (10.20) and (10.22), respectively. It can be shown (see [56], [59], [16], [39]) that the limits

$$\lim_{\Omega^\pm \ni x \to z \in S} T(D_x, n(x))w^{(0)}(\widetilde{g})(x) = \mathcal{L}^{(0)}\widetilde{g}(z), \tag{10.40}$$

$$\lim_{\Omega^\pm \ni x \to z \in S} \partial_{n(x)} w_4^{(0)}(g_4)(x) = \mathcal{L}_4^{(0)} g_4(z) \tag{10.41}$$

exist for any $g_k \in \mathrm{C}^{1,\alpha}(S)$, $k = 1, \dots, 4$, and that the operators $\mathcal{L}^{(0)}$ and $\mathcal{L}_4^{(0)}$ are non-negative, formally self-adjoint singular integro-differential operators with positive definite principal symbols

$$\sigma(\mathcal{L}^{(0)}) = -\tfrac{1}{2\pi} \int_{l^\mp} T(a\xi, n)\, [C(a\xi)]^{-1}\, T^\top(a\xi, n)\, d\xi_3, \tag{10.42}$$

$$\sigma(\mathcal{L}_4^{(0)}) = -\tfrac{1}{2\pi} \int_{l^\mp} \lambda^2(a\xi, n)\, [\Lambda(a\xi)]^{-1}\, d\xi_3 = -[4\, \sigma(\mathcal{H}_4^{(0)})]^{-1}. \tag{10.43}$$

Here the contours $l^\mp$ are the same as in formulae (10.27)-(10.30).

The operators $\mathcal{L}^{(0)}$ and $\mathcal{L}_4^{(0)}$ are elliptic ΨDOs of order 1 with index equal to zero and they possess mapping property (10.37) (for details see [16]).

Further, Remark 10.3 yields that there exist limits on S from $\Omega^\pm$ of the second term in the right-hand side expression of (10.39)

$$\lim_{\Omega^\pm \ni x \to z \in S} \int_S [\widetilde{K}_2'(x,y,x-y) + \omega T_0 \widetilde{K}_2''(x,y,x-y)]\, g(y)\, dS_y = \\ = [\alpha_\pm'(z) + \omega T_0 \alpha_\pm''(z)]g(z) + \widetilde{\mathcal{K}}_2'\, g(z) + \omega T_0\, \widetilde{\mathcal{K}}_2''\, g(z),$$

where $\widetilde{\mathcal{K}}_2'$ and $\widetilde{\mathcal{K}}_2''$ are singular integral operators with singular kernels $\widetilde{K}_2'$ and $\widetilde{K}_2''$, respectively; $\alpha_\pm'$ and $\alpha_\pm''$ are some smooth matrices independent of ω (we do not need their explicit expressions for our purposes).

The existence of the limits on S (from $\Omega^\pm$) of the third term in the right-hand side of (10.39) is evident. It is also obvious that these limits are equal to each other and that the boundary operator $\widetilde{\mathcal{K}}_1$, generated by this term, is a weakly singular integral operator (ΨDO of order $s \leq -1$).

Thus, the existence of the operators $\mathcal{L}^\pm$ is proved in the space $\mathrm{C}^{1,\alpha}(S)$ and we have

$$\mathcal{L}^\pm g(z) = \begin{bmatrix} [\mathcal{L}^{(0)}\,\widetilde{g}(z)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(0)}\,g_4(z) \end{bmatrix}_{4\times 4} +$$

$$+[\alpha_\pm'(z) + \omega T_0\,\alpha_\pm''(z)]g(z) + \widetilde{\mathcal{K}}_2'\,g(z) + \omega T_0\,\widetilde{\mathcal{K}}_2''\,g(z) + \widetilde{\mathcal{K}}_1\,g(z). \tag{10.44}$$

We also see that the operators (10) possess the mapping property (10.37).

It remains to show $\mathcal{L}^+ = \mathcal{L}^-$.

The integral representation formulae (3.2) and (3.3) of a regular vector U we rewrite as follows

$$U(x) = \pm\{W([U]^\pm)(x) - V([BU]^\pm)(x)\}, \quad x \in \Omega^\pm, \tag{10.45}$$

provided $A(D,-i\omega)U(x) = 0$ in $\Omega^\pm$ and $U \in \mathrm{SK}_r^m(\Omega^-)$; here W and V are double and single layer potentials operators (see (10.1) and (10.2)).

Due to Lemma 10.1 from (10.45) we have

$$(-2^{-1}I_4 + \mathcal{K}_2)[U]^+ = \mathcal{H}[BU]^+, \quad (2^{-1}I_4 + \mathcal{K}_2)[U]^- = \mathcal{H}[BU]^-,$$

where the operators $\mathcal{H}$ and $\mathcal{K}_2$ are defined by (10.3) and (10.5), respectively.

If in these equations we substitute $U(x) = W(g)(x)$ with an arbitrary $g \in \mathrm{C}^{1,\alpha}(S)$, apply the same Lemma 10.1 and the above results concerning the limits (10.6), we arrive at the following relations

$$\begin{aligned} &(-2^{-1}I_4 + \mathcal{K}_2)(2^{-1}I_4 + \mathcal{K}_2)\,g = \mathcal{H}\mathcal{L}^+\,g, \\ &(2^{-1}I_4 + \mathcal{K}_2)(-2^{-1}I_4 + \mathcal{K}_2)\,g = \mathcal{H}\mathcal{L}^-\,g. \end{aligned} \tag{10.46}$$

Whence

$$\mathcal{H}(\mathcal{L}^+\,g - \mathcal{L}^-\,g) = 0. \tag{10.47}$$

By (10) we have $\mathcal{L}^+\,g - \mathcal{L}^-\,g =: h \in \mathrm{C}^\alpha(S)$ and, therefore, $V(h)$ is a regular vector in $\Omega^\pm$.

Now, on one side, (10.47) yields that $V(h)$ is a regular solution to the homogeneous roblem $(\mathcal{P}_1)_\omega^-$ and we conclude $V(h)(x) = 0, \quad x \in \Omega^-$, due to Theorem 9.5.

On the other side, the same equation (10.47) implies that $V(h)$ is a regular solution to the homogeneous problem $(\mathcal{P}_1)_\omega^+$ as well, and, by Corollary 10.6, we get $V(h)(x) = 0, \quad x \in \Omega^+$, provided $\omega \notin \Sigma[(\mathcal{P}_1)_\omega^+]$.

The above equations imply $h = [BV(h)]^- - [BV(h)]^+ = 0$.

Thus, we have proved that $\mathcal{L}^+ g = \mathcal{L}^- g$ for all $g \in \mathrm{C}^{1,\alpha}(S)$ if $\omega \notin \Sigma[(\mathcal{P}_1)^+_\omega]$, which according to (10) leads to the equation

$$[\alpha'_+(z) - \alpha'_-(z)]g(z) + \omega T_0[\alpha''_+(z) - \alpha''_-(z)]g(z) = 0.$$

Consequently, $\alpha'_+(z) = \alpha'_-(z)$, $\quad \alpha''_+(z) = \alpha''_-(z)$, and (10.36) holds for an arbitrary value of the parameter ω.

It is also evident that the dominant singular part $(\mathcal{L})_0$ of the operator $\mathcal{L}$ and the corresponding principal homogeneous symbol matrix read

$$(\mathcal{L})_0 = \begin{bmatrix} [\mathcal{L}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(0)} \end{bmatrix}_{4\times 4}, \tag{10.48}$$

$$\sigma(\mathcal{L}) = \begin{bmatrix} [\sigma(\mathcal{L}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{L}_4^{(0)}) \end{bmatrix}_{4\times 4} \tag{10.49}$$

(see (10.40)–(10.43)). Whence the positive definiteness of the matrix (10.49) and the formally self-adjointness of the operator (10.48) follow immediately, since the matrix $\sigma(\mathcal{L}^{(0)})$ is positive definite and, as formulae (10.46), (10.29), and (10.30) show

$$\sigma(\mathcal{L}_4^{(0)}) = -[4\sigma(\mathcal{H}_4^{(0)})]^{-1} > 0. \tag{10.50}$$

The proof is completed. $\square$

10.3. In this subsection we collect the known results concerning some properties of the above introduced single and double layer potentials in Besov and Bessel-potential spaces. The proof of the theorem below is, in fact, the same as proof of analogous theorem in the elasticity theory (or even in the theory of harmonic functions). One has to relay on the fact that regular function spaces are densely embedded in Besov and Bessel-potential functional spaces, and apply the usual limiting extension procedure together with the duality and interpolation principles (for details we refer to, for example, [16], [17], [13], [53]).

Theorem 10.8. *The operators* (10.12), (10.13), (10.10), (10.11), *and* (10.37) *can be extended by continuity to the following bounded operators*

$$\begin{aligned}
V \ &: \ B^s_{p,p}(S) \to H_p^{s+1+1/p}(\Omega^+) \ [B^s_{p,p}(S) \to H^{s+1+1/p}_{p,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-)],\\
&: \ B^s_{p,q}(S) \to B^{s+1+1/p}_{p,q}(\Omega^+) \ [B^s_{p,q}(S) \to B^{s+1+1/p}_{p,q,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-)],\\
W \ &: \ B^s_{p,p}(S) \to H_p^{s+1/p}(\Omega^+) \quad [B^s_{p,p}(S) \to H^{s+1/p}_{p,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-)],\\
&: \ B^s_{p,q}(S) \to B^{s+1/p}_{p,q}(\Omega^+) \quad [B^s_{p,q}(S) \to B^{s+1/p}_{p,q,\mathrm{loc}}(\Omega^-) \cap \mathrm{SK}^m_r(\Omega^-)],\\
\mathcal{H} \ &: \ H^s_p(S) \to H^{s+1}_p(S) \qquad [B^s_{p,q}(S) \to B^{s+1}_{p,q}(S)],\\
\mathcal{K}_1, \mathcal{K}_2 \ &: \ H^s_p(S) \to H^s_p(S) \qquad [B^s_{p,q}(S) \to B^s_{p,q}(S)],\\
\mathcal{L} \ &: \ H^{s+1}_p(S) \to H^s_p(S) \qquad [B^{s+1}_{p,q}(S) \to B^s_{p,q}(S)],
\end{aligned}$$

for arbitrary $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, *provided* $S \in \mathrm{C}^\infty$. *Moreover,*

i) *for these extended operators the formulae* (10.7), (10.8), (10.9), *and* (10.36) *remain valid in the corresponding spaces;*

ii) *the integral representation formula* (3.3) *remains valid for* $U \in W_p^1(\Omega^-) \cap \mathrm{SK}_r^m(\Omega^-)$ *with* $A(D,-i\omega)U = 0$ *in* Ω^-*; the integral representation formula* (3.2) *in* Ω^+ *remains valid for* $U \in W_p^1(\Omega^+)$ *with* $\tau = -i\omega$ *and* $A(D,-i\omega)U = 0$ *in* Ω^+.

11. Thermoelastic Pseudo-Oscillation Potentials

In this section we deal with the single and double layer pseudo-oscillation potentials which are defined as follows

$$V_\tau(g)(x) := \int\limits_S \Gamma(x-y,\tau)\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{11.1}$$

$$W_\tau(g)(x) := \int\limits_S [Q(D_y, n(y), \tau)\Gamma^\top(x-y,\tau)]^\top\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{11.2}$$

where $\Gamma(x-y,\tau)$ is the fundamental matrix defined by (2.2), $S = \partial\Omega^\pm$, $g = (g_1, \dots, g_4)^\top = (\widetilde{g}, g_4)^\top$, $\widetilde{g} = (g_1, g_2, g_3)^\top$; the operator $Q(D,n,\tau)$ is defined by (1.26) with $\varkappa = \tau$.

Due to the results of Section 2 it is evident that the mapping properties and the jump relations of the above pseudo-oscillation potentials and the steady state oscillation potentials (10.1)–(10.2) are the same. It is also obvious that the asymptotic behaviour of the potentials (11.1)–(11.2) at infinity is quite similar to the asymptotic behaviour of the fundamental matrix $\Gamma(x-y,\tau)$ since S is a compact surface.

Next, we introduce the boundary integral (pseudodifferential) operators generated by the pseudo-oscillation potentials

$$\mathcal{H}_\tau\, g(z) = \int_S \Gamma(z-y,\tau)\, g(y)\, dS_y, \quad z \in S, \tag{11.3}$$

$$\mathcal{K}_{1,\tau}\, g(z) = \int_S [B(D_z, n(z))\Gamma(z-y,\tau)]\, g(y)\, dS_y, \quad z \in S, \tag{11.4}$$

$$\mathcal{K}_{2,\tau}\, g(z) = \int_S [Q(D_y, n(y), \tau)\Gamma^\top(z-y,\tau)]^\top\, g(y)\, dS_y, \quad z \in S, \tag{11.5}$$

$$\mathcal{L}_\tau^\pm\, g(z) = \lim_{\Omega^\pm \ni x \to z \in S} B(D_x, n(z)) W_\tau(g)(x), \quad z \in S, \tag{11.6}$$

where the boundary differential operator $B(D,n)$ is given again by (1.25), and the integrals (11.4) and (11.5) are understood in the Cauchy principal value sense.

The properties of the above introduced operators are described by the following propositions.

Theorem 11.1. *Let* $k \geq 0$ *be an integer and* $S \in \mathrm{C}^{k+1,\alpha'}$. *Then for an arbitrary summable* g *the potentials* $V_\tau(g)$ *and* $W_\tau(g)$ *are* C^∞*-smooth solutions to the equation* (1.9) *in* $\Omega^\pm$ *and together with all derivatives they decrease more rapidly then any negative power of* $|x|$ *as* $|x| \to +\infty$.

Moreover, if $0 \leq l \leq k$, *then*

i) *the operators*

$$V_\tau \ :\ \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l+1,\alpha}(\overline{\Omega^\pm}), \tag{11.7}$$

$$W_\tau \ :\ \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l,\alpha}(\overline{\Omega^\pm}) \tag{11.8}$$

are bounded, and

$$[V_\tau(g)(z)]^+ = [V_\tau(g)(z)]^- = \mathcal{H}_\tau\, g(z), \;\; g \in \mathrm{C}(S), \tag{11.9}$$
$$[B(D,n)V_\tau(g)(z)]^\pm = (\mp 2^{-1} I_4 + \mathcal{K}_{1,\tau})\, g(z), \;\; g \in \mathrm{C}^\alpha(S), \tag{11.10}$$
$$[W_\tau(g)(z)]^\pm = (\pm 2^{-1} I_4 + \mathcal{K}_{2,\tau})\, g(z), \;\; g \in \mathrm{C}^\alpha(S), \tag{11.11}$$
$$\mathcal{L}_\tau^+\, g = \mathcal{L}_\tau^-\, g =: \mathcal{L}_\tau\, g, \;\; g \in \mathrm{C}^{1,\alpha}(S), \;\; k \geq 1; \tag{11.12}$$

ii) *the operators*

$$\mathcal{H}_\tau \quad : \quad \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l+1,\alpha}(S), \tag{11.13}$$
$$\mathcal{K}_{1,\tau},\ \mathcal{K}_{2,\tau} \quad : \quad \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l,\alpha}(S), \tag{11.14}$$
$$\mathcal{L}_\tau \quad : \quad \mathrm{C}^{l+1,\alpha}(S) \to \mathrm{C}^{l,\alpha}(S), \tag{11.15}$$

are bounded.

Theorem 11.2. *The operators $\mathcal{H}_\tau$, $\pm 2^{-1} I_4 + \mathcal{K}_{1,\tau}$, $\pm 2^{-1} I_4 + \mathcal{K}_{2,\tau}$, and $\mathcal{L}_\tau$ are elliptic ΨDOs of order -1, 0, 0, and 1, respectively, with index equal to zero. Moreover, the principal homogeneous symbol matrices of the operators $-\mathcal{H}_\tau$ and $\mathcal{L}_\tau$ are positive definite.*

Theorem 11.3. *The operators* (11.7), (11.8), *and* (11.13)–(11.15) *can be extended by continuity to the following bounded operators*

$$\begin{array}{rcll}
V_\tau & : & B^s_{p,p}(S) \to H^{s+1+1/p}_p(\Omega^\pm) & [B^s_{p,q}(S) \to B^{s+1+1/p}_{p,q}(\Omega^\pm)], \\
W_\tau & : & B^s_{p,p}(S) \to H^{s+1/p}_p(\Omega^\pm) & [B^s_{p,q}(S) \to B^{s+1/p}_{p,q}(\Omega^\pm)], \\
\mathcal{H}_\tau & : & H^s_p(S) \to H^{s+1}_p(S) & [B^s_{p,q}(S) \to B^{s+1}_{p,q}(S)], \\
\mathcal{K}_{1,\tau}, \mathcal{K}_{2,\tau} & : & H^s_p(S) \to H^s_p(S) & [B^s_{p,q}(S) \to B^s_{p,q}(S)], \\
\mathcal{L}_\tau & : & H^{s+1}_p(S) \to H^s_p(S) & [B^{s+1}_{p,q}(S) \to B^s_{p,q}(S)],
\end{array}$$

for arbitrary $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, provided $S \in \mathrm{C}^\infty$. Moreover,

i) *for these extended operators the formulae* (11.9)–(11.12) *remain valid in the corresponding spaces;*

ii) *the integral representation formula* (3.2) *remains valid for $U \in W^1_p(\Omega^\pm)$ with $A(D,\tau)U = 0$ in $\Omega^\pm$, provided that U satisfies the decay condition* (1.30) *at infinity in the case of the domain Ω^-.*

Clearly, the proofs of these theorems are verbatim the proofs of the analogous propositions in the previous section and, therefore, we omit them (for details see [16]).

We note here that the formula similar to (10.46) holds also for the pseudo-oscillation operators and read as

$$(-2^{-1} I_4 + \mathcal{K}_{2,\tau})(2^{-1} I_4 + \mathcal{K}_{2,\tau}) = \mathcal{H}_\tau \mathcal{L}_\tau. \tag{11.16}$$

Applying the general integral representation formula (3.2) for $U(x) = V_\tau(g)(x)$ we can also easily derive the following identity

$$(-2^{-1} I_4 + \mathcal{K}_{1,\tau})(2^{-1} I_4 + \mathcal{K}_{1,\tau}) = \mathcal{L}_\tau \mathcal{H}_\tau. \tag{11.17}$$

Remark 11.4. The results of Section 2 imply that the dominant singular parts and the principal homogeneous symbol matrices of the operators $\mathcal{H}_\tau$,

$\pm 2^{-1}I_4 + \mathcal{K}_{1,\tau}$, $\pm 2^{-1}I_4 + \mathcal{K}_{2,\tau}$, and $\mathcal{L}_\tau$ read as (cf. (10.23)-(10.25), (10.48), (10.31), (10.32), (10.49))

$$(\mathcal{H}_\tau)_0 = \begin{bmatrix} [\mathcal{H}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{H}_4^{(0)} \end{bmatrix}_{4\times 4}, \tag{11.18}$$

$$(\pm 2^{-1}I_4 + \mathcal{K}_{1,\tau})_0 = \begin{bmatrix} [\pm 2^{-1}I_3 + \mathcal{K}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \pm 2^{-1}I_1 + \mathcal{K}_4^{(0)} \end{bmatrix}_{4\times 4}, \tag{11.19}$$

$$(\pm 2^{-1}I_4 + \mathcal{K}_{2,\tau})_0 = \begin{bmatrix} [\pm 2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \pm 2^{-1}I_1 + \overset{*}{\mathcal{K}}{}_4^{(0)} \end{bmatrix}_{4\times 4}, \tag{11.20}$$

$$(\mathcal{L}_\tau)_0 = \begin{bmatrix} [\mathcal{L}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(0)} \end{bmatrix}_{4\times 4}, \tag{11.21}$$

and

$$\begin{aligned} \sigma(\pm 2^{-1}I_4 + \mathcal{K}_{1,\tau}) &= [\overline{\sigma(\pm 2^{-1}I_4 + \mathcal{K}_{2,\tau})}]^\top = \\ &= \begin{bmatrix} [\sigma(\pm 2^{-1}I_3 + \mathcal{K}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\pm 2^{-1}I_1 + \mathcal{K}_4^{(0)}) \end{bmatrix}_{4\times 4}, \end{aligned} \tag{11.22}$$

$$\sigma(\mathcal{H}_\tau) = \begin{bmatrix} [\sigma(\mathcal{H}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{H}_4^{(0)}) \end{bmatrix}_{4\times 4}, \tag{11.23}$$

$$\sigma(\mathcal{L}_\tau) = \begin{bmatrix} [\sigma(\mathcal{L}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{L}_4^{(0)}) \end{bmatrix}_{4\times 4}. \tag{11.24}$$

The matrices (11.22)–(11.24), as it has been shown in the previous section, are nonsingular. Moreover, $\sigma(-\mathcal{H}_\tau)$ and $\sigma(\mathcal{L}_\tau)$ are positive definite.

CHAPTER V
REGULAR BOUNDARY VALUE AND INTERFACE PROBLEMS

Here we consider the nonhomogeneous regular basic boundary value and interface problems formulated in Chapter II for the pseudo-oscillation and steady state oscillation equations of the thermoelasticity theory of anisotropic bodies. The existence theorems will be proved in the Hölder continuous and Sobolev functional spaces with the help of the boundary integral equation method.

12. Basic BVPs of Pseudo-Oscillations

12.1. Let us first consider the regular problem $(\mathcal{P}_1)^+_\tau$ (see (5.1) and (5.2)) $S \in \mathrm{C}^{2,\alpha'}$.

We look for a solution in the form of the double layer potential (see (11.2))

$$U(x) = W_\tau(g)(x), \quad x \in \Omega^+, \tag{12.1}$$

where $g = (g_1, \cdots, g_4)^\top \in \mathrm{C}^{1,\alpha}(S)$ is the unknown density. As above, here and in what follows we again provide that $0 < \alpha < \alpha' \leq 1$.

Applying the jump formula for a double layer potential (see Theorem 11.1, item i)) and taking into account the boundary conditions of the problem in question we arrive at the boundary integral equation (BIE)

$$\mathcal{N}^+_{1,\tau} g(x) := [2^{-1} I_4 + \mathcal{K}_{2,\tau}]\, g(x) = G^{(1)}(x), \quad x \in S, \tag{12.2}$$

where $G^{(1)} = (f_1, \cdots, f_4)^\top \in \mathrm{C}^{1,\alpha}(S)$ is the given vector function on S (see (5.1)–(5.2)), and $\mathcal{K}_{2,\tau}$ is defined by (11.5).

Due to Theorem 11.2 the singular integral operator in the left-hand side of (12.2) is an elliptic ΨDO with zero index.

Further, we show that the homogeneous version of the equation (12.2) (i.e., when $G^{(1)} = 0$) has only the trivial solution. Let $g_0 \in \mathrm{C}^{1,\alpha}(S)$ be an arbitrary solution of the equation

$$[2^{-1} I_4 + \mathcal{K}_{2,\tau}]\, g(x) = 0, \quad x \in S. \tag{12.3}$$

It is evident that the vector function

$$U_0(x) = W_\tau(g_0)(x) \in \mathrm{C}^{1,\alpha}(\overline{\Omega^+}) \tag{12.4}$$

represents then a regular solution of the homogeneous problem $(\mathcal{P}_1)^+_\tau$ due to (12.3). Therefore, by the uniqueness Theorem 8.1 we conclude $U_0(x) = 0$ in Ω^+ which, in turn, implies

$$[B(D,n)U_0]^+ = \mathcal{L}_\tau\, g_0 = 0 \quad \text{on} \quad S,$$

where $\mathcal{L}_\tau = \mathcal{L}^\pm_\tau$ is defined by (11.6).

In accordance with equation (11.12) we get

$$[B(D,n)U_0]^- = 0 \quad \text{on} \quad S, \tag{12.5}$$

where U_0 is given again by (12.4) in Ω^-.

Thus, we have obtained that the vector function

$$U_0(x) = W_\tau(g_0)(x) \in \mathrm{C}^{1,\alpha}(\overline{\Omega^-}) \tag{12.6}$$

represents a regular solution to the problem $(\mathcal{P}_2)^-_\tau$. Therefore, $U_0(x) = 0$ in Ω^- due to Theorem 8.1.

As a result we have for arbitrary $x \in S$

$$[U_0(x)]^+ - [U_0(x)]^- = [W_\tau(g_0)(x)]^+ - [W_\tau(g_0)(x)]^- = g_0 = 0$$

which proves that the equation (12.3) has only the trivial solution.

According to the general theory of singular integral equations (see, e.g., [51], [45], Ch.IV), the nonhomogeneous equation (12.2) is uniquely solvable for an arbitrary right-hand side. Moreover, the corresponding embedding theorems for the solution of SIE on closed manifold yield that, if $S \in \mathrm{C}^{k+1,\alpha'}$ and $f \in \mathrm{C}^{k,\alpha}(S)$, then $g \in \mathrm{C}^{k,\alpha}(S)$.

Finally, we arrive at the following existence theorem.

Theorem 12.1. *Let $S \in \mathrm{C}^{k+1,\alpha'}$ and $f_j \in \mathrm{C}^{k,\alpha}(S)$ where $j = \overline{1,4}$ and $k \geq 1$ is an arbitrary integer. Then the problem $(\mathcal{P}_1)^+_\tau$ (i.e., (1.9), (5.1), (5.2)) is uniquelly solvable in the space $\mathrm{C}^{k,\alpha}(\overline{\Omega^+})$ and the solution is representable in the form (12.1), where $g \in \mathrm{C}^{k,\alpha}(S)$ solves the BIE (12.2).*

Remark 12.2. Note that, if one looks for a regular solution to the BVP problem $(\mathcal{P}_1)^+_\tau$ in the form of a single layer potential (see (11.1))

$$U(x) = V_\tau(h)(x), \quad x \in \Omega^+, \tag{12.7}$$

then one gets the ΨDE

$$\mathcal{H}_\tau h(x) = G^{(1)}(x), \quad x \in S, \tag{12.8}$$

due to Theorem 11.1 (see (11.9)).

Applying again the uniqueness Theorem 8.1 and properties of the single layer potential, by the arguments similar to the above ones it can be easily shown that $\ker \mathcal{H}_\tau$ is trivial. Note that $-\mathcal{H}_\tau$ is an elliptic ΨDO of order -1 (with positive definite principal homogeneous symbol matrix) and its index equals zero. Invoking the general theory of ΨDO on closed smooth manifolds (see,e.g., [77]) we conclude that the operator

$$\mathcal{H}_\tau \,:\, \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l+1,\alpha}(S), \quad S \in \mathrm{C}^{k,\alpha'} \quad 0 \leq l \leq k-1, \quad k \geq 1, \tag{12.9}$$

is an isomorphism. Therefore, the equation (12.8) is uniquely solvable in the space $\mathrm{C}^{k-1,\alpha}(S)$ provided that $S \in \mathrm{C}^{k,\alpha'}$ and $f \in \mathrm{C}^{k,\alpha}(S)$ $(k \geq 1)$. As a result we obtain that the solution of the problem $(\mathcal{P}_1)^+_\tau$ can also be uniquely represented as a single layer potential (12.7), where $h \in \mathrm{C}^{k-1,\alpha}(S)$ is the unique solution of the equation (12.8). Clearly, we again have $U = V_\tau(h) \in \mathrm{C}^{k,\alpha}(\overline{\Omega^+})$.

We remark that applying the equation (11.17) one can show that, in fact, the operator

$$\mathcal{H}_\tau^{-1} \,:\, \mathrm{C}^{l+1,\alpha}(S) \to \mathrm{C}^{l,\alpha}(S), \quad S \in \mathrm{C}^{k,\alpha'} \quad 0 \leq l \leq k-1, \quad k \geq 1, \tag{12.10}$$

which is inverse to the operator (12.9), is a singular integro-differential operator (i.e., a ΨDO of order 1). Obviously, the principal homogeneous symbol matrix of the operator $-\mathcal{H}_\tau^{-1}$ is also positive definite.

It should be noted that to prove the existence of a regular solution by the single layer approach, as it is evident from the above arguments, $\mathrm{C}^{1,\alpha'}$-smoothness of the boundary surface $\partial\Omega^+ = S$ is sufficient, while by the double layer approach we need $S \in \mathrm{C}^{2,\alpha'}$.

12.2. Let us look for a regular solution of the problem $(\mathcal{P}_2)_\tau^+$ (see (5.3)–(5.4)) again in the form (12.1). The boundary conditions of the problem in question and the properties of the double layer potential lead to the following system of equations for the unknown density g on S

$$\{[2^{-1}I_4 + \mathcal{K}_{2,\tau}]\, g(x)\}_j = f_j(x), \quad j = 1, 2, 3, \tag{12.11}$$

$$\{\mathcal{L}_\tau\, g(x)\}_4 = F_4(x). \tag{12.12}$$

Note that the operators involved in the first three equations are singular integral operators (SIO), i.e., ΨDOs of zero order, while in the fourth equation we have singular integro-differential operators, i.e., ΨDOs of order 1.

In order to rewrite these equations in the matrix form we set

$$\mathcal{N}_{2,\tau}^+ := \begin{bmatrix} [(2^{-1}I_4 + \mathcal{K}_{2,\tau})_{pq}]_{3\times 4} \\ [(\mathcal{L}_\tau)_{4q}]_{1\times 4} \end{bmatrix}_{4\times 4} \tag{12.13}$$

with $p = 1, 2, 3$ and $q = \overline{1, 4}$.

Clearly, then (12.11) and (12.12) are equivalent to the equation

$$\mathcal{N}_{2,\tau}^+\, g(x) = G^{(2)}(x), \quad x \in S, \quad G^{(2)} = (f_1, f_2, f_3, F_4)^\top. \tag{12.14}$$

We assume that $G^{(2)} \in [\mathrm{C}^{k,\alpha}(S)]^3 \times [\mathrm{C}^{k-1,\alpha}(S)]$, i.e.,

$$S \in \mathrm{C}^{k+1,\alpha'}, \;\; f_j \in \mathrm{C}^{k,\alpha}(S), \; j = 1, 2, 3, \;\; F_4 \in \mathrm{C}^{k-1,\alpha}(S), \tag{12.15}$$

where $k \geq 1$, $0 < \alpha < \alpha' \leq 1$. Moreover, we seek the unknown density vector g in the space $[\mathrm{C}^{k,\alpha}(S)]^4$.

The system of ΨDEs (12.13) is elliptic in the sense of Douglis–Nirenberg (cf. [3], [2], [85]) and its principal symbol matrix

$$\sigma(\mathcal{N}_{2,\tau}^+) = \begin{bmatrix} [\sigma(2^{-1}I_3 + \mathcal{K}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{L}_4^{(0)}) \end{bmatrix}_{4\times 4} \tag{12.16}$$

is nonsingular for arbitrary $x \in S$ and $|\widetilde{\xi}| = 1$ (see Remark 11.4, the formulae (10.26), (10.28), (10.41), (10.43), and the proofs of Lemmata 10.2 and 10.7).

The index of the operator $\mathcal{N}_{2,\tau}^+$ is equal to zero, since the index of the corresponding dominant singular part is zero.

Next, we show that the system (12.11)-(12.12) (i.e., (12.14)) can be equivalently reduced to the system of singular integral equations (SIEs). To this end we formulate the following lemma which will be frequently used in the sequel (see, e.g., [60], [20]).

Lemma 12.3. *The scalar operator*

$$\mathcal{R}\, h(z) = \frac{1}{2\pi} \int\limits_S |z-y|^{-1}\, h(y)\, dS_y, \quad z \in S, \quad S \in \mathrm{C}^{1,\alpha'}, \tag{12.17}$$

generated by the harmonic single layer potential, is a formally self-adjoint, equivalent smoothing lifting ΨDO of order -1, (i.e., $\mathcal{R}\, h = 0$ implies $h = 0$) with the principal homogeneous symbol equal to $|\tilde{\xi}|^{-1}$ (i.e., $\sigma(\mathcal{R})(x,\tilde{\xi}) = |\tilde{\xi}|^{-1}$, $x \in S$, $\tilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$).

Due to this lemma it is evident that the system (12.11)–(12.12) is equivalent to the system of SIEs on S

$$\{[2^{-1} I_4 + \mathcal{K}_{2,\tau}]\, g(x)\}_j = f_j(x), \quad j = 1,2,3, \tag{12.18}$$

$$\mathcal{R}\{\mathcal{L}_\tau\, g(x)\}_4 = \mathcal{R} F_4(x), \tag{12.19}$$

which can also be written as

$$\mathcal{R}_2 \mathcal{N}^+_{2,\tau}\, g(x) = G^{(2)}_*, \tag{12.20}$$

where

$$\mathcal{R}_2 = \begin{bmatrix} [I_3]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{R} \end{bmatrix}_{4\times 4} \tag{12.21}$$

and

$$G^{(2)}_* = (f_1, f_2, f_3, \mathcal{R} F_4)^\top. \tag{12.22}$$

Clearly, (12.20) is an elliptic SIE with index zero.

Further, we prove that the nonhomogeneous system (12.11)-(12.12) (i.e., (12.14) and (12.20)) is uniquely solvable. Invoking again the theory of SIEs on smooth manifolds ([51], [45]), we have to show that the homogeneous version of the system (12.11)-(12.12) admits only the trivial solution. It is an easy consequence of the corresponding uniqueness theorem and the jump relations of the double layer potential, and can be shown by the same arguments as in the previous subsection. These results imply that the equation (12.20) has a unique solution $g \in \mathrm{C}^{k,\alpha}(S)$ for arbitrary $G^{(2)}_* \in \mathrm{C}^{k,\alpha}(S)$. This immediately leads to the following assertion.

Theorem 12.4. *Let conditions* (12.15) *be fulfilled. Then the problem $(\mathcal{P}_2)^+_\tau$ (i.e.,* (1.9), (5.3), (5.4)) *is uniquely solvable in the space $\mathrm{C}^{k,\alpha}(\overline{\Omega^+})$ and the solution is representable in the form* (12.1)*, where $g \in \mathrm{C}^{k,\alpha}(S)$ solves the system of BIEs* (12.11)–(12.12) (*i.e.,* (12.20)).

Let us note here that the single layer aproach is again applicable and leads to the existence of a unique solution in the space $\mathrm{C}^{k,\alpha}(\overline{\Omega^+})$ (cf. Remark 12.2).

12.3. In this subsection we consider the nonhomogeneous problem $(\mathcal{P}_3)^+_\tau$ (see (5.5)–(5.6)). We look for a regular solution U again in the form (12.1) which yields the following system of BIEs on S:

$$\{\mathcal{L}_\tau\, g(x)\}_j = F_j(x), \quad j = 1,2,3, \tag{12.23}$$

$$\{[2^{-1} I_4 + \mathcal{K}_{2,\tau}]\, g(x)\}_4 = f_4(x), \tag{12.24}$$

where we provide

$$S \in \mathrm{C}^{k+1,\alpha'}, \;\; F_j \in \mathrm{C}^{k-1,\alpha}(S), \; j=1,2,3, \;\; f_4 \in \mathrm{C}^{k,\alpha}(S) \tag{12.25}$$

with the same k, α', and α as in (12.15). The unknown density g is again assumed to belong to the class $\mathrm{C}^{k,\alpha}(S)$.

We set

$$\mathcal{N}^{+}_{3,\tau} := \left[\begin{array}{c} [(\mathcal{L}_\tau)_{pq}]_{3\times 4} \\ [(2^{-1}I_4 + \mathcal{K}_{2,\tau})_{4q}]_{1\times 4} \end{array}\right]_{4\times 4} \tag{12.26}$$

with $p = 1,2,3$, and $q = \overline{1,4}$.

The equations (12.23)-(12.24) can be then written in the matrix form as

$$\begin{gathered} \mathcal{N}^{+}_{3,\tau}\, g(x) = G^{(3)}(x), \;\; x \in S, \\ G^{(3)} = (F_1, F_2, F_3, f_4)^\top \in [\mathrm{C}^{k-1,\alpha}(S)]^3 \times \mathrm{C}^{k,\alpha}(S). \end{gathered} \tag{12.27}$$

The operator $\mathcal{N}^{+}_{3,\tau}$ is elliptic (again in the sense of Douglis-Nirenberg) with the nonsingular principal symbol matrix

$$\sigma(\mathcal{N}^{+}_{3,\tau}) = \left[\begin{array}{cc} [\sigma(\mathcal{L}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(2^{-1}I_1 + \mathcal{K}^{(0)}_4) \end{array}\right]_{4\times 4} \tag{12.28}$$

(see Section 10 and Remark 11.4) and the index equal to zero.

Introduce the matrix operator

$$\mathcal{R}_3 = \left[\begin{array}{cc} [I_3\mathcal{R}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & I_1 \end{array}\right]_{4\times 4}, \tag{12.29}$$

where $\mathcal{R}$ is the equivalent lifting operator (12.17).

Now it can be easily seen that

$$\mathcal{R}_3\mathcal{N}^{+}_{3,\tau}\, g(x) = G^{(3)}_*, \quad G^{(3)}_* = (\mathcal{R}F_1, \mathcal{R}F_2, \mathcal{R}F_3, f_4)^\top \in \mathrm{C}^{k,\alpha}(S), \tag{12.30}$$

is an elliptic system of SIEs equivalent to (12.23)–(12.24), due to Lemma 12.3.

As in the previous subsection we can easily establish that the homogeneous version of the system (12.23)–(12.24) admits only the trivial solution. Therefore, the nonhomogeneous system (12.30) and, consequently, (12.23)–(12.24) are uniquely solvable in the class $\mathrm{C}^{k,\alpha}(S)$ if the boundary data meet the conditions (12.25). Thus, we have proved the following existence result.

Theorem 12.5. *Let conditions* (12.25) *be fulfilled. Then the problem* $(\mathcal{P}_3)^+_\tau$ (*i.e.,* (1.9), (5.5), (5.6)) *is uniquely solvable in the space* $\mathrm{C}^{k,\alpha}(\overline{\Omega^+})$ *and the solution is representable in the form* (12.1), *where* $g \in \mathrm{C}^{k,\alpha}(S)$ *solves the system of BIEs* (12.23)–(12.24) (*i.e.,* (12.30)).

We emphasize that the single layer aproach is again applicable.

12.4. Here we consider the nonhomogeneous boundary value problem $(\mathcal{P}_4)^+_\tau$ (see (5.7), (5.8)). We look for a regular solution U again in the form

(12.1) which now leads to the hypersingular BIE (ΨDE of order +1) on S

$$\begin{aligned} &\mathcal{N}^{+}_{4,\tau}\, g(x) := \mathcal{L}_\tau\, g(x) = G^{(4)}(x), \\ &G^{(4)} = (F_1, \cdots, F_4)^\top \in [\mathrm{C}^{k-1,\alpha}(S)]^4. \end{aligned} \tag{12.31}$$

Due to Remark 11.4 the dominant singular part and the principal homogeneous positive definite symbol matrix of the singular integro-differential operator $\mathcal{N}^{+}_{4,\tau} := \mathcal{L}_\tau$ are given by formulae (11.21) and (11.24), respectively. Moreover, the index of $\mathcal{L}_\tau$ is equal to zero.

The ΨDE (12.31) is equivalent to the elliptic system of SIEs

$$\mathcal{R}_4 \mathcal{N}^{+}_{4,\tau}\, g(x) = G^{(4)}_*, \quad G^{(4)}_* = (\mathcal{R}F_1, \cdots, \mathcal{R}F_4)^\top \in \mathrm{C}^{k,\alpha}(S), \tag{12.32}$$

where

$$\mathcal{R}_4 = [I_4 \mathcal{R}]_{4\times 4} \tag{12.33}$$

with $\mathcal{R}$ defined by (12.17).

Applying uniqueness Theorem 8.1 and formula (11.12) we conclude that the homogeneous version of equation (12.31) has only the trivial solution. Therefore, the nonhomogeneous systems (12.32) and (12.31) are uniquely solvable in the space $\mathrm{C}^{k,\alpha}(S)$. This implies the following proposition.

Theorem 12.6. *Let $S \in \mathrm{C}^{k+1,\alpha'}$ and $F \in [\mathrm{C}^{k-1,\alpha}(S)]^4$ with the same k, α', and α as in* (12.15). *Then the problem $(\mathcal{P}_4)^+_\tau$ (i.e.,* (1.9), (5.7), (5.8)) *is uniquely solvable in the space $\mathrm{C}^{k,\alpha}(\overline{\Omega^+})$ and the solution is representable in the form* (12.1), *where $g \in \mathrm{C}^{k,\alpha}(S)$ solves the system of BIEs* (12.31) *(i.e.,* (12.32)).

Remark 12.7. The classical single layer approach for the problem $(\mathcal{P}_4)^+_\tau$ (see (12.7)) reduces the BVP to the system of SIEs on $S \in \mathrm{C}^{k,\alpha'}$ $(k \geq 1)$

$$\begin{aligned} &(-2^{-1} I_4 + \mathcal{K}_{1,\tau})\, h(x) = G^{(4)}, \\ &G^{(4)} = (F_1, \cdots, F_4)^\top \in \mathrm{C}^{k-1,\alpha}(S). \end{aligned} \tag{12.34}$$

The SIO in the left-hand side is elliptic with index zero. Moreover, Theorems 8.1 and 11.1, item i) imply $\ker(-2^{-1} I_4 + \mathcal{K}_{1,\tau}) = \{0\}$. Therefore, the mapping

$$-2^{-1} I_4 + \mathcal{K}_{1,\tau} \;:\; \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l,\alpha}(S), \quad 0 \leq l \leq k-1, \tag{12.35}$$

is an isomorphism.

These arguments show that the equation (12.34) is always solvable in the space $\mathrm{C}^{k-1,\alpha}(S)$. This, in turn, proves that the unique solution to the BVP $(\mathcal{P}_4)^+_\tau$ is representable also in the form of a single layer potential

$$U(x) = V_\tau(h)(x) \in \mathrm{C}^{k,\alpha}(\overline{\Omega^+}),$$

where $h \in \mathrm{C}^{k-1,\alpha}(S)$ solves the SIE (12.34).

12.5. The existence theorems of solutions to the basic exterior BVPs for the pseudo-oscillation equations of thermoelasticity theory can be proved by the word for word repetition of the arguments outlined in the previous subsections. Therefore, we confine oureselves by formulation the final results.

Theorem 12.8. *The basic exterior nonhomogeneous BVPs* $(\mathcal{P}_n)^-_\tau$ $(n = \overline{1,4})$, *formulated in Section* 5 (*see* (5.1)–(5.8)) *are uniquely solvable in the space* $\mathrm{C}^{k,\alpha}(\overline{\Omega^-})$ *provided that*

$$S \in \mathrm{C}^{k+1,\alpha'}, \ \ f_j \in \mathrm{C}^{k,\alpha}(S), \ \ F_j \in \mathrm{C}^{k-1,\alpha}(S), \ \ j = \overline{1,4}, \tag{12.36}$$

where $0 < \alpha < \alpha' \leq 1$ *and* $k \geq 1$ *is an arbitrary integer. The solutions are representable in the form of a double layer potential*

$$U(x) = W_\tau(g)(x), \quad x \in \Omega^-, \tag{12.37}$$

where $g \in \mathrm{C}^{k,\alpha}(S)$ *solves the elliptic* (*in general, in the sense of Douglis-Nirenberg*) *system of boundary integral* (*pseudodifferential*) *equation on* S

$$\mathcal{N}^-_{n,\tau}\, g(x) = G^{(n)}(x). \tag{12.38}$$

Here the BIOs are defined as follows

$$\mathcal{N}^-_{1,\tau} := -2^{-1} I_4 + \mathcal{K}_{2,\tau}, \qquad \mathcal{N}^-_{4,\tau} := \mathcal{L}_\tau, \tag{12.39}$$

$$\begin{gathered}\mathcal{N}^-_{2,\tau} := \begin{bmatrix} [(-2^{-1} I_4 + \mathcal{K}_{2,\tau})_{pq}]_{3\times 4} \\ [(\mathcal{L}_\tau)_{4q}]_{1\times 4} \end{bmatrix}_{4\times 4}, \\ \mathcal{N}^-_{3,\tau} := \begin{bmatrix} [(\mathcal{L}_\tau)_{pq}]_{3\times 4} \\ [(-2^{-1} I_4 + \mathcal{K}_{2,\tau})_{4q}]_{1\times 4} \end{bmatrix}_{4\times 4},\end{gathered} \tag{12.40}$$

where $p = \overline{1,3}$, $q = \overline{1,4}$, *and* $\mathcal{K}_{2,\tau}$ *and* $\mathcal{L}_\tau$ *are given by* (11.5) *and* (11.12), *respectively.*

The right-hand side vector functions $G^{(n)}$ *in* (12.38) *are constructed by the boundary data of the BVPs under consideration and read as*

$$\begin{aligned} G^{(1)} &= (f_1, \cdots, f_4)^\top \in [\mathrm{C}^{k,\alpha}(S)]^4, \\ G^{(2)} &= (f_1, f_2, f_3, F_4)^\top \in [\mathrm{C}^{k,\alpha}(S)]^3 \times \mathrm{C}^{k-1,\alpha}(S), \\ G^{(3)} &= (F_1, F_2, F_3, f_4)^\top \in [\mathrm{C}^{k-1,\alpha}(S)]^3 \times \mathrm{C}^{k,\alpha}(S), \\ G^{(4)} &= (F_1, \cdots, F_4)^\top \in [\mathrm{C}^{k-1,\alpha}(S)]^4. \end{aligned} \tag{12.41}$$

Note that the mappings

$$\begin{aligned} \mathcal{N}^-_{1,\tau} \ &: \ [\mathrm{C}^{l,\alpha}(S)]^4 \to [\mathrm{C}^{l,\alpha}(S)]^4, \quad 0 \leq l \leq k, \\ \mathcal{N}^-_{2,\tau} \ &: \ [\mathrm{C}^{l,\alpha}(S)]^4 \to [\mathrm{C}^{l,\alpha}(S)]^3 \times \mathrm{C}^{l-1,\alpha}(S), \quad 1 \leq l \leq k, \\ \mathcal{N}^-_{3,\tau} \ &: \ [\mathrm{C}^{l,\alpha}(S)]^4 \to [\mathrm{C}^{l-1,\alpha}(S)]^3 \times \mathrm{C}^{l,\alpha}(S), \quad 1 \leq l \leq k, \\ \mathcal{N}^-_{4,\tau} \ &: \ [\mathrm{C}^{l,\alpha}(S)]^4 \to [\mathrm{C}^{l-1,\alpha}(S)]^4, \quad 1 \leq l \leq k, \end{aligned}$$

are again isomorphisms. Moreover, the equations (12.38) (n=2,3,4) can be equivalently reduced to the corresponding elliptic SIEs by the same lifting procedure as above with the help of the lifting operators $\mathcal{R}_n$.

Finally, we remark that one can apply the single layer approach in the all above exterior BVPs to prove the existence theorems.

12.6. In this subsection we shall study the above considered problems in the weak setting. Let us first treat the problems $(\mathcal{P}_1)^{\pm}_{\tau}$. We again look for the solutions $U \in W^1_p(\Omega^{\pm})$, $1 < p < \infty$, in the form of double layer potentials (12.1) and (12.37). Now the unknown density vector function g should be found in the natural space $B^{1-1/p}_{p,p}(S)$ since $W_\tau : B^{1-1/p}_{p,p}(S) \to W^1_p(\Omega^{\pm})$ (see Theorem 11.3 and Section 4).

In what follows, for simplicity, we illustrate our approach for the case $S \in \mathrm{C}^\infty$, and at the same time notice that, actually, some finite smoothness is sufficient for our purposes (for details see [59]).

Applying again Theorem 11.3 and taking into account the boundary conditions (5.1)–(5.2) we arrive at the BIEs on S

$$\mathcal{N}^{\pm}_{1,\tau}\, g(x) := [\pm 2^{-1} I_4 + \mathcal{K}_{2,\tau}]\, g(x) = G^{(1)}(x),\ G^{(1)} = (f_1, \cdots, f_4)^\top, \tag{12.42}$$

which formally coincide with the equations (12.2) and (12.38) (for $n = 1$). But now here

$$G^{(1)} \in B^{1-1/p}_{p,p}(S) \tag{12.43}$$

and we look for the unknown vector function g in the same space, i.e.,

$$g \in B^{1-1/p}_{p,p}(S), \quad 1 < p < \infty. \tag{12.44}$$

Now we prove the following proposition.

Lemma 12.9. *The operators*

$$\mathcal{N}^{\pm}_{1,\tau} : [B^s_{p,q}(S)]^4 \to [B^s_{p,q}(S)]^4 \tag{12.45}$$

are isomorphisms for arbitrary $s \in \mathbb{R}$, $1 < p < \infty$, *and* $1 \le q \le \infty$.

Proof. We outline the proof for the operator $\mathcal{N}^{+}_{1,\tau}$. For $\mathcal{N}^{-}_{1,\tau}$ it is verbatim.

The mapping property (12.45) follows from Theorem 11.3. Since $\mathcal{N}^{+}_{1,\tau}$ is an elliptic ΨDO on closed smooth manifold S, the null-space $\ker\mathcal{N}^{+}_{1,\tau}$ and the index $\operatorname{ind}\mathcal{N}^{+}_{1,\tau}$ are the same for arbitrary two pairs (s_1, p_1) and (s_2, p_2), where $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2 \in (1, \infty)$, and for arbitrary $1 \le q \le \infty$ (see [4], [43], [77], Ch.2). Let $s = 0$ and $p = q = 2$, and prove that in this particular case the null-space of the operator $\mathcal{N}^{+}_{1,\tau}$ is trivial and the index equals zero. In fact, let $g_0 \in B^0_{2,2}(S) = L_2(S)$ be some solution to the homogeneous equation $\mathcal{N}^{+}_{1,\tau}\, g_0 = 0$. The embedding theorems for solutions of elliptic SIEs (see, e.g., [45], Ch.4) imply that, actually, $g_0 \in \mathrm{C}^{k,\alpha}(S)$ for any $k \ge 0$, due to the smoothness of the boundary surface S and the right-hand side of the homogeneous SIE in question. The double layer potential $U_0(x) = W_\tau(g_0)(x)$ represents then a regular vector function of the class $\mathrm{C}^{1,\alpha}(\overline{\Omega^+})$ which solves the homogeneous BVP $(\mathcal{P}_1)^{+}_{\tau}$. Therefore, in the same way as above (see Subsection 12.1) we conclude that $g_0 = 0$ on S, which proves that $\ker\mathcal{N}^{+}_{1,\tau}$ is trivial in $L_2(S)$. According to the above remark it then follows that $\ker\mathcal{N}^{+}_{1,\tau}$ is trivial also in the space $B^s_{p,q}(S)$ for arbitrary $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$.

Finally we note that the equality $\operatorname{ind}\mathcal{N}_{1,\tau}^{+}=0$ follows from Theorem 11.2 which completes the proof. □

This lemma yields the following existence results.

Theorem 12.10. *Let the boundary data meet the condition* (12.43). *Then the BVP* $(\mathcal{P}_1)_\tau^+$ $[(\mathcal{P}_1)_\tau^-]$ *is uniquelly solvable in the Sobolev space* $W_p^1(\Omega^+)$ $[W_p^1(\Omega^-)]$ *with* $1<p<\infty$ *and the solution is representable in the form of a double layer potential* (12.1) [(12.37)] *with the density* $g\in B_{p,p}^{1-1/p}(S)$ *which solves the corresponding SIE* (12.42).

Proof. Solvability of the problems $(\mathcal{P}_1)_\tau^\pm$ is a ready consequence of Lemma 12.9 (for $s=1-1/p$ and $q=p$).

Now let us prove that the homogeneous BVP $(\mathcal{P}_1)_\tau^+$ has only the trivial solution in the space $W_p^1(\Omega^+)$ for $1<p<\infty$. Obviously, this implies that the corresponding nonhomogeneous problem is uniquely solvable in the same space. Note that the case $p=2$ has already been considered in Section 8.

We proceed as follows. Let $U\in W_p^1(\Omega^+)$ be some solution to the homogeneous problem $(\mathcal{P}_1)_\tau^+$. Then by Theorem 11.3, item ii), U can be represented as (cf. (3.2))

$$\begin{aligned} U(x)&=W_\tau([U]^+)(x)-V_\tau([B(D,n)U]^+)(x)=\\ &=-V_\tau([B(D,n)U]^+)(x),\ \ x\in\Omega^+, \end{aligned}\tag{12.46}$$

since by assumption $[U]^+=0$ on S.

On the other hand the same homogeneous boundary condition and the representation (12.46) together with Theorem 11.3, item i) imply

$$[U]^+=-\mathcal{H}_\tau([B(D,n)U]^+)=0\quad\text{on}\quad S,\tag{12.47}$$

where $[B(D,n)U]^+\in B_{p,p}^{-1/p}(S)$.

Noting that $-\mathcal{H}_\tau\,:\,B_{p,q}^{s}(S)\to B_{p,q}^{s+1}(S)$ is an elliptic ΨDO on the closed smooth surface S (with the positive definite principal homogeneous symbol matrix) we conclude that the null-space $\ker\mathcal{H}_\tau$ and the index $\operatorname{ind}\mathcal{H}_\tau$ in the spaces $B_{p,q}^{s}(S)$ do not depend on $s\in\mathbb{R}$, $1<p<\infty$, and $1\le q\le\infty$, and are the same as, for example, in the sapce $B_{2,2}^{-1/2}(S)=H_2^{-1/2}(S)$. Applying the embeding theorem for the solution of the elliptic ΨDEs on closed smooth manifold (see, e.g., [77], Ch.2) we easily show that $\ker\mathcal{H}_\tau$ is trivial in $B_{2,2}^{-1/2}(S)$. Further, we observe that the operator $-\mathcal{H}_\tau\,:\,B_{2,2}^{-1/2}(S)\to B_{2,2}^{1/2}(S)$ and its adjoint $-\mathcal{H}_\tau^*$ have the same mapping properties, i.e., $-\mathcal{H}_\tau^*\,:\,B_{2,2}^{-1/2}(S)\to B_{2,2}^{1/2}(S)$. Since the dominant singular part of the operator $\mathcal{H}_\tau$ is self-adjoint we conclude that $\operatorname{ind}\mathcal{H}_\tau=0$ in $B_{2,2}^{-1/2}(S)$. Therefore, the equation (12.47) has only the trivial solution in the space $B_{p,p}^{-1/p}(S)$ for arbitrary $p>1$. Thus, $[B(D,n)U]^+=0$, which shows that $U=0$ in Ω^+ due to (12.46).

The proof for the BVP $(\mathcal{P}_1)_\tau^-$ is verbatim. □

The analogous theorems hold valid for the problems $(\mathcal{P}_n)_\tau^-$, $n = 2,3,4$. The proofs rely upon the following assertions which can be proved by the arguments quite similar to that ones applied in the proof of Lemma 12.9.

Lemma 12.11. *Let $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$.*
Then the mappings

$$\begin{aligned} \mathcal{N}_{2,\tau}^{\pm} &: [B_{p,q}^s(S)]^4 \to [B_{p,q}^s(S)]^3 \times B_{p,q}^{s-1}(S), \\ \mathcal{N}_{3,\tau}^{\pm} &: [B_{p,q}^s(S)]^4 \to [B_{p,q}^{s-1}(S)]^3 \times B_{p,q}^{s}(S), \\ \mathcal{N}_{4,\tau}^{\pm} &: [B_{p,q}^s(S)]^4 \to [B_{p,q}^{s-1}(S)]^4 \end{aligned}$$

are isomorphisms.

Here $\mathcal{N}_{2,\tau}^{\pm}$, $\mathcal{N}_{3,\tau}^{\pm}$, $\mathcal{N}_{4,\tau}^{\pm}$ are defined as in Subsections 12.1–12.5.

Proof. One needs only to apply the equivalent lifting operator $\mathcal{R}_n$, defined by formulae (12.21), (12.29), and (12.33), to the operators $\mathcal{N}_{l,\tau}^{\pm}$ and show that the mappings

$$\mathcal{R}_n\mathcal{N}_{n,\tau}^{\pm} : [B_{p,q}^s(S)]^4 \to [B_{p,q}^s(S)]^4, \;\; n = 2,3,4,$$

are isomorphisms. Since the operators $\mathcal{R}_n\mathcal{N}_{n,\tau}^{\pm}$ are elliptic singular operators (i.e., ΨDOs of order 0) on the closed smooth manifold S, we can use the same arguments as in the proof of Lemma 12.9 to see that $\ker\mathcal{R}_n\mathcal{N}_{n,\tau}^{\pm} = \{0\}$ and $\mathrm{ind}\mathcal{R}_n\mathcal{N}_{n,\tau}^{\pm} = 0$ in the space $[B_{p,q}^s(S)]^4$. Whence $\ker\mathcal{N}_{n,\tau}^{\pm} = \{0\}$ and $\mathrm{ind}\mathcal{N}_{n,\tau}^{\pm} = 0$ (in the corresponding functional space) follow immediately. □

This lemma (for $s = 1 - 1/p$ and $q = p$) together with Theorem 8.2 implies the following existence theorem.

Theorem 12.12. *Let $1 < p < \infty$ and the boundary data in* (5.3)–(5.8) *meet the conditions*

$$f_j \in B_{p,p}^{1-1/p}(S), \quad F_j \in B_{p,p}^{-1/p}(S), \quad j = \overline{1,4}. \tag{12.48}$$

Then the BVP $(\mathcal{P}_n)_\tau^{\pm}$ ($n = 2,3,4$) are uniquelly solvable in the Sobolev spaces $W_p^1(\Omega^{\pm})$ and the solutions are representable in the form of double layer potentials (12.1) *and* (12.37) *with the density $g \in B_{p,p}^{1-1/p}(S)$ which solves the corresponding ΨDE on S*

$$\mathcal{N}_{n,\tau}^{\pm}\, g = G^{(n)}. \tag{12.49}$$

Here $\mathcal{N}_{n,\tau}^{\pm}$ are the same as in Subsections 12.1–12.5.

Proof. For illustration of the method we outline the proof in the case of BVP $(\mathcal{P}_4)_\tau^-$. For the other problems it is quite analogous.

Let us look for a solution in the form of a double layer potential (12.37), where g belongs to the natural space $B_{p,p}^{1-1/p}(S)$. Then due to Theorem 11.3 and the boundary conditions (5.7)–(5.8) we get the following ΨDE on S for the unknown density g

$$\mathcal{N}_{4,\tau}^{-}\, g := \mathcal{L}_\tau\, g(x) = G^{(4)}, \tag{12.50}$$

where $G^{(4)} := (F_1, \cdots, F_4)^\top \in B_{p,p}^{-1/p}(S)$.

By Lemma 12.11 (for $s = 1 - 1/p$ and $q = p$) the equation (12.50) is uniquely solvable in the space $g \in B^{1-1/p}_{p,p}(S)$. Whence $W_\tau(g) \in H^1_p(\Omega^-) = B^1_{p,p}(\Omega^-) = W^1_p(\Omega^-)$ by Theorem 11.3. Moreover, $W_\tau(g)$ represents a solution of the BVP in question due to (12.50). Now by virtue of Theorems 8.2 and 11.3, and the arguments in the final part of the proof of Theorem 12.10, we conclude that the vector function $U(x) = W_\tau(g) \in W^1_p(\Omega^-)$ is a unique solution of the problem $(\mathcal{P}_4)^-_\tau$ which completes the proof. $\square$

Remark 12.13. It is evident that one can apply a single layer approach to obtain the same existense results in the Sobolev spaces $W^1_p(\Omega^\pm)$ (see Remarks 12.2 and 12.7).

We illustrate this alternative approach for the problem $(\mathcal{P}_1)^\pm_\tau$. We look for a solution in the form of a single layer potential (12.7) where the density h is to be found in the appropriate space $B^{-1/p}_{p,p}(S)$. We recall that $V_\tau : B^{-1/p}_{p,p}(S) \to W^1_p(\Omega^\pm)$ (see Theorem 11.3). Taking into account the boundary conditions (5.1)–(5.2) and applying the trace properties of a single layer potential, we arrive at the elliptic BIE (elliptic ΨDE of order -1)

$$\mathcal{H}_\tau\, h = G^{(1)}, \tag{12.51}$$

where

$$G^{(1)} := f = (f_1, \cdots, f_4)^\top \in B^{1-1/p}_{p,p}(S). \tag{12.52}$$

By the same arguments as above we can easily show that the mapping

$$-\mathcal{H}_\tau \,:\, B^s_{p,q}(S) \to B^{s+1}_{p,q}(S), \tag{12.53}$$

where $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$, is an isomorphism.

Therefore, there exists the unique solution $h \in B^{-1/p}_{p,p}(S)$ of the equation (12.51) with the right-hand side (12.52). Further, invoking Theorem 8.2 it can be established that the single layer potential $U(x) = V_\tau(h)(x)$ represents the unique solution to the problems $(\mathcal{P}_1)^\pm_\tau$ in the space $W^1_p(\Omega^\pm)$.

We note that the elliptic ΨDO of order $+1$

$$-\mathcal{H}_\tau^{-1} \,:\, B^{s+1}_{p,q}(S) \to B^s_{p,q}(S) \tag{12.54}$$

is a singular integro-differential operator with a positive definite principal homogeneous symbol matix. Here $\mathcal{H}_\tau^{-1}$ stands for the inverse of $\mathcal{H}_\tau$, and $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$.

A ready consequence of the above results is that every solution $U \in W^1_p(\Omega^\pm)$, $1 < p < \infty$, of the homogeneous equation (1.9) can be uniquely represented in the form of the single layer potential

$$U(x) = V_\tau(\mathcal{H}_\tau^{-1}[U]^\pm)(x), \quad x \in \Omega^\pm, \tag{12.55}$$

where $[U]^\pm$ are the traces of the solution U on S from $\Omega^\pm$.

13. Basic exterior BVPs of Steady State Oscillations

In this section we shall investigate the basic exterior BVPs for steady state oscillation equations of thermoelasticity theory. In what follows we provide that $r = 1$ for $\omega > 0$ and $r = 2$ for $\omega < 0$.

13.1. First we present the following lemma which will essentially be used below in the proof of existence theorems.

Lemma 13.1. *Let* $g \in \mathrm{C}^{1,\alpha}(S)$, $S \in \mathrm{C}^{2,\alpha'}$, *and*

$$U(x) = W(g)(x) + p_0\, V(g)(x), \quad x \in \mathbb{R}^3 \setminus S,\ S = \partial\Omega^{\pm}, \tag{13.1}$$

$$p_0 = p_1 + ip_2, \quad p_1 \geq 0,\ p_2\,\omega < 0, \tag{13.2}$$

where V *and* W *are single and double layer potentials defined by* (10.1) *and* (10.2), *respectively, while* ω *is the frequency parameter.*

If the vector U *vanishes in* Ω^-, *then the density* $g = 0$ *on* S.

Proof. Due to Lemmata 10.1 and 10.7 we have

$$\begin{aligned} g = [U]^+ - [U]^- = [U]^+, \\ -p_0\, g = [B(D,n)U]^+ - [B(D,n)U]^- = [B(D,n)U]^+, \end{aligned} \tag{13.3}$$

whence

$$[B(D,n)U]^+ = -p_0\,[U]^+ \quad \text{on} \quad S \tag{13.4}$$

follows.

Since U is a regular vector in Ω^+ we can apply the identity (1.23). Taking into account (13.4) and separating the imaginary part, we arrive at the equation

$$\tfrac{1}{\omega T_0} \int_{\Omega^+} \lambda_{kj}\, D_k u_4\, D_j \overline{u}_4\, dx - p_2 \int_S |[u]^+|^2\, dS + \tfrac{p_1}{\omega T_0} \int\limits_S |[u_4]^+|^2\, dS = 0.$$

In view of (1.18), (13.2), and (13.4) from this equality it follows that $[U]^+ = 0$ and by (13.3) we get $g = 0$. □

In the sequel we fix the complex number p_0 as follows

$$p_0 = 1 - i\omega. \tag{13.5}$$

Remark 13.2. In what follows we shall use the representation (13.1) to prove the existence of solutions to the exterior BVPs for the steady state oscillation equations of the thermoelasticity theory. The similar representation for the Helmholtz equation has been first applied in the papers [6], [64], [46]. This type of representation of solutions proved to be very useful since it reduces the exterior BVPs to the uniquely solvable BIEs for arbitrary values of the frequency parameter ω (for details see below).

Remark 13.3. In contrast to the pseudo-oscillation case the classical single layer or double layer approach reduces the exterior BVPs of steady state oscillations to the BIEs which for a countable set of the so-called exceptional values of the frequency parameter ω are not solvable for arbitrary boundary data (see [83], [45], [10], [11]). To investigate the solvability of these BIEs one needs to find explicitly all eigenvalues and eigenfunctions of the corresponding boundary integral operators and their adjoint ones (for details see [83], [45]).

13.2. We start with the problem $(\mathcal{P}_1)^-_\omega$. We look for a solution of the problem in the form (13.1) with p_0 defined by (13.5). By virtue of the boundary conditions (5.1)–(5.2) and Lemma 10.1, we get the following ΨDE on S for the unknown density vector g

$$\mathcal{N}_1^- \, g := (-2^{-1} I_4 + \mathcal{K}_2 + p_0 \, \mathcal{H}) \, g \; = \; G^{(1)} \tag{13.6}$$

with $G^{(1)} = (f_1, \ldots, f_4)^\top \in \mathrm{C}^{k,\alpha}(S)$.

Lemma 13.4. *Let*

$$S \in \mathrm{C}^{k+1,\alpha'} \quad \text{with integer} \;\; k \geq 1 \;\; \text{and} \;\; 0 < \alpha < \alpha' \leq 1. \tag{13.7}$$

Then the Ψ*DE* (13.6) *is an elliptic SIO with index zero, while the mapping*

$$\mathcal{N}_1^- := -2^{-1} I_4 + \mathcal{K}_2 + p_0 \, \mathcal{H} \, : \, \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l,\alpha}(S), \;\; 0 \leq l \leq k, \tag{13.8}$$

is an isomorphism.

Proof. First let us note that the operator $\mathcal{N}_1^-$ is an elliptic singular integral operator with index equal to zero and possesses the mapping property (13.8) due to Lemmata 10.1 and 10.2. Therefore, it remains to prove that

$$\mathcal{N}_1^- \, g = 0 \tag{13.9}$$

has only the trivial solution in $\mathrm{C}^{l,\alpha}(S)$.

Let g be some solution of (13.9) and construct the vector U by formula (13.1). Applying the embedding theorems for solutions to a singular integral equation of normal type on closed smooth manifold we infer that $g \in \mathrm{C}^{k,\alpha}(S)$ (see, e.g., [45], Ch. 4). This implies that U is a regular vector in $\Omega^\pm$. Now the equation (13.9) yields that $[U]^- = 0$ on S, and, consequently, $U(x) = 0$ in Ω^- follows immediately by Theorem 9.5, since $U \in \mathrm{SK}^m_r(\Omega^-)$. Then $g = 0$ by Lemma 13.1. Therefore (13.8) is a one-to-one correspondence and $\mathcal{N}_1^-$ is invertible. □

The material collected until now is enough to prove the existence theorem.

Theorem 13.5. *Let S, k, α', and α be as in* (13.7) *and let $f_j \in \mathrm{C}^{k,\alpha}(S)$* $(j = 1, \ldots, 4)$. *Then Problem $(\mathcal{P}_1)^-_\omega$ has a unique regular solution of the class $\mathrm{C}^{k,\alpha}(\overline{\Omega^-}) \cap \mathrm{SK}^m_r(\Omega^-)$ and the solution is representable in the form* (13.1) *with the density $g \in \mathrm{C}^{k,\alpha}(S)$ defined by the uniquely solvable SIE* (13.6).

Proof. It follows from Lemmata 10.1, 13.4, and Theorem 9.5. □

Remark 13.6. We note that the special representation (13.1) reduces the BVP $(\mathcal{P}_1)^-_\omega$ to the equivalent boundary integral equation (13.6) for an arbitrary value of the frequency parameter ω. If one seeks the solution in the form of either single or double layer potential then such equivalence will be, in general, violated (see Remark 13.3).

13.3. We look for a regular solution to the problem $(\mathcal{P}_2)^-_\omega$ again in the form (13.1). Then the boundary conditions (5.3) and (5.4) lead to the following system of ΨDEs on S for the unknown density g

$$\mathcal{N}_2^-\, g := \{B_{(2)}(D,n)[W(g)+p_0\,V(g)]\}^- = G^{(2)},\ \ G^{(2)} = (f_1,f_2,f_3,F_4)^\top,$$

i.e.,

$$\{[-2^{-1}I_4 + \mathcal{K}_2 + p_0\mathcal{H}]\, g\}_q = f_q, \quad q=1,2,3, \tag{13.10}$$

$$\{[\mathcal{L} + p_0\,(2^{-1}I_4 + \mathcal{K}_1)]\, g\}_4 = F_4, \tag{13.11}$$

where (13.12)

$$f_q \in \mathrm{C}^{k,\alpha}(S), \quad F_4 \in \mathrm{C}^{k-1,\alpha}(S), \quad q=1,2,3. \tag{13.12}$$

Therefore, the operator $\mathcal{N}_2^-$ is represented as

$$\mathcal{N}_2^- = \begin{bmatrix} [\{-2^{-1}I_4 + \mathcal{K}_2 + p_0\mathcal{H}\}_{ql}]_{3\times 4} \\ [\{\mathcal{L} + p_0\,(2^{-1}I_4 + \mathcal{K}_1)\}_{4l}]_{1\times 4} \end{bmatrix}_{4\times 4} = (\mathcal{N}_2^-)_0 + \widetilde{\mathcal{N}}_2^-, \tag{13.13}$$

$$q=1,2,3,\ \ l=1,\dots,4,$$

where $(\mathcal{N}_2^-)_0$ is the dominant singular part of $\mathcal{N}_2^-$. Due to (10.25), (10.48), and Lemma 10.1 we have

$$(\mathcal{N}_2^-)_0 = \begin{bmatrix} [-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(0)} \end{bmatrix}_{4\times 4}. \tag{13.14}$$

The entries of the first three rows of the matrix $\widetilde{\mathcal{N}}_2^-$ are weakly singular integral operators (ΨDOs of order $s \le -1$), while the fourth row contains singular integral operators (ΨDOs of order $s \le 0$). It is easy to see that (13.14) is a ΨDO elliptic in the sense of Douglis-Nirenberg.

Now it is also evident that the operator $\mathcal{R}_2$, defined by (12.21), is an equivalent lifting operator which reduces the system (13.10)-(13.11) to the equivalent system of singular integral equations

$$\mathcal{R}_2\mathcal{N}_2^-\, g = G_*^{(2)},\ \ G_*^{(2)} = (f_1,f_2,f_3,\mathcal{R}F_4)^\top.$$

For the principal homogeneous symbol matrix we have

$$\sigma(\mathcal{R}_2\mathcal{N}_2^-) = \begin{bmatrix} [\sigma(-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(\mathcal{R}\mathcal{L}_4^{(0)}) \end{bmatrix}_{4\times 4},$$

which is nonsingular due to Lemmata 10.2, 10.7, and 12.3.

Lemma 13.7. *Let conditions* (13.7) *be fulfilled. Then the* Ψ*DO*

$$\mathcal{N}_2^- \;:\; [\mathrm{C}^{l,\alpha}(S)]^4 \to [\mathrm{C}^{l,\alpha}(S)]^3 \times \mathrm{C}^{l-1,\alpha}(S),\ \ 1 \le l \le k, \tag{13.15}$$

is an isomorphism.

Proof. The mapping property (13.15) of the operator $\mathcal{N}_2^-$ is an easy consequence of Lemmata 10.1 and 10.7. Clearly, the invertibility of the operator (13.15) is equivalent to the invertibility of the operator

$$\mathcal{R}_2\mathcal{N}_2^- \;:\; [\mathrm{C}^{l,\alpha}(S)]^4 \to [\mathrm{C}^{l,\alpha}(S)]^4,\ \ 0 \le l \le k, \tag{13.16}$$

according to Lemma 12.3.

Now from Lemmata 10.2, 10.7, and 12.3 it follows that $\mathcal{R}_2\mathcal{N}_2^-$ is an elliptic singular integral operator with index zero. By the arguments applied in the proof of Lemma 13.4 we can show that the homogeneous equation $\mathcal{N}_2^- g = 0$, where $g \in \mathrm{C}^{l,\alpha}(S)$, has only the trivial solution $g = 0$. Further, by Lemma 12.3 we conclude that the null-space of the operator $\mathcal{R}_2\mathcal{N}_2^-$ in $\mathrm{C}^{l,\alpha}(S)$ is trivial, which completes the proof. □

Theorem 13.8. *Let conditions* (13.7) *and* (13.12) *be fulfilled. Then the problem* $(\mathcal{P}_2)_\omega^-$ *has a unique regular solution of the class* $\mathrm{C}^{k,\alpha}(\overline{\Omega^-}) \cap \mathrm{SK}_r^m(\Omega^-)$ *and the solution is representable in the form* (13.1) *with the density* $g \in \mathrm{C}^{k,\alpha}(S)$ *defined by the uniquely solvable* Ψ*DEs* (13.10)–(13.11).

Proof. It is a ready consequence of Lemmata 10.1, 13.7 and Theorem 9.5. □

13.4. Here we consider the problem $(\mathcal{P}_3)_\omega^-$. Applying again the same representation formula (13.1) and taking into account the boundary conditions (5.7) and (5.8), we arrive at the following system of ΨDEs for the unknown density g on S:

$$\mathcal{N}_3^-\, g := \{B_{(3)}(D,n)[W(g) + p_0\, V(g)]\}^- = G^{(3)}, \;\; G^{(3)} = (F_1, F_2, F_3, f_4)^\top,$$

i.e.,

$$\{[\mathcal{L} + p_0(2^{-1}I_4 + \mathcal{K}_1)]\, g\}_q = F_q, \quad q = 1,2,3, \tag{13.17}$$

$$\{[-2^{-1}I_4 + \mathcal{K}_2 + p_0\mathcal{H}]\, g\}_4 = f_4, \tag{13.18}$$

where

$$F_q \in \mathrm{C}^{k-1,\alpha}(S), \quad f_4 \in \mathrm{C}^{k,\alpha}(S), \quad q = 1,2,3. \tag{13.19}$$

Clearly, $\mathcal{N}_3^-$ is representable in the form

$$\mathcal{N}_3^- = \begin{bmatrix} [\{\mathcal{L} + p_0(2^{-1}I_4 + \mathcal{K}_1)\}_{ql}]_{3\times 4} \\ [\{-2^{-1}I_4 + \mathcal{K}_2) + p_0\mathcal{H}\}_{4l}]_{1\times 4} \end{bmatrix}_{4\times 4} = (\mathcal{N}_3^-)_0 + \widetilde{\mathcal{N}}_3^-, \tag{13.20}$$

$$q = 1,2,3, \;\; l = 1,\dots,4,$$

where

$$(\mathcal{N}_3^-)_0 = \begin{bmatrix} [\mathcal{L}^{(0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & -2^{-1}I_1 + \overset{*}{\mathcal{K}}{}_4^{(0)} \end{bmatrix}_{4\times 4}$$

is the dominant singular part of $\mathcal{N}_3^-$ due to (10.25) and (10.48); the operator $\widetilde{\mathcal{N}}_3^-$ contains ΨDOs of order $s \le 0$ in the first three rows and ΨDOs of order $s \le -1$ in the fourth row. Obviously, $\mathcal{N}_3^-$ is again an elliptic ΨDO in the sense of Douglis-Nirenberg.

The diagonal operator $\mathcal{R}_3$, defined by (12.29), is an equivalent lifting operator which reduces (13.17)-(13.18) to the equivalent system of singular integral equations

$$\mathcal{R}_3\mathcal{N}_3^-\, g = G_*^{(3)}, \;\; G_*^{(3)} = (\mathcal{R}F_1, \mathcal{R}F_2, \mathcal{R}F_3, f_4)^\top.$$

The principal homogeneous symbol matrix of the operator $\mathcal{R}_3\mathcal{N}_3^-$ reads

$$\sigma(\mathcal{R}_3\mathcal{N}_3^-) = \begin{bmatrix} [\sigma(\mathcal{R}\mathcal{L}^{(0)})]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(-2^{-1}I_1 + \overset{*}{\mathcal{K}}{}_4^{(0)}) \end{bmatrix}_{4\times 4}$$

and is nonsingular according to the results of Section 10.

Now in the same way as in the previous subsection we can prove the following assertions.

Lemma 13.9. *Let the conditions* (13.7) *be fulfilled. Then the* Ψ*DO*

$$\mathcal{N}_3^- \; : \; [\mathrm{C}^{l,\alpha}(S)]^4 \to [\mathrm{C}^{l-1,\alpha}(S)]^3 \times \mathrm{C}^{l,\alpha}(S), \;\; 1 \le l \le k,$$

is an isomorphism.

Theorem 13.10. *Let the conditions* (13.7) *and* (13.19) *be fulfilled. Then the problem* $(\mathcal{P}_3)_\omega^-$ *has a unique regular solution of the class* $\mathrm{C}^{k,\alpha}(\overline{\Omega^-}) \cap \mathrm{SK}_r^m(\Omega^-)$ *and the solution is representable in the form* (13.1) *with the density* $g \in \mathrm{C}^{k,\alpha}(S)$ *defined by the uniquely solvable* Ψ*DEs* (13.17)–(13.18).

13.5. The representation (13.1) of a regular solution and the boundary conditions (5.7), (5.8) reduce the BVP $(\mathcal{P}_4)_\omega^-$ to the system of ΨDEs on S

$$\mathcal{N}_4^- \, g := [\mathcal{L} + p_0(2^{-1}I_4 + \mathcal{K}_1)] \, g = G^{(4)}, \;\; G^{(4)} = (F_1, \dots, F_4)^\top. \quad (13.21)$$

For the dominant singular part we have the following elliptic ΨDO (of order 1) $(\mathcal{N}_4^-)_0 = (\mathcal{L})_0$, where $(\mathcal{L})_0$ is given by (10.48). It is easy to check that the diagonal operator $\mathcal{R}_4 = I_4\mathcal{R}$ with $\mathcal{R}$ defined by (12.17), is a lifting operator, which reduces equivalently the equations (13.21) to the following elliptic system of singular integral equations with index equal to zero

$$\mathcal{R}_4\mathcal{N}_4^- \, g = G_*^{(4)}, \quad G_*^{(4)} = (\mathcal{R}F_1, \dots, \mathcal{R}F_4)^\top.$$

The proofs of the next lemma and theorem are quite similar to the proofs of Lemma 13.4 and Theorem 13.5.

Lemma 13.11. *Let the conditions* (13.7) *be fulfilled. Then the* Ψ*DO*

$$\mathcal{N}_4^- \; : \; \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l-1,\alpha}(S), \;\; 1 \le l \le k,$$

is an isomorphism.

Theorem 13.12. *Let the conditions* (13.7) *be fulfilled and* $F_j \in \mathrm{C}^{k-1,\alpha}(S)$, $j = \overline{1,4}$. *Then the problem* $(\mathcal{P}_4)_\omega^-$ *has a unique regular solution of the class* $\mathrm{C}^{k,\alpha}(\overline{\Omega^-}) \cap \mathrm{SK}_r^m(\Omega^-)$ *and the solution is representable in the form* (13.1) *with the density* $g \in \mathrm{C}^{k,\alpha}(S)$ *defined by the uniquely solvable* Ψ*DE* (13.21).

13.5. In this subsection we consider the problems $(\mathcal{P}_n)_\omega^-$ $(n = \overline{1,4})$ in the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega^-)$. The corresponding existence theorems can be proved with the help of the following lemma (cf. Lemmata 12.9 and 12.11).

Lemma 13.13. *Let* S *be a* C^∞*-regular surface and let* $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$. *Then the mappings*

$$\mathcal{N}_1^- \quad : \quad [B^s_{p,q}(S)]^4 \to [B^s_{p,q}(S)]^4$$

$$
\begin{aligned}
\mathcal{N}_2^- &: [B^s_{p,q}(S)]^4 \to [B^s_{p,q}(S)]^3 \times B^{s-1}_{p,q}(S),\\
\mathcal{N}_3^- &: [B^s_{p,q}(S)]^4 \to [B^{s-1}_{p,q}(S)]^3 \times B^{s}_{p,q}(S),\\
\mathcal{N}_4^- &: [B^s_{p,q}(S)]^4 \to [B^{s-1}_{p,q}(S)]^4
\end{aligned}
$$

are isomorphisms.

Here the Ψ*DOs* $\mathcal{N}_1^-$, $\mathcal{N}_2^-$, $\mathcal{N}_3^-$,*and* $\mathcal{N}_4^-$ *are given by formulae* (13.8), (13.13), (13.20), *and* (13.21), *respectively.*

Proof. The mapping properties indicated in the lemma follow from Theorem 10.8. The operators $\mathcal{N}_n^-$ $(n=\overline{1,4})$ have zero indices since $\mathcal{N}_n^- - \mathcal{N}_{n,\tau}^-$ are compact operators in the corresponding functional spaces due to the results of Section 2 and since $\mathrm{ind}\mathcal{N}_{n,\tau}^- = 0$ $(n=\overline{1,4})$ (see Lemmata 12.9 and 12.11). Here the operators $\mathcal{N}_{n,\tau}^-$ are the same as in Section 12.

It remains to prove that $\ker\mathcal{N}_n^-$ is trivial. To see this, let us consider the homogeneous equations $\mathcal{N}_n^-\, g = 0$ which are equivalent to the SIEs $\mathcal{R}_n\mathcal{N}_n^-\, g = 0$, where $\mathcal{R}_n$ $(n=\overline{2,4})$ are the same invertible lifting operators as in Section 12, $\mathbb{R}_1 = I_4$, and $g \in B^s_{p,q}(S)$. Bearing in mind that $\mathcal{R}_n\mathcal{N}_n^-$ $(n=\overline{1,4})$ are elliptic SIOs on the closed smooth manifold S we infer that any solution $g \in L_2(S)$ to the above SIEs, actually, belongs to the space $\mathrm{C}^{1,\alpha}(S)$ due to the embedding theorems. Moreover, by the above mentioned equivalence we get $\mathcal{N}_n^-\, g = 0$. These relations imply that the linear combination of the double and single layer potentials $W(g)(x) + p_0\, V(g)(x)$ constructed by the density $g \in \mathrm{C}^{1,\alpha}(S)$ and p_0 given by (13.5), belong to the class $\mathrm{C}^{1,\alpha}(\overline{\Omega^-}) \cap \mathrm{SK}^m_r(\Omega^-)$ and solves the homogeneous exterior BVP $(\mathcal{P}_n)^-_\omega$. By the uniqueness theorems (see Section 9) $W(g)(x) + p_0\, V(g)(x) = 0$ in Ω^- whence $g = 0$ on S follows by Lemma 13.1. Thus, $\ker\mathcal{R}_n\mathcal{N}_n^-$ is trivial in the space $L_2(S)$. It is then trivial also in the space $B^s_{p,q}(S)$ for arbitrary $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$ (see the reasonings in the proof of Lemma 12.9). Terefore, $\ker\mathcal{R}_n\mathcal{N}_n^- = \{0\}$ again due to the invertibility of the operator $\mathcal{R}_n$ $(n=\overline{1,4})$ which completes the proof. $\square$

This lemma implies the following existence results.

Theorem 13.14. *Let* $1 < p < \infty$ *and the boundary data in* (5.1)–(5.8) *satisfy the conditions*

$$
f_j \in B^{1-1/p}_{p,p}(S), \quad F_j \in B^{-1/p}_{p,p}(S), \quad j = \overline{1,4}.
$$

Then the BVP $(\mathcal{P}_n)^-_\omega$ $(n=\overline{1,4})$ *are uniquely solvable in the class* $W^1_{p,\mathrm{loc}}(\Omega^-)$ $\cap\, \mathrm{SK}^m_r(\Omega^-)$ *and the solutions are representable in the form* (13.1)*, where the density* $g \in B^{1-1/p}_{p,p}(S)$ *solves the corresponding* Ψ*DE on* S

$$
\mathcal{N}_n^-\, g = G^{(n)}, \quad n = \overline{1,4}.
$$

Here $G^{(n)}$ *are the vectors given by* (12.41)*.*

Proof. It is quite similar to the proof of Theorems 12.10 and 12.12. Indeed, the solvability of the BVPs indicated in the theorem follows from Lemma 13.13. To prove the uniqueness of solutions in the class $W^1_{p,\mathrm{loc}}(\Omega^-) \cap$ $\mathrm{SK}^m_r(\Omega^-)$, we can again apply the general integral representation formula (see Theorem 10.8, item ii)) and show that all solutions to the homogeneous BVPs $(\mathcal{P}_n)^-_\omega$ of this class, actually, belong to the class of regular vector functions $\mathrm{C}^1(\overline{\Omega^-}) \cap \mathrm{SK}^m_r(\Omega^-)$ due to the ellipticity of the corresponding ΨDEs on closed smooth surface S. This completes the proof. $\square$

14. Basic Interface Problems of Pseudo-Oscillations

In this section we shall construct an "explicit" solution to the basic nonhomogeneous interface problem $(\mathcal{C})_\tau$ which will essentially be employed afterwards in the study of the other regular and mixed interface problems.

14.1. Let us consider the problem $(\mathcal{C})_\tau$, i.e., we look for four-dimensional vector functions $U^{(1)} = (u^{(1)}, u_4^{(1)})^\top \in \mathrm{C}^1(\overline{\Omega^1})$ and $U^{(2)} = (u^{(2)}, u_4^{(2)})^\top \in \mathrm{C}^1(\overline{\Omega^2})$ which are solutions of the pseudo-oscillation equations

$$A^{(1)}(D,\tau)U^{(1)}(x) = 0 \quad \text{in} \quad \Omega^1, \tag{14.1}$$

$$A^{(2)}(D,\tau)U^{(2)}(x) = 0 \quad \text{in} \quad \Omega^2, \tag{14.2}$$

and satisfy the transmission conditions on the interface S

$$[u^{(1)}]^+ - [u^{(2)}]^- = \widetilde{f}, \quad [u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4, \tag{14.3}$$

$$\left.\begin{aligned} &[P^{(1)}(D,n)U^{(1)}]^+ - [P^{(2)}(D,n)U^{(2)}]^- = \widetilde{F}, \\ &[\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4, \end{aligned}\right\} \tag{14.4}$$

where $P^{(\mu)}(D,n)$ and $\lambda^{(\mu)}(D,n)$ are the thermostress and heat flux operators defined by (1.13) and (1.24), respectively. Here

$$\begin{gathered} S \in \mathrm{C}^{k+1,\alpha'}, \;\; f_j \in \mathrm{C}^{k,\alpha}(S), \;\; F_j \in \mathrm{C}^{k-1,\alpha}(S), \;\; j = \overline{1,4}, \\ f = (f_1, \dots, f_4)^\top, \quad F = (F_1, \dots, F_4)^\top, \end{gathered} \tag{14.5}$$

where as above $k \geq 1$ is an integer and $0 < \alpha < \alpha' \leq 1$.

Making use of the notation (1.25) the above transmission conditions can be written as follows

$$[U^{(1)}]^+ - [U^{(2)}]^- = f, \tag{14.6}$$

$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F. \tag{14.7}$$

We look for a solution to the problem $(\mathcal{C})_\tau$ in the form of single layer potentials

$$U^{(1)}(x) = V_\tau^{(1)}[(\mathcal{H}_\tau^{(1)})^{-1} g^{(1)}](x), \quad x \in \Omega^1, \tag{14.8}$$

$$U^{(2)}(x) = V_\tau^{(2)}[(\mathcal{H}_\tau^{(2)})^{-1} g^{(2)}](x), \quad x \in \Omega^2, \tag{14.9}$$

where $g^{(\mu)} = (\widetilde{g}^{(\mu)}, g_4^{(\mu)})^\top$, $\widetilde{g}^{(\mu)} = (g_1^{(\mu)}, g_2^{(\mu)}, g_3^{(\mu)})^\top$, $\mu = 1, 2$, are unknown densities and $(\mathcal{H}_\tau^{(\mu)})^{-1}$ is the operator inverse to $\mathcal{H}_\tau^{(\mu)}$ (see Remark 12.2). Here and in what follows the superscript μ ($\mu = 1, 2$) denotes that the corresponding operator is constructed by the thermoelastic characteristics of the elastic material occupying the domain Ω^μ.

Due to Theorem 11.1, the transmission conditions (14.3) and (14.4), i.e., (14.6) and (14.7), lead to the following system of boundary equations on S:

$$g^{(1)} - g^{(2)} = f, \tag{14.10}$$

$$(-2^{-1}I_4 + \mathcal{K}^{(1)}_{1,\tau})(\mathcal{H}^{(1)}_\tau)^{-1} g^{(1)} - (2^{-1}I_4 + \mathcal{K}^{(2)}_{1,\tau})(\mathcal{H}^{(2)}_\tau)^{-1} g^{(2)} = F, \tag{14.11}$$

where $\mathcal{K}^{(\mu)}_{1,\tau}$, $\mu = 1,2$, are defined by (11.4).

Let

$$\mathcal{N}_{1,\tau} = (-2^{-1}I_4 + \mathcal{K}^{(1)}_{1,\tau})(\mathcal{H}^{(1)}_\tau)^{-1}, \;\; \mathcal{N}_{2,\tau} = -(2^{-1}I_4 + \mathcal{K}^{(2)}_{1,\tau})(\mathcal{H}^{(2)}_\tau)^{-1},$$
$$\mathcal{N}_\tau = \mathcal{N}_{1,\tau} + \mathcal{N}_{2,\tau}. \tag{14.12}$$

Then equations (14.10) and (14.11) yield:

$$g^{(1)} = f + g^{(2)}, \tag{14.13}$$

$$\mathcal{N}_\tau\, g^{(2)} = F - \mathcal{N}_{1,\tau}\, f. \tag{14.14}$$

Now we will study properties of the boundary operators $\mathcal{N}_{1,\tau}, \mathcal{N}_{2,\tau}$, and $\mathcal{N}_\tau$.

Lemma 14.1. *Let S be as in* (14.5). *Then*

$$\mathcal{N}_\tau,\ \mathcal{N}_{j,\tau}\ :\ \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l-1,\alpha}(S), \quad j = 1,2, \quad 1 \le l \le k, \tag{14.15}$$

are bounded operators with the trivial null-spaces.

Operators $\mathcal{N}_\tau$, $\mathcal{N}_{j,\tau}$, $j = 1,2$, defined by (14.12) *and* (14.15), *are isomorphisms.*

Proof. The mapping property (14.15) is an easy consequence of Theorem 11.1, item ii), since the operator $(\mathcal{H}^{(\mu)}_\tau)^{-1}\ :\ \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l-1,\alpha}(S)$ is an isomorphism due to Remark 12.2.

From Remark 12.7 it follows also that the equations $\mathcal{N}_{j,\tau}\, h = 0$ $(j = 1,2)$ have only the trivial solutions. Therefore, the operators $\mathcal{N}_{j,\tau}$, $(j = 1,2)$ defined by (14.12), (14.15) are invertible and their inverses are bounded.

It remains to prove that the null-space of the operator $\mathcal{N}_\tau$ is trivial as well. Let $h = (h_1, \dots, h_4)^\top \in \mathrm{C}^{1,\alpha}(S)$ be an arbitrary solution of the equation $\mathcal{N}_\tau\, h = 0$, i.e., $\mathcal{N}_{1,\tau}\, h + \mathcal{N}_{2,\tau}\, h = 0$. Then it can be easily seen that the vectors $U^{(1)}(x) = V^{(1)}_\tau[(\mathcal{H}^{(1)}_\tau)^{-1}h](x)$, $x \in \Omega^1$ and $U^{(2)}(x) = V^{(2)}_\tau[(\mathcal{H}^{(2)}_\tau)^{-1}h](x)$, $x \in \Omega^2$, are regular and they solve the homogeneous problem $(\mathcal{C})_\tau$, since $[U^{(1)}]^+ = h$, $[U^{(2)}]^- = h$, and $[B^{(1)}U^{(1)}]^+ - [B^{(2)}U^{(2)}]^- = N_\tau\, h = 0$. Therefore, by Theorem 8.6 we have $U^{(1)} = 0$ in Ω^1 and $U^{(2)} = 0$ in Ω^2, whence $h = 0$ on S follows immediately. $\square$

Lemma 14.2. *The principal homogeneous symbol matrices of the operators $\mathcal{N}_{1\tau}$, $\mathcal{N}_{2,\tau}$, and $\mathcal{N}_\tau$ are positive definite.*

Proof. Here again $\sigma(\mathcal{K})(x,\xi)$ with $x \in S$ and $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$ denotes the principal homogeneous symbol of the pseudodifferential operator $\mathcal{K}$.

Equations (14.12) imply

$$\sigma(\mathcal{N}_\tau) = \sigma(\mathcal{N}_{1,\tau}) + \sigma(\mathcal{N}_{2,\tau}), \;\; \sigma(\mathcal{N}_{1,\tau}) = \sigma(-2^{-1}I_4 + \mathcal{K}^{(1)}_{1,\tau})\,[\sigma(\mathcal{H}^{(1)}_\tau)]^{-1},$$
$$\sigma(\mathcal{N}_{2,\tau}) = -\sigma(2^{-1}I_4 + \mathcal{K}^{(2)}_{1,\tau})\,[\sigma(\mathcal{H}^{(2)}_\tau)]^{-1}. \tag{14.16}$$

In the same way as in the proof of Lemma 10.2 we can easily show that $\sigma(\mathcal{H}_\tau^{(\mu)}) = \sigma((\mathcal{H}^{(\mu)})_0)$, $\sigma(\mathcal{K}_{1,\tau}^{(\mu)}) = \sigma((\mathcal{K}^{(\mu)})_0)$, where $(\mathcal{H}^{(\mu)})_0$ and $(\mathcal{K}^{(\mu)})_0$ are 4×4 matrix boundary operators on S:

$$(\mathcal{H}^{(\mu)})_0\, g(x) := \int_S \Gamma^{(\mu)}(x-y)\, g(y)\, dS_y,\ \ x\in S,$$
$$(\mathcal{K}^{(\mu)})_0\, g(x) := \int_S [B_0^{(\mu)}(D_x, n(x))\Gamma^{(\mu)}(x-y)]\, g(y)\, dS_y,\ \ x\in S,$$

with $g = (\widetilde{g}, g_4)^\top$ and $\widetilde{g} = (g_1, g_2, g_3)^\top$; here $\Gamma^{(\mu)}(x)$ is given by (2.8) and

$$B_0^{(\mu)}(D,n) = \begin{bmatrix} [T^{(\mu)}(D,n)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \lambda^{(\mu)}(D,n) \end{bmatrix}_{4\times 4}.$$

Therefore,

$$(\mathcal{H}^{(\mu)})_0 = \begin{bmatrix} [\mathcal{H}^{(\mu,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{H}_4^{(\mu,0)} \end{bmatrix}_{4\times 4}, \tag{14.17}$$
$$(\mathcal{K}^{(\mu)})_0 = \begin{bmatrix} [\mathcal{K}^{(\mu,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{K}_4^{(\mu,0)} \end{bmatrix}_{4\times 4}, \tag{14.18}$$

where $\mathcal{H}^{(\mu,0)}$, $\mathcal{K}^{(\mu,0)}$, and $\mathcal{H}_4^{(\mu,0)}$, $\mathcal{K}_4^{(\mu,0)}$ are 3×3 matrix and scalar operators, respectively, generated by the single layer potentials constructed by the fundamental matrix $\Gamma^{(\mu,0)}(x)$ and the fundamental function $\gamma^{(\mu,0)}(x)]$ (see (2.6), (2.7), (10.19)–(10.22), (10.26)):

$$\begin{aligned} \mathcal{H}^{(\mu,0)}\, \widetilde{g}(x) &= \int_S \Gamma^{(\mu,0)}(x-y)\, \widetilde{g}(y)\, dS_y, \\ \mathcal{H}_4^{(\mu,0)}\, g_4(x) &= \int_S \gamma^{(\mu,0)}(x-y)\, g_4(y)\, dS_y, \\ \mathcal{K}^{(\mu,0)}\, \widetilde{g}(x) &= \int_S [T^{(\mu)}(D_x, n(x))\Gamma^{(\mu,0)}(x-y)]\, \widetilde{g}(y)\, dS_y, \\ \mathcal{K}_4^{(\mu,0)}\, g_4(x) &= \int_S \lambda^{(\mu)}(D_x, n(x))\gamma^{(\mu,0)}(x-y)\, g_4(y)\, dS_y. \end{aligned} \tag{14.19}$$

Taking into account the structure of the matrices (14.17) and (14.18) we get from (14.16)

$$\sigma(\mathcal{N}_{1,\tau}) = \sigma\left(-2^{-1}I_4 + (\mathcal{K}^{(1)})_0\right)[\sigma((\mathcal{H}^{(1)})_0)]^{-1} = \tag{14.20}$$
$$= \begin{bmatrix} [\sigma(-2^{-1}I_3 + \mathcal{K}^{(1,0)})[\sigma(\mathcal{H}^{(1,0)})]^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(-2^{-1}I_1 + \mathcal{K}_4^{(1,0)})[\sigma(\mathcal{H}_4^{(1,0)})]^{-1} \end{bmatrix}_{4\times 4},$$
$$\sigma(\mathcal{N}_{2,\tau}) = -\sigma\left(2^{-1}I_4 + (\mathcal{K}^{(2)})_0\right)[\sigma((\mathcal{H}^{(2)})_0)]^{-1} = \tag{14.21}$$
$$= -\begin{bmatrix} [\sigma(2^{-1}I_3 + \mathcal{K}^{(2,0)})[\sigma(\mathcal{H}^{(2,0)})]^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(2^{-1}I_1 + \mathcal{K}_4^{(2,0)})[\sigma(\mathcal{H}_4^{(2,0)})]^{-1} \end{bmatrix}_{4\times 4}.$$

Next, let us note that the following Green formulae hold for regular solutions to the system of classical elastostatics $C^{(\mu)}(D)\, u^{(\mu)} = 0$ and to the elliptic scalar equation $\lambda_{kj}^{(\mu)} D_k D_j\, u_4^{(\mu)} = 0$ in Ω^μ:

$$\begin{aligned}
\int_{\Omega^1} E_0^{(1)}(u^{(1)}, u^{(1)})\, dx &= \int_{\partial\Omega^1} [u^{(1)}]^+ [T^{(1)}(D,n)u^{(1)}]^+\, dS,\\
\int_{\Omega^2} E_0^{(2)}(u^{(2)}, u^{(2)})\, dx &= -\int_{\partial\Omega^2} [u^{(2)}]^- [T^{(2)}(D,n)u^{(2)}]^-\, dS,\\
\int_{\Omega^1} \lambda_{kj}^{(1)} D_k u_4^{(1)}\, D_j u_4^{(1)}\, dx &= \int_{\partial\Omega^1} [u_4^{(1)}]^+ [\lambda^{(1)}(D,n) u_4^{(1)}]^+\, dS,\\
\int_{\Omega^2} \lambda_{kj}^{(2)} D_k u_4^{(2)}\, D_j u_4^{(2)}\, dx &= -\int_{\partial\Omega^2} [u_4^{(2)}]^- [\lambda^{(2)}(D,n) u_4^{(2)}]^-\, dS,
\end{aligned} \tag{14.22}$$

where $E_0^{(\mu)}(u^{(\mu)}, u^{(\mu)}) = c_{kjpq}^{(\mu)} D_k u_j^{(\mu)}\, D_j u_k^{(\mu)} \geq 0$ (see (1.15)), the classical stress operator $T^{(\mu)}(D,n)$ and the co-normal derivative (the heat flux operator) $\lambda^{(\mu)}(D,n)$ are given by (1.12) and (1.24), respectively; moreover, $u^{(2)} = o(1)$ and $u_4^{(2)} = o(1)$ at infinity.

Further, if we substitute in these formulae the corresponding single layer potentials $v^{(\mu,0)}$ and $v_4^{(\mu,0)}$ (see (10.19), (10.21)) with densities $(\mathcal{H}^{(\mu,0)})^{-1}\,\widetilde{g}$ and $(\mathcal{H}_4^{(\mu,0)})^{-1} g_4$, respectively, in the place of $u^{(\mu)}$ and $u_4^{(\mu)}$, we can show that $(-2^{-1}I_3 + \mathcal{K}^{(1,0)})(\mathcal{H}^{(1,0)})^{-1}$ and $-(2^{-1}I_3 + \mathcal{K}^{(2,0)})(\mathcal{H}^{(2,0)})^{-1}$ are non-negative 3×3 matrix pseudodifferential operators with positive definite principal symbol matrices, while $(-2^{-1}I_1 + \mathcal{K}_4^{(1,0)})(\mathcal{H}_4^{(1,0)})^{-1}$ and $-(2^{-1}I_1 + \mathcal{K}_4^{(2,0)})(\mathcal{H}_4^{(2,0)})^{-1}$ are non-negative scalar ΨDOs with positive principal symbol functions (here we note that the Fourier transform is unitary and that the principal symbol of the product of two operators is equal to the product of the principal symbols of these operators; for details see the proof of Lemma 4.2 in [41]).

Therefore, the equations (14.20) and (14.21) together with (14.16) yield that $\sigma(\mathcal{N}_{1,\tau})$, $\sigma(\mathcal{N}_{2,\tau})$, and $\sigma(\mathcal{N}_\tau)$ are positive definite matrices for arbitrary $x \in S$ and $\widetilde{\xi} \in \mathbb{R}^2\backslash\{0\}$. $\square$

Corollary 14.3. *Let S, k, α', and α be as in* (14.5). *Then the operator $\mathcal{N}_\tau^{-1}$, inverse to the operator $\mathcal{N}_\tau$ defined by* (14.15)*, is an isomorphism; consequently, $\mathcal{N}_\tau^{-1} : \mathrm{C}^{l-1,\alpha}(S) \to \mathrm{C}^{l,\alpha}(S)$, $1 \leq l \leq k$, is a bounded operator.*

Applying the above results we get from (14.13) and (14.14):

$$g^{(1)} = \mathcal{N}_\tau^{-1}(F + \mathcal{N}_{2,\tau}\, f), \quad g^{(2)} = \mathcal{N}_\tau^{-1}(F - \mathcal{N}_{1,\tau}\, f). \tag{14.23}$$

Clearly, $g^{(\mu)} \in \mathrm{C}^{k,\alpha}(S)$, $(\mu = 1,2)$ if conditions (14.5) are fulfilled. Now we are ready to formulate the following existence results.

Theorem 14.4. *Let S, k, α', α, f and F meet the conditions* (14.5).

Then the nonhomogeneous problem $(\mathcal{C})_\tau$ is uniquely solvable, and the solution is representable in the form of potentials

$$U^{(1)}(x) = V_\tau^{(1)}\Big[(\mathcal{H}_\tau^{(1)})^{-1}\mathcal{N}_\tau^{-1}(F + \mathcal{N}_{2,\tau}\, f)\Big](x), \quad x \in \Omega^1, \tag{14.24}$$

$$U^{(2)}(x) = V_\tau^{(2)}\Big[(\mathcal{H}_\tau^{(2)})^{-1}\mathcal{N}_\tau^{-1}(F - \mathcal{N}_{1,\tau}\, f)\Big](x), \quad x \in \Omega^2. \tag{14.25}$$

Moreover,

$$U^{(\mu)} \in \mathrm{C}^{k,\alpha}(\overline{\Omega^\mu}),\ \mu = 1,2, \tag{14.26}$$

and

$$||U^{(\mu)}||_{(\Omega^\mu,k,\alpha)} \le C_0 \;\big[\,||f||_{(S,k,\alpha)} + ||F||_{(S,k-1,\alpha)}\,\big]\,, \;\; C_0 = \mathrm{const} > 0, \quad (14.27)$$

where $||\cdot||_{(M,k,\alpha)}$ *denotes the norm in the space* $\mathrm{C}^{k,\alpha}(M)$.

Proof. It follows from (14.8), (14.9), (14.23), Corollary 14.3 and Remark 12.2. □

14.2. In this subsection we assume $S \in \mathrm{C}^\infty$, and establish the existence results for the problem $(\mathcal{C})_\tau$ in the weak setting with $1 < p < \infty$.

First we prove the following statement.

Lemma 14.5. *The operators* (14.15) *can be extended by continuity to the following bounded elliptic* ΨDOs (*of order* -1)

$$\mathcal{N}_\tau,\, \mathcal{N}_{j,\tau} \,:\, H^{s+1}_p(S) \to H^s_p(S) \;\; [B^{s+1}_{p,q}(S) \to B^s_{p,q}(S)] \qquad (14.28)$$

for arbitrary $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$. *Moreover, the operator* $\mathcal{N}_\tau$ *defined by* (14.28) *is invertible.*

Proof. The boundedness, ellipticity, and mapping properties (14.28) of the operators $\mathcal{N}_\tau$ and $\mathcal{N}_{j,\tau}$ easily follow from Theorem 11.3 and Lemma 14.2.

The invertibility of the operator $\mathcal{N}_\tau$ is a consequense of the embedding theorems for solutions of elliptic pseudodifferential equations on closed smooth manifold (see the proof of the analogous assertions in Section 12). In fact, any solution $h \in H^{s+1}_p(S)$ $[B^{s+1}_{p,q}(S)]$ of the homogeneous pseudodifferential equation $\mathcal{N}_\tau\, h = 0$, belongs also to the space $\mathrm{C}^{k,\alpha}(S)$, where $k \ge 1$ is an arbitrary integer and $0 < \alpha < 1$. Therefore, we can derive $h = 0$ on S, due to Corollary 14.3. Thus $\ker \mathcal{N}_\tau = \{0\}$. Moreover, $\mathrm{ind} \mathcal{N}_\tau = 0$, since the principal homogeneous symbol matrix of $\mathcal{N}_\tau$ is positive definite. These results imply the unique solvability of the nonhomogeneous equation $\mathcal{N}_\tau\, h = f$ in the spaces $H^{s+1}_p(S)$ $[B^{s+1}_{p,q}(S)]$ for the arbitrary right-hand side vector $f \in H^s_p(S)$ $[B^s_{p,q}(S)]$. □

Now we are able to prove the existence theorem.

Theorem 14.6. *Let*

$$S \in \mathrm{C}^\infty,\;\; f_j \in B^{1-1/p}_{p,p}(S),\;\; F \in B^{-1/p}_{p,p}(S),\;\; j = \overline{1,4},\;\; 1 < p < \infty. \quad (14.29)$$

Then the problem $(\mathcal{C})_\tau$ *is uniquely solvable in the space* $(W^1_p(\Omega^1), W^1_p(\Omega^2))$ *and the solution is representable by formulae* (14.24)–(14.25).

Proof. Let conditions (14.29) be fulfilled. Then Lemma 14.5 and Theorem 11.3 imply that the pair of vectors $(U^{(1)}, U^{(2)})$ defined by (14.24) and (14.25) represent a solution to the problem $(\mathcal{C})_\tau$ of the class $(W^1_p(\Omega^1), W^1_p(\Omega^2))$.

Next we show the uniqueness of solution to the problem $(\mathcal{C})_\tau$ in the Sobolev spaces $(W^1_p(\Omega^1), W^1_p(\Omega^2))$.

Let $(U^{(1)}, U^{(2)}) \in (W^1_p(\Omega^1), W^1_p(\Omega^2))$ be some solution to the homogeneous problem $(\mathcal{C})_\tau$. We recall that $U^{(\mu)} \in \mathrm{C}^\infty(\Omega^\mu)$. Then Theorem 11.3, item ii) yield

$$U^{(1)}(x) = W^{(1)}_\tau\left([U^{(1)}]^+\right)(x) - V^{(1)}_\tau\left([B^{(1)}(D,n)U^{(1)}]^+\right)(x),\;\; x \in \Omega^1, \quad (14.30)$$

$$U^{(2)}(x) = -W_\tau^{(2)}\Big([U^{(2)}]^-\Big)(x) + V_\tau^{(2)}\Big([B^{(2)}(D,n)U^{(2)}]^-\Big)(x),\ x \in \Omega^2, \quad (14.31)$$

where $[U^{(1)}]^+$, $[U^{(2)}]^- \in B_{p,p}^{1-1/p}(S)$, $[B^{(1)}(D,n)U^{(1)}]^+$, $[B^{(2)}(D,n)U^{(2)}]^- \in B_{p,p}^{-1/p}(S)$. The homogeneous transmission conditions read as (see (14.6), (14.7))

$$[U^{(1)}]^+ = [U^{(2)}]^-, \quad [B^{(1)}(D,n)U^{(1)}]^+ = [B^{(2)}(D,n)U^{(2)}]^-. \qquad (14.32)$$

Denote

$$[U^{(1)}]^+ =: g, \quad [B^{(1)}(D,n)U^{(1)}]^+ =: h. \qquad (14.33)$$

Then (14.32) along with (14.30), (14.31), and Theorem 11.3 implies that the vector functions h and g solve the homogeneous system of boundary ΨDEs:

$$-(\mathcal{H}_\tau^{(1)} + \mathcal{H}_\tau^{(2)})\, h + (\mathcal{K}_{2,\tau}^{(1)} + \mathcal{K}_{2,\tau}^{(2)})\, g = 0, \qquad (14.34)$$

$$-(\mathcal{K}_{1,\tau}^{(1)} + \mathcal{K}_{1,\tau}^{(2)})\, h + (\mathcal{L}_\tau^{(1)} + \mathcal{L}_\tau^{(2)})\, g = 0. \qquad (14.35)$$

From the positive definiteness of the principal symbol matrices $-\sigma(\mathcal{H}_\tau^{(\mu)})$, $\sigma(\mathcal{L}_\tau^{(\mu)})$ (see Theorem 11.2), and the equation $\sigma(\mathcal{K}_{2,\tau}^{(\mu)}) = \overline{[\sigma(\mathcal{K}_{1,\tau}^{(\mu)})]^\top}$, it follows that the system of ΨDEs (14.34) and (14.35) is strongly elliptic in the sense of Douglis-Nirenberg. Therefore, by the embedding theorems we conclude that h and g are smooth vector functions on S, i.e. $h \in \mathrm{C}^{k-1,\alpha}(S)$ and $g \in \mathrm{C}^{k,\alpha}(S)$ for any $k \geq 1$ and $0 < \alpha < 1$. But then the vectors $U^{(\mu)}$, $\mu = 1, 2$, given by (14.30) and (14.31), are regular due to the formulae (14.32), (14.33), and Theorem 11.1. Now the conditions (14.32) and Theorem 8.6 complete the proof. □

Remark 14.7. Using the representation formulae (14.30) and (14.31) we can solve the problem $(\mathcal{C})_\tau$ by the so-called direct boundary integral equation method. This method reduces the transmission problem in question to the strongly elliptic (in the sense of Douglis-Nirenberg) system of ΨDEs on S

$$G_\tau\, \psi = Q, \qquad (14.36)$$

where $\psi = (\psi', \psi'')^\top$ is the unknown vector with $\psi' = [B^{(1)}(D,n)U^{(1)}]^+$ and $\psi'' = [U^{(1)}]^+$; the matrix operator G_τ is given by formula

$$G_\tau = \begin{bmatrix} [-\mathcal{H}_\tau^{(1)} - \mathcal{H}_\tau^{(2)}]_{4\times 4} & [\mathcal{K}_{2,\tau}^{(1)} + \mathcal{K}_{2,\tau}^{(2)}]_{4\times 4} \\ [-\mathcal{K}_{1,\tau}^{(1)} - \mathcal{K}_{1,\tau}^{(2)}]_{4\times 4} & [\mathcal{L}_\tau^{(1)} + \mathcal{L}_\tau^{(2)}]_{4\times 4} \end{bmatrix}_{8\times 8},$$

while the given on S right hand-side 8-vector Q reads as

$$Q = \Big((2^{-1}I_4 + \mathcal{K}_{2,\tau}^{(2)})\, f - \mathcal{H}_\tau^{(2)}\, F\,,\, \mathcal{L}_\tau^{(2)}\, f + (2^{-1}I_4 - \mathcal{K}_{1,\tau}^{(2)})\, F\Big)^\top.$$

Actually, in the proof of Theorem 14.6 we have shown that the operators

$$\begin{aligned} G_\tau \ &: \ [\mathrm{C}^{k-1,\alpha}(S)]^4 \times [\mathrm{C}^{k,\alpha}(S)]^4 \to [\mathrm{C}^{k,\alpha}(S)]^4 \times [\mathrm{C}^{k-1,\alpha}(S)]^4 \\ &: \ [H_p^s(S)]^4 \times [H_p^{s+1}(S)]^4 \to [H_p^{s+1}(S)]^4 \times [H_p^s(S)]^4 \\ &: \ [B_{p,q}^s(S)]^4 \times [B_{p,q}^{s+1}(S)]^4 \to [B_{p,q}^{s+1}(S)]^4 \times [B_{p,q}^s(S)]^4 \end{aligned}$$

are invertible.

Therefore, the unique solution to the problem $(\mathcal{C})_\tau$ can be represented also in the form

$$
\begin{aligned}
U^{(1)}(x) &= W_\tau^{(1)}(\psi'')(x) - V_\tau^{(1)}(\psi')(x),\\
U^{(2)}(x) &= -W_\tau^{(2)}(\psi'' - f)(x) + V_\tau^{(2)}(\psi' - F)(x),
\end{aligned}
\tag{14.37}
$$

where ψ solves the system of ΨDEs (14.36).

Note that the conclusions of Theorems 14.4 and 14.6 remain valid for the vectors defined by (14.37) if the conditions (14.5) and (14.29) are fulfilled.

14.3. In this subsection we investigate the problem $(\mathcal{G})_\tau$.

First let us rewrite the transmission conditions (7.5)–(7.8) in the following equivalent form

$$[P^{(1)}(D,n)U^{(1)} \cdot l]^+ + [P^{(2)}(D,n)U^{(2)} \cdot l]^- = \widetilde{F}_l^{(+)} + \widetilde{F}_l^{(-)}, \tag{14.38}$$

$$[P^{(1)}(D,n)U^{(1)} \cdot m]^+ + [P^{(2)}(D,n)U^{(2)} \cdot m]^- = \widetilde{F}_m^{(+)} + \widetilde{F}_m^{(-)}, \tag{14.39}$$

$$[P^{(1)}(D,n)U^{(1)} \cdot l]^+ - [P^{(2)}(D,n)U^{(2)} \cdot l]^- = \widetilde{F}_l^{(+)} - \widetilde{F}_l^{(-)}, \tag{14.40}$$

$$[P^{(1)}(D,n)U^{(1)} \cdot m]^+ - [P^{(2)}(D,n)U^{(2)} \cdot m]^- = \widetilde{F}_m^{(+)} - \widetilde{F}_m^{(-)}, \tag{14.41}$$

$$[u^{(1)} \cdot n]^+ - [u^{(2)} \cdot n]^- = \widetilde{f}_n, \tag{14.42}$$

$$[P^{(1)}(D,n)U^{(1)} \cdot n]^+ - [P^{(2)}(D,n)U^{(2)} \cdot n]^- = \widetilde{F}_n, \tag{14.43}$$

$$[u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4, \quad [\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4. \tag{14.44}$$

Clearly, due to (14.40), (14.41), (14.43), and (14.44), the vector $[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F$ is a given vector on S with

$$F = \Big((\widetilde{F}_l^{(+)} - \widetilde{F}_l^{(-)})\, l + (\widetilde{F}_m^{(+)} - \widetilde{F}_m^{(-)})\, m + \widetilde{F}_n\, n,\ F_4 \Big)^\top. \tag{14.45}$$

Denote

$$[u^{(1)} \cdot l]^+ - [u^{(2)} \cdot l]^- = \psi_1, \quad [u^{(1)} \cdot m]^+ - [u^{(2)} \cdot m]^- = \psi_2, \tag{14.46}$$

where ψ_1 and ψ_2 are the unknown scalar functions. Equations (14.42), (14.44), and (14.46) imply $[U^{(1)}]^+ - [U^{(2)}]^- = f$, where

$$f = (\psi_1 l + \psi_2 m + \widetilde{f}_n n, f_4)^\top. \tag{14.47}$$

Now let us look for a solution to the problem $(\mathcal{G})_\tau$ in the form (14.24) and (14.25), where F and f are given by (14.45) and (14.47), respectively. Then from the results of the previous subsection it follows that the transmission conditions (14.40)-(14.44) are automatically satisfied. It remains to satisfy only the conditions (14.38) and (14.39). Taking into account Theorem 11.1 and the equations (14.12), we get from (14.24) and (14.25):

$$
\begin{aligned}
[B^{(1)}(D,n)U^{(1)}]^+ &= [(P^{(1)}(D,n)U^{(1)}, \lambda^{(1)}(D,n)u_4)^\top]^+ =\\
&= \mathcal{N}_{1,\tau}\mathcal{N}_\tau^{-1}(F + \mathcal{N}_{2,\tau} f),\\
[B^{(2)}(D,n)U^{(2)}]^- &= [(P^{(2)}(D,n)U^{(2)}, \lambda^{(2)}(D,n)u_4)^\top]^- =\\
&= -\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}(F - \mathcal{N}_{1,\tau} f).
\end{aligned}
$$

Further, we put

$$l^* = [(l,0)^\top]_{4\times 1}, \quad m^* = [(m,0)^\top]_{4\times 1}, \quad n^* = [(n,0)^\top]_{4\times 1}, \tag{14.48}$$

where l, m and n are again the tangent and the normal vectors introduced in Subsection 7.2.

Conditions (14.38) and (14.39) then imply

$$\begin{aligned}
&[P^{(1)}(D,n)U^{(1)}\cdot l]^+ + [P^{(2)}(D,n)U^{(2)}\cdot l]^- = \\
&\quad = [B^{(1)}(D,n)U^{(1)}\cdot l^*]^+ + [B^{(2)}(D,n)U^{(2)}\cdot l^*]^- = \\
&\quad =(\mathcal{N}_{1,\tau}-\mathcal{N}_{2,\tau})\mathcal{N}_\tau^{-1}F\cdot l^* + 2\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau}f\cdot l^* = \widetilde{F}_l^{(+)} + \widetilde{F}_l^{(-)}, \\
&[P^{(1)}(D,n)U^{(1)}\cdot m]^+ + [P^{(2)}(D,n)U^{(2)}\cdot m]^- = [B^{(1)}(D,n)U^{(1)}\cdot m^*]^+ + \\
&\quad +[B^{(2)}(D,n)U^{(2)}\cdot m^*]^- = (\mathcal{N}_{1,\tau}-\mathcal{N}_{2,\tau})\mathcal{N}_\tau^{-1}F\cdot m^* + \\
&\quad +2\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau}f\cdot m^* = \widetilde{F}_m^{(+)} + \widetilde{F}_m^{(-)},
\end{aligned} \tag{14.49}$$

since $\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau} = \mathcal{N}_{1,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{2,\tau}$. By virtue of (14.47) from (14.49) we have the following system of ΨDEs for the unknown functions ψ_1 and ψ_2:

$$\sum_{k,j=1}^{3}[(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{kj}(\psi_1 l_j + \psi_2 m_j)]l_k = q_1, \tag{14.50}$$

$$\sum_{k,j=1}^{3}[(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{kj}(\psi_1 l_j + \psi_2 m_j)]m_k = q_2, \tag{14.51}$$

where

$$\begin{aligned}
&q_1 = 2^{-1}\{\widetilde{F}_l^{(+)} + \widetilde{F}_l^{(-)} - (\mathcal{N}_{1,\tau}-\mathcal{N}_{2,\tau})\mathcal{N}_\tau^{-1}F\cdot l^*\} - \\
&-\sum_{k=1}^{3}[(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{k4}f_4]l_k - \sum_{k,j=1}^{3}[(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{kj}(\widetilde{f}_n n_j)]l_k, \\
&q_2 = 2^{-1}\{\widetilde{F}_m^{(+)} + \widetilde{F}_m^{(-)} - (\mathcal{N}_{1,\tau}-\mathcal{N}_{2,\tau})\mathcal{N}_\tau^{-1}F\cdot m^*\} - \\
&-\sum_{k=1}^{3}[(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{k4}f_4]m_k - \sum_{k,j=1}^{3}[(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{kj}(\widetilde{f}_n n_j)]m_k
\end{aligned} \tag{14.52}$$

are given functions on S.

Now let

$$\mathcal{M}_{G,\tau} := \begin{bmatrix} l_k(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{kj}l_j & l_k(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{kj}m_j \\ m_k(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{kj}l_j & m_k(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau})_{kj}m_j \end{bmatrix}_{2\times 2}.$$

We recall that the summation over repeated indices is meant from 1 to 3. Clearly, (14.50) and (14.51) can be written in the matrix form as

$$\mathcal{M}_{G,\tau}\,\psi = q^* \tag{14.53}$$

with the unknown vector $\psi = (\psi_1,\psi_2)^\top$ and the right-hand side $q^* = (q_1,q_2)^\top$ given by formulae (14.52).

Lemma 14.8. *The operator $\mathcal{M}_{G,\tau}$ is an elliptic ΨDO of order* 1 *with a positive definite principal homogeneous symbol matrix and the index equal to zero.*

Proof. The equations (14.12), (14.20), and (14.21) imply that $\mathcal{M}_{G,\tau}$ is a ΨDO of order 1 with the principal homogeneous symbol matrix

$$\sigma(\mathcal{M}_{G,\tau}) = \begin{bmatrix} l_k l_j E_{kj} & l_k m_j E_{kj} \\ m_k l_j E_{kj} & m_k m_j E_{kj} \end{bmatrix}_{2\times 2} = E_1 E E_1^\top, \tag{14.54}$$

where

$$\begin{aligned} E_1 &= \begin{bmatrix} l_1, & l_2, & l_3, & 0 \\ m_1, & m_2, & m_3, & 0 \end{bmatrix}_{2\times 4}, \\ E &= \sigma(\mathcal{N}_{2,\tau}\mathcal{N}_\tau^{-1}\mathcal{N}_{1,\tau}) = \sigma(\mathcal{N}_{2,\tau})\sigma(\mathcal{N}_\tau^{-1})\sigma(\mathcal{N}_{1,\tau}) = \\ &= \sigma(\mathcal{N}_{2,\tau})[\sigma(\mathcal{N}_{1,\tau}) + \sigma(\mathcal{N}_{2,\tau})]^{-1}\sigma(\mathcal{N}_{1,\tau}). \end{aligned}$$

Due to Lemma 14.2 the matrices $\sigma(\mathcal{N}_{j,\tau})$, $j = 1, 2$, are positive definite for arbitrary $x \in S$ and $\widetilde{\xi} \in \mathbb{R}^2 \setminus 0$ (see (14.20), (14.21)). Therefore, the matrix E is positive definite as well. Next, for arbitrary $\eta = (\eta_1, \eta_2)^\top \in \mathbb{C}^2$ we have

$$\begin{gathered} \sigma(\mathcal{M}_{G,\tau})\eta \cdot \eta = (E_1 E E_1^\top)\eta \cdot \eta = E(E_1^\top \eta) \cdot (E_1^\top \eta) = \\ = E(l^*\eta_1 + m^*\eta_2) \cdot (l^*\eta_1 + m^*\eta_2) \geq c|\widetilde{\xi}|\, |\eta_1 l^* + \eta_2 m^*|^2 = \\ = c\,|\widetilde{\xi}|\,(|\eta_1|^2 + |\eta_2|^2),\ c > 0, \end{gathered}$$

whence the positive definiteness of the matrix (14.54) follows. This implies that the index of the operator $\mathcal{M}_{G,\tau}$ is equal to zero since the positive definiteness of $\sigma(\mathcal{M}_{G,\tau})$ yields the formally self-adjointness of the dominant singular part of the $\mathcal{M}_{G,\tau}$. □

Lemma 14.9. *Let S, k, α, and α' be as in* (14.5). *Then the operator*

$$\mathcal{M}_{G,\tau} : \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l-1,\alpha}(S), \quad 1 \leq l \leq k, \tag{14.55}$$

is an isomorphism.

If $S \in \mathrm{C}^\infty$, then (14.55) *can be extended by continuity to the following bounded, invertible, elliptic ΨDO* (*of order* 1)

$$\begin{gathered} \mathcal{M}_{G,\tau} : H_p^{s+1}(S) \to H_p^s(S)\ \ [B_{p,q}^{s+1}(S) \to B_{p,q}^s(S)], \\ s \in \mathbb{R},\ 1 < p < \infty,\ 1 \leq q \leq \infty. \end{gathered}$$

Proof. It is quite similar to the proofs of Lemmata 14.1 and 14.5. □

The above results yield the following existence theorems.

Theorem 14.10. *Let S, k, α', and α be as in* (14.5), *and let*

$$\widetilde{F}_l^{(\pm)},\ \widetilde{F}_m^{(\pm)},\ \widetilde{F}_n,\ F_4 \in \mathrm{C}^{k-1,\alpha}(S), \quad \widetilde{f}_n,\ f_4 \in \mathrm{C}^{k,\alpha}(S).$$

Then the problem $(\mathcal{G})_\tau$ is uniquelly solvable, and the solution is representable in the form (14.24) – (14.25) *with F and f given by* (14.45) *and* (14.47), *where ψ_1, $\psi_2 \in \mathrm{C}^{k,\alpha}(S)$ are defined by the system of ΨDEs* (14.50) *and* (14.51) (*i.e.,* (14.53)). *Moreover,* (14.26) *and the inequality* (14.27) *hold.*

Theorem 14.11. *Let* $S \in \mathrm{C}^\infty$ *and*

$$\widetilde{F}_l^{(\pm)},\ \widetilde{F}_m^{(\pm)},\ \widetilde{F}_n,\ F_4 \in B_{p,p}^{-1/p}(S), \quad \widetilde{f}_n,\ f_4 \in B_{p,p}^{1-1/p}(S).$$

Then the problem $(\mathcal{G})_\tau$ *is uniquely solvable in the space* $(W_p^1(\Omega^1), W_p^1(\Omega^2))$, *and the solutions are representable by the formulae* (14.24)–(14.25) *with* F *and* f *given by* (14.45) *and* (14.47), *where* $\psi_1,\, \psi_2 \in B_{p,p}^{1-1/p}(S)$ *are defined by the system of* Ψ*DEs* (14.50) *and* (14.51) (*i.e.,* (14.53)).

The proof of these theorems are quite similar (in fact, verbatim) to the proofs of Theorems 14.4 and 14.6. □

14.4. In this subsection we shall study the problem $(\mathcal{H})_\tau$. As in the previous subsection let us rewrite the transmission conditions of the problem (see Subsection 7.2) in the equivalent form

$$[u^{(1)}\cdot l]^+ + [u^{(2)}\cdot l]^- = \widetilde{f}_l^{(+)} + \widetilde{f}_l^{(-)}, \tag{14.56}$$

$$[u^{(1)}\cdot m]^+ + [u^{(2)}\cdot m]^- = \widetilde{f}_m^{(+)} + \widetilde{f}_m^{(-)}, \tag{14.57}$$

$$[u^{(1)}\cdot l]^+ - [u^{(2)}\cdot l]^- = \widetilde{f}_l^{(+)} - \widetilde{f}_l^{(-)}, \tag{14.58}$$

$$[u^{(1)}\cdot m]^+ - [u^{(2)}\cdot m]^- = \widetilde{f}_m^{(+)} - \widetilde{f}_m^{(-)}, \tag{14.59}$$

$$[u^{(1)}\cdot n]^+ - [u^{(2)}\cdot n]^- = \widetilde{f}_n, \tag{14.60}$$

$$[P^{(1)}(D,n)U^{(1)}\cdot n]^+ - [P^{(2)}(D,n)U^{(2)}\cdot n]^- = \widetilde{F}_n, \tag{14.61}$$

$$[u_4^{(1)}]^+ - [u_4^{(2)}]^- = f_4,\ \ [\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4. \tag{14.62}$$

Equations (14.58)–(14.60) imply $[U^{(1)}]^+ - [U^{(2)}]^- = f$, where f is a given vector on S

$$f = \Big((\widetilde{f}_l^{(+)} - \widetilde{f}_l^{(-)})\, l + (\widetilde{f}_m^{(+)} - \widetilde{f}_m^{(-)})\, m + \widetilde{f}_n\, n,\ f_4 \Big)^\top. \tag{14.63}$$

It is also evident that $[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F$ with

$$F = (\psi_1 l + \psi_2 m + \widetilde{F}_n n, F_4)^\top, \tag{14.64}$$

where $\widetilde{F}_n$ and F_4 are given functions on S, while $\psi_1 = [P^{(1)}(D,n)U^{(1)}\cdot l]^+ - [P^{(2)}(D,n)U^{(2)}\cdot l]^-$ and $\psi_2 = [P^{(1)}(D,n)U^{(1)}\cdot m]^+ - [P^{(2)}(D,n)U^{(2)}\cdot m]^-$, are yet unknown scalar functions.

We look for a solution to the problem $(\mathcal{H})_\tau$ again in the form (14.24)-(14.25), with F and f defined by (14.63) and (14.64), respectively. It can be easily checked that the transmission conditions (14.58)-(14.62) are then automatically satisfied, while the equations (14.56) and (14.57) lead to the following system of ΨDEs for the unknown vector $\psi = (\psi_1, \psi_2)^\top$ on S:

$$\mathcal{M}_{H,\tau}\, \psi = q^*, \tag{14.65}$$

where

$$\mathcal{M}_{H,\tau} = \begin{bmatrix} l_k(\mathcal{N}_\tau^{-1})_{kj} l_j & l_k(\mathcal{N}_\tau^{-1})_{kj} m_j \\ m_k(\mathcal{N}_\tau^{-1})_{kj} l_j & m_k(\mathcal{N}_\tau^{-1})_{kj} m_j \end{bmatrix}_{2\times 2} \tag{14.66}$$

and the right hand-side vector $q^* = (q_1, q_2)^\top$ is defined by formulae:

$$\begin{aligned} q_1 &= 2^{-1}\{\widetilde{f}_l^{(+)} + \widetilde{f}_l^{(-)} - [\mathcal{N}_\tau^{-1}(\mathcal{N}_{2,\tau} - \mathcal{N}_{1,\tau})f \cdot l^*]\} \\ &- [(\mathcal{N}_\tau^{-1})_{kj}(\widetilde{F}_n n_j)]l_k - [(\mathcal{N}_\tau^{-1})_{k4}F_4]l_k, \\ q_2 &= 2^{-1}\{\widetilde{f}_m^{(+)} + \widetilde{f}_m^{(-)} - [\mathcal{N}_\tau^{-1}(\mathcal{N}_{2,\tau} - \mathcal{N}_{1,\tau})f \cdot m^*]\} \\ &- [(\mathcal{N}_\tau^{-1})_{kj}(\widetilde{F}_n n_j)]m_k - [(\mathcal{N}_\tau^{-1})_{k4}F_4]m_k; \end{aligned}$$

here l^* and m^* are given by (14.48).

By quite the same arguments as in Subsection 14.3 we can easily show that $\mathcal{M}_{H,\tau}$ is an elliptic invertible ΨDO of order -1 with a positive definite principal symbol matrix.

Therefore the operators

$$\begin{aligned} \mathcal{M}_{H,\tau} &: \mathrm{C}^{k-1,\alpha}(S) \to \mathrm{C}^{k,\alpha}(S), \quad S \in \mathrm{C}^{k+1,\alpha'}, \\ &: H_p^s(S) \to H_p^{s+1}(S), \quad S \in \mathrm{C}^\infty, \\ &: B_{p,q}^s(S) \to B_{p,q}^{s+1}(S), \quad S \in \mathrm{C}^\infty, \end{aligned}$$

are isomorphisms.

These results lead us to the following existence theorems.

Theorem 14.12. *Let S, k, α, and α' be as in* (14.5) *and let*

$$\widetilde{f}_l^{(\pm)},\ \widetilde{f}_m^{(\pm)},\ \widetilde{f}_n,\ f_4 \in \mathrm{C}^{k,\alpha}(S), \quad \widetilde{F}_n,\ F_4 \in \mathrm{C}^{k-1,\alpha}(S).$$

Then the problem $(\mathcal{H})_\tau$ has the unique solution representable in the form (14.24)–(14.25) *with f and F given by* (14.63) *and* (14.64), *where ψ_1, $\psi_2 \in \mathrm{C}^{k-1,\alpha}(S)$ in* (14.64) *are defined by the system of ΨDEs* (14.65).

Theorem 14.13. *Let $S \in \mathrm{C}^\infty$ and*

$$\widetilde{f}_l^{(\pm)},\ \widetilde{f}_m^{(\pm)},\ \widetilde{f}_n,\ f_4 \in B_{p,p}^{1-1/p}(S), \quad \widetilde{F}_n,\ F_4 \in B_{p,p}^{-1/p}(S).$$

Then the problem $(\mathcal{H})_\tau$ is uniquely solvable in the space $(W_p^1(\Omega^1), W_p^1(\Omega^2))$, and the solution is representable by the formulae (14.24) *and* (14.25) *with f and F given by* (14.63) *and* (14.64), *where ψ_1, $\psi_2 \in B_{p,p}^{-1/p}(S)$ in* (14.64) *are defined by the system of ΨDEs* (14.65).

Again proofs are verbatim the proofs of Theorems 14.4 and 14.6.

15. Basic Interface Problems of Steady State Oscillations

In this section we deal with the basic interface problems $(\mathcal{C})_\omega$, $(\mathcal{G})_\omega$, and $(\mathcal{H})_\omega$ of steady state thermoelastic oscillations formulated in Section 7. In contrast to the pseudo-oscillation case, one can not here apply the single layer approach to obtain the "explicit" solution to the basic interface problem $(\mathcal{C})_\omega$ for an arbitrary value of the frequency parameter ω, since the integral operator $\mathcal{H}$ (see (10.3)) is not invertible for the so-called exceptional values of ω. Therefore, we offer another approach which relays on the representation of a solution in the form of a complex linear combination of the single and double layer potentials (see Section 13).

15.1. Here we again assume that the conditions (14.5) are fulfilled and look for the solution to the nonhomogeneous interface problem $(\mathcal{C})_\omega$ (see (7.3)–(7.4) or (7.11)–(7.12)) in the following form

$$U^{(1)}(x) = W^{(1)}(g^{(1)})(x), \quad x \in \Omega^1, \tag{15.1}$$

$$U^{(2)}(x) = W^{(2)}(g^{(2)})(x) + p_0\, V^{(2)}(g^{(2)})(x), \quad x \in \Omega^2, \tag{15.2}$$

where $W^{(\mu)}$ and $V^{(\mu)}$ are the double and single layer potentials constructed by the fundamental solution $\Gamma^{(\mu)}(x-y,\omega,r)$ (see (10.1)–(10.2)), $g^{(\mu)} = (g_1^{(\mu)},\cdots,g_4^{(\mu)})^\top$ $(\mu=1,2)$ are unknown densities, and p_0 is given by (13.5). Moreover, in the sequel we again provide that

$$r = 1 \quad \text{for} \quad \omega > 0 \quad \text{and} \quad r = 2 \quad \text{for} \quad \omega < 0. \tag{15.3}$$

Taking into account the properties of the above potentials and inserting the representations (15.1)-(15.2) into the transmission conditions (7.11)–(7.12), we get the system of ΨDEs on S for $g^{(\mu)}$ $(\mu = 1,2)$:

$$[2^{-1}I_4 + \mathcal{K}_2^{(1)}]\, g^{(1)} - [-2^{-1}I_4 + \mathcal{K}_2^{(2)} + p_0\mathcal{H}^{(2)}]\, g^{(2)} = f, \tag{15.4}$$

$$\mathcal{L}^{(1)}\, g^{(1)} - [\mathcal{L}^{(2)} + p_0(2^{-1}I_4 + \mathcal{K}_1^{(2)})]\, g^{(2)} = F, \tag{15.5}$$

where $\mathcal{H}^{(\mu)}$, $\mathcal{K}_1^{(\mu)}$, $\mathcal{K}_2^{(\mu)}$, and $\mathcal{L}^{(\mu)}$ $(\mu = 1,2)$ are defined by (10.3), (10.4), (10.5), and (10.6), respectively.

To investigate the solvability of the above system of ΨDEs we first prove the following lemma.

Lemma 15.1. *Let* $g^{(\mu)} \in \mathrm{C}^{1,\alpha}(S)$ $(\mu=1,2)$ *and let the vector functions, represented by* (15.1)–(15.2)*, vanish in* Ω^1 *and* Ω^2*, respectively.*

Then $g^{(\mu)} = 0$ $(\mu = 1,2)$ *on* S.

Proof. Obviously, the regular vector function $U^{(1)}$, defined by (15.1), can be extended by the same formula from the domain Ω^1 into Ω^2. Denote the extended vector function again by $U^{(1)}$. By Lemmata 10.1 and 10.7 then we have

$$[U^{(1)}]^- = -g^{(1)} \quad \text{and} \quad [B^{(1)}(D,n)U^{(1)}]^- = 0 \quad \text{on } S, \tag{15.6}$$

in accordance with the assumption $U^{(1)} = 0$ in Ω^1. Since $U^{(1)}$ is a (m,r)-thermo-radiating regular vector function, we deduce by virtue of Theorem 9.5 and the second equation in (15.6) that $U^{(1)} = 0$ in Ω^2, whence $g^{(1)} = 0$ on S follows.

The assertion for $g^{(2)}$ is a ready consequence of Lemma 13.1. □

In the matrix form the system (15.4)-(15.5) reads

$$\mathcal{M}_C\, g = Q, \tag{15.7}$$

where $g = (g^{(1)}, g^{(2)})^\top$, $Q = (f,F)^\top$, and

$$\mathcal{M}_C = \begin{bmatrix} [2^{-1}I_4 + \mathcal{K}_2^{(1)}]_{4\times4} & [2^{-1}I_4 - \mathcal{K}_2^{(2)} - p_0\mathcal{H}^{(2)}]_{4\times4} \\ [\mathcal{L}^{(1)}]_{4\times4} & [-\mathcal{L}^{(2)} - p_0(2^{-1}I_4 + \mathcal{K}_1^{(2)})]_{4\times4} \end{bmatrix}_{8\times8}. \tag{15.8}$$

Next, let us introduce the following operators

$$\Phi_1 := 2^{-1} I_4 + \mathcal{K}_2^{(1)}, \qquad \Psi_1 := \mathcal{L}^{(1)}, \tag{15.9}$$

$$\Phi_2 := -2^{-1} I_4 + \mathcal{K}_2^{(2)} + p_0 \mathcal{H}^{(2)}, \quad \Psi_2 := \mathcal{L}^{(2)} + p_0 (2^{-1} I_4 + \mathcal{K}_1^{(2)}), \tag{15.10}$$

and rewrite the system (15.4)-(15.5) as

$$\Phi_1 \, g^{(1)} - \Phi_2 \, g^{(2)} = f, \tag{15.11}$$

$$\Psi_1 \, g^{(1)} - \Psi_2 \, g^{(2)} = F. \tag{15.12}$$

Note that the mappings

$$\Phi_2 \quad : \quad \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l,\alpha}(S), \quad 0 \le l \le k, \tag{15.13}$$

$$\Psi_2 \quad : \quad \mathrm{C}^{l,\alpha}(S) \to \mathrm{C}^{l-1,\alpha}(S), \quad 1 \le l \le k, \tag{15.14}$$

are isomorphisms due to Lemmata 13.4 and 13. 11. Therefore, (15.11)-(15.12) equivalently can be reduced to the system

$$g^{(2)} = \Phi_2^{-1} \, \Phi_1 \, g^{(1)} - \Phi_2^{-1} \, f, \tag{15.15}$$

$$[\Psi_1 - \Psi_2 \, \Phi_2^{-1} \, \Phi_1] \, g^{(1)} = F - \Psi_2 \, \Phi_2^{-1} \, f. \tag{15.16}$$

Remark 15.2. Note that the system (15.4)-(15.5) (i.e., (15.11)-(15.12)) is equivalent to the following system of SIEs

$$\Phi_1 \, g^{(1)} - \Phi_2 \, g^{(2)} = f, \tag{15.17}$$

$$\mathcal{R}_4 \, \Psi_1 \, g^{(1)} - \mathcal{R}_4 \, \Psi_2 \, g^{(2)} = \mathcal{R}_4 \, F, \tag{15.18}$$

where the equivalent lifting matrix operator $\mathcal{R}_4$ is given by (12.33).

Lemma 15.3. *The operator $\mathcal{M}_C$ is elliptic in the sense of Douglis-Nirenberg with index equal to zero. The mapping*

$$\mathcal{M}_C \; : \; [\mathrm{C}^{l,\alpha}(S)]^8 \to [\mathrm{C}^{l,\alpha}(S)]^4 \times [\mathrm{C}^{l-1,\alpha}(S)]^4, \quad 1 \le l \le k, \tag{15.19}$$

is an isomorphism.

Proof. First we show that $\mathcal{M}_C$ is an elliptic ΨDO in the sense of Douglis-Nirenberg. To this end let us remark that, due to the results of Section 10 (see (10.23)–(10.30),(10.48), (10.49)), for the principal homogeneous symbol matrices of the operators (15.9) and (15.10) we have the following expressions:

$$\sigma(\Phi_1) = \sigma((2^{-1} I_4 + \mathcal{K}_2^{(1)})_0) =: \begin{bmatrix} [K^{(1)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & K_{44}^{(1)} \end{bmatrix}_{4\times 4}, \tag{15.20}$$

$$\sigma(\Phi_2) = \sigma((-2^{-1} I_4 + \mathcal{K}_2^{(2)})_0) =: \begin{bmatrix} [K^{(2)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & K_{44}^{(2)} \end{bmatrix}_{4\times 4}, \tag{15.21}$$

$$\sigma(\Psi_1) = \sigma((\mathcal{L}^{(1)})_0) =: \begin{bmatrix} [L^{(1)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & L_{44}^{(1)} \end{bmatrix}_{4\times 4}, \tag{15.22}$$

$$\sigma(\Psi_2) = \sigma((\mathcal{L}^{(2)})_0) =: \begin{bmatrix} [L^{(2)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & L_{44}^{(2)} \end{bmatrix}_{4\times 4}, \tag{15.23}$$

where $(\mathcal{K})_0$ denotes again the dominant singular part of the operator $\mathcal{K}$; here we employed the notations:

$$K^{(1)} = \sigma(2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(1,0)}) = [\overline{\sigma(2^{-1}I_3 + \mathcal{K}^{(1,0)})}]^\top, \tag{15.24}$$

$$K^{(2)} = \sigma(-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(2,0)}) = [\overline{\sigma(-2^{-1}I_3 + \mathcal{K}^{(2,0)})}]^\top, \tag{15.25}$$

$$K^{(1)}_{44} = \sigma(2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(1,0)}_4) = \tfrac{1}{2}, \tag{15.26}$$

$$K^{(2)}_{44} = \sigma(-2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(2,0)}_4) = -\tfrac{1}{2}, \tag{15.27}$$

$$L^{(\mu)} = \sigma(\mathcal{L}^{(\mu,0)}), \;\; \mu = 1, 2, \tag{15.28}$$

$$L^{(\mu)}_{44} = \sigma(\mathcal{L}^{(\mu,0)}_4) = -[4\sigma(\mathcal{H}^{(\mu,0)}_4)]^{-1} \; > \; 0, \;\; \mu = 1, 2, \tag{15.29}$$

where by $\overset{*}{\mathcal{K}}{}^{(\mu,0)}$, $\mathcal{K}^{(\mu,0)}$, $\overset{*}{\mathcal{K}}{}^{(\mu,0)}_4$, $\mathcal{K}^{(\mu,0)}$, $\mathcal{L}^{(\mu,0)}$, and $\mathcal{L}^{(\mu,0)}_4$ are denoted again the operatos (10.26), (10.40), and (10.41) corresponding to the thermoelastic characteristics of the medium occupying the domain Ω^μ (cf. (14.19)).

In Lemma 3.3 of the reference [41] it has been proved that

$$\sigma_* = \det \begin{bmatrix} [K^{(1)}]_{3\times 3} & -[K^{(2)}]_{3\times 3} \\ [L^{(1)}]_{3\times 3} & [-L^{(2)}]_{3\times 3} \end{bmatrix}_{6\times 6} \neq 0 \tag{15.30}$$

for arbitrary $x \in S$ and $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$.

Let us now consider the symbol matrix of the operator $\mathcal{M}_C$

$$\sigma(\mathcal{M}_C) = \begin{bmatrix} \sigma(\Phi_1) & -\sigma(\Phi_2) \\ \sigma(\Psi_1) & -\sigma(\Psi_2) \end{bmatrix}_{8\times 8} \tag{15.31}$$

and show that the corresponding determinant does not vanish for arbitrary $x \in S$ and $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$, which in turn implies the usual ellipticity of the system (15.17)-(15.18) (or the ellipticity of the system (15.4)-(15.5) in the sense of Douglis-Nirenberg). By virtue of formulae (15.20)-(15.29) we get from (15.31) after some simple rearrangements

$$\det \sigma(\mathcal{M}_C) = \det \begin{bmatrix} [K^{(1)}]_{3\times 3} & [0]_{3\times 1} & -[K^{(2)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \frac{1}{2} & [0]_{1\times 3} & -\frac{1}{2} \\ [L^{(1)}]_{3\times 3} & [0]_{3\times 1} & [-L^{(2)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & L^{(1)}_{44} & [0]_{1\times 3} & -L^{(2)}_{44} \end{bmatrix}_{8\times 8} =$$

$$= \det \begin{bmatrix} [K^{(1)}]_{3\times 3} & -[K^{(2)}]_{3\times 3} \\ [L^{(1)}]_{3\times 3} & [-L^{(2)}]_{3\times 3} \end{bmatrix}_{6\times 6} \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ L^{(1)}_{44} & -L^{(2)}_{44} \end{bmatrix}_{2\times 2} =$$

$$= -\tfrac{1}{2} \left(L^{(1)}_{44} + L^{(2)}_{44} \right) \sigma_* \; \neq \; 0, \tag{15.32}$$

due to (15.29) and (15.30).

Next we show that the index of the operator $\mathcal{M}_C$ equals zero. To see this, let us note that the index does not depend on a compact pertubation, and consider the following operator

$$\widetilde{\mathcal{M}}_C = \begin{bmatrix} [\widetilde{\mathcal{M}}_C^{(1)}]_{4\times 4} & [\widetilde{\mathcal{M}}_C^{(2)}]_{4\times 4} \\ [\widetilde{\mathcal{M}}_C^{(3)}]_{4\times 4} & [\widetilde{\mathcal{M}}_C^{(4)}]_{4\times 4} \end{bmatrix}_{8\times 8}, \tag{15.33}$$

where

$$\widetilde{\mathcal{M}}_C^{(1)} = \begin{bmatrix} [2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(1,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2^{-1}I_1 + \overset{*}{\mathcal{K}}{}_4^{(1,0)} \end{bmatrix}_{4\times 4},$$

$$\widetilde{\mathcal{M}}_C^{(2)} = \begin{bmatrix} [2^{-1}I_3 - \overset{*}{\mathcal{K}}{}^{(2,0)} - \{\mathcal{H}^{(2,0)}\}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2^{-1}I_1 - \overset{*}{\mathcal{K}}{}_4^{(2,0)} - \{\mathcal{H}_4^{(2,0)}\} \end{bmatrix}_{4\times 4},$$

$$\widetilde{\mathcal{M}}_C^{(3)} = \begin{bmatrix} [\mathcal{L}^{(1,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(1,0)} \end{bmatrix}_{4\times 4},$$

$$\widetilde{\mathcal{M}}_C^{(4)} = \begin{bmatrix} [-\mathcal{L}^{(2,0)} - \{2^{-1}I_3 + \mathcal{K}^{(2,0)}\}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & -\mathcal{L}_4^{(2,0)} - \{2^{-1}I_1 + \mathcal{K}_4^{(2,0)}\} \end{bmatrix}_{4\times 4}.$$

Clearly, the dominant singular parts $(\mathcal{M}_C)_0$ and $(\widetilde{\mathcal{M}}_C)_0$ coincide. Indeed, these dominant singular parts in the both cases can be represented in the form (15.33) where the summands in curly brackets are removed.

The corresponding formally adjoint operator to $\widetilde{\mathcal{M}}_C$ reads as

$$\widetilde{\mathcal{M}}_C^* = \begin{bmatrix} [\widetilde{\mathcal{M}}_C^{(1)*}]_{4\times 4} & [\widetilde{\mathcal{M}}_C^{(2)*}]_{4\times 4} \\ [\widetilde{\mathcal{M}}_C^{(3)*}]_{4\times 4} & [\widetilde{\mathcal{M}}_C^{(4)*}]_{4\times 4} \end{bmatrix}_{8\times 8}, \tag{15.34}$$

where

$$\widetilde{\mathcal{M}}_C^{(1)*} = \begin{bmatrix} [2^{-1}I_3 + \mathcal{K}^{(1,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2^{-1}I_1 + \mathcal{K}_4^{(1,0)} \end{bmatrix}_{4\times 4},$$

$$\widetilde{\mathcal{M}}_C^{(2)*} = \begin{bmatrix} [\mathcal{L}^{(1,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{L}_4^{(1,0)} \end{bmatrix}_{4\times 4},$$

$$\widetilde{\mathcal{M}}_C^{(3)*} = \begin{bmatrix} [2^{-1}I_3 - \mathcal{K}^{(2,0)} - \mathcal{H}^{(2,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2^{-1}I_1 - \mathcal{K}_4^{(2,0)} - \mathcal{H}_4^{(2,0)} \end{bmatrix}_{4\times 4},$$

$$\widetilde{\mathcal{M}}_C^{(4)*} = \begin{bmatrix} [-\mathcal{L}^{(2,0)} - 2^{-1}I_3 - \overset{*}{\mathcal{K}}{}^{(2,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & -\mathcal{L}_4^{(2,0)} - 2^{-1}I_1 - \overset{*}{\mathcal{K}}{}_4^{(2,0)} \end{bmatrix}_{4\times 4}.$$

We again recall that the operators involved in (15.33) and (15.34) are defined in Section 10. Moreover, here we have applied that the operators $\mathcal{L}^{(\mu,0)}$, $\mathcal{L}_4^{(\mu,0)}$, $\mathcal{H}^{(\mu,0)}$, and $\mathcal{H}_4^{(\mu,0)}$ are formally self-adjoint (see [34], [59]).

In what follows we prove that the homogeneous equations

$$\widetilde{\mathcal{M}}_C\,\varphi = 0,\ \varphi = (\varphi^{(1)}, \varphi^{(2)})^\top,\ \varphi^{(j)} = (\varphi_1^{(j)}, \cdots, \varphi_4^{(j)})^\top,\ j = 1, 2, \tag{15.35}$$

and

$$\widetilde{\mathcal{M}}_C^* \psi = 0,\ \psi = (\psi^{(1)}, \psi^{(2)})^\top,\ \psi^{(j)} = (\psi_1^{(j)}, \cdots, \psi_4^{(j)})^\top,\ j = 1, 2, \quad (15.36)$$

have only the trivial solutions.

Due to the above established ellipticity we consider these equations in the regular space of $\mathrm{C}^{1,\alpha}$-smooth vector functions.

Note that the system (15.35) can be decomposed into the following two independent systems:

$$\left.\begin{aligned} &[2^{-1} I_3 + \overset{*}{\mathcal{K}}{}^{(1,0)}]\, \widetilde{\varphi}^{(1)} - [\,-2^{-1} I_3 + \overset{*}{\mathcal{K}}{}^{(2,0)} + \mathcal{H}^{(2,0)}]\, \widetilde{\varphi}^{(2)} = 0,\\ &\mathcal{L}^{(1,0)}\, \widetilde{\varphi}^{(1)} - [\mathcal{L}^{(2,0)} + 2^{-1} I_3 + \mathcal{K}^{(2,0)}]\, \widetilde{\varphi}^{(2)} = 0, \end{aligned}\right\} \quad (15.37)$$

$$\left.\begin{aligned} &[2^{-1} I_1 + \overset{*}{\mathcal{K}}_4{}^{(1,0)}]\, \varphi_4^{(1)} - [\,-2^{-1} I_1 + \overset{*}{\mathcal{K}}_4{}^{(2,0)} + \mathcal{H}_4^{(2,0)}]\, \varphi_4^{(2)} = 0,\\ &\mathcal{L}_4^{(1,0)}\, \varphi_4^{(1)} - [\mathcal{L}_4^{(2,0)} + 2^{-1} I_1 + \mathcal{K}_4^{(2,0)}]\, \varphi_4^{(2)} = 0, \end{aligned}\right\} \quad (15.38)$$

where $\widetilde{\varphi}^{(j)} = (\varphi_1^{(j)}, \varphi_2^{(j)}, \varphi_3^{(j)})^\top$, $j = 1, 2$.

These systems are generated by the following interface problems for the equations of elastostatics and the stationary distribution of temperature

$$\left.\begin{aligned} &C^{(\mu)}(D) u^{(\mu)} = 0 \text{ in } \Omega^\mu,\ u^{(\mu)} = (u_1^{(\mu)}, u_2^{(\mu)}, u_3^{(\mu)})^\top,\ \mu = 1, 2,\\ &[u^{(1)}]^+ - [u^{(2)}]^- = 0 \text{ and } [T^{(1)}(D, n) u^{(1)}]^+ -\\ &\qquad -[T^{(2)}(D, n) u^{(2)}]^- = 0 \ \text{ on } \ S,\\ &u^{(2)}(x) = o(1) \ \text{ as } \ |x| \to +\infty, \end{aligned}\right\} \quad (15.39)$$

and

$$\left.\begin{aligned} &\lambda_{pq}^{(\mu)} D_p D_q u_4^{(\mu)} = 0 \ \text{ in } \ \Omega^\mu,\ \ \mu = 1, 2,\\ &[u_4^{(1)}]^+ - [u_4^{(2)}]^- = 0 \ \text{ and } \ [\lambda^{(1)}(D, n) u_4^{(1)}]^+ -\\ &\qquad -[\lambda^{(2)}(D, n) u_4^{(2)}]^- = 0 \ \text{ on } \ S,\\ &u_4^{(2)}(x) = o(1) \ \text{ as } \ |x| \to +\infty, \end{aligned}\right\} \quad (15.40)$$

where $C^{(\mu)}(D)$, $T^{(\mu)}(D, n)$, and $\lambda^{(\mu)}(D, n)$ are given by (1.7), (1.12), and (1.24), respectively.

If one looks for solutions $(u^{(1)}, u^{(2)})$ and $(u_4^{(1)}, u_4^{(2)})$ in the form of following potentials (see (10.19)–(10.22))

$$\begin{aligned} u^{(1)}(x) = \textstyle\int_S [T^{(1)}(D_y, n(y)) \Gamma^{(1,0)}(y - x)]^\top\, \widetilde{\varphi}^{(1)}(y)\, dS_y =:\\ =: w^{(1,0)}(\widetilde{\varphi}^{(1)})(x), \end{aligned} \quad (15.41)$$

$$\begin{aligned} u^{(2)}(x) = \textstyle\int_S [T^{(2)}(D_y, n(y)) \Gamma^{(2,0)}(y - x)]^\top\, \widetilde{\varphi}^{(2)}(y)\, dS_y +\\ + \textstyle\int_S \Gamma^{(2,0)}(y - x)\, \widetilde{\varphi}^{(2)}(y)\, dS_y =: w^{(2,0)}(\widetilde{\varphi}^{(2)})(x) + v^{(2,0)}(\widetilde{\varphi}^{(2)})(x), \end{aligned} \quad (15.42)$$

$$\begin{aligned} u_4^{(1)}(x) = \textstyle\int_S \lambda^{(1)}(D_y, n(y)) \gamma^{(1,0)}(y - x)\, \varphi_4^{(1)}(y)\, dS_y =:\\ =: w_4^{(1,0)}(\varphi_4^{(1)})(x), \end{aligned} \quad (15.43)$$

$$\begin{aligned} u_4^{(2)}(x) = \textstyle\int_S \lambda^{(2)}(D_y, n(y)) \gamma^{(2,0)}(y - x)\, \varphi_4^{(2)}(y)\, dS_y +\\ + \textstyle\int_S \gamma^{(2,0)}(y - x)\, \varphi_4^{(2)}(y)\, dS_y =: w_4^{(2,0)}(\varphi_4^{(2)})(x) + v_4^{(2,0)}(\varphi_4^{(2)})(x), \end{aligned} \quad (15.44)$$

one arrives at the systems (15.37) and (15.38).

Using the usual Green identities (14.22) it can be easily shown that the homogeneous problems (15.39) and (15.40) have only the trivial solutions.

These uniqueness results and standard arguments of the potential theory imply that the systems (15.37) and (15.38) possess only the trivial solutions as well.

Indeed, let $(\varphi^{(1)},\varphi^{(2)})^\top$ be some solution to the homogeneous system (15.37), and let us construct by these densities the potentials (15.41) in Ω^1 and (15.42) in Ω^2. Due to the above uniqueness $u^{(\mu)}(x)=0$ in Ω^μ, $\mu=1,2$. Applying the jump properties of the single and double layer potentials of elastostatics (see [8], [34], [56]) we conclude that $\varphi^{(1)}=\varphi^{(2)}=0$ on S. For the system (15.38) the proof is verbatim. Thus, $\ker\widetilde{\mathcal{M}}_C=\{0\}$.

To prove that $\ker\widetilde{\mathcal{M}}^*_C=\{0\}$, we decompose analogously the system (15.36) into the two systems

$$\left.\begin{aligned}
&[2^{-1}I_3+\mathcal{K}^{(1,0)}]\,\widetilde{\psi}^{(1)}+\mathcal{L}^{(1,0)}\,\widetilde{\psi}^{(2)}=0,\\
&[-2^{-1}I_3+\mathcal{K}^{(2,0)}+\mathcal{H}^{(2,0)}]\,\widetilde{\psi}^{(1)}+\\
&\qquad+[\mathcal{L}^{(2,0)}+2^{-1}I_3+\overset{*}{\mathcal{K}}{}^{(2,0)}]\,\widetilde{\psi}^{(2)}=0,
\end{aligned}\right\}\tag{15.45}$$

$$\left.\begin{aligned}
&[2^{-1}I_1+\mathcal{K}_4^{(1,0)}]\,\psi_4^{(1)}+\mathcal{L}_4^{(1,0)}\,\psi_4^{(2)}=0,\\
&[-2^{-1}I_1+\mathcal{K}_4^{(2,0)}+\mathcal{H}_4^{(2,0)}]\,\psi_4^{(1)}+\\
&\qquad+[\mathcal{L}_4^{(2,0)}+2^{-1}I_1+\overset{*}{\mathcal{K}}{}_4^{(2,0)}]\,\psi_4^{(2)}=0.
\end{aligned}\right\}\tag{15.46}$$

Denote by $(\widetilde{\psi}^{(1)},\widetilde{\psi}^{(2)})^\top$ some solution of the homogeneous system (15.45) and by these densities construct the vectors (see (15.41)–(15.44))

$$u_*^{(1)}(x)=v^{(1,0)}(\widetilde{\psi}^{(1)})(x)+w^{(1,0)}(\widetilde{\psi}^{(2)})(x)\ \text{ in }\ \Omega^-=\Omega^2,\tag{15.47}$$

$$u_*^{(2)}(x)=v^{(2,0)}(\widetilde{\psi}^{(1)})(x)+w^{(2,0)}(\widetilde{\psi}^{(2)})(x)\ \text{ in }\ \Omega^+=\Omega^1.\tag{15.48}$$

Obviously, $C^{(1)}(D)u_*^{(1)}=0$ in $\Omega^-=\Omega^2$ and $C^{(2)}(D)u_*^{(2)}=0$ in $\Omega^+=\Omega^1$. It can be also easily verified that the equations (15.45) correspond to the conditions

$$[T^{(1)}(D,n)u_*^{(1)}]^-=0,\tag{15.49}$$

$$[T^{(2)}(D,n)u_*^{(2)}]^++[u_*^{(2)}]^+=0.\tag{15.50}$$

Therefore, $u_*^{(1)}$ is a solution of the homogeneous exterior stress problem in Ω^-, while $u_*^{(2)}$ represents a solution to the Robin type problem in Ω^+. By uniqueness theorems, which can be established again with the help of (14.22), we conclude $u_*^{(1)}=0$ in Ω^-, and $u_*^{(2)}=0$ in Ω^+. The jump relations then lead to the equations

$$\begin{aligned}
&[u_*^{(1)}]^+=\widetilde{\psi}^{(2)},\quad [T^{(1)}(D,n)u_*^{(1)}]^+=-\widetilde{\psi}^{(1)},\\
&[u_*^{(2)}]^-=-\widetilde{\psi}^{(2)},\quad [T^{(2)}(D,n)u_*^{(2)}]^-=\widetilde{\psi}^{(1)},
\end{aligned}\tag{15.51}$$

whence

$$[u_*^{(1)}]^+ + [u_*^{(2)}]^- = 0,$$
$$[T^{(1)}(D,n)u_*^{(1)}]^+ + [T^{(2)}(D,n)u_*^{(2)}]^- = 0. \tag{15.52}$$

Making use once again of Green formulae (14.22) together with homogeneous conditions (15.52) we obtain that $u_*^{(1)} = 0$ in Ω^+ and $u_*^{(2)} = 0$ in Ω^-. Now (15.51) shows $\widetilde{\psi}^{(1)} = \widetilde{\psi}^{(2)} = 0$ on S. In the same way we can show that the system (15.46) has also only the trivial solution. Thus, $\ker\widetilde{\mathcal{M}}_C^* = \{0\}$ as well, and, therefore, $\mathrm{ind}\widetilde{\mathcal{M}}_C = 0$, which proves the first part of the lemma.

Next we prove that the mapping (15.19) is an isomorphism. Due to the first part of the lemma it remains to check that the homogeneous equation $\widetilde{\mathcal{M}}_C\, g = 0$ admits only the trivial solution. Let $g = (g^{(1)}, g^{(2)})^\top$ be an arbitrary solution of this equation. Then the potentials (15.1) and (15.2) solve the homogeneous problem $(\mathcal{C})_\omega$ and by Theorem 9.8 they vanish in the corresponding domains. Now Lemma 15.1 completes the proof. $\square$

Corollary 15.4. *Let $S \in \mathrm{C}^\infty$ and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$. Then the operators*

$$\begin{aligned} \mathcal{M}_C \quad &: \quad [H_p^s(S)]^8 \to [H_p^s(S)]^4 \times [H_p^{s-1}(S)]^4, \\ &: \quad [B_{p,q}^s(S)]^8 \to [B_{p,q}^s(S)]^4 \times [B_{p,q}^{s-1}(S)]^4 \end{aligned}$$

are isomorphisms.

Proof. It follows from the fact that, due to the general theory of elliptic ΨDEs on closed smooth manifolds, the uniqueness of solution implies the corresponding existence results for the nonhomogeneous equation (15.7) in the Besov $B_{p,q}^s(S)$ and the Bessel-potential $H_p^s(S)$ spaces (see the proof of Lemma 12.9). $\square$

We are now ready to present the solution of the system (15.4)-(15.5) (i.e., (15.17)-(15.18)) in terms of explicitly given boundary integral operators and their inverses. To this end we need the following lemma.

Lemma 15.5. *Let S, k, and α be as in* (14.5). *Then the mapping*

$$[\Psi_1 - \Psi_2\, \Phi_2^{-1}\, \Phi_1] \,:\, [\mathrm{C}^{l,\alpha}(S)]^4 \to \times[\mathrm{C}^{l-1,\alpha}(S)]^4, \quad 1 \le l \le k, \tag{15.53}$$

is an elliptic invertible ΨDO of order $+1$.

Proof. First we show the ellipticity of the principal homogeneous symbol matrix of the operator in question. Due to the equations (15.20)-(15.29) we have

$$\begin{aligned} M := \sigma(\Psi_1 - \Psi_2\, \Phi_2^{-1}\, \Phi_1) &= \sigma(\Psi_1) - \sigma(\Psi_2)[\sigma(\Phi_2)]^{-1}\sigma(\Phi_1) = \\ &= \big\{\sigma(\Psi_1)[\sigma(\Phi_1)]^{-1} - \sigma(\Psi_2)[\sigma(\Phi_2)]^{-1}\big\}\ \sigma(\Phi_1) = \\ &= \left\{ \begin{bmatrix} [L^{(1)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & L_{44}^{(1)} \end{bmatrix} \begin{bmatrix} [(K^{(1)})^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2 \end{bmatrix} - \right. \\ &\quad \left. - \begin{bmatrix} [L^{(2)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & L_{44}^{(2)} \end{bmatrix} \begin{bmatrix} [(K^{(2)})^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & -2 \end{bmatrix} \right\} \times \end{aligned}$$

$$
\times \begin{bmatrix} [K^{(1)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2^{-1} \end{bmatrix} =
$$

$$
= \begin{bmatrix} [L^{(1)}(K^{(1)})^{-1} - L^{(2)}(K^{(2)})^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2L_{44}^{(1)} + 2L_{44}^{(2)} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} [K^{(1)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2^{-1} \end{bmatrix}. \tag{15.54}
$$

We used here that the matrices $K^{(1)}$ and $K^{(2)}$ defined by (15.24) and (15.25) are not singular (see, e.g., [34], [56]) and employed the following simple facts: if

$$
X = \begin{bmatrix} [\widetilde{X}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & x_{44} \end{bmatrix}_{4\times 4} \quad \text{and} \quad Y = \begin{bmatrix} [\widetilde{Y}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & y_{44} \end{bmatrix}_{4\times 4},
$$

then

$$
XY = \begin{bmatrix} [\widetilde{X}\widetilde{Y}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & x_{44}y_{44} \end{bmatrix}_{4\times 4} \quad \text{and} \quad X^{-1} = \begin{bmatrix} [(\widetilde{X})^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & (x_{44})^{-1} \end{bmatrix}_{4\times 4}
$$

where $\det\widetilde{X} \neq 0$ and $x_{44} \neq 0$ are assumed.

We recall that the matrices (15.28) are nonsingular, too. Moreover, by the arguments similar to that of applied in the proof of Lemma 14.2 we can show that the matrices

$$
L^{(1)}(K^{(1)})^{-1} \quad \text{and} \quad - L^{(2)}(K^{(2)})^{-1} \tag{15.55}
$$

are positive definite (for details see [41], [59], [34], [57]). Therefore, the matrix

$$
M_0 := \begin{bmatrix} [L^{(1)}(K^{(1)})^{-1} - L^{(2)}(K^{(2)})^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2L_{44}^{(1)} + 2L_{44}^{(2)} \end{bmatrix}_{4\times 4}, \tag{15.56}
$$

is positive definite. Consequently, the matrix M defined by (15.54), which represents the principal homogeneous symbol matrix of the operator (15.53), is nonsingular. Thus, the operator (15.53) is an elliptic ΨDO.

Further, from (15.54) it follows that the dominant singular part of the operator (15.53) can be represented as the composition of two operators where the first one is the operator with the positive definite principal symbol matrix (15.56), while the second one is the following invertible operator

$$
\begin{bmatrix} [2^{-1}I_3 + \overset{*}{\mathcal{K}}{}^{(1,0)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & 2^{-1} \end{bmatrix}_{4\times 4},
$$

which corresponds to the second matrix multiplyer in (15.54). These facts yield that the index of the operator (15.53) is equal to zero.

Next we prove that the operator (15.53) has the trivial null-space . Let the homogeneous equation

$$
[\Psi_1 - \Psi_2\, \Phi_2^{-1}\, \Phi_1]\, g' = 0, \quad g' = (g_1', \cdots, g_4')^\top, \tag{15.57}
$$

admits a nontrivial solution $g' \neq 0$. Then the nontrivial vector $(g', \Phi_2^{-1}\Phi_1 g')^\top \neq 0$ solves the system (15.11)–(15.12) (with $f = 0$, $F = 0$). This contradicts to Lemma 15.3. Therefore, (15.57) has only the trivial solution, which completes the proof. □

Corollary 15.6. *Let $S \in \mathrm{C}^\infty$ and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$. Then the operators*

$$\begin{aligned}\Psi_1 - \Psi_2\,\Phi_2^{-1}\,\Phi_1 \quad &: \quad [H_p^s(S)]^4 \to [H_p^{s-1}(S)]^4,\\ &: \quad [B_{p,q}^s(S)]^4 \to [B_{p,q}^{s-1}(S)]^4\end{aligned}$$

are elliptic invertible ΨDOs of order $+1$.

Proof. It is verbatim the proof of Corollary 15.4. □

Let us introduce the following ΨDO of order -1

$$\Psi := [\Psi_1 - \Psi_2\,\Phi_2^{-1}\,\Phi_1]^{-1}. \tag{15.58}$$

From Lemma 15.5 it follows that we can represent the solution of the system (15.7) "explicitly" by formulae

$$g^{(1)} = \Psi\,F - \Psi\,\Psi_2\,\Phi_2^{-1}\,f, \tag{15.59}$$

$$g^{(2)} = \Phi_2^{-1}\,\Phi_1\,\Psi\,F - \Phi_2^{-1}\,(\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,+\,I)\,f, \tag{15.60}$$

where I is again the identity operator.

Substituting (15.59) and (15.60) into (15.1) and (15.2) we obtain the following representation of solution of the problem $(\mathcal{C})_\omega$:

$$U^{(1)}(x) = W^{(1)}\left(\Psi\,F - \Psi\,\Psi_2\,\Phi_2^{-1}\,f\right)(x), \tag{15.61}$$

$$\begin{aligned}U^{(2)}(x) = \Big(W^{(2)} + p_0\,V^{(2)}\Big)\,\big(&\Phi_2^{-1}\,\Phi_1\,\Psi\,F-\\ &-\Phi_2^{-1}\,[\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,+\,I]\,f\big)\,(x),\end{aligned} \tag{15.62}$$

where F and f are the boundary data of the interface problem under consideration (see (7.3)–(7.4) or (7.11)–(7.12)).

Now we are in the position to formulate the basic existence results in the form of the following propositions.

Theorem 15.7. *Let conditions* (14.5) *be fulfilled. Then the formulae* (15.61)–(15.62) *define the unique regular solution to the problem $(\mathcal{C})_\omega$ of the class*

$$(U^{(1)}\,,\,U^{(2)}) \in ([\mathrm{C}^{k,\alpha}(\overline{\Omega^1})]^4\,,\,[\mathrm{C}^{k,\alpha}(\overline{\Omega^2}) \cap \mathrm{SK}_r^m(\Omega^2)]^4) \tag{15.63}$$

(*with r and ω as in* (15.3)).

Proof. It is a ready consequence of the uniqueness Theorem 9.8 and Lemmata 10.1, 15.3, and 15.5. □

Theorem 15.8. *Let $S \in \mathrm{C}^\infty$, $1 < p < \infty$, and*

$$f \in [B_{p,p}^{1-1/p}(S)]^4, \quad F \in [B_{p,p}^{-1/p}(S)]^4. \tag{15.64}$$

Then the formulae (15.61)–(15.62) *represent the unique solution to the problem* $(\mathcal{C})_\omega$ *of the class*

$$(U^{(1)}\,,\,U^{(2)}) \in ([W^1_p(\Omega^1)]^4\,,\,[W^1_{p,\mathrm{loc}}(\Omega^2)\cap \mathrm{SK}^m_r(\Omega^2)]^4) \tag{15.65}$$

(*with* r *and* ω *as in* (15.3)).

Proof. Solvability of the problem $(\mathcal{C})_\omega$ in the class indicated in the theorem is an immediate consequence of the formulae (15.61)-(15.62), and Theorem 10.8 (with $s = 1 - 1/p$).

To prove the uniqueness of solution to the problem $(\mathcal{C})_\omega$ for arbitrary $p \in (1, \infty)$, we have to repeate word for word the arguments of the proof of Theorem 14.6. The case is that the key integral representation formulae similar to (14.30)-(14.31) we can also write for a solution $(U^{(1)}\,,\,U^{(2)})$ to the homogeneous problem $(\mathcal{C})_\omega$ of the class (15.65) (see Theorem 10.8, item ii)). □

15.2. In this subsection we present the existence results for the problem $(\mathcal{G})_\omega$. First we transform the interface conditions (7.5)–(7.8) to the equivalent equations on S (cf. Subsection 14.3):

$$[B^{(1)}(D,n)U^{(1)}]^+ - [B^{(2)}(D,n)U^{(2)}]^- = F, \tag{15.66}$$

$$[u^{(1)}\cdot n]^+ - [u^{(2)}\cdot n]^- = \widetilde{f}_n,\ \ [u^{(1)}_4]^+ - [u^{(2)}_4]^- = f_4, \tag{15.67}$$

$$[P^{(1)}(D,n)U^{(1)}\cdot l]^+ + [P^{(2)}(D,n)U^{(2)}\cdot l]^- = \widetilde{F}^{(+)}_l + \widetilde{F}^{(-)}_l, \tag{15.68}$$

$$[P^{(1)}(D,n)U^{(1)}\cdot m]^+ + [P^{(2)}(D,n)U^{(2)}\cdot m]^- = \widetilde{F}^{(+)}_m + \widetilde{F}^{(-)}_m, \tag{15.69}$$

where

$$F = \Big((\widetilde{F}^{(+)}_l - \widetilde{F}^{(-)}_l)\,l + (\widetilde{F}^{(+)}_m - \widetilde{F}^{(-)}_m)\,m + \widetilde{F}_n\,n,\ F_4\Big)^\top, \tag{15.70}$$

and l, m, and n are as in Subsection 7.2.

We seek the solution of the problem $(\mathcal{G})_\omega$ in the form of potentials (15.61)-(15.62), where F is given by (15.70), and

$$[U^{(1)}]^+ - [U^{(2)}]^- = f = (\varphi\, l + \psi\, m + \widetilde{f}_n\, n,\ f_4)^\top. \tag{15.71}$$

Here φ and ψ are unknown scalar functions of the space $\mathrm{C}^{k,\alpha}(S)$, while $\widetilde{F}^{(\pm)}_l$, $\widetilde{F}^{(\pm)}_m$, $\widetilde{F}_n$, F_4, $\widetilde{f}_n$, and f_4 are given functions on S. We assume that

$$\begin{gathered}\widetilde{F}^{(\pm)}_l,\ \widetilde{F}^{(\pm)}_m,\ \widetilde{F}_n,\ F_4\ \in \mathrm{C}^{k-1,\alpha}(S),\ \ \widetilde{f}_n, f_4\ \in \mathrm{C}^{k,\alpha}(S),\\ S \in \mathrm{C}^{k+1,\alpha'},\ \ k\geq 1,\ \ 0<\alpha<\alpha'\leq 1.\end{gathered} \tag{15.72}$$

From the results of the previous subsection it is evident that the vectors $U^{(1)}$ and $U^{(2)}$ given by (15.61) and (15.62) are regular solutions to the steady state oscillation equations of thermoelasticity theory (7.2). Moreover, they automatically satisfy the conditions (15.66) and (15.67). It remains to fulfil the conditions (15.68) and (15.69) by choosing the unknown functions φ and ψ appropriately.

Due to the jump relations of the single and double layer potentials (see Lemmata 10.1 and 10.7) we have from (15.61)-(15.62) (see also (15.9), (15.10) and (15.58))

$$[B^{(1)}(D,n)U^{(1)}]^{+} = \mathcal{L}^{(1)}\,\Psi\,[F-\Psi_2\,\Phi_2^{-1}\,f] = \Psi_1\,\Psi\,[F-\Psi_2\,\Phi_2^{-1}\,f] = \\ = \Psi_1\,\Psi\,F - \Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,(\varphi\,l+\psi\,m+\widetilde{f}_n\,n,\,f_4)^{\top}, \tag{15.73}$$

$$[B^{(2)}(D,n)U^{(2)}]^{-} = [\mathcal{L}^{(2)} + p_0(2^{-1}I_4 + \mathcal{K}_1^{(2)})]\,\Phi_2^{-1}\,[\Phi_1\,\Psi\,F - \\ -(\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}+I)f] = \Psi_2\,\Phi_2^{-1}[\Phi_1\,\Psi\,F - (\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}+I)f] = \\ = \Psi_2\Phi_2^{-1}\Phi_1\Psi F - \Psi_2\Phi_2^{-1}(\Phi_1\Psi\Psi_2\Phi_2^{-1}+I)(\varphi l+\psi m+\widetilde{f}_n n, f_4)^{\top}. \tag{15.74}$$

Now let l^*, m^*, and n^*, be the 4-vectors defined by (14.48) and let

$$e^* = (0,0,0,1)^{\top}. \tag{15.75}$$

Then

$$(\varphi\,l+\psi\,m+\widetilde{f}_n\,n,\,f_4)^{\top} = \varphi\,l^* + \psi\,m^* + \widetilde{f}_n\,n^* + f_4\,e^*. \tag{15.76}$$

Next we set

$$\begin{aligned} \widetilde{q}_1 &= \Psi_1\,\Psi\,F - \Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,(\widetilde{f}_n\,n^* + f_4\,e^*), \\ \widetilde{q}_2 &= \Psi_2\,\Phi_2^{-1}\,\Phi_1\,\Psi\,F - \Psi_2\,\Phi_2^{-1}\,(\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}+I)(\widetilde{f}_n\,n^* + f_4\,e^*). \end{aligned} \tag{15.77}$$

Applying these notations in (15.73) and (15.74) we get

$$[B^{(1)}(D,n)U^{(1)}]^{+} = \Big([P^{(1)}(D,n)U^{(1)}, \lambda^{(1)}(D,n)u_4^{(1)}]^{+}\Big)^{\top} = \\ = -\Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,(\varphi\,l^* + \psi\,m^*) + \widetilde{q}_1, \tag{15.78}$$

$$[B^{(2)}(D,n)U^{(2)}]^{-} = \Big([P^{(2)}(D,n)U^{(2)}, \lambda^{(2)}(D,n)u_4^{(2)}]^{-}\Big)^{\top} = \\ = -\Psi_2\,\Phi_2^{-1}\,(\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}+I)\,(\varphi\,l^* + \psi\,m^*) + \widetilde{q}_2 = \\ = -\Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,(\varphi\,l^* + \psi\,m^*) + \widetilde{q}_2, \tag{15.79}$$

since

$$-\Psi_2\,\Phi_2^{-1}\,[\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}+I] = -[\Psi_2\,\Phi_2^{-1}\,\Phi_1\,\Psi\,+I]\Psi_2\,\Phi_2^{-1} = \\ = -[(\Psi_1-\Psi^{-1})\,\Psi + I]\,\Psi_2\,\Phi_2^{-1} = -\Psi_1\,\Psi\,\Psi_2\,\Phi_2^{-1} \tag{15.80}$$

due to (15.58).

Substitution of the formulae (15.78)–(15.79) into the interface conditions (15.68)–(15.69) leads to the following system of ΨDEs on S for the unknown functions φ and ψ:

$$-[\Psi_1\Psi\Psi_2\Phi_2^{-1}(\varphi l^*+\psi m^*)]\cdot l^* = 2^{-1}(\widetilde{F}_l^{(+)} + \widetilde{F}_l^{(-)} - \widetilde{q}_1\cdot l^* - \widetilde{q}_2\cdot l^*), \tag{15.81}$$

$$-[\Psi_1\Psi\Psi_2\Phi_2^{-1}(\varphi l^*+\psi m^*)]\cdot m^* = 2^{-1}(\widetilde{F}_m^{(+)} + \widetilde{F}_m^{(-)} - \widetilde{q}_1\cdot m^* - \widetilde{q}_2\cdot m^*). \tag{15.82}$$

This system can also be rewritten as

$$\mathcal{M}_G\,h = q, \tag{15.83}$$

where $h = (\varphi, \psi)^\top$ is the sought for 2-vector, $q = (q_1, q_2)^\top$ is the given 2-vector,

$$\begin{aligned} q_1 &= 2^{-1}(\widetilde{F}_l^{(+)} + \widetilde{F}_l^{(-)} - \widetilde{q}_1 \cdot l^* - \widetilde{q}_2 \cdot l^*), \\ q_2 &= 2^{-1}(\widetilde{F}_m^{(+)} + \widetilde{F}_m^{(-)} - \widetilde{q}_1 \cdot m^* + \widetilde{q}_2 \cdot m^*), \end{aligned} \tag{15.84}$$

$$\mathcal{M}_G = \begin{bmatrix} l_k(\mathcal{K}_G)_{kj} l_j & l_k(\mathcal{K}_G)_{kj} m_j \\ m_k(\mathcal{K}_G)_{kj} l_j & m_k(\mathcal{K}_G)_{kj} m_j \end{bmatrix}_{2\times 2}, \tag{15.85}$$

$$\mathcal{K}_G = -\Psi_1 \, \Psi \, \Psi_2 \, \Phi_2^{-1}, \tag{15.86}$$

in (15.85) the summation over repeated indices k and j is meant from 1 to 3.

Note that $\mathcal{K}_G$ is a 4×4 matrix ΨDO of order 1. As in the proof of Lemma 15.5 we easily derive that the principal homogeneous symbol matrix of the operator $\mathcal{K}_G$ reads as

$$\sigma(\mathcal{K}_G) = -\sigma(\Psi_1)\sigma(\Psi)\sigma(\Psi_2)[\sigma(\Phi_2)]^{-1} = -\begin{bmatrix} [L^{(1)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & L_{44}^{(1)} \end{bmatrix} \times$$

$$\times M^{-1} \begin{bmatrix} [L^{(2)}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & L_{44}^{(2)} \end{bmatrix} \begin{bmatrix} [(K^{(2)})^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & -2 \end{bmatrix}$$

with the same M, $K^{(j)}$, $L^{(j)}$, and $L_{44}^{(j)}$ as in (15.54), due to formulae (15.20)-(15.29) and (15.54). The last equation together with (15.56) implies

$$\sigma(\mathcal{K}_G) = \begin{bmatrix} [Z]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & Z_{44} \end{bmatrix}_{4\times 4}, \tag{15.87}$$

where

$$Z_{44} = 2\, L_{44}^{(1)} \, L_{44}^{(2)} \, [\, L_{44}^{(1)} + L_{44}^{(2)} \,]^{-1} \tag{15.88}$$

is a positive function, while

$$\begin{aligned} Z &= -L^{(1)}(K^{(1)})^{-1}\,[L^{(1)}(K^{(1)})^{-1} - L^{(2)}(K^{(2)})^{-1}]^{-1} L^{(2)}(K^{(2)})^{-1} = \\ &= \{-K^{(2)}(L^{(2)})^{-1}[L^{(1)}(K^{(1)})^{-1} - L^{(2)}(K^{(2)})^{-1}]K^{(1)}(L^{(1)})^{-1}\}^{-1} = \\ &= [K^{(1)}(L^{(1)})^{-1} - K^{(2)}(L^{(2)})^{-1}]^{-1} \end{aligned} \tag{15.89}$$

is a positive definite 3×3 matrix (since the matrices (15.55) are positive definite). Whence for arbitrary $x \in S$, $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$, and $\eta \in \mathbb{C}^3$ there hold the inequalities

$$Z_{44}(x, \widetilde{\xi}) \geq c' \, |\widetilde{\xi}|, \;\; Z(x, \widetilde{\xi})\eta \cdot \eta \geq c'' \, |\widetilde{\xi}| \, |\eta|^2, \tag{15.90}$$

with positive constants c' and c''.

Lemma 15.9. *The principal homogeneous symbol matrices of the ΨDOs $\mathcal{K}_G$ amd $\mathcal{M}_G$ are positive definite.*

Proof. The positive definiteness of $\sigma(\mathcal{K}_G)$ follows from (15.87)-(15.90). In the case of the matrix $\mathcal{M}_G$, for arbitrary $x \in S$, $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$, and $\eta \in \mathbb{C}^2$, we have

$$\sigma(\mathcal{M}_G)\eta \cdot \eta =$$

$$= \begin{bmatrix} l_k(x)l_j(x)[\sigma(\mathcal{K}_G)]_{kj} & l_k(x)m_j(x)[\sigma(\mathcal{K}_G)]_{kj} \\ m_k(x)l_j(x)[\sigma(\mathcal{K}_G)]_{kj} & m_k(x)m_j(x)[\sigma(\mathcal{K}_G)]_{kj} \end{bmatrix}_{2\times 2} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \cdot \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} =$$

$$= [l_k(x)l_j(x)Z_{kj}\eta_1 + l_k(x)m_j(x)Z_{kj}\eta_2]\overline{\eta}_1 +$$

$$+[m_k(x)l_j(x)Z_{kj}\eta_1 + m_k(x)m_j(x)Z_{kj}\eta_2]\overline{\eta}_2 =$$

$$= Z_{kj}[l_j(x)\eta_1 + m_j(x)\eta_2][l_k(x)\overline{\eta}_1 + m_k(x)\overline{\eta}_2] =$$

$$= Z[\eta_1 l(x) + \eta_2 m(x)] \cdot [\eta_1 l(x) + \eta_2 m(x)] \geq$$

$$\geq c''|\widetilde{\xi}|\,|\eta_1 l(x) + \eta_2 m(x)|^2 = c''|\widetilde{\xi}|\,|\eta|^2,$$

due to the second inequality in (15.90). Therefore, $\sigma(\mathcal{M}_G)$ is a positive definite matrix as well. □

Corollary 15.10. *The dominant singular parts of the operators* (15.85) *and* (15.86) *are formally self-adjoint elliptic* Ψ*DOs of order* 1 *with indices equal to zero.*

Next we recall that $J_G(\Omega^1)$ denotes the set of Jones eigenfrequencies for the problem $(\mathcal{G})_\omega$ (see (9.54)–(9.55)) and prove the following assertion.

Lemma 15.11. *If* $\omega \notin J_G(\Omega^1)$, *then the operators*

$$\begin{aligned} \mathcal{M}_G \quad &: \quad [\mathrm{C}^{l,\alpha}(S)]^2 \to [\mathrm{C}^{l-1,\alpha}(S)]^2, \quad 1 \leq l \leq k, \\ &: \quad [H^s_p(S)]^2 \to [H^{s-1}_p(S)]^2, \quad S \in \mathrm{C}^\infty, \ s \in \mathbb{R}, \ 1 < p < \infty, \\ &: \quad [B^s_{p,q}(S)]^2 \to [B^{s-1}_{p,q}(S)]^2, \quad S \in \mathrm{C}^\infty, \ s \in \mathbb{R}, \\ &\qquad\qquad 1 < p < \infty, \ 1 \leq q \leq \infty, \end{aligned}$$

are isomorphisms.

Proof. Again due to the general theory of ΨDOs on closed smooth manifolds, it suffices to show that the homogeneous version of equation (15.83) ($q = 0$) has only the trivial solution in the space $\mathrm{C}^{1,\alpha}(S)$. Let $h = (\varphi, \psi)^\top \in [\mathrm{C}^{1,\alpha}(S)]^2$ be some solution of the homogeneous equation and construct the vectors $U^{(1)}$ and $U^{(2)}$ by formulae (15.61)–(15.62), where $F = 0$ and $f = l^*\varphi + m^*\psi$. Clearly, to the nontrivial pair (φ, ψ) there corresponds the nontrivial vector f since l^* and m^* are orthonormal (see (14.48)). On the other hand it is evident that $(U^{(1)}, U^{(2)}) \in (\mathrm{C}^{1,\alpha}(\overline{\Omega^1})\,,\, \mathrm{C}^{1,\alpha}(\overline{\Omega^2}) \cap \mathrm{SK}^m_r(\Omega^2))$ and they satisfy the homogeneous conditions (15.66)-(15.69), which are equivalent to the homogeneous version of equations (7.5)–(7.8). Therefore, by Theorem 9.9 we conclude $U^{(\mu)} = 0$ in Ω^μ ($\mu = 1, 2$). Now, from the equation $[U^{(1)}]^+ - [U^{(2)}]^- = f = l^*\varphi + m^*\psi = 0$, it follows that $\varphi = \psi = 0$. □

With quite the same arguments as in the previous subsection (see proofs of Theorems 15.7 and 15.8) we derive the following propositions.

Theorem 15.12. *Let* $\omega \notin J_G(\Omega^1)$ *and conditions* (15.72) *be fulfilled. Then the problem* $(\mathcal{G})_\omega$ *is uniquely solvable in the class* $([\mathrm{C}^{k,\alpha}(\overline{\Omega^1})]^4\,,\, [\mathrm{C}^{k,\alpha}(\overline{\Omega^2}) \cap \mathrm{SK}^m_r(\Omega^2)]^4)$ *and the solution is representable in the form of potentials* (15.61) –(15.62)*, where* F *and* f *are given by* (15.70) *and* (15.71)*, respectively, and where* $(\varphi, \psi)^\top \in [\mathrm{C}^{k,\alpha}(S)]^2$ *is the unique solution of the system of* Ψ*DEs* (15.83) *with the right-hand side* $q \in [\mathrm{C}^{k-1,\alpha}(S)]^2$.

Theorem 15.13. *Let* $\omega \notin J_G(\Omega^1)$, $S \in \mathrm{C}^\infty$, *and*

$$\widetilde{F}_l^{(\pm)}\, \widetilde{F}_m^{(\pm)},\, \widetilde{F}_n,\, F_4 \,\in B_{p,p}^{-1/p}(S), \;\; \widetilde{f}_n,\, f_4 \,\in B_{p,p}^{1-1/p}(S), \;\; 1 < p < \infty.$$

Then the problem $(\mathcal{G})_\omega$ *is uniquely solvable in the class* $([W_p^1(\Omega^1)]^4,$ $[W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2)]^4)$ *and the solution is representable in the form of potentials* (15.61)–(15.62)*, where* F *and* f *are given by* (15.70) *and* (15.71)*, respectively, and where* $(\varphi,\psi)^\top \in [B_{p,p}^{1-1/p}(S)]^2$ *is the unique solution of the system of* Ψ*DEs* (15.83) *with the right-hand side* $q \in [B_{p,p}^{-1/p}(S)]^2$.

15.3. Here we investigate the nonhomogeneous problem $(\mathcal{H})_\omega$ applying the same approach as above. Again we start with the reformulation of the interface conditions (7.7)–(7.10) to the equivalent equations

$$[U^{(1)}]^+ - [U^{(2)}]^- = f, \quad [\lambda^{(1)}(D,n)u_4^{(1)}]^+ - [\lambda^{(2)}(D,n)u_4^{(2)}]^- = F_4, \tag{15.91}$$

$$[P^{(1)}(D,n)U^{(1)} \cdot n]^+ - [P^{(2)}(D,n)U^{(2)} \cdot n]^- = \widetilde{F}_n, \tag{15.92}$$

$$\begin{aligned} &[u^{(1)} \cdot l]^+ + [u^{(2)} \cdot l]^- = \widetilde{f}_l^{(+)} + \widetilde{f}_l^{(-)}, \\ &[u^{(1)} \cdot m]^+ + [u^{(2)} \cdot m]^- = \widetilde{f}_m^{(+)} + \widetilde{f}_m^{(-)}, \end{aligned} \tag{15.93}$$

where

$$f = \Big([\widetilde{f}_l^{(+)} - \widetilde{f}_l^{(-)}]\, l + [\widetilde{f}_m^{(+)} - \widetilde{f}_m^{(-)}]\, m + \widetilde{f}_n\, n,\, f_4 \Big)^\top. \tag{15.94}$$

Next we set

$$F = (\varphi\, l + \psi\, m + \widetilde{F}_n\, n,\, F_4)^\top = \varphi\, l^* + \psi\, m^* + \widetilde{F}_n\, n^* + F_4\, e^*, \tag{15.95}$$

where φ and ψ are unknown scalar functions, while l^*, m^*, n^*, and e^* are the same 4-vectors as in the previous subsection. Here we assume either

$$\begin{aligned} &\widetilde{f}_l^{(\pm)},\, \widetilde{f}_m^{(\pm)},\, \widetilde{f}_n, f_4 \,\in \mathrm{C}^{k,\alpha}(S), \;\; \widetilde{F}_n,\, F_4 \,\in \mathrm{C}^{k-1,\alpha}(S), \\ &S \in \mathrm{C}^{k+1,\alpha'}, \;\; k \geq 1, \;\; 0 < \alpha < \alpha' \leq 1, \end{aligned} \tag{15.96}$$

or

$$\begin{aligned} &\widetilde{f}_l^{(\pm)},\, \widetilde{f}_m^{(\pm)},\, \widetilde{f}_n,\, f_4 \,\in B_{p,p}^{1-1/p}(S), \\ &\widetilde{F}_n,\, F_4 \,\in B_{p,p}^{-1/p}(S), \;\; S \in \mathrm{C}^\infty, \;\; 1 < p < \infty. \end{aligned} \tag{15.97}$$

Now we look for the solution to the nonhomogeneous problem $(\mathcal{H})_\omega$ in the form of potentials (15.61)–(15.62), where f and F are defined by (15.94) and (15.95), respectively.

One can easily check that the conditions (15.91) and (15.92) are automatically fulfilled. It remains to satisfy conditions (15.93).

Note that (see (15.10), (15.11), (15.58))

$$[U^{(1)}]^+ = \Phi_1\, \Psi\, (F - \Psi_2\, \Phi_2^{-1}\, f) = \Phi_1\, \Psi\, (\varphi\, l^* + \psi\, m^*) + \widetilde{q}_3, \tag{15.98}$$

$$\begin{aligned} [U^{(2)}]^- &= \Phi_2\, (\Phi_2^{-1}\, \Phi_1\, \Psi\, F - \Phi_2^{-1}\, [\Phi_1\, \Psi\, \Psi_2\, \Phi_2^{-1} + I]\, f) = \\ &= \Phi_1\, \Psi\, (\varphi l^* + \psi m^*) + \widetilde{q}_4, \end{aligned} \tag{15.99}$$

where $\widetilde{q}_3$ and $\widetilde{q}_4$ are given 4-vectors:

$$\begin{aligned} \widetilde{q}_3 &= \Phi_1\,\Psi\,(\widetilde{F}_n\,n^* + F_4\,e^*) - \Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1}\,f, \\ \widetilde{q}_4 &= \Phi_1\,\Psi\,(\widetilde{F}_n\,n^* + F_4\,e^*) - [\Phi_1\,\Psi\,\Psi_2\,\Phi_2^{-1} + I]\,f. \end{aligned} \tag{15.100}$$

Therefore, the interface conditions (15.93) lead to the system of ΨDEs for φ and ψ on S:

$$\begin{aligned} \Phi_1\,\Psi\,(\varphi\,l^* + \psi\,m^*)\cdot l^* &= 2^{-1}[\widetilde{f}_l^{(+)} + \widetilde{f}_l^{(-)} - \widetilde{q}_3\cdot l^* - \widetilde{q}_4\cdot l^*], \\ \Phi_1\,\Psi\,(\varphi\,l^* + \psi\,m^*)\cdot m^* &= 2^{-1}[\widetilde{f}_m^{(+)} + \widetilde{f}_m^{(-)} - \widetilde{q}_3\cdot m^* - \widetilde{q}_4\cdot m^*]. \end{aligned} \tag{15.101}$$

We rewrite these equations in matrix form

$$\mathcal{M}_H\,h = q', \tag{15.102}$$

where $h = (\varphi,\psi)^\top$ is the sought for 2-vector, $q' = (q_1', q_2')^\top$ is the given 2-vector,

$$\begin{aligned} q_1' &= 2^{-1}[\widetilde{f}_l^{(+)} + \widetilde{f}_l^{(-)} - \widetilde{q}_3\cdot l^* - \widetilde{q}_4\cdot l^*], \\ q_2' &= 2^{-1}[\widetilde{f}_m^{(+)} + \widetilde{f}_m^{(-)} - \widetilde{q}_3\cdot m^* - \widetilde{q}_4\cdot m^*], \end{aligned} \tag{15.103}$$

$$\mathcal{M}_H = \begin{bmatrix} l_k(\mathcal{K}_H)_{kj}l_j & l_k(\mathcal{K}_H)_{kj}m_j \\ m_k(\mathcal{K}_H)_{kj}l_j & m_k(\mathcal{K}_H)_{kj}m_j \end{bmatrix}_{2\times 2}, \tag{15.104}$$

$$\mathcal{K}_H = \Phi_1\,\Psi; \tag{15.105}$$

here again the summation over repeated indices k and j is meant from 1 to 3.

By formulae (15.20)-(15.29) and (15.54) we get

$$\sigma(\mathcal{K}_H) = \sigma(\Phi_1)\sigma(\Psi) = \begin{bmatrix} [X]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & X_{44} \end{bmatrix}_{4\times 4}, \tag{15.106}$$

where

$$\begin{aligned} X &= K^{(1)}\Big([L^{(1)}(K^{(1)})^{-1} - L^{(2)}(K^{(2)})^{-1}]K^{(1)}\Big)^{-1} = \\ &= [L^{(1)}(K^{(1)})^{-1} - L^{(2)}(K^{(2)})^{-1}]^{-1} \end{aligned} \tag{15.107}$$

is a positive definite 3×3 matrix and $X_{44} = 2^{-1}[L_{44}^{(1)} + L_{44}^{(2)}]^{-1} > 0$ for arbitrary $x \in S$ and $\widetilde{\xi} \in \mathbb{R}^2 \setminus \{0\}$.

Now by the same reasonings as in the previous subsection one can prove the following propositions.

Lemma 15.14. *The principal homogeneous symbol matrices of the ΨDOs $\mathcal{K}_H$ amd $\mathcal{M}_H$ are positive definite.*

Corollary 15.15. *The dominant singular parts of the operators* (15.104) *and* (15.105) *are formally self-adjoint elliptic ΨDOs of order -1 with indices equal to zero.*

Lemma 15.16. *If $\omega \notin J_H(\Omega^1)$ (i.e., see* (9.56), (9.57)*), then the operators*

$$\begin{aligned} \mathcal{M}_H \ &: \ [\mathrm{C}^{l-1,\alpha}(S)]^2 \to [\mathrm{C}^{l,\alpha}(S)]^2, \ \ 1 \le l \le k, \\ &: \ [H_p^s(S)]^2 \to [H_p^{s+1}(S)]^2, \ \ S \in \mathrm{C}^\infty, \ s \in \mathbb{R}, \ 1 < p < \infty, \\ &: \ [B_{p,q}^s(S)]^2 \to [B_{p,q}^{s+1}(S)]^2, \ \ S \in \mathrm{C}^\infty, \ s \in \mathbb{R}, \end{aligned}$$

$$1 < p < \infty,\ 1 \le q \le \infty,$$

are isomorphisms.

Theorem 15.17. *Let* $\omega \notin J_H(\Omega^1)$, $S \in \mathrm{C}^\infty$, *and conditions* (15.96) [(15.97)] *be fulfilled. Then the nonhomogeneous problem* $(\mathcal{H})_\omega$ *is uniquely solvable in the class*

$$(U^{(1)}, U^{(2)}) \in ([\mathrm{C}^{k,\alpha}(\overline{\Omega^1})]^4\,,\ [\mathrm{C}^{k,\alpha}(\overline{\Omega^2}) \cap \mathrm{SK}_r^m(\Omega^2)]^4)$$

$$\Big[(U^{(1)}, U^{(2)}) \in ([W_p^1(\Omega^1)]^4\,,\ [W_{p,\mathrm{loc}}^1(\Omega^2) \cap \mathrm{SK}_r^m(\Omega^2)]^4)\Big]$$

and the solution is representable in the form of potentials (15.61)–(15.62), *where* f *and* F *are given by* (15.94) *and* (15.95)*, and where*

$$(\varphi, \psi)^\top \in [\mathrm{C}^{k,\alpha}(S)]^2 \quad \Big[(\varphi, \psi)^\top \in [B_{p,p}^{1-1/p}(S)]^2\Big]$$

is the unique solution to the system of ΨDEs (15.102).

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Authors' addresses:

Lothar Jentsch
Fakultät für Mathematik
Technische Univertät Chemnits
D-09107 Chemnitz
Germany

David Natroshvili
Department of Mathematics
Georgian Technical University
77, Kostava St., Tbilisi 380075
Georgia