Mem. Differential Equations Math. Phys. 12(1997), 178–183

František Neuman

GLOBAL TRANSFORMATIONS OF LINEAR DIFFERENTIAL EQUATIONS

Abstract. A survey of the theory of global transformations of linear differential equations is presented including a criterion of global equivalence and global canonical forms. A program for further investigations in the area of functional-differential equations is also discussed.

რეზიუმე. ნაშრომში წარმოდგენილია წრფივი დიფერენციალური განტოლებების გლობალური გარდაქმნების თეორიის მიმოხილვა, რომელიც, სხვა საკითხებთან ერთად, ეხება გლობალური ექვივალენტობის ნიშნებს და გლობალურ კანონიკურ ფორმებს. დასახულია შემდგომი კვლევის პროგრამა ფუნქციონალურ-დიფერენციალურ განტოლებათა სფეროში.

1. History

In several areas of mathematics we may observe that after studying single objects, connections between them are considered. For linear differential equations, this happened in the middle of the last century, when E. E. Kummer [6] studied transformations of linear differential equations of the second order in the form $z(t) = f(t) \cdot y(h(t))$.

Then several mathematicians dealt with transformations, invariants and canonical forms of higher order linear differential equations. Perhaps the best known result is the canonical form of these equations, characterized by the vanishing of the coefficients by the $(n-1)$st and $(n-2)$nd derivatives (E. Laguerre [7], A. R. Forsyth [4]). P. Stäckel [17] and S. Lie [8] proved to the end of the last century that Kummer has already considered the most general pointwise transformation of the second and higher order equations. In 1910, G. D. Birkhoff [1] pointed out that these investigations were of local character by presenting an example of a third order equation that cannot be transformed into the Laguerre-Forsyth form. Sansone [16] presented his example of a third order linear differential equation with all oscillatory solutions 17 years after Mammana's result [9] of 1931 showing the nonexistence

1991 *Mathematics Subject Classification.* 34A30, 34C20, 34K05

Key words and phrases. Linear differential equations, functional-differential equations, global investigations, transformation.

of factorization of linear differential operators in general. There were also results of a global nature, but a systematic study of global transformations was started in the fifties by O. Borůvka [2] for the second order equations. Results of a global character for linear differential equations of an arbitrary order are summarized in [10].

2. General approach to transformations

A class is called a *category* if to each pair of its elements P and Q called objects, a set $\mathrm{Hom}(P,Q)$ of morphisms is assigned such that the following axioms are satisfied:

1. The sets $\mathrm{Hom}(P,Q)$ are disjoint for different pairs (P,Q).

2. A composition $\alpha\beta \in \mathrm{Hom}(P,T)$ is defined for each $\alpha \in \mathrm{Hom}(P,Q)$ and $\beta \in \mathrm{Hom}(Q,T)$ such that

a) the associativity $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds for each $\gamma \in \mathrm{Hom}(T,U)$,

b) there exists an identity for each object.

A category is an *Ehresmann groupoid* if each morphism has an inverse.

Moreover, an Ehresmann groupoid is a *Brandt groupoid* if $\mathrm{Hom}(P,Q)$ is not empty for any pair (P,Q).

The Ehresmann groupoid is a collection of connected components, Brandt groupoids, also called classes of equivalent objects. The set $\mathrm{Hom}(P,P)$ is a group, a *stationary group* of the object P. The following problems are studied when an Ehresmann groupoid is considered:

Criterion of equivalence.

Canonical objects in Brandt groupoids.

Structure of stationary groups.

Special problems depending on a particular Ehresmann groupoid (*e.g., zeros of solutions if differential equations are objects*).

3. Ordinary linear differential equations

Let us consider a linear differential equation

$$P_n \equiv y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0 \quad \text{on} \quad I,$$

I being an open interval of the reals and p_i real continuous functions defined on I for $i = 0, 1, \ldots, n-1$, i.e., $p_i \in C^0(I)$, $p_i : I \to \mathbb{R}$, $n \geq 2$. Denote also

$$Q_n \equiv z^{(n)} + q_{n-1}(t)z^{(n-1)} + \cdots + q_0(t)z = 0 \quad \text{on} \quad J,$$

another equation of this type. We say that the equation P_n is globally equivalent to the equation Q_n if there exist two functions $f \in C^n(J)$, $f(t) \neq 0$ for each $t \in J$, and $h \in C^n(J)$, $h'(t) \neq 0$ for each $t \in J$, such that $h(J) = I$ and whenever $y : I \to \mathbb{R}$ is a solution of P_n, then

$$z : J \to \mathbb{R}, \quad z(t) := f(t) \cdot y(h(t)), \quad t \in J, \tag{1}$$

is a solution of Q_n. This equivalency is *global* in the sense that solutions are transformed on their whole interval of definition, because $h(J) = I$.

Let $\mathbf{y}(x) = (y_1(x), \dots, y_n(x))^T$ denote an n-tuple of linearly independent solutions of the equation P_n considered as a column vector function or as a curve in n-dimensional Euclidean space $\mathbb{E}_n$ with the independent variable x as the parameter and $y_1(x), \dots, y_n(x)$ as its coordinate functions; M^T denotes the transpose of the matrix M.

If $\mathbf{z}(t) = (z_1(t), \dots, z_n(t)^T$ denotes an n-tuple of linearly independent solutions of the equation Q_n, then the global transformation (1) can be equivalently written as $\mathbf{z}(t) = f(t) \cdot \mathbf{y}(h(x))$ or, for an arbitrary regular constant $n \times n$ matrix A,

$$\mathbf{z}(t) = Af(t) \cdot \mathbf{y}(h(x)) \tag{2}$$

expressing only that another n-tuple of linearly independent solutions of the *same* equation Q_n is taken. Global transformations (2) attached to the equation (P) are considered as morphisms in $\mathrm{Hom}(P, Q)$ of the Ehresmann groupoid of linear differential equations as its objects.

There is also a geometric representation of classes of equivalent equations, i.e., Brandt groupoids of the Ehresmann groupoid:

All curves representing all n-tuples of linearly independent solutions of all equations globally equivalent to an equation P with an n-tuple of linearly independent solutions (a curve) $\mathbf{y}$ are obtained as sections (given by f) of a centroaffine image (determined by A) of the fixed cone K formed by halflines going from the origin and the points of the curve $\mathbf{y}$ in a certain parametrization (given by h).

4. Solutions of the problems

Criterion of global equivalence.

For second order equations, this criterion was proved by O. Borůvka in 1967 [2]: *two equations are globally equivalent if and only if their solutions have the same number of zeros.* In general it is not effective.

In 1984, the criterion was also given for n-th order equations [10]. In general, this criterion is effective and can be expressed by quadratures from coefficients of given equations.

Global canonical forms.

For second order equations, the following canonical forms were considered by O. Borůvka [2]: $y'' + y = 0$ on $(0, \pi/2)$, $(0, \pi)$, $(0, 3\pi/2), \dots, (0, k\pi/2)$, $\dots, (0, \infty)$, $(-\infty, \infty)$. These intervals of definition express the precise meaning of the number of zeros of solutions.

As canonical forms for n-th order equations, we may take

$$y^{(n)} + y^{(n-2)} + p_{n-3}(x) y^{(n-3)} + \cdots + p_0(x) y = 0$$

on a certain set of intervals. These equations are characterized by the first three coefficients: $(1, 0, 1)$. If Laguerre and Forsyth had taken 1 instead of their 0, they would have obtained global canonical forms.

Another approach by using a geometrical interpretation and Cartan's moving-frame-of reference method gives the following global canonical forms:

$$y'' + y = 0,$$

$$y''' - \frac{k'(x)}{k(x)} y'' + (1 + k^2(x)) y' - \frac{k'(x)}{k(x)} y = 0, \quad k > 0, \ \ k \in C^1,$$

$$\dots .$$

Stationary groups.

These groups were described in 1967 by O. Borůvka [2]. He calls their elements the dispersions.

A characterization of stationary groups for the n-th order equations was given in 1984 by using a criterion of global equivalence for these equations, see [11] or [10]. There are, up to conjugacy, 10 types of these groups (with countable many subtypes), from 3-parameter groups to the trivial one.

5. Zeros of solutions

The essence of our approach to the distribution of zeros is based on two readings of the following relation:

$$\mathbf{c}^T \cdot \mathbf{y}(x_0) = c_1 y_1(x_0) + \dots + c_n y_n(x_0) = 0.$$

The first meaning of the relation: the solution $\mathbf{c}^T \cdot \mathbf{y}(x)$ has a zero at x_0.

Secondly, equivalently, a hyperplane $c_1 \eta_1 + \dots + c_n \eta_n = 0$ intersects the curve $\mathbf{y}(x)$ at the point corresponding to the parameter x_0.

If $\mathbf{y}$ is considered in the n-dimensional Euclidean space and its central projection $\mathbf{v}$ onto the unit sphere is taken without change of parametrization, then the parameters of intersections of $\mathbf{v}$ with great circles correspond to zeros of the corresponding solutions; even multiplicities of zeros occur as orders of contacts $+1$.

By using this approach, we can see what is possible and what impossible, simply by drawing a curve $\mathbf{v}$ on the sphere, without lengthy and sometimes tiresome $\varepsilon - \delta$ calculations. Only $\mathbf{v}$ must be sufficiently smooth, i.e., of the class C^n for the n-th order equations with the nonvanishing Wronskian $\det(\mathbf{v}, \mathbf{v}', \dots, \mathbf{v}^{(n-1)}) \neq 0$ at each point. As examples, let us mention the Separation Theorem for the second order equations, equations of the third order with solutions all oscillatory (Sansone's result), or a third order equation with just one-dimensional subset of oscillatory solutions that cannot occur for equations with constant coefficients.

6. Functional differential equations

The general approach to transformations can be applied also for nonlinear functional differential equations

$$F\big(x, y(x), \dots, y^{(n)}(x), y(\xi_1(x)), \dots, y^{(n)}(\xi_1(x)), \dots, \\ y(\xi_k(x)), \dots, y^{(n)}(\xi_k(x))\big) = 0$$

as objects, and substitutions $x = h(t)$, $z(t) = y(h(t))$ as morphisms transforming the above equation into

$$G\big(t, z(t), \dots, z^{(n)}(t), z(\eta_1(t)), \dots, z^{(n)}(\eta_1(t)), \dots, \\ z(\eta_k(t)), \dots, z^{(n)}(\eta_k(t))\big) = 0.$$

Then $y(\xi_i(x)) = yh(h^{-1}\xi h(t)) = z(\eta_i(t))$, i.e., $h^{-1}(\xi_i(h(t)) = \eta_i(t)$, or

$$h\eta_i(t) = \xi_i h(t), \quad i = 1, \dots, k,$$

expressing the fact that the deviating arguments ξ_i, and η_i are conjugate functions [12].

If $k = 1$, i.e., when we have a single Abel's equation, there were lot of results in the literature, e.g. see [5]; for $k > 1$ there has recently been an intensive research in Brno, Katowice and Krakow, [12], [13], [18] etc.

If we consider the possibility of the special choice of *canonical* deviations $\xi_i(x) = x + c_i$, $c_i = \text{const}$ [14], then we come to the problem of the general solution h of a *system of Abel's functional equations* for prescribed η_i: $h(\eta_i(t)) = h(t) + c_i$, $i = 1, \dots, k$.

In the simplest case of linear functional equations of the first order with one delay $y'(x) + a(x)y(x) + b(x)y(\xi(x)) = 0$, we may consider their *canonical form* as $z'(t) + c(t)z(t-1) = 0$.

For linear functional differential equations, we may take even more general morphisms of Kummer's type $z(t) = f(t)y(h(t))$ which enable us to impose one more condition on the coefficients because of a rather arbitrary function f in the transformations. For another choice of special deviations, e.g., of the form $\xi_i(x) = c_i x$, we get a system of Schröder's functional equations [3], $h(\eta_i(t)) = c_i h(t)$, $i = 1, \dots, k$.

In general, zeros of solutions are preserved and may be studied on canonical forms only. Since the factor f in the transformation can be explicitly evaluated from the coefficients, asymptotic properties of solutions of equations, their boundedness, classes L^p, convergency to zero, or the rate of growth, can be obtained from these properties of canonical equations.

For some cases we have also a *criterion of equivalence*, see [15].

7. Final remark

We would like to propose the following approach. In each area of study there are some objects: nonlinear differential equations of a certain type, functional differential equations in a certain form, etc. Let us consider some transformations, substitutions, as morphisms. They may be given by a definition. However, it would be better to define them as "the most general form" satisfying some properties (like pointwise transformations, or keeping some properties of solutions unchanged, e.g., boundedness, etc.). If we get an Ehresmann groupoid, besides special questions for this area, there are general problems mentioned in Section 2 to be solved: a criterion of equivalence, canonical forms, and the structure of stationary groups.

Acknowledgement

Research was supported by the grant N. 201/96/0410 of the Czech Republic.

References

1. G. D. Birkhoff, On the solutions of ordinary linear homogeneous differential equations of the third order. *Ann. Math.* 12(1910/11), 103–127.

2. O. Borůvka, Lineare Differentialtransformationen 2 Ordnung. *VEB Verlag, Berlin,* 1967*; extended English version:* Linear Differential Transformations of the Second Order, *English Universities Press, London*, 1973.

3. J. Čermák, Note on simultaneous solutions of a system of Schröder equations. *Math. Bohemica* 120(1995), 225–236.

4. A. R. Forsyth, Invariants, covariants and quotient-derivatives associated with linear differential equations. *Philos. Trans. Roy. Soc. London Ser. A*, 179(1899), 377–489.

5. M. Kuczma, B. Choczewski, and R. Ger, Iterative Functional Equations. *Cambridge Univ. Press, Cambridge – New York*, 1989.

6. E. E. Kummer, De generali quadam aequatione differentiali tertii ordinis. *Progr. Evang. Königl. Stadtgymnasium Liegnitz* 1834 (*reprinted in J. Reine Angew. Math.* 100(1887), 1–10).

7. E. Laguerre, Sur les équations differérentielles linéaires du troisième ordre. *C. R. Acad. Sci. Paris* 88(1879), 116–118.

8. S. Lie, F. Engel, Theorie der Transformationgruppe. *Teubner, Leipzig*, 1930.

9. G. Mammana, Decomposizione delle espressioni differenziali lineari omogenee in prodotti di fattori simbolici e applicazione rellativa allo studio delle equazioni differenziali lineari. *Math. Z.* 33(1931), 186–231.

10. F. Neuman, Global Properties of Linear Ordinary Differential Equations. *Mathematics and Its Applications, East European Series* 52*, Kluwer Academic Publishers* (*with Academia Praha*) *Dordrecht–Boston–London,* 1991.

11. F. Neuman, Criterion of global equivalence of linear differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* 97(1984), 217–221.

12. F. Neuman, Simultaneous solutions of a system of Abel equations and differential equations with several deviations. *Czechoslovak Math. J.* 32(107)(1982), 488–494.

13. F. Neuman, On transformations of differential equations and systems with deviating argument. *Czechoslovak Math. J.* 31(1981), 87–90.

14. F. Neuman, Transformations and canonical forms of functional-differential equations. *Proc. Roy. Soc. Edinburgh Sect. A* 115(1990), 349–357.

15. F. Neuman, On equivalence of linear functional-differential equations. *Results Math.* 26(1994), 354–359.

16. G. Sansone, Studi sulle equazioni differenziali lineari omogenee di terzo ordine nel campo reale. *Revista Mat. Fis. Teor. Tucuman* 6(1948), 195–253.

17. P. Stäckel, Über Transformationen von Differentialgleichungen, *Leipzig*, 1897.

18. M. C. Zdun, On simultaneous Abel equations. *Aequationes Math.* 38(1989), 163–177.

(Received 4.06.1997)

Author's address:
Žižkova 22, CZ-616 62, Brno
Czech Republic