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TO THE THEORY OF BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC TYPE EQUATIONS AND SYSTEMS

Abstract. Most well–posed boundary value problems, if considered in spaces of higher dimension, have a lot of different variants which, unlike the original problems, do not obey the standard existence and uniqueness conditions. This observation is, in particular, typical of equations and systems of hyperbolic type. Just such problems are needed to be investigated for the purpose of applications. A. V. Bitsadze obtained a number of results in this direction both for strictly hyperbolic and degenerating linear and nonlinear equations. In the subsequent years, these results stimulated investigations of his followers. Some of the results are discussed in the paper.

რეზიუმე. მაღალი განზომილების სივრცეზე გადასვლისას კორექტულად დასმულ სასაზღვრო ამოცანათა უმეტესობას შეესაბამება მრავალი სხვადასხვა ვარიანტი, რომლებიც საწყისი ამოცანებისაგან განსხვავებით, უკვე აღარ ემორჩილებიან ამონახსნთა არსებობის და ერთადერთობის სტანდარტულ პირობებს. ეს კანონზომიერება განსაკუთრებით ნიშანდობლივია ჰიპერბოლური ტიპის განტოლებებისა და სისტემებისათვის. პრაქტიკა მოითხოვს სწორედ ასეთი ამოცანების გამოკვლევას. ა. ბიწაძის მიერ მიღებული იყო მთელი რიგი შედეგებისა, როგორც მკაცრად ჰიპერბოლური, აგრეთვე გადაგვარებული წრფივი და არაწრფივი განტოლებებისათვის, რომლებმაც შემდგომში განაპირობეს მისი მიმდევრების გამოკვლევები. ჩვენს მოხსენებაში შევეხებით სწორედ ამ მიმართულებით მიღებულ ზოგიერთ შედეგს.

It is common knowledge that the initial and characteristic problems for model hyperbolic equations on a plane are uniquely solvable and their solutions can be obtained explicitly. These problems, when passing from one equation to a hyperbolic type system, as it is shown by A.V. Bitsadze [1, 2], can have even infinite number of linearly independent solutions. He also discovered the effect of influence offered by lower derivatives on the correctness of some variants of the Goursat problem. These facts impelled mathematicians to investigate different variants of boundary-value and characteristic problems for hyperbolic systems. In the present paper, we will consider

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some results connected with the problem with oblique derivatives. Because of the fact that in this case the results for a system and for one equation are comparable, for the sake of clearness we will present this result for the equation

$$u_{xy} + au_x + bu_y + cu = F, \tag{1}$$

which is given in a curvilinear quadrangle bounded by arcs of the curves γ_1 and γ_2 coming out of the origin and lying in the first quadrant and also by two characteristic segments coming out of the ends of the above-mentioned arcs. The Poincaré conditions

$$\left(\frac{\partial u}{\partial l_i} + \lambda_i u\right)\Big|_{\gamma_i} = f_i, \quad i = 1, 2, \tag{2}$$

are given on the arcs γ_1 and γ_2.

As it turned out, the directions l_1 and l_2 at the origin and the angle between γ_1 and γ_2 influence greatly the solvability of the problem (1),(2). Under certain conditions, this problem may appear to be well-posed in one case and admit an infinite number of solutions in the other [3].

In an analogous angular domain we can consider the Poincaré problem for the normal hyperbolic system of general type:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + A_1 u_x + B_1 u_y + C_1 u = F. \tag{3}$$

It will be assumed that the characteristics of the system (3) passing through the origin do not get into the angle formed by the arcs γ_1 and γ_2. A number of different variants is available for formulation of the Poincaré problem. We are dwelling upon one of them.

Let on the arc γ_1 as many boundary conditions of the type (2) be given as there are the characteristics coming out of the end of the arc γ_2 and intersecting γ_1. The number of boundary conditions on the arc γ_2 is defined analogously. The character of solvability of such a problem both for a system and for an equation does not differ in principle. In the case of the system (3), the domain of definition for the solution of the problem has an interesting structure. Along with the arcs γ_1 and γ_2, it is bounded by the characteristics coming out of the ends of these arcs. The question is: by what kind of characteristics? If we put the characteristics coming out of the end of the arc $\gamma_1(\gamma_2)$ in order with regard to the distance from the origin to the point of their intersection with the arc $\gamma_2(\gamma_1)$, then we have to take the first not intersecting $\gamma_2(\gamma_1)$ characteristic.

In passing to the domains of higher dimension, the situation changes in principble. One of the reasons of this phenomenon is probably the following fact: for second order hyperbolic equations, the characteristics on the plane are presented by two different sets. But in the space they appear as bicharacteristics of the characteristic cones and represent connected sets. Envelopes of these cones may have multiform configurations. This implies that formulation of various well-posed problems, including characteristic ones, is quite possible. A wide range of well-posed multidimensional analogues of the Darboux and Goursat problems for the wave equation

$$u_{tt} - u_{xx} - u_{yy} = 0 \tag{4}$$

belongs to A. V. Bitsadze.

It should be noted that we do not intend to elaborate well-posed formulations of the problem but we are going to find those ones which would be most adequate to certain physical features. One example of such problems is a multidimensional analogue of the first Darboux problem for the equation (4) in the half-plane $y > 0$ bounded by the cones $S^{\pm} : t \pm t_0 = \pm\sqrt{x^2 + y^2}$, $t_0 > 0$ and by the plane $y = 0$. It is required to determine a solution by the boundary conditions [4]

$$u|_{S^+} = f_1, \quad u|_{y=0} = f_2. \tag{5}$$

The first Darboux problem has many analogues; some of them are ill–posed. For example, there is the problem formulated as follows: find in the domain $0 < t < \frac{1}{2}$, $t < \sqrt{x^2 + y^2} < 1 - t$, a solution of the equation (4) by the boundary conditions

$$u|_{t=0} = f_1, \quad u|_{t=r} = f_2, \quad r = \sqrt{x^2 + y^2}. \tag{6}$$

As it appeared, the corresponding homogeneous problem has an infinite number of linearly independent solutions, but nevertheless the problem (4), (6) itself is solvable [5]. It is of interest that if we assume the outer cone to be the data support, then the problem becomes overdetermined. The problem (4), (6) itself turns into a well-posed one if we transfer the vertex of the inner cone below the plane $t = 0$ [6].

As a direct extension of the Goursat problem to the space, one can consider the problem with data on a finite or infinite light characteristic cone for the wave equation (4). The well-posedness of this problem is well-known. The well-posedness of an analogous problem when data support is a cone lying strictly inside the characteristic cone, has been established in [7, 8]. In fact, this problem is a direct analogue of the second two–dimensional Darboux problem. Similar multidimensional problem for higher order hyperbolic equations with constant coefficients in the principal part has been investigated in [3].

The two-dimensional Darboux and Goursat problems have also other natural generalizations [3], in particular, when the data supports are half-planes bounding dihedral angles and having definite orientation. If both faces are characteristic, then we deal with the analogue of the Goursat problem. If, however, one of the faces is characteristic and the other one is time-oriented, then we get the direct generalization of the first Darboux problem. On the faces we can specify the values of: (a) the solution itself;

(b) the normal derivative or (c) the conditions of the type (2). The conditions ensuring the unique solvability of all these problems are obtained. Moreover, in the case of the Dirichlet problem the solutions are constructed in quadratures.

The analogue of the second Darboux problem, i.e., when the both faces are of time type, turned out to be more complicated. In this case it is impossible to construct the solution in quadratures, and the unique solvability can be proved in Sobolev's space.

In the dihedral angles, the analogues of the Darboux problem have been considered for equations of higher order as well. Formulation of one such analogue for the third order equation

$$u_{xyt} + L(u) = F \tag{7}$$

with a general second order linear differential operator L containing out of the second order derivatives only mixed ones, can be found in [9]. The problem consists in finding a solution of the equation (7) by two boundary conditions on one face and by one condition on the other,

$$M_k(u) = f_k, \quad k = 1, 2, 3, \tag{8}$$

where M_k is also a general linear second order operator of an analogous to L structure. For the problem (7), (8) to be well-posed, sufficient conditions are established among which there appear requirements regarding the given domain's geometry. In particular, the angle edge is parallel to none of the coordinate planes, and bicharacteristics of the equation (7) passing through the edge get neither on the faces nor in the angle formed the by them. These sufficient conditions are close to the necessary ones: the problem (7), (8) may turn out ill-posed if one of these conditions violates.

As is noted above, the Goursat problem for the wave equation (4) with data on a characteristic cone $S^- : t - 1 = -\sqrt{x^2 + y^2}$, $0 \le t \le 1$, is well-posed and its solution is defined in the domain contained between S^- and $S^+ : t + 1 = \sqrt{x^2 + y^2}$, $-1 \le t \le 0$. If instead of (4) we consider the equation

$$(|t|^m u_t)_t - u_{xx} - u_{yy} + L(u) = F, \tag{9}$$

$L(u) \equiv au_x + bu_y + cu_t + du$ with given $m \in (1, 2)$, then instead of the cones $S^\pm$ we have to introduce $S^\pm_m$. The Goursat problem for the equation (9) with data on S^-_m is also well-posed but its solution extends only up to the circle $S_0 : x^2 + y^2 < \left(\frac{2}{2-m}\right)^2$ of the plane $t = 0$. Although when $m = 0$ the initial conditions given on that circle define a solution of the Cauchy problem for the equation (4) in the whole conic domain, the same problem for equation (9) is overdetermined. The problem becomes well-posed if we prescribe on the circle only the values of the unknown solution, and as for the coefficients, require

$$(d - a_x - b_y - c_t)|_{\overline{D}} > 0, \quad c|_{S_0} > 0.$$

where D is the domain contained between S_m^- and S_0.

The reason of all this is the characteristic parabolic degeneration of the equation (9) on the plane $t = 0$ which itself is characteristic.

Parabolic degeneration for hyperbolic equations causes various and very interesting effects even on the plane. In the plane case, the principal parts of a mixed type equation can be represented either by $|t|^m \operatorname{sgn} t u_{xx} + u_{tt}$ or by $u_{xx} + |t|^m \operatorname{sgn} t u_{tt}$. The above-mentioned operators can be continued to the space in different ways. Likewise in different ways can be extended the results known for the two-dimensional case. Then the equation (9) is one of the variants of such a continuation. Not less interesting is the equation

$$u_{tt} - u_{xx} - (|y|^m u_y)_y + L(u) = F, \tag{10}$$

which is by its nature close to the equation (9) and has a characteristic parabolic degeneration on the plane $y = 0$. It turns out that the first Darboux problem with the data on the strip $0 < t < 1$, $y = 0$ and on the charcteristic surface plane $(2-m)t - 2y^{\frac{2-m}{2}} = 0$, $0 < t < \frac{1}{2}$, is overdetermined. In the class of generalized solutions in the space W_2^1 with weighted norm

$$\|u\| = \int_G (u_t^2 + u_x^2 + y^m u_y^2 + u^2) dG,$$

$G : \alpha^{-1} y^\alpha < t < 1 - \alpha^{-1} y^\alpha$, $0 < y < (\frac{\alpha}{2})^{\alpha^{-1}}$, $\alpha = \frac{2-m}{2}$, this problem can be well-posed if we get rid of the data on the strip of the plane $y = 0$ [10].

As is known, a free (non-characteristic) parabolic degeneration has no so considerable influence on the statements of classical problems and on their solvability as it is the case for the equations (9) and (10). For example, for the equations $u_{tt} - |y|^m u_{xx} - u_{yy} + L(u) = F$, $u_{tt} - t^m(u_{xx} + u_{yy}) + L(u) = F$, $m > 0$, with regard for the influence of the lower terms, the above arguments concerning the wave equation remain valid.

We have mentioned above the passage from the two-dimensional case to the multi-dimensional one. We deal in fact with the passage from the multidimensional case to the spaces of lesser dimension because real processes take place both in the space and in time. Mathematical models of such processes are very complicated and therefore we have to consider these processes which possess some kind of symmetry. This is done for the sake of the dimension lowering and model simplification. The idea of the dimension lowering is not new. On this basis, successfully applicable powerful methods have been constructed. The results we mentioned above were achieved in part by realization of this idea. It has successfully been applied [4] to the construction of whole classes of exact solutions of nonlinear equations and of the second order special systems of the kind

$$\sum_{i,j=1}^{n} a^{ij}(x)[u_{x_i x_j} - b(u)u_{x_i}u_{x_j}] + \sum_{i=1}^{n} c^i(x)u_{x_i} + d(x,u) = 0. \qquad (11)$$

Such equations appear in modelling different processes. A.V. Bitsadze's method is based on the description of the structure of the family of level manifolds of solutions of the equation under consideration. He seeks for the solution in the class of functions with the same families of level manifolds. This correspondence is expressed by a simple relation $u = \varphi(v)$ which allows one by a suitable choice of the function φ to reduce the equation (11) to a linear one with respect to v.

It often is more convenient to consider level manifolds referring not to a solution of the given equation but to special combinations of its values and derivatives. For hyperbolic equations, it is more efficient to introduce as such combinations analogues of Riemann's characteristic invariants which follow from differential relations on the characteristics. For some classes of second order nonlinear equations with real characteristics whose differential relations are quite integrable, one can construct general integrals in the sense of Monge–Darboux. In [11, 12], for the Dubreil–Jacotin equation on the plane

$$(u_y^2 - 1)u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0, \qquad (12)$$

the general integral with two arbitrary, smooth enough, functions f and g is constructed which has the form $f(u+y) + g(u-y) + x = 0$. The latter made it possible to discover a number of new facts for the Cauchy problem. In particular, it is shown that discontinuous initial data may cause discontinuites not only of solutions but also of their domains of definition. It is proved that in many cases initial data may become a reason of parabolic degeneration of the equation (12) in the domain of definition of a solution even if the data support is free from such a degeneration. Moreover, in such domains there may exist subdomains which are free from the influence of initial perturbations. As a rule, these subdomains are bounded by the envelopes of the both families of characteristics and therefore, along their boundaries the equation has a strong parabolic degeneration.

The general integral of the equation (12) has been applied to the investigation of the inverse Cauchy problems. The problem of finding the initial values of a solution and its derivatives by means of two arbitrarily given one-parametric families of characteristics is considered in [13]. Sufficient conditions of unique solvability of such a problem are etablished. The dependence of characteristics on the solutions of the equation (12) allows one to formulate these problems. The same dependence makes difficult the consideration of characteristic problems, the Darboux and Goursat problems among them, even in the process of their posing. For the characteristic problem to be well-posed, it is necessary to take into account the structure of the corresponding invariants. The general integral has been constructed on their basis, and therefore it can serve as a basis in posing the problems. Different variants of characteristic problems have been presented in [11, 12]. For the equation (12), for example, there is a problem formulated as follows: two arbitrary arcs, one being characteristic and belonging to one of the families and the other being free, on which the values of solutions are given, come out of a certain fixed point. It is required to determine a solution and its domain of definition by the above-mentioned data. In the given posing, representation of curves and their inclusion into the characteristic families or into the set of free curves is of importance. Sufficient conditions for solvability of such problems are established and in some cases we managed to count the solutions.

As is shown in [14], analogous posings are also possible for the equation of nonlinear oscillations $u_y^4 u_{xx} - u_{yy} = cuu_y^4 x^{-2}$. But such an approach is far from being universal, since for some classes of nonlinear equations the characteristics can form families with quite specific properties. In particular, the family of curves corresponding to an independent of the solution characteristic root is completely determined. Then, naturally, there is no need in attributing an arbitrarily taken arc to some family of characteristics. One has to take a curve endowed with the appropriate properties.

Many similar well-known facts regarding the solvability of multidimensional and nonlinear problems are available in literature. Our present information is far from the full description. Separate results cannot be considered as independent ones. They are interconnected and are the subject of a theory in the process of its formation.

References

1. A. V. Bitsadze, To the analysis of the formulation of characteristic problems for second order hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* 223(1975), No. 6, 1289–1292.

2. A. V. Bitsadze, Influence of lower terms on the well-posedness of characteristic problems for second order hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* 225(1975), No. 1, 31–34.

3. S. Kharibegashvili, Goursat and darboux type problems for linear hyperbolic partial differential equations and systems. *Mem. Differential Equations Mat. Phys.,* 4 (1995), 1–127.

4. A. V. Bitsadze, Some classes of partial differential equations. (Russian) *Nauka, Moscow,* 1981.

5. Kwang-Chang Tong, On a boundary value problem for the wave equation. *Sci. Records* 1(1957), 1–2.

6. S. A. Aldashev, On some local and nonlocal boundary value problems for the wave equation. (Russian) *Differentsial'nye Uravneniya* 19 (1983), No. 1, 3–8.

7. S. L. Sobolev, Some new problems of the theory of partial differential equations of hyperbolic type. (Russian) *Mat. Sb.* 11(53)(1942), No. 3, 155–200.

8. L. Gårding, Linear hyperbolic partial differential equations with constant coefficients. *Acta Math.* 85(1951), 1–62.

9. O. Jokhadze, Spatial problem of Darboux type for one model equation of third order. *Georgian Math. J.* 3(1996), No. 6, 543–560.

10. S. Kharibegashvili, On the solvability of a multidimensional version of the first Darboux problem for a model second order degenerating hyperbolic equation. *Georgian Math. J.* 3(1997), No. 1

11. J. K. Gvazava, Some classes of hyperbolic and mixed type equations. (Georgian) *Metsniereba, Tbilisi,* 1992.

12. J. K. Gvazava, On a class of quasilinear mixed type equations. (Russian) *Metsniereba, Tbilisi,* 1982, 1–94.

13. M. Z. Menteshashvili, Method of characteristics in the initial and inverse problems formulated for some classes of second order quasilinear equations. (Russian) *(Dissertation), Tbilisi,* 1995.

14. R. G. Bitsadze, On the characteristic Goursat problem for a class of nonlinear vibration equations. (Russian) *Dokl. Rasshir. Zased. Sem. Inst. Prikl. Mat. I. N. Vekua* 4(1989), No. 1, 11–14.

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