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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF n-th ORDER DIFFERENTIAL EQUATIONS

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Consider the differential equation

$$y^{(n)}=\sum_{i=1}^{m}\alpha_i p_i(t)|y|^{\sigma_{0i}}|y'|^{\sigma_{1i}}\dots|y^{(l-1)}|^{\sigma_{l-1i}}\operatorname{sign} y, \tag{1}$$

where $n\geq 2$, $\alpha_i\in\{-1,1\}$ $(i=1,\dots,m)$, $l\in\{1,\dots,n\}$; $\sigma_{0i},\dots,\sigma_{l-1i}$ are real constants such that $\sum\limits_{k=0}^{l-1}\sigma_{ki}\neq 1$ for any $i\in\{1,\dots,m\}$, $\sum\limits_{k=0}^{l-1}\sigma_{ki}\neq\sum\limits_{k=0}^{l-1}\sigma_{kj}$ for $i\neq j$, and $p_i:[a,\omega[\longrightarrow]0,+\infty[$ $(i=1,\dots,m)$ are continuous functions with $-\infty<a<\omega\leq+\infty$.

We call a solution of the equation (1) defined on some interval $[t_0,+\infty[\subset[a,+\infty[$ a P_ω–solution if it satisfies the following three conditions:

a) $y^{(n)}(t)\neq 0$;

b) for any $k\in\{0,\dots,n-1\}$, either $\lim\limits_{t\uparrow\omega}y^{(k)}(t)=0$ or $\lim\limits_{t\uparrow\omega}y^{(k)}(t)=\pm\infty$;

c) the limit

$$\lim_{t\uparrow\omega}\frac{\left[y^{(n-1)}(t)\right]^2}{y^{(n)}(t)\,y^{(n-2)}(t)}=\lambda^0_{n-1}$$

exists and is finite or infinite.

Some asymptotic properties of solutions of the equations (1) satisfying the conditions a), b) can be obtained according to results from the monograph of I. T. Kiguradze and T. A. Chanturia [1] and in the scientific works of A. V. Kostin [2]. Exact asymptotic formulas for all types of P_ω– solutions are established only in the case $m=1$ (see [3], [4]).

In this paper, the question of asymptotics of all P_ω–solutions of the equation (1) with $\lambda^0_{n-1}\neq 1$ is considered.

We introduce the auxiliary notation:

$$M=\{1,2,\dots,m\},\quad J=\{(i,j)\in M\times M\;:\;i\neq j\};$$

$$\gamma_i=1-\sum_{k=0}^{l-1}\sigma_{ki}\,,\quad \mu_{ri}=(n-r)+\sum_{k=0}^{l-1}(r-1-k)\sigma_{ki}\,,$$

$$i=1,\dots,m,\qquad r=1,\dots,n;$$

$$a_k=(n-k)\lambda^0_{n-1}-(n-k-1),\quad k=1,\dots,n-1;$$

$$\pi_\omega(t)=\begin{cases} t & \text{for}\quad \omega=+\infty\\ t-\omega & \text{for}\quad \omega<+\infty\end{cases},\quad I_{ri}(t)=\int_{A_{ri}}^{t}p_i(\tau)|\pi_\omega(\tau)|^{\mu_{ri}}d\tau,$$

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$$A_{ri} = \begin{cases} a & \text{for} \quad \int\limits_a^\omega p_i(\tau)|\pi_\omega(\tau)|^{\mu_{ri}} d\tau = +\infty \\ \omega & \text{for} \quad \int\limits_a^\omega p_i(\tau)|\pi_\omega(\tau)|^{\mu_{ri}} d\tau < +\infty \end{cases}, \quad i = 1, \ldots m\,, \quad r = 1, \ldots, n.$$

Further, we investigate the case where for each fixed pair $(i, j) \in J$ the following two conditions are fulfilled:

1) the functions

$$\varphi_{rs}(t) = p_s(t) \left[\frac{p_i(t)}{p_j(t)}\right]^{\frac{\gamma_s}{\gamma_j - \gamma_i}} |\pi_\omega(t)|^{\mu_{rs} + \frac{(\mu_{ri} - \mu_{rj})\gamma_s}{\gamma_j - \gamma_i}} \qquad (s = 1, \ldots, m)$$

are asymptotically comparable if $t \uparrow \omega$;

2) the limit $\lim\limits_{t\uparrow\omega} \dfrac{h_{rij}(t)}{\varphi_{rg}(t)} = h_g^0$ exists and is finite or infinite for the function

$$h_{rij}(t) = \frac{1}{\gamma_i - \gamma_j} \left[\frac{p_i'(t)}{p_i(t)} - \frac{p_j'(t)}{p_j(t)} + \frac{\mu_{ri} - \mu_{rj}}{\pi_\omega(t)}\right],$$

where $r \in \{1, \ldots, n\}$ and φ_{rg} $(g \in M)$ are such that $\lim\limits_{t\uparrow\omega} \dfrac{\varphi_{ri}(t)}{\varphi_{rg}(t)} = \Phi_{rig}^0 = \text{const}$ for any $i \in M$.

Theorem 1. *Let the conditions* 1), 2) *be true for each fixed pair* $(i, j) \in J$ *and for* $r = n$. *Then any positive* P_ω*-solution of the equation* (1) *with* $\lambda_{n-1}^0 \notin \{0, \frac{1}{2}, \ldots, \frac{n-2}{n-1}, 1, \pm\infty\}$ *assumes either the asymptotic representations*

$$\begin{aligned} y^{(k-1)}(t) &\sim b_{0k}[\pi_\omega(t)]^{n-k} y^{(n-1)}(t), \quad k = 1, \ldots, n-1, \\ y^{(n-1)}(t) &\sim c_{ij} \left[\frac{p_i(t)}{p_j(t)} |\pi_\omega(t)|^{\mu_{ni} - \mu_{nj}}\right]^{\frac{1}{\gamma_i - \gamma_j}}, \quad (i, j) \in J; \end{aligned} \tag{2}$$

or the representations

$$\begin{aligned} y^{(k-1)}(t) &\sim b_{0k}[\pi_\omega(t)]^{n-k} y^{(n-1)}(t), \quad k = 1, \ldots, n-1, \\ y^{(n-1)}(t) &\sim \alpha_i |\gamma_i d_{oi} I_{ni}(t)|^{\frac{1}{\gamma_i}} \operatorname{sign}[\gamma_i I_{ni}(t)], \quad i \in M \end{aligned} \tag{3}$$

as $t \uparrow \omega$, *where*

$$c_{ij} \neq 0, \quad b_{0k} = \frac{[\lambda_{n-1}^0 - 1]^{n-k}}{\prod\limits_{i=k}^{n-1} a_i}, \quad d_{0i} = |\lambda_{n-1}^0 - 1|^{\mu_{ni}} \prod_{k=0}^{l-1} \left|\prod_{j=k+1}^{n-1} a_j\right|^{-\sigma_{ki}}.$$

Theorem 2. *Let the conditions* 1), 2) *be true for each fixed pair* $(i, j) \in J$ *and for* $r = n$. *Then any positive* P_ω*-solution of the equation* (1) *with* $\lambda_{n-1}^0 = \pm\infty$ *assumes either the asymptotic representations* (2) *or the representations* (3), *in which the constants* b_{0k}, d_{0i} *are replaced by* b_k^0, d_i^0, *where*

$$b_k^0 = \lim_{\lambda_{n-1}^0 \to \infty} b_{0k} = \frac{1}{(n-k)!}, \qquad d_i^0 = \lim_{\lambda_{n-1}^0 \to \infty} d_{0i} = \prod_{k=0}^{l-1} [(n-1-k)!]^{-\sigma_{ki}}.$$

Theorem 3. *Let $r \in \{1, \dots, n-1\}$, $l \le r$ and the conditions 1), 2) be true for each fixed pair $(i,j) \in J$. Then any positive P_ω–solution of the equation (1) with $\lambda^0_{n-1} = \dfrac{n-r-1}{n-r}$ assumes either the asymptotic representations*

$$
\begin{aligned}
y^{(k-1)}(t) &\sim \frac{[\pi_\omega(t)]^{r-k}}{(r-k)!}\, y^{(r-1)}(t), \quad k = 1, \dots, r-1, \\
y^{(r-1)}(t) &\sim c_{ij} \left[\frac{p_i(t)}{p_j(t)} |\pi_\omega(t)|^{\mu_{ri}-\mu_{rj}} \right]^{\frac{1}{\gamma_i - \gamma_j}}, \quad (i,j) \in J, \\
y^{(r)}(t) &= \varphi_{rg}(t) y^{(r-1)}(t)\,[h^0_g + 0(1)], \\
y^{(k)}(t) &\sim (-1)^{k-r} \frac{(k-r)!}{[\pi_\omega(t)]^{k-r}} y^{(r)}(t), \quad k = r+1, \dots, n,
\end{aligned}
\tag{4}
$$

or the representations

$$
\begin{aligned}
y^{(k-1)}(t) &\sim \frac{[\pi_\omega(t)]^{r-k}}{(r-k)!}\, y^{(r-1)}(t), \quad k = 1, \dots, r-1, \\
y^{(r-1)}(t) &\sim \alpha_i \gamma_i |\gamma_i d_{ri} I_{ri}(t)|^{\frac{1}{\gamma_i}} \operatorname{sign} I_{ri}, \quad i \in M, \\
y^{(r)}(t) &\sim \alpha_i d_{ri} p_i(t) |\pi_\omega(t)|^{\mu_{ri}} |y^{(r-1)}(t)|^{1-\gamma_i}, \\
y^{(k)}(t) &\sim (-1)^{k-r} \frac{(k-r)!}{[\pi_\omega(t)]^{k-r}}\, y^{(r)}(t), \quad k = r+1, \dots, n,
\end{aligned}
\tag{5}
$$

as $t \uparrow \omega$, *where*

$$
c_{ij} \neq 0, \qquad d_{ri} = \frac{\prod\limits_{k=o}^{l-1} [(r-k-1)!]^{-\sigma_{ki}}}{(n-r)!}.
$$

In addition, we have obtained necessary and sufficient conditions for the existence of all types of P_ω–solutions of the equation (1) defined by Theorems $1-3$.

We formulate Theorems 4 and 5 for the case where $\lambda^0_{n-1} = \dfrac{n-r-1}{n-r}$.

Theorem 4. *Let the hypotheses of Theorem 3 be true. Then for the existence of positive P_ω–solutions of the type (5) with $\lambda^0_{n-1} = \dfrac{n-r-1}{n-r}$, it is necessary and sufficient that*

$$
\alpha_i \gamma_i I_{ri}(t) [\pi_\omega(t)]^{r-1} > 0 \quad \text{for } t \in (a, \omega),
$$

$$
\lim_{t \uparrow \omega} \frac{p_s(t) |\pi_\omega(t)|^{\mu_{rs}-\mu_{ri}}}{p_i(t)} \cdot \frac{[I_{rs}(t)]^{\frac{1-\gamma_s}{\gamma_s}}}{[I_{ri}(t)]^{\frac{1-\gamma_i}{\gamma_i}}} = 0 \quad \text{for any } s \neq i,
$$

$$
\lim_{t \uparrow \omega} \frac{\pi_\omega(t)\, G'_{ri}(t)}{G_{ri}(t)} = r + 1 - n,
$$

where

$$
G_{ri}(t) = \int_{Q_{ri}}^{t} p_i(\tau) |\pi_\omega(\tau)|^{\mu_{ri}-n+r+1} |I_{ri}(\tau)|^{\frac{1-\gamma_i}{\gamma_i}}\, d\tau
$$

and Q_{ri} are defined by analogy with A_{ri}.

Theorem 5. *Let the hypotheses of Theorem* 3 *be true. Then for the existence of positive* P_ω*– solutions of the equation* (1) *of the type* (4) *with* $\lambda_{n-1}^0 = \dfrac{n-r-1}{n-r}$*, it is necessary that*

$$\int_a^\omega \varphi_{rg}(t)\,dt = \infty\,, \quad h_g^0 = \text{const}, \quad c_{ij}[\pi_\omega(t)]^{r-1} > 0\,, \tag{6}$$

$$-h_g^0 c_{ij}\,\text{sign}[-\pi_\omega(t)]^{n-r} + \sum_{s=1}^m \alpha_s d_{rs}\Phi_{rsg}^0 |c_{ij}|^{1-\gamma_s} = 0\,, \tag{7}$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) I'_{rij}(t)}{I_{rij}(t)} = r+1-n\,, \tag{8}$$

where

$$I_{rij}(t) = \int_{A_{rij}}^t \varphi_{rg}(\tau)|\pi_\omega(\tau)|^{r+1-n+\frac{\mu_{ri}-\mu_{rj}}{\gamma_i-\gamma_j}} \left[\frac{p_i(\tau)}{p_j(\tau)}\right]^{\frac{1}{\gamma_i-\gamma_j}} d\tau$$

and A_{rij} *are defined by analogy with* A_{ri}*.*

Suppose that along with (6) – (8)*, the conditions*

$$h_g^0 \neq 0\,, \qquad \sum_{k=1}^m \alpha_k d_{rk}\Phi_{rkg}^0 |c_{ij}|^{-\gamma_k}[\sigma_{0k} + \dots + \sigma_{l-1k} - 1] \neq 0$$

are fulfilled. Then the equation (1) *has positive* P_ω*-solutions of the type* (4)*.*

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