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ZEROS' ASYMPTOTIC DISTRIBUTION OF POLYNOMIALS ORTHOGONAL WITH RESPECT TO VARYING WEIGHTS

Dedicated to the memory of Prof. Gaetano Fichera

Abstract. By using the WKB method, we found the asymptotic distribution of the zeros of some sets of polynomials orthogonal with respect to varying weights, including the so called relativistic orthogonal polynomials.

რეზიუმე. *WKB* მეთოდის გამოყენებით ნაპოვნია ცვალებადი წონის მიმართ ორთოგონალურ პოლინომთა ზოგიერთი კლასის ნულების ასიმპტოტური განაწილება, მათ შორის ე.წ. რელატივისტურ პოლინომთა კლასისაც.

1. Introduction

Given a real function $f : I \to \mathbf{R}$ of the real variable x, the *normalized distribution of zeros of f in I* is defined by

$$\tilde{\rho}_f(x; I) := \frac{1}{N_f(I)} \sum_{\substack{f(x_k)=0 \\ (x_k \in I)}} \delta(x - x_k), \tag{1.1}$$

where $\delta(x - x_k)$ is the Dirac distribution concentrated in x_k, and $N_f(I)$ is the total number of zeros of f in I. We consider second order differential equations of hypergeometric type

$$\sigma(x) y_n'' + \tau(x; n) y_n' + \lambda_n y_n = 0, \tag{1.2}$$

i.e., second order homogeneous linear differential equations in which σ and τ are polynomials of degree not greater than 2 and 1, respectively, and λ_n is a constant defined by

$$\lambda_n = -n\tau' - (n(n-1)/2)\sigma''. \tag{1.3}$$

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We deal with the problem of finding the *zeros' asymptotic distribution of the sequence* $\{P_n(x)\}$ in I, i.e. the limit

$$\lim_{n\to\infty} \tilde{\rho}_{P_n}(x;I) = [\mu(x;I)]' \tag{1.4}$$

by using the WKB (Wentzel-Kramers-Brillouin) approximation method [3], where I is the limiting support of the measures with respect to which the polynomial sequence $\{P_n(x)\}$ satisfy a varying orthogonality property. Following an approach of A. Zarzo [2], we will study the problem of finding the asymptotic distribution of zeros of the set of polynomial solutions of the hypergeometric differential equations of the type (1.2) in the orthogonality interval, by using the WKB method (see [3]). We will consider in detail some particular cases of asymptotic distribution of zeros of some sets of polynomials orthogonal with respect to varying weights (see [4]), and mainly the so called relativistic polynomials (see [5]-[11]). These polynomials have been studied in several recent papers (see [12]-[15]). Some other sets have been also introduced (see [16]-[17]).

2. Definitions and Notation

We recall three canonical forms for the second homogenous linear differential equation which are used in what follows.

1. Selfadjoint form:

$$[a(x)y']' + c(x)y = 0, \ (a(x) > 0). \tag{2.1}$$

By putting $y(x) = [a(x)]^{-1/2}u(x)$, we pass to the

2. Jacobi form:

$$u'' + S(x)u = 0. \tag{2.2}$$

If $S(x) \in C^2(\Delta)$ and $S(x) > 0$, in $\Delta \equiv [a, b]$, setting

$$\omega(x) = \int_a^x \sqrt{S(t)}dt, \quad v(x) = [S(x)]^{1/4}u(x),$$
$$V(\omega(x)) = v(x), \ i.e. \ V(\omega) = v(x(\omega)),$$

we obtain the

3. Liouville-Green form:

$$\frac{d^2}{d\omega^2}V(\omega) + [1 - \Phi(\omega)]V(\omega) = 0, \tag{2.3}$$

where

$$\Phi(\omega(x)) \equiv \phi(x) = \frac{4S(x)S''(x) - 5[S'(x)]^2}{16[S(x)]^3}.$$

We recall the following

Definition 1. The differential equation (2.1) ((2.2)) is called disconjugate in the interval I if its only solution having more than one zero in I is the trivial solution.

Definition 2. The differential equation (2.2) is called oscillatory in the interval I if there exists a nontrivial solution having infinitely many zeros in I.

A simple result of M. Picone [18] ensures that dealing with oscillatory differential equations in the selfadjoint form (2.1), it is sufficient to consider equations in intervals where

$$a(x)c(x) \geq 0,$$

otherwise the equation should be disconjugate.

3. WKB Approximation

We follow here somewhat closely the exposition and notation of A. Zarzo [2]. The WKB (Wentzel-Kramers-Brillouin) approximation can be applied to the equation (2.3) if $|\phi(x)| = |\phi(\omega(x))| < 1$ in $\Delta \equiv [a,b] \subseteq I$. In this case, the WKB approximation leads to the equation

$$\frac{d^2}{d\omega^2} V(\omega) + V(\omega) = 0. \tag{3.1}$$

Solutions of this equation are given by

$$V_{wkb}(\omega) = A \sin[\omega + B], \tag{3.2}$$

where A, B are arbitrary constants. Then, returning to the variable x, an approximate solution of eq. (2.2) is given by

$$u_{wkb}(\omega) = A[S(x)]^{-\frac{1}{4}} \sin[\omega + B], \; (x \in \Delta). \tag{3.3}$$

But $\omega(x)$ is a positive increasing function in Δ, so the zeros of $u_{wkb}(x)$ can be ordered in the form

$$x_1 < x_2 < x_3 < \cdots < x_k < \cdots,$$

where $\omega(x_k) + B$ must be a zero of $\sin(x)$, and therefore

$$\omega(x_k) = k\pi - B, \; (k = 0, 1, 2, \dots). \tag{3.4}$$

Consider the function

$$N_{u_{wkb}}(x) := \frac{1}{\pi}\omega(x). \tag{3.5}$$

Therefore

$$N_{u_{wkb}}(x_{k+j}) - N_{u_{wkb}}(x_k) = \frac{1}{\pi}[(k+j)\pi - B - k\pi + B] = j,$$

and consequently, the associated set function

$$N_{u_{wkb}}(\Delta) = N_{u_{wkb}}(b) - N_{u_{wkb}}(a),\ if\ \Delta := [a,b],$$

can be considered as a counting function of the zeros of u_{wkb} in $[a,b]$.

Definition 3. The function

$$\rho_{wkb}[u(x)] := \rho_{wkb}(x;\Delta) = \frac{d}{dx}N_{u_{wkb}}(x) = \frac{1}{\pi}\sqrt{S(x)}, \tag{3.6}$$

is called the WKB approximation of the density of zeros in Δ of solutions of the differential equation (2.2).

Consider the equation (2.2). Let $I = I_{S>0}$ be an interval in which the function $S(x) > 0$ belongs to C^2, and let $\Delta \equiv [a,b] \subseteq I$. Consider the Liouville-Green form of (2.2) denoting

$$\Phi(\omega(x)) \equiv \phi(x) = \frac{4S(x)S''(x) - 5[S'(x)]^2}{16[S(x)]^3}, \tag{3.7}$$

and by M_ϕ and m_ϕ respectively the maximum and minimum value of $\phi(x)$ in Δ. By using the Sturm comparison theorem, Zarzo proved the following

Proposition 1 (Zarzo). *Under the preceding hypotheses and notation, if* $\forall x \in \Delta$

$$\phi(x) < 1, \tag{3.8}$$

then there exists an absolute constant $K > 0$ *such that*

$$\sqrt{1 - M_\phi}N_{wkb}(\Delta) - K \le N_u(\Delta) \le \sqrt{1 - m_\phi}N_{wkb}(\Delta) + K,$$

where $N_u(\Delta)$ *is the number of zeros of any solution of* (2.2) *in* Δ *and*

$$N_{wkb}(\Delta) := N_{u_{wkb}}(\Delta) = \frac{1}{\pi}\omega(b) = \frac{1}{\pi}\int_a^b \sqrt{S(x)}dx = \int_a^b \rho_{wkb}[u(x)]dx.$$

Then it is possible to deduce a necessary and sufficient condition in order that (2.2) to be oscillatory in $\Delta \equiv [a,b]$. Namely, this is true if and only if the corresponding $u_{wkb}(x)$ has the same property in Δ. A general procedure in order to find the asymptotic distribution of zeros of polynomial solutions of the hypergeometric differential equation (1.2) has been found. In order to include the case of unbounded intervals, suppose that it is possible to choose two sequences of positive real numbers $\{s_n\}$ and $\{r_n\}$ such that $\lim_{n\to\infty} \frac{r_n s_n}{n} = 0$ and there exist the finite limits:

$$\lim_{n\to\infty}\left\{s_n r_n^2 \sigma(\frac{x}{r_n};n)\right\} =: \sigma(x;\infty),$$
$$\lim_{n\to\infty}\left\{\frac{s_n r_n}{n} \tau(\frac{x}{r_n};n)\right\} =: \tau(x;\infty).$$

If we consider the polynomial

$$P(x) := -\{[\tau(x;\infty)]^2 + 2[2\tau'(0;\infty) + \sigma''(0;\infty)]\sigma(0;\infty)\},$$

then the following general result holds true.

Proposition 2 (Zarzo). *Under the above hypotheses and notation, the weak-star convergence property holds true:*

$$\tilde{\rho}[y_n(\frac{x}{r_n})] := \frac{1}{n} \sum_{y_n(\xi)=0} \delta(t - r_n\xi) \xrightarrow{*} \tilde{\rho}'(t), \ (n \to \infty),$$

where $t := \frac{x}{r_n}$,

$$\tilde{\rho}'(t) = \lim_{n\to\infty} \frac{\rho_{wkb}[y_n(t)]}{N_{wkb}(I^*_{S>0}(n))} = \frac{1}{2\pi} \frac{\sqrt{P(t)}}{|\sigma(t;\infty)|}, \ (t \in I^*_{S>0}(\infty))$$

and, by definition,

$$I^*_{S>0}(\infty) = \lim_{n\to\infty} I^*_{S>0}(n) := \{t \in \mathbf{R} : P(t) > 0\}.$$

3.1. Examples.

Example 1. Relativistic Hermite polynomials. The differential equation is

$$(1 + \frac{\xi^2}{N})y'' - \frac{2}{N}(N + n - 1)\xi y' + \frac{n}{N}(2N + n - 1)y = 0.$$

We have

$$\begin{aligned} &\tau(\xi;\infty) = -\frac{2}{N}\xi, \ \sigma(\xi;\infty) = 1 + \frac{\xi^2}{N}, \\ &s_n = r_n = 1, \\ &P(\xi) = -\left\{\frac{4}{N^2}\xi^2 + 2[2(-\frac{2}{N}) + \frac{2}{N}](1 + \frac{\xi^2}{N})\right\} = \frac{4}{N}, \\ &\tilde{\rho}'(\xi) = \frac{1}{2\pi} \frac{\sqrt{\frac{4}{N}}}{1 + \frac{\xi^2}{N}} = \frac{\sqrt{N}}{\pi} \frac{1}{N + \xi^2}. \end{aligned}$$

This formula is always true, but it is necessary to distinguish many cases in relation to the sign of $S(x)$ (see A. Zarzo [2], p. 184)).

Example 2. Relativistic Jacobi polynomials. The differential equation is

$$\sigma(x;N)y_n'' + \tau(x;n;N)y_n' + \lambda_n(N)y_n = 0.$$

We have

$$\begin{aligned}\sigma(x;N) &= \left[\frac{N+\beta}{N-\beta} - \frac{2\beta}{N-\beta}x - x^2\right],\\ \tau(x;n;N) &= \left[\frac{N(\beta-\alpha)+\beta(2n-3/2)}{N-\beta} - \frac{N(\alpha+\beta+2)+\beta(2n-3/2)}{N-\beta}x\right],\\ \lambda_n(N) &= \frac{n}{N-\beta}(N(\alpha+\beta+n+1)+\beta(n-1/2)).\end{aligned}$$

Putting $s_n = r_n = 1$, we obtain

$$\begin{aligned}&\tau(x;\infty) = \frac{2\beta}{N}(1-x),\;\; \sigma(x;\infty) = \left[\frac{N+\beta}{N-\beta} - \frac{2\beta}{N-\beta}x - x^2\right],\\ &P(x) = 4\frac{(N+\beta)^2}{N^2}(1-x)\left[x + \frac{N+\beta}{N-\beta}\right],\\ &\tilde{\rho}'(x) = \frac{1}{2\pi}\frac{\sqrt{P(x)}}{|(1-x)(x+\frac{N+\beta}{N-\beta})|} = \frac{N+\beta}{\pi N}\frac{1}{\sqrt{(1-x)[x+\frac{N+\beta}{N-\beta}]}},\end{aligned}$$

and this density tends to the density of classical Jacobi polynomials in the non relativistic limit $N \to \infty$.

Example 3. A class of Jacobi polynomials orthogonal with respect to varying weights. We consider here the polynomials: $J_n^{(\alpha_n,\beta_n)}$, satisfying the differential equation

$$\begin{gathered}(1-x^2)y'' + (\beta_n - \alpha_n - (\alpha_n+\beta_n+2)x)y' + n(\alpha_n+\beta_n+n+1)y = 0,\\ \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'' = n(\alpha_n+\beta_n+n+1).\end{gathered}$$

We will limit ourselves to the case essentially considered by W. Gawronski and B. Shawyer [4]:

$$\begin{aligned}\alpha_n &= An\alpha + \epsilon(n),\; \epsilon(n) = O(1),\; (n\to\infty),\\ \beta_n &= Bn\beta + \chi(n),\; \chi(n) = O(1),\; (n\to\infty).\end{aligned}$$

For $\forall\alpha,\beta$, we suppose

$$\begin{aligned}An\alpha + \epsilon(n) > -1 &\Rightarrow A\alpha \geq 0,\\ Bn\beta + \chi(n) > -1 &\Rightarrow B\beta \geq 0.\end{aligned}$$

Then

$$\begin{aligned}\tau(x;\infty) =: \lim_{n\to\infty}\frac{\tau(x;n)}{n} &= \\ &= \lim_{n\to\infty}\frac{\beta_n-\alpha_n-(\alpha_n+\beta_n+2)x}{n} = \\ &= \frac{n(B\beta-A\alpha)-[n(A\alpha+B\beta)x+2x]}{n}+o(1) = \\ &= (B\beta-A\alpha)-(A\alpha+B\beta)x,\\ &\quad \sigma(x;\infty) =: 1-x^2.\end{aligned}$$

By putting $s_n = r_n = 1$, we obtain

$$\begin{aligned}P(x) = &-[(A\alpha+B\beta)^2+4(A\alpha+B\beta+1)]x^2+2(B^2\beta^2-A^2\alpha^2)x-\\ &-[(B\beta-A\alpha)^2-4(A\alpha+B\beta+1)],\\ &\tilde{\rho}'(x) = \frac{1}{2\pi}\frac{\sqrt{P(x)}}{|1-x^2|}.\end{aligned}$$

In particular, if $A\alpha = B\beta$, we find

$$P(x) = 4(2A\alpha+1)-4(A\alpha+1)^2x^2,$$

$$\tilde{\rho}'(x) = \frac{1}{\pi}\frac{\sqrt{(2A\alpha+1)-(A\alpha+1)^2x^2}}{|1-x^2|}.$$

E.g., if $A = 1, \alpha > -\frac{1}{2}$ the numerator has two zeros, symmetric with respect to the origin, which can belong or not to the interval $[-1,1]$. As a consequence, the asymptotic distribution of zeros of such polynomial set is certainly different from the standard (arcsin) one.

Remark. The preceding results do not work in the case of the so called Relativistic Laguerre Polynomials ([9]–[10]). The problem of finding the asymptotic distribution of zeros of this set of polynomials is still open.

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