Mem. Differential Equations Math. Phys. 12(1997), 210–218

Milan Tvrdý

LINEAR INTEGRAL EQUATIONS IN THE SPACE OF REGULATED FUNCTIONS

Abstract. In this paper, we investigate the existence of solutions to a wide class of systems of linear integral equations with solutions which can have in the closed interval [0,1] only discontinuities of the first kind and are left–continuous on the corresponding open interval (0,1). The results cower, e.g., the results knownfor systems of linear generalized differential equations as well as systems of Stieltjes Integral equations. Some possible applications to functional differential equationsare discussed as well.

რეზიუმე. ნაშრომში წრფივ ინტეგრალურ განტოლებათა სისტემების ფართო კლასისათვის გამოკვლეულია ისეთ ამონახსნთა არსებობა, რომლებსაც ჩაკეტილ $[0,1]$ ინტერვალში მხოლოდ პირველი გვარის წყვეტა შეიძლება ჰქონდეთ, ხოლო $(0,1)$ ინტერვალში მარცხნიდან უწყვეტებია. მიღებული შედეგები მოიცავს ცნობილ შედეგებს როგორც წრფივი განზოგადებული დიფერენციალური განტოლებებისათვის, ასევე სტილტიესის ინტეგრალური განტოლებებისათვის. განხილულია ზოგიერთი შესაძლო გამოყენება ფუნქციონალურ-დიფერენციალური განტოლებებისათვის.

0. Introduction

In this note we will describe some of the recent results concerning linear operator equations in the space of regulated functions. The equations considered cover as special cases e.g. the generalized linear differential equations in the sense of J.Kurzweil (cf. [4], [7] and [5]) as well as the linear Volterra–Stieltjes integral equations

$$\mathbf{x}(t) - \int_0^t \mathrm{d}_s[\mathbf{K}(t,s)]\mathbf{x}(s) = \mathbf{f}(t), \quad t \in [0,1], \tag{0.1}$$

and various types of functional differential equations.

Solutions are sought in the space of n-vector valued functions which are regulated on $[0,1]$.

1991 *Mathematics Subject Classification.* 45A05, 47G10, 26A39, 26A42.

Key words and phrases. Regulated function, Fredholm–Stieltjes integral equation, Voltera–Stieltjes integral equation, compact operator, Perron–Stieltjes integral, Kurzweil integral.

1. Regulated functions

First, let us recall some of the basic properties of regulated functions.

Real functions $x : [0,1] \to \mathbb{R}$ possessing for all $t \in [0,1)$ and $s \in (0,1]$ finite one-sided limits $x(t+) = \lim_{\tau \to t+} x(\tau)$ and $x(s-) = \lim_{\tau \to s-} x(\tau)$ are said to be *regulated* on $[0,1]$. The linear space of functions $x : [0,1] \to \mathbb{R}$ regulated on $[0,1]$ is denoted by $\mathbb{G}$, while $\mathbb{G}_L$ stands for the set of all functions regulated on $[0,1]$ and left-continuous on $(0,1)$. Both $\mathbb{G}$ and $\mathbb{G}_L$ become Banach spaces when equipped with the norm $x \in \mathbb{G} \to \|x\| = \sup_{t\in[0,1]} |x(t)|$ (cf. [2]).

Furthermore, for any $x \in \mathbb{G}$ there is a sequence $\{x_k\}_{k=1}^{\infty}$ of *finite step functions* on $[0,1]$ such that $\|x_k - x\| \to 0$ as $k \to \infty$, i.e., $\mathbb{G}$ is the closure of the set $\mathbb{S}$ of finite step functions ($\mathbb{G} = \mathrm{cl}(\mathbb{S})$). Moreover,

$$\mathbb{G}_L = \mathrm{cl}\Bigl(\mathrm{Lin}\Bigl(\bigl\{\chi_{[0,1]}, \chi_{(\tau,1]}(\tau \in (0,1)), \chi_{[1]}\bigr\}\Bigr)\Bigr), \tag{1.1}$$

where as usual χ_M stands for the characteristic function of the set M and $\mathrm{Lin}(\Xi)$ stands for the space of all finite linear combinations of the elements of the set Ξ.

The integrals are considered in the *Perron–Stieltjes* sense. We work with the following equivalent summation definition due to J. Kurzweil (cf. [4]) which is now usually called the *Kurzweil–Henstock integral* or the *gauge integral*:

For a given partition $D = \{t_0, \tau_1, t_1, \ldots, \tau_m, t_m\}$ of $[0,1]$ (i.e. $0 = t_0 \le \tau_1 \le t_1 \le \cdots \le \tau_m \le t_m = 1$ and $t_0 < t_1 < \cdots < t_m$) we denote $|D| = m$ and

$$\Sigma(f\Delta g, D) = \sum_{j=1}^{|D|} f(\tau_j)[g(t_j) - g(t_{j-1})].$$

If an arbitrary positive valued function (a *gauge*) $\delta : [0,1] \to (0,\infty)$ ($\delta \in \mathfrak{G}$) is given, then D is said to be δ-fine ($D \in A(\delta)$) if $[t_{j-1}, t_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j))$ holds for any $j = 1, 2, \ldots, |D|$. We define

$$\int_0^1 f\,\mathrm{d}[g] = I \in \mathbb{R} \Leftrightarrow \forall\, \varepsilon > 0 \;\; \exists \delta \in \mathfrak{G} \;\; \forall D \in A(\delta) \; : \; \bigl|\Sigma(f\Delta g, D) - I\bigr| < \varepsilon.$$

Let us recall some of the basic results concerning Stieltjes integration with respect to regulated functions.

If $f : [0,1] \to \mathbb{R}$ has a bounded variation on $[0,1]$ ($f \in \mathbb{BV}$) and $g \in \mathbb{G}$, then (cf. [8], Theorem 2.8) both the integrals $\int_0^1 f\mathrm{d}[g]$ and $\int_0^1 g\mathrm{d}[f]$ exist and the following inequalities are true:

$$\left|\int_0^1 f[\mathrm{d}g]\right| \le 2\|f\|_{\mathbb{BV}}\,\|g\| \quad \text{and} \quad \left|\int_0^1 g[\mathrm{d}f]\right| \le \|g\|\,\|f\|_{\mathbb{BV}},$$

where $\|f\|_{\mathbb{BV}} = |f(0)| + \operatorname{var}_0^1 f$. Hence

$$\|g_k - g\| \to 0 \implies \int_0^1 f \mathrm{d}[g_k] \to \int_0^1 f \mathrm{d}[g] \quad \text{for all} \quad f \in \mathbb{BV}$$

and

$$\|f_k - f\|_{\mathbb{BV}} \to 0 \implies \int_0^1 f_k \mathrm{d}[g] \to \int_0^1 f \mathrm{d}[g] \quad \text{for all} \quad g \in \mathbb{G}.$$

In particular, if $q \in \mathbb{R}$ and $p \in \mathbb{BV}$, then the mapping

$$x \in \mathbb{G} \to qx(0) + \int_0^1 p \mathrm{d}[x] \in \mathbb{R} \tag{1.2}$$

is a linear bounded functional on $\mathbb{G}$. It may be shown (cf. [8], Theorem 3.8) that (1.2) is the general form of a linear bounded functional on $\mathbb{G}_L$.

The above notions and results are extended to the case of matrix-valued or vector-valued functions in a natural way. The corresponding spaces of $p \times q$-matrix valued regulated functions are denoted by $\mathbb{G}^{p\times q}$ and $\mathbb{G}_L^{p\times q}$. ($\mathbb{G}^{n\times 1} = \mathbb{G}^n$ and $\mathbb{G}_L^{n\times 1} = \mathbb{G}_L^n$.)

2. Linear Operators on $\mathbb{G}_L^n$

The following characterizations of linear bounded and linear compact mappings of $\mathbb{G}_L^n$ into $\mathbb{G}^n$ are due to Š. Schwabik (cf. [6]).

Theorem 2.1. L *is a linear bounded mapping of* $\mathbb{G}_L^n$ *into* $\mathbb{G}^n$ ($\mathrm{L} \in B(\mathbb{G}_L^n, \mathbb{G}^n)$) *iff*

$$(\mathrm{L}\,\mathbf{x})(t) = \mathbf{A}(t)\mathbf{x}(0) + \int_0^1 \mathbf{B}(t,s)\,d[\mathbf{x}(s)] \tag{2.1}$$
$$\textit{for all} \quad \mathbf{x} \in \mathbb{G}_L^n \quad \textit{and} \quad t \in [0,1],$$

where

(A_1) $\mathbf{A} \in \mathbb{G}^{n\times n}$,
(B_1) $\mathbf{B}(t,\cdot) \in \mathbb{BV}^{n\times n}$ *for all* $t \in [0,1]$,
(B_2) *there is a* $\varkappa \in \mathbb{R}$ *such that* $\|\mathbf{B}(t,\cdot)\|_{\mathbb{BV}} \le \varkappa$ *for all* $t \in [0,1]$,
(B_3) $\mathbf{B}(\cdot,s) \in \mathbb{G}^{n\times n}$ *for all* $s \in [0,1]$.

Theorem 2.2. L *is a linear compact mapping of* $\mathbb{G}_L^n$ *into* $\mathbb{G}^n$ ($\mathrm{L} \in K(\mathbb{G}_L^n, \mathbb{G}^n)$) *iff it is given by* (2.1), *where the functions* $\mathbf{A}$ *and* $\mathbf{B}$ *fulfil* (A_1), (B_1) *and*

(B_4) *the mapping* $m_{\mathbf{B}} : t \in [0,1] \to m_{\mathbf{B}} = \mathbf{B}(t,\cdot) \in \mathbb{BV}^{n\times n}$ *is regulated.*

Definition 2.3. If $n \times n$-matrix valued functions $\mathbf{A}$ and $\mathbf{B}$ satisfy (A_1), (B_1) and (B_4) we write $(\mathbf{A},\mathbf{B}) \in \mathrm{K}$ and $\mathbf{B} \in \mathrm{K}$.

Theorem 2.4. L *is a linear bounded mapping of* $\mathbb{G}_L^n$ *into* $\mathbb{G}_L^n$ $(\mathrm{L} \in B(\mathbb{G}_L^n))$ *iff it is given by* (2.1), *where the functions* $\mathbf{A}$ *and* $\mathbf{B}$ *fulfil* (B_1), (B_2) *and*

(A_1^L) $\mathbf{A} \in \mathbb{G}_L^{n\times n}$,

(B_3^L) $\mathbf{B}(\cdot, s) \in \mathbb{G}_L^{n\times n}$ *for all* $s \in [0,1]$.

Moreover, a linear bounded mapping L *of* $\mathbb{G}_L^n$ *into* $\mathbb{G}_L^n$ *is compact (*$\mathrm{L} \in K(\mathbb{G}_L^n)$*) iff it is given by* (2.1), *where the functions* $\mathbf{A}$ *and* $\mathbf{B}$ *fulfil* (A_1^L), (B_1) *and*

(B_4^L) *the mapping* $m_{\mathbf{B}} : t \in [0,1] \to m_{\mathbf{B}} = \mathbf{B}(t,\cdot) \in \mathbb{BV}^{n\times n}$ *is regulated and left continuous on* $(0,1)$.

Remark 2.5. Let us notice that the condition (B_4) implies (B_2) and (B_3), while the condition (B_4^L) implies (B_2) and (B_3^L).

Definition 2.6. If $n \times n$-matrix valued functions $\mathbf{A}$ and $\mathbf{B}$ satisfy (A_1^L), (B_1) and (B_4^L) we write $(\mathbf{A},\mathbf{B}) \in \mathrm{K}_L$ and $\mathbf{B} \in \mathrm{K}_L$.

An important tool for the study of linear compact operators on $\mathbb{G}_L^n$ is provided by the following theorem usually called the Bray Theorem.

Theorem 2.7. *If* $\mathbf{B} \in K$, *then*

$$\int_0^1 \mathbf{y}^T(t)\left[d_t \int_0^1 \mathbf{B}(t,s)\, d[\mathbf{x}(s)]\right] = \int_0^1 \left(\int_0^1 \mathbf{y}^T(t)[d_t\mathbf{B}(t,s)]\right) d[\mathbf{x}(s)] \tag{2.2}$$

for all $\mathbf{x} \in \mathbb{G}_L^n$ *and* $\mathbf{y} \in \mathbb{BV}^n$.

Sketch of the proof. (For the detailed proof see [10], Theorem 5.5.) The proof is based on the fact that the formula (2.2) may be easily verified if $x \in \mathbb{S}$. Making use of the density of $\mathbb{S}$ in $\mathbb{G}$ and of the basic convergence theorems for the Perron–Stieltjes integral mentioned above the validity of the formula (2.2) is then extended to the whole space $\mathbb{G}$. Extension of the proof to the vector case $(n >= 1)$ is obvious. ∎

3. Linear Volterra Type Operator Equations in $\mathbb{G}_L^n$

It is well known that if $\mathrm{L} \in K(\mathbb{G}_L^n)$, then the linear operator equation

$$\mathbf{x} - \mathrm{L}\,\mathbf{x} = \mathbf{f} \tag{3.1}$$

possesses a unique solution $\mathbf{x} \in \mathbb{G}_L^n$ for any $\mathbf{f} \in \mathbb{G}_L^n$ iff $\dim N(\mathbf{I} - \mathrm{L}) = 0$, i.e.,

$$\mathbf{x} - \mathrm{L}\,\mathbf{x} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}. \tag{3.2}$$

Usually one can expect that (3.2) is true if L is *strongly causal*, i.e., $(\mathrm{L}\,\mathbf{x})(0) = 0$ for all $\mathbf{x} \in \mathbb{G}_L^n$ and $(\mathrm{L}\,\mathbf{x})(t) = 0$ for all $t \in (0,1]$ and $\mathbf{x} \in \mathbb{G}_L^n$ such that $\mathbf{x}(\tau) = \mathbf{0}$ on $[0,t)$. The operator L given by (2.1) is strongly causal iff

$$\mathbf{A}(0) = 0 \quad \text{and} \quad \mathbf{B}(t,s) = \mathbf{0} \quad \text{whenever} \quad 0 \le t \le s \le 1. \tag{3.3}$$

Indeed, taking into account the general form of linear bounded functionals on $\mathbb{G}_L^n$ mentioned in Section 1 this follows mainly from the fact that for a given $t \in (0,1]$ the relation

$$\int_t^1 \mathbf{B}(t,s)\mathrm{d}[\mathbf{x}(s)] = \mathbf{0} \quad \text{for all} \quad \mathbf{x} \in \mathbb{G}_L^n(t,1)$$

holds iff $\mathbf{B}(t,s) = \mathbf{0}$ for all $s \in [t,1]$. (For more details see [10], Lemma 5.2.)

Our main result is the following theorem.

Theorem 3.1. *Let* $(\mathbf{A},\mathbf{B}) \in K_L$ *fulfil* (3.3). *Then the equation*

$$\mathbf{x}(t) - \mathbf{A}(t)\mathbf{x}(0) - \int_0^t \mathbf{B}(t,s)\,d[\mathbf{x}(s)] = \mathbf{f}(t) \tag{3.4}$$

possesses for any $\mathbf{f} \in \mathbb{G}_L^n$ *a unique solution* $\mathbf{x} \in \mathbb{G}_L^n$ *iff the conditions*

$$\det\big[\mathbf{I} - \mathbf{B}(t+,t)\big] \neq 0 \quad \textit{for all} \quad t \in [0,1) \tag{3.5}$$

are satisfied.

Sketch of the proof. (For the complete proof see [10], Theorem 5.5.) Let $(\mathbf{A},\mathbf{B}) \in \mathrm{K}_L$ and (3.3) and (3.5) be satisfied. To show that the equation (3.4) possesses a unique solution for any $\mathbf{f} \in \mathbb{G}_L^n$, it is sufficient to show that the relation

$$\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(0) + \int_0^t \mathbf{B}(t,s)\mathrm{d}[\mathbf{x}(s)] \quad \text{on} \quad [0,1] \tag{3.6}$$

implies $\mathbf{x} \equiv \mathbf{0}$ on $[0,1]$. Let $\mathbf{x} \in \mathbb{G}_L^n$ be such that (3.6) holds. Then obviously $\mathbf{x}(0) = \mathbf{0}$ and in virtue of (3.3) we have

$$\mathbf{x}(0+) = \mathbf{B}(0+,0)\Delta^+\mathbf{x}(0) = \mathbf{B}(0+,0)\mathbf{x}(0+) \quad \text{or} \quad \big[\mathbf{I} - \mathbf{B}(0+,0)\big]x(0+) = \mathbf{0}.$$

Consequently, the conditions (3.5) imply that the relation $\mathbf{x}(0+) = \mathbf{0}$ holds, as well. Hence the equation (3.6) can be rewritten as follows:

$$\mathbf{x}(t) = \int_0^t \left(\mathbf{B}(t,s) - \mathbf{B}(0+,s)\right)\mathrm{d}[\mathbf{x}(s)].$$

In virtue of [8], Theorem 2.8, this yields that the inequality

$$|\mathbf{x}(t)| \leq 2\big\|\mathbf{B}(t,\cdot) - \mathbf{B}(0+,\cdot)\big\|_{\mathbb{BV}}\Big(\sup_{s\in[0,t]}|\mathbf{x}(s)|\Big)$$

is true for any $t \in [0,1]$. Furthermore, it may be shown that if $\mathbf{B} \in \mathrm{K}$, then the relations $\lim\limits_{\tau\to t+}\big\|\mathbf{B}(\tau,\cdot) - \mathbf{B}(t+,\cdot)\big\|_{\mathbb{BV}} = 0$ for all $t \in [0,1)$ and $\lim\limits_{\tau\to t-}\|\big\|\mathbf{B}(\tau,\cdot) - \mathbf{B}(t-,\cdot)\big\|_{\mathbb{BV}} = 0$ for all $t \in (0,1]$ are true. Hence there is a $\delta > 0$ such that $\|\mathbf{B}(t,\cdot) - \mathbf{B}(0+,\cdot)\|_{\mathbb{BV}} < \frac{1}{4}$ whenever $t \in (0,\delta]$ and hence also

$$\sup_{t\in[0,\delta]}|\mathbf{x}(s)| < \frac{1}{2}\sup_{t\in[0,\delta]}|\mathbf{x}(s)|$$

wherefrom the relation $\mathbf{x} \equiv 0$ on $[0,\delta]$ immediately follows. By making use of the properties of the kernels $\mathbf{B} \in \mathrm{K}_L$ and of the assumption (3.5) it is possible to extend this equality to the whole interval $[0,1]$ in a rather standard way.

In the case that the conditions (3.5) are not satisfied it is possible to construct a right hand side $\mathbf{f} \in \mathbb{G}_L^n$ such that the given equation (3.4) does not possess any solution in $\mathbb{G}_L^n$. ∎

Furthermore, making use of the Banach Bounded Inverse Theorem we can show (cf. [10], Corollary 5.7) the existence of the resolvent couple $\mathbf{U}$, $\mathbf{V}$ with the properties summarized in the following assertion.

Theorem 3.2. *Let* $(\mathbf{A},\mathbf{B}) \in K_L$ *and let the conditions* (3.3) *and* (3.5) *be satisfied. Then there exist matrix valued functions* $(\mathbf{U},\mathbf{V}) \in K_L$ *such that for any* $\mathbf{f} \in \mathbb{G}_L^n$ *the unique solution* $\mathbf{x} \in \mathbb{G}_L^n$ *of* (3.4) *is given by*

$$\mathbf{x}(t) = \mathbf{f}(t) + \mathbf{U}(t)f(0) + \int_0^1 \mathbf{V}(t,s)d[\mathbf{f}(s)], \quad t \in [0,1].$$

The functions $\mathbf{U}$ *and* $\mathbf{V}$ *satisfy in addition the relations*

$$\mathbf{U}(0) = 0, \quad \mathbf{V}(t,s) = 0 \quad \textit{whenever} \quad 0 \le t \le s \le 1,$$

$$\mathbf{U}(t) - \int_0^t \mathbf{B}(t,\tau)d[\mathbf{U}(\tau)] = \mathbf{A}(t) \quad \textit{for all} \quad t \in [0,1],$$

and

$$\mathbf{V}(t,s) - \int_0^t \mathbf{B}(t,\tau)d_\tau[\mathbf{V}(\tau,s)] = \mathbf{B}(t,s) \quad \textit{for all} \quad t,s \in [0,1].$$

Remark 3.3. Let $\mathbf{K} \in \mathrm{K}_L$ be such that $\mathbf{K}(1,1-) = \mathbf{K}(1,1)$ and let us put $\mathbf{A}(t) = \mathbf{I} + \mathbf{K}(t,t) - \mathbf{K}(t,0)$ for $t \in [0,1]$ and

$$\mathbf{B}(t,s) = \begin{cases} \mathbf{K}(t,t) - \mathbf{K}(t,s+) & \text{if} \quad 0 \le s < t \le 1, \\ \mathbf{0} & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

It may be shown (cf. [10], cf. Example 5.10) that then $(\mathbf{A},\mathbf{B}) \in \mathrm{K}_L$, the conditions (3.3) and (3.5) are satisfied and the relation

$$\int_0^t \mathrm{d}_s[\mathbf{K}(t,s)]\mathbf{x}(s) = \mathbf{A}(t)\mathbf{x}(0) + \int_0^t \mathbf{B}(t,s)[\mathrm{d}\mathbf{x}(s)]$$

is true for all $\mathbf{x} \in \mathbb{G}_L^n$ and $t \in [0,1]$. Thus Theorem 3.2 implies that under the above assumptions the equation (0.1) possesses for any $\mathbf{f} \in \mathbb{G}_L^n$ a unique solution $\mathbf{x} \in \mathbb{G}_L^n$. Moreover, this solution may be expressed in the form

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^t \mathrm{d}_s[\mathbf{R}(t,s)]\mathbf{f}(s), \quad t \in [0,1],$$

where $\mathbf{R} \in \mathrm{K}_L$ satisfies the equation

$$\mathbf{R}(t,s) = \mathbf{K}(t,s) - \mathbf{K}(t,0) + \int_0^t \mathrm{d}_\tau[\mathbf{K}(t,\tau)]\mathbf{R}(\tau,s) \quad \text{for} \quad 0 \le s \le t \le 1$$

and $\mathbf{R}(t,s) = \mathbf{R}(t,t)$ for $0 \le t < s \le 1$. These results are supplementary to analogous results from [7], where instead of our present assumption $\mathbf{K} \in \mathrm{K}$ a stronger assumption that $\mathbf{K}$ has a strongly bounded Vitali variation was used and solutions were sought in the space $\mathbb{BV}^n$. On the other hand, the results in [7] concern the cases that $\mathbf{K}(\cdot,s)$ need not be left continuous on $(0,1)$, as well.

Remark 3.4. Theorems 3.1 and 3.2 may be extended (cf. [10], Theorem 5.5 and Corollary 5.7) to the case that instead of the strong causality only the causality of the operator L is required (i.e., $(\mathrm{L}\,\mathbf{x})(0) = \mathbf{0}$ for any $\mathbf{x} \in \mathbb{G}_L^n$ and $(\mathrm{L}\,\mathbf{x})(t) = \mathbf{0}$ for all $t \in (0,1]$ and all $\mathbf{x} \in \mathbb{G}_L^n$ such that $\mathbf{x} \equiv \mathbf{0}$ on $[0,t]$).

Remark 3.5. Similar problems with the Dushnik interior integral in the place of the Perron–Stieltjes integral were treated by Ch. S. Hönig (cf. e.g. [2]).

4. Linear Volterra Type Operator Equations in $\mathbb{G}^n$

Let us assume that $(\mathbf{A},\mathbf{B}) \in \mathrm{K}$ (i.e., $\mathbf{A}$ fulfils (A_1), $\mathbf{B}$ fulfils (B_1) and (B_4), while the mapping $m_{\mathbf{B}}$ from (B_4) and (B_4^L) need not be left-continuous on $(0,1)$) and let the operator L be given by (2.1). Then obviously $\mathrm{L} \in B(\mathbb{G}^n)$. Moreover, L is compact as well. Indeed, if $\mathrm{N} = \{\mathbf{x} \in \mathbb{G}^n;\ \|\mathbf{x}\| \le 1\}$, then $\|\,\mathrm{L}\,\mathbf{x}\| \le \|\,\mathrm{L}\,\| < \infty$ holds for any $\mathbf{x} \in \mathrm{N}$. In particular, the set $\{|(\mathrm{L}\,\mathbf{x})(t)| : \mathbf{x} \in \mathrm{N}\}$ is bounded for every $t \in [0,1]$. Moreover, we have for arbitrary $\mathbf{x} \in \mathrm{N}$, $t_1 \in [0,1]$ and $t_2 \in [0,1]$

$$\big|(\mathrm{L}\,\mathbf{x})(t_2) - (\mathrm{L}\,\mathbf{x})(t_1)\big| \le |\mathbf{A}(t_2) - \mathbf{A}(t_1)| + 2\|\mathbf{B}(t_2,\cdot) - \mathbf{B}(t_1,\cdot)\|_{\mathbb{BV}},$$

wherefrom we easily obtain that the set L(N) is equiregulated and hence also compact (cf. [1]). Analogously as in the case $(\mathbf{A},\mathbf{B}) \in \mathrm{K}_L$ we could verify that L is strongly causal iff (3.3) is true. Moreover, making use of only slightly modified procedure from the proof of Theorem 3.1, we could show that the following extension of Theorem 3.1 is true.

Theorem 4.1. *Let* $(\mathbf{A},\mathbf{B}) \in K$ *fulfil* (3.3). *Then the equation* (3.4) *possesses for any* $\mathbf{f} \in \mathbb{G}^n$ *a unique solution* $\mathbf{x} \in \mathbb{G}$ *iff the conditions* (3.5) *are satisfied.*

Contrary to the case $(\mathbf{A},\mathbf{B}) \in \mathrm{K}_L$ described in the previous section, now the equation (0.1) can not be in general converted to the form (3.4) ((2.1) is not the general form of a linear compact operator on $\mathbb{G}^n$) and Theorem 4.1 in general does not include the equation (0.1). However, using the methods described above, we could prove the following assertion which extends the corresponding existence and uniqueness result from [7].

Theorem 4.2. *Let* $\mathbf{K} \in K$. *Then the equation* (0.1) *possesses for any* $\mathbf{f} \in \mathbb{G}^n$ *a unique solution* $\mathbf{x} \in \mathbb{G}^n$ *iff the conditions*

$$\det\left[\mathbf{I} - (\mathbf{K}(t,t) - \mathbf{K}(t,t-)\right] \neq 0 \quad \textit{for all} \quad t \in (0,1] \tag{4.1}$$

are satisfied.

Remark 4.3. If $\mathbf{K} \in \mathrm{K}$, then the operator

$$\mathrm{K}:\ \mathbf{x} \in \mathbb{G}^n \to \int_0^1 \mathrm{d}_s[\mathbf{K}(t,s)]\mathbf{x}(s) \in \mathbb{G}^n$$

from the left-hand side of the equation (0.1) is obviously causal but in general it is not strongly causal. It is easy to verify that K is strongly causal iff $(\mathbf{K}(t,t) - \mathbf{K}(t,t-))\mathbf{c} = \mathbf{0}$ for all $\mathbf{c} \in \mathbb{R}^n$ and $t \in (0,1]$, i.e., iff $\mathbf{K}(t,t) = \mathbf{K}(t,t-)$ for all $t \in (0,1]$. The conditions (4.1) are satisfied whenever K is strongly causal, of course.

5. Applications to Functional Differential Equations

The equations considered in this note cover as special cases various types of functional differential equations. For example, let us consider the system

$$\dot{\mathbf{x}}(t) = (P\mathbf{x})(t) + \mathbf{q}(t) \quad \text{a.e. on} \quad [0,1], \tag{5.1}$$

where P is a linear bounded mapping of the space $\mathbb{C}^n$ of continuous n-vector valued functions continuous on $[0,1]$ into the space $\mathbb{L}_1^n$ of n-vector valued functions Lebesgue integrable on $[0,1]$ and $\mathbf{q} \in \mathbb{L}_1^n$. Such systems were recently treated (together with the corresponding boundary value problems) e.g. by I. Kiguradze and B. Půža (cf. [3]) and include as special cases systems of the form

$$\dot{\mathbf{x}}(t) = \mathbf{P}(t)\mathbf{x}(\tau(t)) + \mathbf{g}(t) \ \text{ a.e. on } \ [0,1], \quad \mathbf{x}(t) = \mathbf{u}(t) \ \text{ for } \ t \in \mathbb{R} \setminus [0,1],$$

where the continuous and bounded function $\mathbf{u} : \mathbb{R} \to \mathbb{R}^n$ and the measurable function $\tau : [0,1] \to \mathbb{R}$ are given and fixed. Obviously, the system (5.1) may be rewritten as

$$\mathbf{x}(t) - \mathbf{x}(0) - \int_0^t (P\mathbf{x})(s)\mathrm{d}s = \int_0^t \mathbf{q}(s)\mathrm{d}s, \quad t \in [0,1]. \tag{5.2}$$

Analogously as above in the case of the operator L, it follows from the characterization of compact sets in $\mathbb{G}^n$ due to D. Fraňková (cf. [1]) that the operator

$$\mathbf{x} \in \mathbb{G}_L^n \to \mathbf{x}(0) + \int_0^t (P\mathbf{x})(s)\mathrm{d}s \in \mathbb{G}_L^n \tag{5.3}$$

is linear and compact and hence may be represented in the form (2.1). In particular, Fredholm type theorems are valid for any boundary value problem generated by the equation (5.2) (with finite dimensional additional "boundary" conditions, of course). Moreover, whenever the operator (5.3) is causal, any initial value problem for the equation (5.2) possesses a unique solution for any $\mathbf{q} \in \mathbb{L}_1^n$.

Neutral functional differential equations may be converted to operator equations of the form studied in this note and in [10], as well. However, e.g. the problem

$$\dot{\mathbf{x}}(t) - \mathbf{P}\dot{\mathbf{x}}(t-h) = \mathbf{Q}\mathbf{x}(t) \ \text{ a.e. on } \ [0,1], \quad \mathbf{x}(t) = \mathbf{u}(t) \ \text{ for } \ t < 0$$

with $h > 0$ and the initial function $\mathbf{u}$ fixed leads to an operator equation $\mathbf{x} - \mathrm{L}\,\mathbf{x} = \mathbf{0}$, where L is a linear bounded operator on $\mathbb{G}_L^n$ but it can not be compact since the functions of the form

$$h(t,s) = \begin{cases} 0, & \text{if } \ 0 \le t \le h \ \text{ or } \ 0 \le t-h < s \\ 1, & \text{if } \ 0 \le t \ \text{ and } \ 0 \le s \le t-h \end{cases}$$

do not belong to the class K.

References

1. D. Fraňková, Regulated functions. *Mathematica Bohemica* 116(1991), 20–59.
2. Ch. S. Hönig, Volterra–Stieltjes Integral Equations. *North–Holland, Mathematics Studies, vol.16 Amsterdam–Oxford*, 1975 .
3. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional differential equations. *Czechoslovak Math. J.* 47(122)(1997), 341–374.
4. J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter. *Czechoslovak Math. J.* 7(82)(1957), 418–449.
5. Š. Schwabik, Generalized Ordinary Differential Equations. *World Scientific, Singapore*, 1992.
6. Š. Schwabik, Linear operators in the space of regulated functions. *Mathematica Bohemica* 117(1992), 79–92.
7. Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and Integral Equations: Boundary Value Problems and Adjoints. *Academia and D. Reidel, Praha and Dordrecht*, 1979.
8. M. Tvrdý, Regulated functions and the Perron–Stieltjes integral. *Časopis pěst. mat.* 0187–209 114(1989).
9. M. Tvrdý, Generalized differential equations in the space of regulated functions (Boundary value problems and controllability). *Mathematica Bohemica* 116(1991), 225–244.
10. M. Tvrdý, Linear integral equations in the space of regulated functions. *Mathematica Bohemica* 125(1998) (*to appear*).

(Received 22.07.1997)

Author's address:
Matematický ústav AV ČR
Žitná 25, 115 67 Praha 1
Czech Republic