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ON SOME BOUNDARY VALUE PROBLEMS FOR HIGH ORDER ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Singular boundary value problems are considered for high order nonlinear equations in the case where the right-hand side may have singularities both in independent and phase variables. Existence, uniqueness theorems are proved. A priori asymptotic estimates of solutions are obtained. The obtained problems, in the case of the second order, involve those arising while studying the flow of a viscuous fluid when written in the so-called Crocco variables.

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Introduntion

The present paper deals with the problem of finding a solution of the n-th order differential equation

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}) \tag{0.1}$$

satisfying the boundary conditions

$$\lim_{t\to t_0^+} u^{(i-1)}(t) = c_i \quad (i = 1, \dots, n-1), \qquad \lim_{t\to b-} u^{(n-1)}(t) = 0, \tag{0.2}$$

where $-\infty < a \le t_0 < b \le +\infty$ and $c_i \ge 0$ $(i = 1, \dots, n-1)$, and the function $f : [a, b[\times]0, +\infty[^n \to [-\infty, 0]$ satisfies the local Carathéodory conditions. The specific character of this problem is that $f(t, x_1, \dots, x_n)$ may be unbounded both as $x_i \to 0+$ $(i = 1, \dots, n)$ and $t \to b-$.

We call the problem (0.1), (0.2) the Blasius–Crocco type problem, motivating this by the fact that problems of such kind go back to the classical work of Blasius [1]. While studying the flow of a semi-infinite plate by a homogeneous stream of a viscuous incompressible fluid, Blasius arrived at the singular boundary value problem

$$v''' + vv'' = 0,$$
$$v(0) = v'(0) = 0, \quad \lim_{s\to\infty} v'(s) = 1.$$

Written in the so-called Crocco vaiables (see [2]), it takes the form

$$u'' = -\frac{t}{u}, \tag{0.3}$$

$$u'(0) = 0, \quad u(t) > 0 \quad \text{for} \quad 0 < t < 1, \quad \lim_{t\to 1-} u(t) = 0. \tag{0.4}$$

If one gives up the requirement that the fluid should be incompressible, the above mentioned problem on the flow of a plate reduces to the boundary value problem (0.4) for the equation

$$u'' = -\frac{h(t)}{u} \tag{0.5}$$

with a nonnegative coefficint h, which was considered by Callegari and Friedman [2]. The problems (0.3), (0.4) and (0.4), (0.5) are easily reduced to the problem (0.1), (0.2).The problem

$$\lim_{t\to 0-} u(t) = 0, \quad u'(1) = 0$$

studied by Callegari and Nachman [3] is also a special case of (0.1), (0.2).

The Blasius-Crocco type problem with conditions at infinity for the equation

$$u^{(n)} = g(t)|u|^\lambda \operatorname{sign} u$$

with $\lambda < 0$ and g nonpositive, as well as a related problem on a finite interval, was studied by S. Talliaferro [7,8]. We study these problems for higher order equations.

Everywhere below by $K_{\mathrm{loc}}(I \times D, J)$, where $I, J \subset R$ are some intervals and $D \subset R^n$, we denote the class of all functions $f : I \times D \to J$ satisfying the local Carathéodory conditions. This means that $f(\cdot, x) : I \to J$ is measurable for all $x \in D$, $f(t, \cdot) : D \to J$ is continuous for almost all $t \in J$ and $\sup\{|f(\cdot, x)| : x \in D_0\}$ is locally integrable for any compactum $D_0 \subset D$.

1. On Equations with the Property V

In this section, we consider the equation

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}) \tag{1.1}$$

under the assumptions

$$n \geq 1, \quad -\infty < a < +\infty, \quad f \in K_{\mathrm{loc}}([a, +\infty[\times]0, +\infty[^n; R). \tag{1.2}$$

Definition 1.1. We say that the equation (1.1) has *the property V* if for any $t_0 \geq a$ and $r > 0$ there is a positive $\eta(t_0, r)$ such that any solution $u : [t_0, t^*[\to R$ of the equation (8.1) satisfying

$$0 < u^{(i-1)}(t_0) \leq r \quad (i = 1, \dots, n-1), \quad 0 < u^{(n-1)}(t_0) \leq \eta(t_0, r) \tag{1.3}$$

when $n > 1$ and

$$0 < u(t_0) \leq \eta(t_0) \tag{1.4}$$

when $n = 1$ satisfies also

$$t^* < +\infty, \quad \lim_{t \to t^*-} u^{(n-1)}(t) = 0. \tag{1.5}$$

Theorem 1.1. *Let the inequality*

$$-f(t, x_1, \dots, x_n) \geq \varphi(t, x_1, \dots, x_n), \tag{1.6}$$

be fulfilled on $[a, +\infty[\times]0, +\infty[^n$, *where* $\varphi \in K_{\mathrm{loc}}([a, +\infty[\times]0, +\infty[^n; R_+)$ *is nonincreasing in the last n arguments and for some n*

$$\mathrm{mes}\{\tau \geq t : \varphi(\tau, (\tau - a)^{n-1} x_0, \dots, (\tau - a) x_0, x_0) > 0\} > 0 \ \textit{for}\ t \geq a. \tag{1.7}$$

Then the equation (1.1) *has the property V. Moreover, if*

$$\int_a^{+\infty} \varphi(t, (t - a)^{n-1} x, \dots, (t - a)x, x) dt = +\infty \quad \textit{for} \quad x > 0, \tag{1.8}$$

then the equation (8.1) *has no solution defined in the vicinity of* $+\infty$.

Proof. Let $t_0 \ge a$ and $r > 0$. One can find $t_1 > t_0$ such that

$$t_1 - t_0 \ge 1, \quad \frac{x_0}{2}(t_1 - a) \ge r(n-1). \tag{1.9}$$

In view of (1.7), $\alpha \in]0, \frac{x_0}{2}]$ can be chosen such that

$$0 < \alpha < \int\limits_{t_1}^{+\infty} \varphi(t, (t-a)^{n-1}x_0, \dots, (t-a)x_0, x_0)dt. \tag{1.10}$$

Put $\eta(t_0, r) = \alpha$ and show that any solution $u : [t_0, t^*[\to R$ of the equation (1.1) satisfying (1.3) for $n > 1$ and (1.4) for $n = 1$ satisfies also (1.5).

Suppose the contrary. Then $t^* = +\infty$ and $0 < u^{(n-1)}(t) \le \frac{x_0}{2}$ for $t \ge t_0$. By (1.3) ((1.4) for $n = 1$) and (1.9) we have

$$u^{(i-1)}(t) = \sum_{j=0}^{n-j-1} \frac{(t-t_0)^j}{j!} u^{(j)}(t_0) +$$

$$+\frac{1}{(n-i-1)!}\int\limits_{t_0}^{t}(t-\tau)^{n-i-1}u^{(n-1)}(\tau) \le r(n-i)(t-a)^{n-i-1} +$$

$$+\frac{x_0}{2}(t-a)^{n-i} \le (t-a)^{n-i}x_0 \quad \text{for} \quad t \ge t_1 \quad (i = 1, \dots, n-1). \tag{1.11}$$

Therefore, taking (1.6) into account as well as the monotonicity of φ in the last n arguments, we find

$$0 < u^{(n-1)}(t) \le \alpha - \int\limits_{t_1}^{t} \varphi(\tau, (\tau-a)^{n-1}x_0, \dots, (\tau-a)x_0, x_0)d\tau \quad \text{for} \quad t \ge t_1,$$

which is impossible in view of (1.10). The obtained contradiction shows that u satisfies (1.5).

Suppose now that (1.8) is fulfilled. Then, as it is clear from the above reasoning, $\eta(t_0, r)$ can be taken arbitrarily large. Therefore, any solution of (1.1) satisfies (1.5). ■

2. Solvability and Estimates of Solutions of a Blasius-Crocco Type Problem

In this section, we study the Blasius-Crocco type problem

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}), \tag{2.1}$$

$$\lim_{t \to t_0+} u^{(i-1)}(t) = c_i \quad (i = 1, \dots, n-1), \quad \lim_{t \to b-} u^{(n-1)}(t) = 0, \tag{2.2}$$

where

$$n \ge 1, \quad -\infty < a \le t_0 < b \le +\infty, \quad c_i \ge 0 \quad (i = 1, \dots, n-1),$$
$$f \in K_{\mathrm{loc}}([a, b[\times]0, +\infty[^n; R). \tag{2.3}$$

In the first two subsections, a particular case of the problem (2.1), (2.2) is considered, when $c_i > 0$ $(i = 1, \dots, n-1)$ and $b = +\infty$, i.e., the problem

$$u^{(i-1)}(t_0) = c_i \quad (i = 1, \dots, n-1), \quad \lim_{t \to +\infty} u^{(n-1)}(t) = 0. \tag{2.4}$$

Everywhere below we assume that the conditions (1.2) are fulfilled.

2.1. Theorems on Existence and Uniqueness of Solution of the Problem (2.1), (2.4).

Theorem 2.1. *Let $t_0 \geq a$, $c_i > 0$, $(i = 1, \dots, n-1)$, f be nondecreasing in the last n arguments and let for any $c > 0$ the Cauchy problem*

$$u^{(i-1)}(t_0) = c_i \quad (i = 1, \dots, n-1), \quad u^{(n-1)}(t_0) = c \tag{2.5}$$

for the equation (2.1) *be uniquely solvable. Then the problem* (2.1), (2.4) *has at most one solution.*

Proof. Suppose, on the contrary, that the problem (2.1), (2.5) has two different solutions u_1 and u_2. The unique solvability of Cauchy problems implies that, without restriction of generality, we can assume that $u_1^{(n-1)}(t_0) < u_2^{(n-1)}(t_0)$. According to the well known lemma on integral inequalities (see, e.g., [4], Lemma 4.3), hence we have

$$u_1^{(i-1)}(t) < u_2^{(i-1)}(t) \quad \text{for} \quad t \geq t_0 \quad (i = 1, \dots, n).$$

Therefore, taking the monotonicity of f into account, we obtain $u_1^{(n)}(t) \leq u_2^{(n)}(t)$ for $t \geq t_0$, and this fact along with the preceding inequalities contradicts to $u_1^{(n-1)}(+\infty) = u_2^{(n-1)}(+\infty) = 0$. The obtained contradiction proves the theorem. ∎

Theorem 2.2. *Let $t_0 \geq a$, $c_i > 0$ $(i = 1, \dots, n-1)$, the equation* (1) *have the property V, for any $c > 0$ the problem* (2.5), (2.6) *be uniquely solvable and let the inequalities*

$$0 \leq -f(t, x_1, \dots, x_n) \leq \psi(t, x_1, \dots, x_n) \tag{2.6}$$

be fulfilled on $[a, +\infty[\times]0, +\infty[^n$, where $\psi \in K_{\mathrm{loc}}([a, +\infty[\times]0, +\infty[^n; R_+)$ is nonincreasing in the last n arguments and the equation

$$u^{(n)} = -\psi(t, u, u', \dots, u^{(n-1)}) \tag{2.7}$$

have at least one solution u satisfying (2.4). *Then the problem* (2.1), (2.4) *is solvable.*

Proof. Let $u_c : [t_0, b_c[\to R$ be the solution of the problem (2.1), (2.5). We say that $c \in \Gamma_1$ if $b_c = +\infty$ and $\lim\limits_{t \to +\infty} u_c^{(n-1)}(t) > 0$, we say $c \in \Gamma_2$ if $b_c = +\infty$ and $\lim\limits_{t \to +\infty} u_c^{(n-1)}(t) = 0$, and we say $c \in \Gamma_3$ if $b_c < +\infty$ and $\lim\limits_{t \to b_c^-} u_c^{(n-1)}(t) = 0$.

Suppose that, contrary to the assertion of the theorem, $\Gamma_2 = \varnothing$. Let $c > u^{(n-1)}(t_0)$. According to (2.4) and the above mentioned lemma on integral inequalities, we have

$$u_c^{(n-1)}(t) > u^{(n-1)}(t) \quad \text{for} \quad t_0 \le t < b_c, \tag{2.8}$$

Therefore $b_c = +\infty$, and since according to the hypothesis $\Gamma_2 = \varnothing$, we obtain $c \in \Gamma_1$. Thus $]u^{(n-1)}(t_0), +\infty[\subset \Gamma_1$.

Now take an arbitrary $c_0 \in \Gamma_1$. There is $t_1 \in]t_0, +\infty[$ such that $u_{c_0}^{(i-1)}(t_1) > u^{(i-1)}(t_1)$ $(i = 1, \dots, n)$. In view of the continuous dependence of solutions of a differential equation on initial values, the last inequalities remain valid if we change c_0 by any sufficiently close $c > 0$. According to (2.6) and the lemma on integral inequalities, this implies the validity of (2.8) for $t_1 \le t < b_c$. Therefore, since $\Gamma_2 = \varnothing$, all c_0 sufficiently close to $c > 0$ belong to Γ_1. Thus Γ_1 is a nonempty open subset of $]0, +\infty[$.

The nonemptiness of Γ_3 follows from the property V of the equation (2.1). Let $c_0 \in \Gamma_3$. Put

$$\delta = \max\{b_{c_0} - t_0, 1\}, \; r = \max\{c_i : 1 \le i \le n-1\}, \; r_0 = \delta^{n-1}[r(n-1) + c_0 + 1],$$

and choose $t_1 \in [t_0, b_{c_0}[$ such that

$$0 < u_{c_0}^{(n-1)}(t_1) < \eta(b_{c_0}, r_0),$$

where $\eta(b_{c_0}, r_0)$ is the number appearing in the definition of the property V. For any $c > 0$ sufficiently close to c_0, we will have either $b_c > b_{c_0}$, and then $c \in \Gamma_3$, or $b_c > b_{c_0}$ and

$$0 < u_c^{(i-1)}(b_{c_0}) \le r_0 \quad (i = 1, \dots, n-1), \quad 0 < u_c^{(n-1)}(b_{c_0}) < \eta(b_{c_0}, r_0).$$

In the last case, the property V again implies $c \in \Gamma_3$. Thus both Γ_1 and Γ_3 are nonempty open subsets of $]0, +\infty[$.

On the other hand, $]0, +\infty[= \Gamma_1 \cup \Gamma_3$ and $\Gamma_1 \cap \Gamma_3 = \varnothing$. But this is impossible since $]0, +\infty[$ is connected. The obtained contradiction proves the theorem. ■

Theorem 2.3. *Let $t_0 \ge a$, $c_i > 0$ $(i = 1, \dots, n-1)$, $\psi \in K_{\mathrm{loc}}([a, +\infty[\times]0, +\infty[^n; R_+)$ be nonincreasing in the last n arguments, the equation* (2.1) *have the property V, for any $c > 0$ the problem* (2.7), (2.5) *be uniquely solvable, and let the problem*

$$\frac{dx}{dt} = -\psi\Big(t, \frac{(t-a)^{n-1}}{(n-1)!}x, \dots, (t-a)x, x\Big), \tag{2.9}$$

$$\lim_{t \to +\infty} x(t) = 0 \tag{2.10}$$

have at least one solution defined on $[t_0, +\infty[$.[1] *Then the problem* (2.7), (2.4) *has a unique solution.*

[1] According to Theorem 2.1, this solution is unique.

Proof. Let $u_c : [t_0, b_c[\to R$ be a solution of the problem (2.7), (2.4). Writing for u_c the equalities analogous to (8.11) and taking into account the nonincreasingness of $u_c^{(n-1)}$, we find

$$u_c^{i-1}(t) \geq \frac{(t-a)^{n-i}}{(n-i)!} u_c^{(n-1)}(t) \quad \text{for} \quad t_0 \leq t < b_c. \tag{2.11}$$

Therefore

$$u_c^{(n)}(t) \geq -\psi\Big(t, \frac{(t-a)^{n-1}}{(n-1)!} u_c^{(n-1)}(t), \dots, (t-a) u_c^{(n-1)}(t), u_c^{n-1}(t)\Big) \tag{2.12}$$
$$\text{for} \quad t_0 \leq t < b_c.$$

Define the sets Γ_j $(j = 1, 2, 3)$ as in the proof of Theorem 2.2. Repeating the arguments given there and taking Theorem 2.1 into account, we see that it suffices to prove the following: the assumption $\Gamma_2 = \varnothing$ implies that Γ_1 is a nonempty open subset of $]0, +\infty[$.

So let $\Gamma_2 = \varnothing$ and $c > x(t_0)$. According to (2.12) and the lemma on integral inequalities, we have

$$u_c^{(n-1)}(t) > x(t) \quad \text{for} \quad t_0 \leq t < b_c. \tag{2.13}$$

Therefore $c \in \Gamma_1 \cup \Gamma_2 = \Gamma_1$ and thus $\Gamma_1 \neq \varnothing$.

Let now $c_0 \in \Gamma_1$. Then for some $t_1 \geq t_0$ we have $u_{c_0}^{(n-1)}(t_1) > x(t_1)$ which along with (2.12) implies (2.13) for $t_1 \leq t < b_c$ and all $c > 0$ sufficiently close to c_0. ■

Lemma 2.1. *Let $t_0 \geq a$, $\overline{\psi} \in K_{\mathrm{loc}}([a, +\infty[\times]0, +\infty[; R_+)$ be nonincreasing in the second argument, the equation*

$$\frac{dx}{dt} = -\overline{\psi}(t, x) \tag{2.14}$$

have the property V, and let for any $c > 0$ the Cauchy problem

$$x(t_0) = c \tag{2.15}$$

for this equation be uniquely solvable. Let, moreover,

$$\int_a^{+\infty} \overline{\psi}(t, x) dt < +\infty \quad \textit{for} \quad x > 0. \tag{2.16}$$

Then the problem (2.14), (2.10) *has a unique solution defined on* $[t_0, +\infty[$.

Proof. Let $x_c : [t_0, b_c[\to R$ be the solution of the problem (2.14), (2.15). Define the sets Γ_j $(j = 1, 2, 3)$ as in the proof of Theorem 2.2. As above, it suffices to prove that Γ_1 is a nonempty open subset of $]0, +\infty[$.

In view of (2.16) and the monotonicity of $\overline{\psi}$ in the second argument, $\alpha > 0$ can be chosen such that

$$\int_{t_0}^{+\infty} \overline{\psi}(t, \alpha) < \alpha. \tag{2.17}$$

Let $c > 2\alpha$. Show that then $c \in \Gamma_1$. Indeed, if it is not the case, one can find $t_1 \in]t_0, b_c[$ such that

$$x_c(t) > \alpha \quad \text{for} \quad t_0 \le t < t_1, \quad x_c(t_1) = \alpha. \tag{2.18}$$

But then by (2.17), (2.18) and the monotonicity of $\overline{\psi}$, we will have

$$x_c(t_1) = x_c(t_0) - \int_{t_0}^{t_1} \overline{\psi}(t, x(t))dt > 2\alpha - \alpha = \alpha,$$

which contradicts (2.18). The obtained contradiction shows that $]2\alpha, +\infty[\subset \Gamma_1$.

Let now $c_0 \in \Gamma_1$. There exists $\alpha_0 > 0$ such that $x_c(t_1) > 2\alpha_0$ for $t \ge t_0$. Choose $t_1 \ge t_0$ such that

$$\int_{t_1}^{+\infty} \overline{\psi}(t, \alpha_0) < \alpha_0.$$

For all $c > 0$ sufficiently close to c_0, we will have $x_c(t_1) > 2\alpha_0$. Hence, as above, it follows that for all such c the function x_c can not admit the value equal to α_0. Therefore Γ_1 is a nonempty open set. ■

Theorem 2.4. *Let $t_0 \ge a$, $c_i > 0$ $(i = 1, \dots, n-1)$ and the inequality*

$$\varphi(t, x_1, \dots, x_n) \le -f(t, x_1, \dots, x_n) \le \psi(t, x_1, \dots, x_n) \tag{2.19}$$

hold on $[a, +\infty[\times]0, +\infty[^n$, where φ, $\psi \in K_{\mathrm{loc}}([a, +\infty[\times]0, +\infty[^n; R_+)$ are nonincreasing in the last n arguments and

$$0 < \int_t^{+\infty} \varphi(\tau, (\tau - a)^{n-1}x, \dots, (\tau - a)x, x)d\tau \le$$

$$\le \int_t^{+\infty} \psi(\tau, (\tau - a)^{n-1}x, \dots, (\tau - a)x, x)d\tau < +\infty \ \text{ for } t \ge a, \quad x > 0. \tag{2.20}$$

Then the problem (2.1), (2.4) has a solution. If, moreover, f is nondecreasing in the last n arguments, then this solution is unique.

Proof. We can replace ψ by the function ψ^* satisfying all the hypotheses of the theorem and, in addition, the local Lipshitz conditions in the last n arguments. But then the theorem becomes an immediate consequence of Theorems 1.1, 2.1-2.3 and Lemma 2.1. ■

2.2. Asymptotic Behaviour of Solutions of the Problem (2.1), (2.2). Everywhere in this subsection we will have $t_0 \geq a$, $c_i > 0$ $(i = 1, \dots, n-1)$.

First of all, we will mention here some definitions and a lemma on singular differential inqualities which will be of use immediately.

Let $w \in K_{\mathrm{loc}}(]a, b[\times R; R)$, where $a < b \leq +\infty$ and $c_0 \in R$. Consider the problem

$$\frac{dx}{dt} = w(t, x), \quad \lim_{t \to b-} x(t) = c_0. \tag{P}$$

Definition 2.1. A solution x^* (x_*) of the problem (P) defined on $]a_0, b[\subset]a, b[$ is said to be an upper (a lower) solution of the problem (P), if for any solution x of this problem defined on an interval $]a_1, b[\subset]a, b[$, the inequality

$$x(t) \leq x^*(t) \quad (x(t) \geq x_*(t)) \quad \text{for} \quad t \in]a_0, b[\cap]a_1, b[$$

holds.

We have [5]

Lemma 2.2. *Let the problem* (P) *have an upper* (*a lower*) *solution* x^* (x_*) *defined on* $]a_0, b[\subset]a, b[$. *Then for any locally absolutely continuous function* $v :]a_0, b[\to R$ *satisfying*

$$\limsup_{t \to b-} v(t) \leq c_0 \quad (\liminf_{t \to b-} v(t) \geq c_0)$$

and almost everywhere on $]a_0, b[$ *the inequality*

$$v'(t) \geq w(t, v(t)) \quad (v'(t) \leq w(t, v(t)),$$

we have

$$v(t) \leq x^*(t) \quad (x(t) \geq x_*(t)) \quad \textit{for} \quad a_0 < t < b.$$

Theorem 2.5. *Let the inequalities* (2.19) *be fulfilled on* $[a, +\infty[\times]0, +\infty[^n$, *where* $\varphi, \psi \in K_{\mathrm{loc}}([a, +\infty[\times]0, +\infty[^n; R_+)$ *are nonincreasing in* $x_1, \dots, x_{n-1}$, *the problem* (2.9), (2.10) *has an upper solution* x^* *defined on* $[t_0, +\infty[$, *and the problem*

$$\frac{dx}{dt} = -\varphi(t, \beta(t), \dots, \beta^{(n-2)}(t), x), \tag{2.21}$$

$$\lim_{t \to +\infty} x(t) = 0. \tag{2.22}$$

where

$$\beta(t)=\sum_{j=1}^{n-1}\frac{c_j(t-t_0)^{j-1}}{(j-1)!}+\frac{1}{(n-2)!}\int_{t_0}^{t}(t-\tau)^{n-2}x^*(\tau)d\tau \quad for \ t\geq t_0, \tag{2.23}$$

has a lower solution x_* defined on $[t_0,+\infty[$. Then for any solution u of the problem (2.1), (2.4) *we have*

$$\beta_*^{(j-1)}(t)\leq u^{(i-1)}(t)\leq \beta^{(i-1)}(t) \quad for \quad t\geq t_0 \quad (i=1,\dots,n), \tag{2.24}$$

where

$$\beta_*(t)=\sum_{j=1}^{n-1}\frac{c_j(t-t_0)^{j-1}}{(j-1)!}+\frac{1}{(n-2)!}\int_{t_0}^{t}(t-\tau)^{n-2}x_*(\tau)d\tau \tag{2.25}$$

$$for \quad t\geq t_0.$$

Proof. Let u be an arbitrary solution of the problem (2.1), (2.4). Analogously to (2.11), we have

$$u^{(i-1)}(t)\geq \frac{(t-a)^{n-i}}{(n-i)!}u^{(n-1)}(t) \quad \text{for} \quad t\geq t_0 \quad (i=1,\dots,n-1).$$

Taking (2.19) into account as well as the monotonicity of ψ, and applying Lemma 2.2, we find $u^{(n-1)}(t)\leq x^*(t)$ for $t\geq t_0$, i.e.,

$$u^{(i-1)}(t)\leq \beta^{(i-1)}(t) \quad \text{for} \quad t\geq t_0 \quad (i=1,\dots,n). \tag{2.26}$$

From (2.19), (2.26) and the monotonicity condition implied on φ, according to Lemma 2.2 it follows that $u^{(n-1)}(t)\geq x_*(t)$ for $t\geq t_0$ which along with (2.26) proves the estimates (2.24). ■

Remark. Let φ and ψ be nonincreasing in the last n arguments and the conditions (2.19) and (2.20) be fulfilled. Then the problems (2.9), (2.10) and (2.21), (2.22) have unique solutions.

Indeed, the existence of a unique solution of the problem (2.9), (2.10) follows from Lemma 2.1 (taking Theorem 1.1 into account). In order to ascertain the unique solvability of the problem (2.21), (2.22), it suffices to notice that

$$\beta^{(i-1)}(t)\leq \alpha(t-a)^{n-i} \quad (i=1,\dots,n)$$

for large t, where α is a positive number.

Theorem 2.6. *Let $n\geq 2$ and the inequalities*

$$\varphi(t,x_1,\dots,x_{n-1})\leq -f(t,x_1,\dots,x_n)\leq \psi(t,x_1,\dots,x_{n-1}) \tag{2.27}$$

be fulfilled on $[a,+\infty[\times]0,+\infty[^n$, *where* $\varphi,\psi\in K_{\mathrm{loc}}([a,+\infty[\times]0,+\infty[^{n-1};R_+)$ *are nonincreasing in the last* $n-1$ *arguments. Let for at least one solution* u *of the problem* (2.1), (2.4) *a finite limit*

$$\lim_{t\to+\infty} u^{(n-2)}(t)<+\infty \tag{2.28}$$

exist. Then for some $\alpha>c_{n-1}$,

$$\int_a^{+\infty}(t-a)\varphi(t,(t-a)^{n-2})\alpha,\dots,(t-a)\alpha,\alpha)dt<+\infty. \tag{2.29}$$

Moreover, if

$$\int_a^{+\infty}(t-a)\psi(t,(t-a)^{n-2}c_{n-1},\dots,(t-a)c_{n-1},c_{n-1})dt<+\infty,$$

then for any solution u *of the problem* (2.1), (2.4) *we have* (2.28).

Proof. Let a solution u of the problem (2.1), (2.4) satisfy (2.28). Then, denoting this limit by α, we will have

$$u^{(i-1)}(t)\le\alpha(t-a)^{n-i-1}\quad\text{for}\quad t\ge t_1\quad(i=1,\dots,n-1),$$

where $t_1\ge t_0$ is sufficiently large. By (2.27) and the monotonicity of φ

$$u^{(n-1)}(t)\ge\int_t^{+\infty}\varphi(\tau,(\tau-a)^{n-2}\alpha,\dots,(\tau-a)\alpha,\alpha)d\tau\quad\text{for}\quad t\ge t_1.$$

Integrating from t to $+\infty$, we find

$$\int_t^{+\infty}(\tau-t)\varphi(\tau,(\tau-a)^{n-2}\alpha,\dots,(\tau-a)\alpha,\alpha)d\tau\le\alpha<+\infty,$$

whence it follows (2.29). The second part of the theorem can be proved analogously. ■

From Theorems 2.4 –2.6, it immediately follows

Theorem 2.7. *Let the inequalities*

$$p(t)x_1^\lambda\le -f(t,x_1,\dots,x_n)\le q(t)x_1^\lambda,$$

be fulfilled on $[a,+\infty[\times]0,+\infty[^n$, *where* $\lambda<0$, $p,q\in L_{\mathrm{loc}}([a,+\infty[;R_+)$ *and*

$$0<\int_t^{+\infty}(\tau-a)^{(n-1)\lambda}p(\tau)\le\int_t^{+\infty}(\tau-a)^{(n-1)\lambda}q(\tau)d\tau<+\infty\quad\textit{for}\quad t\ge a.$$

Then the problem (2.1), (2.4) *is solvable and any of its solutions satisfies*

$$u^{(n-1)}(t) \le \Big[\frac{1-\lambda}{[(n-1)!]^{\lambda}} \int\limits_{t}^{+\infty} (\tau-a)^{(n-1)\lambda} q(\tau)d\tau\Big]^{\frac{1}{1-\lambda}} \quad for \quad t \ge t_0. \quad (2.30)$$

If, moreover,

$$\int\limits_{a}^{+\infty} (t-a)^{\lambda(n-2)+1} q(t)dt < +\infty,$$

then a finite limit $\lim_{t\to+\infty} u^{(n-2)}(t)$ *exists. If, however,*

$$\int\limits_{a}^{+\infty} (t-a)^{\lambda(n-2)+1} p(t)dt = +\infty,$$

then $\lim_{t\to+\infty} u^{(n-2)}(t) = +\infty$, *and for any* $\gamma > 1$,

$$u^{(n-1)}(t) \ge \Big[\frac{\gamma}{(n-2)!}\Big]^{\lambda} \int\limits_{t}^{+\infty} p(\tau)\Big[\int\limits_{a}^{\tau} (\tau-s)^{n-2}\widetilde{q}(s)ds\Big]^{\lambda} d\tau$$

for large t, *where* $\widetilde{q}(t)$ *is the right-hand side of the inequality* (2.30).

2.3. The Problem (2.1), (2.2). The Case Where $b < +\infty$ and $c_i > 0$ $(i = 1,\dots,n-1)$. The approach which was used for the investigation of the problem (2.1), (2.4) can be likewise applied to the problem

$$u^{(i-1)}(t_0) = c_i \quad (i = 1,\dots,n-1), \quad \lim_{t\to b^-} u^{(n-1)}(t) = 0 \qquad (2.31)$$

for the equation (2.1), where

$$-\infty < a \le t_0 < b < +\infty, \quad c_i > 0 \quad (i = 1,\dots,n-1). \qquad (2.32)$$

The following two theorems will be stated without proof.

Theorem 2.8. *Let the conditions* (2.32) *be fulfilled, for any* $c > 0$ *the problem* (2.1), (2.5) *be uniquely solvable, and let the inequalities* (2.19) *hold on* $[a, b[\times]0, +\infty[^n$, *where* $\varphi, \psi \in K_{\mathrm{loc}}([a, b[\times]0, +\infty[^n; R_+)$ *are nonincreasing in the last* n *arguments and*

$$0 < \int\limits_{t}^{b} \varphi(\tau, x, \dots, x)d\tau \le \int\limits_{t}^{b} \psi(\tau, x, \dots, x)d\tau < +\infty \qquad (2.33)$$
$$for \quad a \le t < b, \quad x > 0.$$

Then the problem (2.1), (2.31) *has a solution. If, moreover,* f *is nondecreasing in the last* n *arguments, then this solution is unique.*

Theorem 2.9. *Let the conditions* (2.32) *be fulfilled and let the inequalities* (2.19) *hold on* $[a,b[\times]0,+\infty[^n$, *where* $\varphi,\psi\in K_{\rm loc}([a,b[\times]0,+\infty[^n;R_+)$ *is nonincreasing in* $x_1,\dots,x_{n-1}$, *the problem*

$$\lim_{t\to b^-} x(t)=0 \tag{2.34}$$

for the equation (2.9) *have an upper solution* x^* *defined on* $[t_0,b[$, *and the problem* (2.21), (2.34), *with the function* β *given by* (2.23), *have a lower solution* x_* *defined on* $[t_0,b[$. *Then any solution* u *of the problem* (2.1), (2.31) *satisfies* (2.2) *on* $[t_0,b[$, *where* β_* *is defined by* (2.25).

Remark. Let φ and ψ be nonincreasing in the last n arguments and (2.19) and (2.33) be fulfilled. Then the problems (2.9), (2.34) and (2.21), (2.34) have unique solutions.

2.4. The Problem (2.1), (2.2). The General Case. In this subsection, we consider the problem (2.1), (2.2) under the the general hypotheses (2.3).

Theorem 2.10. *Let the conditions* (2.3) *be fulfilled and the inequalities* (2.19) *hold on* $[a,b[\times]0,+\infty[^n$, *where* $\varphi,\psi\in K_{\rm loc}([a,b[\times]0,+\infty[^n;R_+)$ *are nonincreasing in the last* n *arguments and satisfy* (2.20). *Let, moreover, any Cauchy problem for the equation* (2.1) *be uniquely solvable. Then the problem* (2.1), (2.2) *is solvable. Moreover, any of its solutions satisfy* (2.24) *on* $]t_0,b[$, *where* β *and* β_* *are defined by* (2.23) *and* (2.25), *and* x^* *and* x_* *are the unique solutions of the problems* (2.9), (2.34) *and* (2.21), (2.34), *respectively.* [2]

Proof. For the sake of definiteness, we will assume that $b<+\infty$. The case $b=+\infty$ can be considered analogously. To ascertain the estimates (2.24), it suffices to consider a sequence $(t_k)_{k=1}^\infty$ such that $t_k\in]a,b[$ $(k=1,2,\dots)$ and $\lim\limits_{k\to\infty} t_k=t_0$, to apply Theorem 2.9, and then to pass to limit as $k\to\infty$.

Prove now the existence of a solution. Let $r=\max\{c_i:1\le i\le n-1\}$, and let $(c_{ik})_{k=1}^\infty$ $(i=1,\dots,n-1)$ be sequences satisfiying

$$0<c_{ik}<r+1\quad (k=1,2,\dots),\quad \lim_{k\to\infty} c_{ik}=c_i\quad (i=1,\dots,n-1). \tag{2.35}$$

In the case of the finite b, (2.20) implies (2.33), so fixing an arbitrary k, according to Theorem 2.8 we see that the problem

$$u^{(i-1)}(t_0)=c_{ik}\quad (i=1,\dots,n-1),\quad \lim_{t\to b^-} u^{(n-1)}(t)=0$$

for the equation (2.1) has a solution u_k. Moreover, by Theorem 2.9 and (2.35), we have

$$\delta_*^{(i-1)}(t)\le u_k^{(i-1)}(t)\le \delta^{(i-1)}(t)\quad \text{for}\quad t_0\le t<b\quad (i=1,\dots,n), \tag{2.36}$$

[2]The existence and uniqueness of x^* and x_* follow from Remarks to Theorems 2.5 and 2.9.

where

$$\delta(t) = (r+1)\sum_{j=0}^{n-2} \frac{(t-a)^j}{j!} + \frac{1}{(n-2)!}\int_{t_0}^{t} (t-\tau)^{n-2} x^*(\tau)d\tau,$$

$$\delta_*(t) = \frac{1}{(n-2)!}\int_{t_0}^{t} (t-\tau)^{n-2} x_*(\tau)d\tau \quad \text{for} \quad t_0 < t < b.$$

From (2.36) we see that the sequences $(u_k^{(j-1)})_{k=1}^{\infty}$ $(i = 1, \dots, n)$ are uniformly bounded and equicontinuous on each subsegment of $]t_0, b[$. Therefore, by the Arzela-Ascoli lemma, we can assume that they converge uniformly on every such subsegment. The function $u(t) = \lim\limits_{k\to\infty} u_k(t)$ for $t_0 < t < b$ obviously is a solution of the problem (2.1), (2.2). ■

2.5. On Solutions of an Emden-Fowler Type Equation with the Negative Exponent. In this subsection, we consider the equation

$$u^{(n)} = g(t)|u|^{\lambda} \operatorname{sign} u, \tag{2.37}$$

with $g \in L_{\mathrm{loc}}([a, +\infty[; R)$ and

$$\lambda < 0, \quad g(t) \le 0 \quad \text{for} \quad t \ge a. \tag{2.38}$$

Note that if

$$\operatorname{mes}\{\tau \ge t : g(\tau) \ne 0\} \quad \text{for} \quad t \ge a \tag{2.39}$$

and for some solution $u : [t_0, t_1[\to R$ of (9.37) there is $t^* \in]t_0, t_1[$ such that (8.5) is fulfilled, then

$$t_1 < +\infty, \quad \lim_{t \to t_1-} u(t) = 0. \tag{2.40}$$

So under the conditions (2.38) and (2.39), the property V of the equation (2.37) is equivalent to the following one: for any $t_0 \ge a$ and $r > 0$, there is $\eta(t_0, r) > 0$ such that for any solution $u : [t_0, t_1[\to R$ of (2.37) satisfying (1.3) if $n > 1$ and (1.4) if $n = 1$, the condition (2.40) holds.

Theorem 1.1 immediately implies

Theorem 2.11. *Let* (2.36) *and* (2.39) *be fulfilled. Then the equation* (2.37) *has the property* V. *If, moreover,*

$$\int_{a}^{+\infty} (t-a)^{(n-1)\lambda} |g(t)| dt = +\infty,$$

then the equation (2.37) *does not have solutions defined in the vicinity of* $+\infty$.

Below we will be interested in the solutions of (2.27) satisfying

$$u(t) > 0, \quad u^{(n-2)}(t) > 0 \quad \text{for} \quad t \geq t_0, \quad \lim_{t \to +\infty} u^{(n-1)}(t) = 0. \quad (2.41)$$

Theorem 2.7 implies

Theorem 2.12. *Let* (2.38) *and* (2.29) *be fulfilled. The equation* (2.37) *has solutions satisfying* (2.41) *if and only if*

$$\int_a^{+\infty} (t-a)^{(n-1)\lambda} |g(t)| dt < +\infty, \tag{2.42}$$

and in this case there exists an $(n-1)$*-parametrical family of such solutions. If, moreover,*

$$\int_a^{+\infty} (t-a)^{(n-2)\lambda+1} |g(t)| dt < +\infty,$$

then for any solution u *of* (2.37) *satisfying* (2.41) *there exists a finite limit* $\lim\limits_{t \to +\infty} u^{(n-2)}(t) = c$ *and*

$$u^{(n-2)}(t) = c - c^{\lambda}(1 + o(1)) \int_t^{+\infty} (\tau - t)(\tau - a)^{\lambda(n-2)} |g(\tau)| d\tau \quad \textit{for} \quad t \to +\infty.$$

If along with (2.42) *it holds*

$$\int_a^{+\infty} (t-a)^{\lambda(n-2)+1} |g(t)| dt = +\infty,$$

then for any solution u *of* (2.37) *satisfying* (2.41), *we have*

$$\lim_{t \to +\infty} u^{(n-2)}(t) = +\infty \tag{2.43}$$

and, given $\gamma > 1$,

$$\left[\frac{\gamma}{(n-2)!}\right]^{\lambda} \int_t^{+\infty} |g(\tau)| \left[\int_a^{\tau} (\tau - s)^{n-2} \widetilde{g}(s) ds\right]^{\lambda} d\tau \leq u^{(n-1)}(t) \leq \widetilde{g}(t)$$

for large t, *where*

$$\widetilde{g}(t) = \left[\frac{1-\lambda}{[(n-1)!]^{\lambda}} \int_t^{+\infty} (\tau - a)^{(n-1)\lambda} |g(\tau)| d\tau\right]^{\frac{1}{1-\lambda}} \quad \textit{for} \quad t \geq t_0.$$

Theorem 2.13. *Let $\lambda < 0$ and $\gamma_1 \le -t^\sigma g(t) \le \gamma_2$ for $t > 0$ with $\sigma \in R$ and γ_1, γ_2 positive constants. Then* (2.37) *has a solution satisfying* (2.41) *and* (2.43) *iff $\lambda(n-1)+1 < \sigma \le \lambda(n-2)+2$. Moreover, any of such solutions u admits the estimate*

$$c_* t^{\frac{n-\sigma}{1-\lambda}} \le u(t) \le c^* t^{\frac{n-\sigma}{1-\lambda}}$$

for large t, where c_* and c^* are positive constants depending only on n, σ, λ, γ_1, and γ_2.

3. On a Modification of the Blasius–Crocco Problem

In this section, we consider the problem

$$u^{(n)} = f(t, u, u', \dots, u^{(n-1)}), \tag{3.1}$$

$$\lim_{t\to t_0+} u^{(i-1)}(t) = c_i \quad (i = 1, \dots, n-1), \quad \lim_{t\to b-} u(t) = 0, \tag{3.2}$$

Everywhere below we will assume, not stating it explicitly, that $n \ge 1$, $-\infty < a \le t_0 < b < +\infty$, $c_i \ge 0$ $(i = 1, \dots, n-1)$ and $f \in K_{\mathrm{loc}}([a, b[\times]0, +\infty[^n; R)$.

Lemma 3.1. *Let $-\infty < t_0 < b < +\infty$, $s_0 = (b-t_0)^{-1}$ and $u : [t_0, b[\to R$ be locally absolutely continuous along with its $(n-1)$-st derivatives inclusively. Then the function $v : [s_0, +\infty[\to R$ defined by*

$$v(s) = s^{n-1} u\Big(b - \frac{1}{s}\Big) \tag{3.3}$$

almost everywhere on $[s_0, +\infty[$ satisfies

$$v^{(n)}(s) = \frac{1}{s^{n-1}}\, u^{(n)}\Big(b - \frac{1}{s}\Big).$$

Moreover,

$$s^{i-1} v^{(i-1)}(s) = \sum_{j=1}^{i} \gamma_{nij} s^{n-j} u^{(j-1)}\Big(b - \frac{1}{s}\Big)$$
$$\text{for} \quad s_0 \le s < +\infty \quad (i = 1, \dots, n+1),$$

where $\gamma_{nij} \ge 0$ $(n = 1, 2, \dots;\ 1 \le j \le i \le n+1)$.

The proof of this lemma can be found in [6].

If f is nonpositive, then by Lemma 3.1, the function $u \in \widetilde{C}^{n-1}_{\mathrm{loc}}$ is a solution of the problem (3.1), (3.2) iff the function $v \in \widetilde{C}^{n-1}_{\mathrm{loc}}(]s_0, +\infty[)$ defined by (3.3) with $s_0 = (b-t_0)^{-1}$ is a solution of the problem

$$v^{(n)} = \widetilde{f}(s, v, v', \dots, v^{(n-1)}), \tag{3.4}$$

$$\lim_{s\to s_0+} v^{(i-1)}(s_0) = \sum_{j=1}^{i} \gamma_{nij} s_0^{n-i-j+1} c_j, \quad \lim_{s\to+\infty} v^{(n-1)}(s) = 0, \tag{3.5}$$

where $\widetilde{f} \in K_{\rm loc}([(b-a)^{-1}, +\infty[\times]0, +\infty[^n; R)$ is nonpositive and the constants γ_{nij} $(i = 1, \dots, n-1;\ j = 1, \dots, i)$ are nonnegative. If, moreover, $n \geq 2$, then

$$\lim_{t \to b-} u'(t) = -\infty \tag{3.6}$$

if and only if

$$\lim_{s \to +\infty} v^{(n-2)}(s) = +\infty. \tag{3.7}$$

Indeed, let (3.6) hold. Then $\lim\limits_{t \to b-} \frac{u(t)}{b-t} = +\infty$, i.e.,

$$\lim_{s \to +\infty} \frac{v(s)}{s^{n-2}} = +\infty. \tag{3.8}$$

By (3.5) and the nonpositiveness of $\widetilde{f}$, the limit $\lim\limits_{s \to +\infty} v^{(n-2)}(s) \in]0, +\infty]$ exists. If this limit is finite, then such will be the limit of $v(s)/s^{n-2}$ as well which contradicts (3.7). Analogously, (3.7) implies (3.6).

Taking the above said into account and applying Theorems 2.10 and 2.6 to the problem (3.4), (3.5), we ascertain the following

Theorem 3.1. *Let the inequalities*

$$\varphi(t, x_1) \leq -f(t, x_1, \dots, x_n) \leq \psi(t, x_1), \tag{3.9}$$

be fulfilled on $[a, b[\times]0, +\infty[^n$, *where* $\varphi, \psi \in K_{\rm loc}([a, b[\times]0, +\infty[; R_+)$ *are nonincreasing in the second argument and*

$$0 < \int_t^b (b-t)^{n-1} \varphi(\tau, x)\, d\tau \leq \int_t^b (b-\tau)^{n-1} \psi(\tau, x)\, d\tau < +\infty \quad for \quad a \leq t < b,\ x > 0.$$

Then the problem (3.1), (3.2) *is solvable.*

Theorem 3.2. *Let* $n \geq 2$ *and the inequalities* (3.8) *be fulfilled on* $[a, b[\times]0, +\infty[^n$, *where* $\varphi, \psi \in K_{\rm loc}([a, b[\times]0, +\infty[; R_+)$ *are nonincreasing in the second argument. Let for at least one solution* u *of* (3.1), (3.2) *a finite limit*

$$\lim_{t \to b-} u'(t) > -\infty \tag{3.10}$$

exist. Then for some $\alpha > 0$ *we have*

$$\int_a^b (b-t)^{n-2} \varphi(t, (b-t)\alpha)\, dt < +\infty.$$

On the other hand, if

$$\int_a^b (b-t)^{n-2} \psi(t, (b-t)x)\, dt < +\infty \quad for \quad x > 0,$$

then any solution u of (3.1), (3.2) *satisfies* (3.10).

Consider the Emden-Fowler type equation

$$u^{(n)} = g(t)|u|^{\lambda} \operatorname{sign} u, \tag{3.11}$$

where $g \in L_{\text{loc}}([a, b[; R)$ and

$$\lambda < 0, \quad g(t) \le 0 \quad \text{for} \quad a \le t < b. \tag{3.12}$$

The transformation (3.3) changes (3.11) into

$$v^{(n)} = s^{-[n+1+\lambda(n-1)]} g\Big(b - \frac{1}{s}\Big) |v|^{\lambda} \operatorname{sign} v.$$

Applying to the last equation Theorems 2.12 and 2.13, we ascertain the following.

Theorem 3.3. *Let $t_0 \in [a, b[$, (3.12) hold and*

$$\operatorname{mes}\{\tau \in [t, b[, \quad g(\tau) \ne 0\} > 0 \quad for \quad a \le t < b.$$

The equation (3.11) *has a solution u satisfying*

$$u(t) > 0, \quad u'(t) < 0 \quad for \quad t \ge t_0, \quad \lim_{t \to b-} u(t) = 0 \tag{3.13}$$

if and only if

$$\int_a^b (b - t)^{n-1} |g(t)| dt < +\infty, \tag{3.14}$$

and in this case there exists an $(n-1)$-parametric family of such solutions. If, moreover,

$$\int_a^b (b - t)^{\lambda+n-2} |g(t)| dt < +\infty,$$

then for any solution u of (3.11) *satisfying* (3.13), *there exists a finite limit $\lim\limits_{t \to +\infty} u'(t) = c < 0$ and $u(t) \sim |c|(b - t)$. If, however, along with* (3.14) *the condition*

$$\int_a^b (b - t)^{\lambda+n-2} |g(t)| dt = +\infty$$

is fulfilled, then for any solution u of (3.11) *satisfying* (3.13) *we have* (3.6) *and, given $\gamma > 1$,*

$$\frac{1}{(n-1)!} \Big[\frac{\gamma}{(n-2)!}\Big]^{\lambda} \int_t^b (b - \tau)^{\lambda+n-1} |g(\tau)| \Big[\int_a^{\tau} \frac{\widetilde{g}(\xi) d\xi}{(b - \xi)^2}\Big]^{\lambda} d\tau \le$$

$$\leq u(t) \leq \frac{\gamma}{(n-2)!}(b-t)\int\limits_a^t \frac{\widetilde{g}(\tau)}{(b-\tau)^2}d\tau$$

for all t sufficiently close to b, where

$$\widetilde{g}(t) = \Big[\frac{1-\lambda}{[(n-1)!]^\lambda}\int\limits_t^b (b-\tau)^{n-1}|g(\tau)|d\tau\Big]^{\frac{1}{1-\lambda}} \quad \textit{for} \quad a \leq t < b.$$

Theorem 3.4. *Let $\lambda < 0$ and $\gamma_1 \leq -(b-t)^\sigma g(t) \leq \gamma_2$ with $\sigma \in R$ and γ_1 and γ_2 positive constants. Then* (3.11) *has a solution u satisfying* (3.13) *and* (3.6) *iff $\lambda + n - 1 \leq \sigma < n$. Moreover, any of such solutions for all t sufficiently close to b admits the estimates*

$$\underline{c}(b-t)^{\frac{n-\sigma}{1-\lambda}} \leq u(t) \leq \overline{c}^{\frac{n-\sigma}{1-\lambda}}(b-t),$$

where $\underline{c}$ and $\overline{c}$ are positive constants depending only on n, σ, λ, γ_1 and γ_2.

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