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ASYMPTOTIC REPRESENTATIONS OF OSCILLATORY SOLUTIONS OF A NONLINEAR EQUATION

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Consider the equation

$$u'' + t^{-(n+3)/2}|u|^n \operatorname{sign} u = 0, \tag{1}$$

where $t > 0$, $n > 0$. In this note asymptotic formulas are derived for oscillatory solutions of (1) which are somewhat different from those given in [1].

Theorem 1. *Let $0 < n < 1$. Then for any nontrivial oscillatory soluition $u(t)$ of the equation (1) the equalities*

$$u(t) = t^{1/2} f^{-1}[\sqrt{c}\sin((c_0 + o(1))\ln t)], \quad t \to +\infty,$$
$$u'(t) = (1/2)t^{-1/2} f^{-1}[\sqrt{c}\sin((c_0 + o(1))\ln t)] +$$
$$+\sqrt{c}t^{-1/2}\cos((c_0 + o(1))\ln t), \quad t \to +\infty,$$

hold, where $c_0 > 0$, f^{-1} is the function inverse to

$$f(z) = \operatorname{sign} z\sqrt{2|z|^{1+n}/(1+n) - z^2/4} \quad \text{for} \quad |z| < 2^{2/(1-n)}$$

and

$$0 < c < ((1-n)/(1+n))2^{2(1+n)/(1-n)}.$$

Proof. By means of the transformation

$$x(s) = t^{1/2}u'(t), \quad y(s) = t^{-1/2}u(t), \quad s = \ln t,$$

the equation (1) can be written as

$$x' = x/2 - |y|^n \operatorname{sign} y, \quad y' = x - y/2. \tag{2}$$

The nontrivial oscillatory solution of (2) defined by the initial conditions $x(0) = \sqrt{c}$, $y(0) = 0$ satisfies

$$x^2(s) - x(s)y(s) + 2|y(s)|^{1+n}/(1+n) \equiv c.$$

Moreover,

$$|x(s)| < 2^{(1+n)/(1-n)}, \quad |y(s)| < 2^{2/(1-n)} \quad \text{for} \quad -\infty < s < +\infty.$$

Introduce the functions

$$w_1(\tau) = \operatorname{sign} y(s)\sqrt{2|y(s)|^{1+n}/(1+n) - y^2(s)/4},$$
$$w_2(\tau) = x(s) - y(s)/2,$$

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where

$$\tau = \int_0^s \frac{|y(\vartheta)|^n - |y(\vartheta)|/4}{\sqrt{2|y(\vartheta)|^{1+n}/(1+n) - y^2(\vartheta)/4}} d\vartheta. \tag{3}$$

If the numbers s_n are such that $y(s_n) = 0$, then (2) implies

$$y'(s_n) = x(s_n) = \pm\sqrt{c} \neq 0.$$

Therefore, $\varepsilon > 0$ and $\delta > 0$ can be found such that

$$|y(s)| \geq \varepsilon_n |s - s_n| \quad \text{for} \quad s \in [s_n - \delta_n, s_n + \delta_n],$$

Hence the integrals $\int_{s_n}^s |y(\vartheta)|^{(n-1)/2} d\vartheta$ converge. This means that the improper integral (3) converges for any s.

Taking into account, for instance, [2, p. 235], it can be easily verified that $w_1(\tau)$, $w_2(\tau)$ is a solution of the problem

$$w_1' = w_2, \quad w_2' = -w_1, \quad w_1(0) = 0, \quad w_2(0) = \sqrt{c}.$$

Therefore,

$$w_1(\tau) = \sqrt{c}\sin\tau, \quad w_2(\tau) = \sqrt{c}\cos\tau.$$

In view of the periodicity of $y(s)$, there exists a finite limit

$$c_0 = \lim_{s\to+\infty} \frac{\tau(s)}{s},$$

and since $\tau'(s) > 0$, we have $c_0 > 0$. ■

Remark 1. The theorem remains true for $t \to 0+$.

Remark 2. Asymptotic representations of oscillatory and nonoscillatory solutions of (1) with $n > 1$ are given in [3].

References

1. Mirzov J. D., *Differentsial'nye Uravneniya* **32**(1996), No. 11, 1576.
2. Filippov V. V., Solution spaces of ordinary differential equations. (Russian) *Moscow University Press, Moscow,* 1993.
3. Mirzov J. D., Asymptotic formulas for solutions of an Emden–Fowler equation. (Russian) *Trudy FORA,* 1997, No. 2, 49–55.

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