Memoirs on Differential Equations and Mathematical Physics

Volume ??, 202?, 1–12

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TWO PARAMETER C_0-GROUPS OF LINEAR OPERATORS ON ULTRAMETRIC BANACH SPACES

Abstract. This paper is concerned with some properties of two-parameter C_0-groups and two-parameter uniformly continuous groups and series of bounded linear operators on ultrametric Banach spaces. Our main result is to extend some known theorems in the classical setting to the ultrametric case. Finally, we give some examples.

2020 Mathematics Subject Classification. 47D03, 47S10.

Key words and phrases. Ultrametric Banach spaces, C_0-groups of bounded linear operators, two-parameter groups of bounded linear operators.

1 Introduction and preliminaries

In this paper, E is an ultrametric Banach space over a non-trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $\mathbb{Q}_p$ is the field of p-adic numbers equipped with $|\cdot|_p$. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic valuation $|\cdot|_p$ by $\mathbb{C}_p$ and $B(E)$ denotes the set of all continuous linear operators on E. For more details, we refer to [6] and [15]. The family of two-parameter C_0-semigroups on the classical Banach space was studied by Hille and Phillips [12], Arora and Sharma [2] and Al-Sharif, Khalil [1]. The definition of two-parameter semigroups on Archimedean Banach space E is given as follows.

Definition 1.1 ([1]). A two-parameter family $(T(s,t))_{t,s\geq 0}$ in $L(E)$ is called a two-parameter semigroup on E if:

(i) $T(0,0)=I$;

(ii) for all $t,s,u,v\in\mathbb{R}^+$, $T((t,s)+(u,v))=T(t,s)T(u,v)$.

A two-parameter semigroup $(T(t,s))_{(t,s)\in\mathbb{R}^+\times\mathbb{R}^+}$ is called of class C_0 or strongly continuous if the following condition holds:

- For each $x\in E$, $\lim\limits_{(t,s)\to(0,0)}\|T(t,s)x-x\|=0$.

A two-parameter semigroup $(T(t,s))_{(t,s)\in\mathbb{R}^+\times\mathbb{R}^+}$ is said to be uniformly continuous if and only if

$$\lim_{(t,s)\to(0,0)}\|T(t,s)-I\|=0.$$

Recently, Diagana [5] introduced the concept of C_0-groups of bounded linear operators on an ultrametric free Banach space E. Also, in [8], El Amrani, Blali, Ettayb and Babahmed introduced and studied the notions of C-groups and a cosine family of bounded linear operators on the ultrametric Banach space. The aim of this paper is to introduce and study the uniformly continuous, strongly continuous two-parameter groups and series of bounded linear operators on the ultrametric Banach spaces. For more basic concepts of ultrametric operators theory, we refer to [3–6, 8–10, 12].

Let $r>0$, Ω_r be the clopen ball of $\mathbb{K}$ centred at 0 with radius $r>0$; that is, $\Omega_r=\{t\in\mathbb{K}:\ |t|<r\}$ and $\Omega_r^*=\Omega_r\setminus\{0\}$. Throughout this paper, $(T(t))_{t\in\Omega_r}$ is a family of bounded linear operators on E. We always suppose that r is suitably chosen so that $t\mapsto T(t)$ is well defined.

Definition 1.2 ([5]). Let $r>0$ be a real number. A one-parameter family $(T(t))_{t\in\Omega_r}$ of bounded linear operators on E is a group on E if:

(i) $T(0)=I$, where I is the identity operator on E;

(ii) for all $t,\ s\in\Omega_r$, $T(t+s)=T(t)T(s)$.

The group $(T(t))_{t\in\Omega_r}$ is called C_0-group or strongly continuous group if the following condition holds:

- For each $x\in E$,
$$\lim_{t\to 0}\|T(t)x-x\|=0.$$

A group of bounded linear operators $(T(t))_{t\in\Omega_r}$ is uniformly continuous if and only if

$$\lim_{t\to 0}\|T(t)-I\|=0.$$

The linear operator A defined by

$$D(A)=\left\{x\in E:\ \lim_{t\to 0}\frac{T(t)x-x}{t}\ \text{ exists}\right\}$$

and

$$\text{for each }\ x\in D(A),\quad Ax=\lim_{t\to 0}\frac{T(t)x-x}{t}$$

is called the infinitesimal generator of the group $(T(t))_{t\in\Omega_r}$.

Remark 1.1 ([5]).

(i) Let E be an ultrametric free Banach space over $\mathbb{K}$, let $(T(t))_{t\in\Omega_r}$ be a group on E and $(e_i)_{i\in\mathbb{N}}$ denote an orthogonal basis for E, then for each $t \in \Omega_r$, $T(t)$ can be expressed as

$$T(t)(x) = \sum_{i\in\mathbb{N}} x_i T(t)(e_i) \text{ for any } x = \sum_{i\in\mathbb{N}} x_i e_i \in E,$$

where

$$(\forall\, j \in \mathbb{N}) \quad T(t)(e_j) = \sum_{i\in\mathbb{N}} a_{ij}(t) e_i \text{ with } \lim_{i\to\infty} |a_{ij}(t)| \|e_i\| = 0.$$

(ii) Using (i), one can easily see that for each $t \in \Omega_r^*$, we have

$$(\forall\, j \in \mathbb{N}) \quad \Big(\frac{T(t) - I}{t}\Big) e_j = \Big(\frac{a_{jj}(t) - 1}{t}\Big) e_j + \sum_{i\neq j} \frac{a_{ij}(t)}{t}\, e_i$$

with

$$\lim_{i\neq j,\, i\to\infty} |a_{ij}(t)|\, \|e_i\| = 0.$$

(iii) If $(T(t))_{t\in\Omega_r}$ is a group on E, then its infinitesimal generator A may or may not be a bounded linear operator on E.

Theorem 1.1 ([5])**.** *Let $(T(t))_{t\in\Omega_r}$ be a C_0-group satisfying for each $t \in \Omega_r$, $\|T(t)\| \le M$, where $M > 0$ and let A be its infinitesimal generator. Then for every $x \in D(A)$, $T(t)x \in D(A)$ for each $t \in \Omega_r$. Furthermore,*

$$\frac{dT(t)}{dt}\, x = AT(t)x = T(t)Ax.$$

We start with the following definitions.

Definition 1.3 ([3])**.** Let $r > 0$ be a real number. A two-parameter family $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ of bounded linear operators on E is said to be a two-parameter group of bounded linear operators on E if:

(i) $T(0,0) = I$;

(ii) for every $t_1, t_2, s_1, s_2 \in \Omega_r$, $\quad T(s_1 + s_2, t_1 + t_2) = T(s_1,t_1)T(s_2,t_2)$.

Definition 1.4 ([3])**.** A two-parameter group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ on E is a strongly continuous two-parameter group or two-parameter C_0-group if

$$(\forall\, x \in E) \qquad \lim_{(s,t)\to(0,0)} \|T(s,t)x - x\| = 0.$$

Definition 1.5 ([3])**.** A two-parameter group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly continuous group if

$$\lim_{(s,t)\to(0,0)} \|T(s,t) - I\| = 0.$$

Definition 1.6 ([3])**.** A two-parameter C_0-group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly bounded if there exists $M > 0$ such that for all $(s,t) \in \Omega_r^2$, $\|T(s,t)\| \le M$.

Definition 1.7 ([3])**.** A two-parameter C_0-group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a C_0-group of contractions if

$$(\forall\, (s,t) \in \Omega_r^2) \quad \|T(s,t)\| \le 1.$$

Remark 1.2 ([3])**.** Let E be an ultrametric Banach space over $\mathbb{K}$.

(i) Let $(T(t))_{t\in\Omega_r}$ and $(S(s))_{s\in\Omega_r}$ be a one-parameter groups of bounded linear operators on E such that for all $t, s \in \Omega_r$, $T(t)S(s) = S(s)T(t)$. Set, for all $t, s \in \Omega_r$, $U(s,t) = T(t)S(s)$. Then $(U(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a two-parameter group of bounded linear operators on E. Moreover, if $(T(t))_{t\in\Omega_r}$ and $(S(s))_{s\in\Omega_r}$ are one-parameter C_0-groups on E, then $(U(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a two-parameter C_0-group of bounded linear operators on E.

(ii) Let $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be an uniformly bounded group on E, if $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly continuous on E, then the map $T : \Omega_r^2 \to B(E)$ is continuous on Ω_r^2.

We have the following statements.

Theorem 1.2 ([3])**.** *Let E be an ultrametric Banach space over $\mathbb{K}$. Let $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be a two-parameter uniformly bounded group on E. Then the following statements hold:*

(i) *$(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly continuous if and only if $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are uniformly continuous.*

(ii) *$(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is strongly continuous if and only if $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are uniformly continuous.*

We have the following proposition.

Proposition 1.1 ([3])**.** *Let E be an ultrametric Banach space over $\mathbb{K}$ and let $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be a two-parameter group on E. Then the following statements hold:*

(i) *$(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly bounded if and only if $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are uniformly bounded.*

(ii) *$(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is C_0-group of contractions if and only if $(T(0,t))_{t\in\Omega_r}$ and $(T(s,0))_{s\in\Omega_r}$ are C_0-group of contractions.*

We continue by recalling the following definition.

Definition 1.8 ([14])**.** Let E be an ultrametric Banach space over $\mathbb{K}$ and let a be an interior point of $U \subset E$. A function $f : U \to E$ is called differentiable at a if there exist a continuous mapping $l_a : E \to E$ (called the derivative of f in a) and a function $\varepsilon_a : U \setminus \{a\} \to E$ such that for all $h \in E$ for which $a + h \in U$,

$$f(a+h) = f(a) + l_a(h) + \varepsilon_a(h),$$

where

$$\lim_{h\to 0} \frac{\|\varepsilon_a(h)\|}{\|h\|} = 0.$$

In this case, l_a is called the differentiable at a and is denoted by $Df(a)$.

Remark 1.3 ([14])**.** The above definition is directly derived from the classical case. The derivative l_a is uniquely determined.

Definition 1.9 ([13])**.** The collection of all p-adic integers $\mathbb{Z}_p$ of $\mathbb{Q}_p$ is

$$\mathbb{Z}_p = \big\{x \in \mathbb{Q}_p : \ |x|_p \le 1\big\}.$$

Definition 1.10. Let $A \in B(\mathbb{K}^n)$. A is said to be nilpotent of index d, if there is an integer number $d \le n$ with $A^d = 0_{\mathbb{K}^n}$ and $A^{d-1} \neq 0_{\mathbb{K}^n}$ (where $0_{\mathbb{K}^n}$ denotes the null operator from $\mathbb{K}^n$ into $\mathbb{K}^n$).

Definition 1.11 ([13])**.** If $u \in \mathbb{K}$ and $m \in \mathbb{N}$, we define $\binom{u}{0} = 1$ and $\binom{u}{m} = \frac{u(u-1)\cdots(u-m+1)}{m!}$. If $k \in \mathbb{N}$ with $k \ge n$, hence $|\binom{k}{n}| \le 1$.

Theorem 1.3 ([13])**.** *Additive, translation invariant and bounded $\mathbb{Q}_p$-valued measure on clopens of $\mathbb{Z}_p$ is the zero measure.*

We denote by $C(\mathbb{Z}_p, \mathbb{Q}_p)$ the space of all functions defined and continuous from $\mathbb{Z}_p$ into $\mathbb{Q}_p$.

Theorem 1.4 ([13]). *Let $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$. The function defined on $\mathbb{N}$ by*

$$F(0) = 0,\ F(n) = f(0) + f(1) + \cdots + f(n-1)$$

is uniformly continuous. The extended function is denoted by $Sf(x)$ (called indefinite sum of f). If f is strictly differentiable, so is Sf.

$C_s^1(\mathbb{Z}_p, \mathbb{Q}_p)$ is the space of all functions defined and strictly differentiable from $\mathbb{Z}_p$ into $\mathbb{Q}_p$. For more details, we refer to [13].

Definition 1.12 ([13]). The Volkenborn integral of $f \in C_s^1(\mathbb{Z}_p, \mathbb{Q}_p)$ is defined by

$$\int_{\mathbb{Z}_p} f(t)\, dt = \lim_{n\to\infty} p^{-n} \sum_{j=0}^{p^n-1} f(j) = \lim_{n\to\infty} \frac{Sf(p^n) - Sf(0)}{p^n} = (Sf)'(0).$$

We have the following statements.

Lemma 1.1 ([13]). *For all $t \in \Omega^*_{p^{\frac{-1}{p-1}}}$,*

$$\int_{\mathbb{Z}_p} e^{tu}\, du = \frac{t}{e^t - 1}\,.$$

Lemma 1.2 ([13]). *For all $k \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} \binom{t}{k}\, dt = \frac{(-1)^k}{k+1}\,.$$

Proposition 1.2 ([13, Proposition 55.3]). *Let $f = \sum\limits_{k=0}^{\infty} a_k \binom{t}{k} \in C_s^1(\mathbb{Z}_p, \mathbb{K})$. Then*

$$\int_{\mathbb{Z}_p} f(t)\, dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}\, a_k.$$

2 Main results

We begin with the following remark.

Remark 2.1. Let E be an ultrametric Banach space over $\mathbb{K}$. Let $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be a two-parameter C_0-group on E. Then for all $x \in E$, the map $(s,t) \mapsto T(s,t)x$ is continuous on $\Omega_r \times \Omega_r$. Let A_1 and A_2 be a linear operators defined by

$$D(A_1) = \Big\{x \in E :\ \lim_{h\to 0} \frac{T(h,0)x - x}{h} \text{ exists in } E\Big\},$$
$$D(A_2) = \Big\{x \in E :\ \lim_{h\to 0} \frac{T(0,h)x - x}{h} \text{ exists in } E\Big\},$$

and

$$A_1 x = \lim_{h\to 0} \frac{T(h,0)x - x}{h} = \frac{\partial T(s,\ t)}{\partial s}\bigg|_{(s,t)=(0,0)} \quad \text{for each } x \in D(A_1),$$
$$A_2 x = \lim_{h\to 0} \frac{T(0,h)x - x}{h} = \frac{\partial T(s,\ t)}{\partial t}\bigg|_{(s,t)=(0,0)} \quad \text{for each } x \in D(A_2).$$

It is easy to see that A_1 and A_2 are the infinitesimal generators of the one-parameter groups $(T(s,0))_{s\in\Omega_r}$ and $(T(0,t))_{t\in\Omega_r}$, respectively. We define the infinitesimal generator of two-parameter group as follows.

Definition 2.1. Let E be an ultrametric Banach space over $\mathbb{K}$. Let $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be a two-parameter group on E. The infinitesimal generator of the two-parameter group $(T(s,t))_{s,t\in\Omega_r}$ is the derivative of T at $(0,0)$.

From the definition of the infinitesimal generator, we have the following theorem.

Theorem 2.1. *Let E be an ultrametric Banach space over $\mathbb{K}$. Let A_1 and A_2 be the infinitesimal generators of the one-parameter groups $(T(s,0))_{s\in\Omega_r}$ and $(T(0,t))_{t\in\Omega_r}$ on E, respectively, and $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ be two-parameter C_0-group on E. Then the infinitesimal generator of the two-parameter C_0-group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is the linear transformation $L:\ \Omega_r\times\Omega_r\to B(D(A_1)\cap D(A_2),E)$ defined by*

$$L(s,t)x=(A_1,A_2)\begin{pmatrix}s\\t\end{pmatrix}x=sA_1x+tA_2x \ \textit{ for all } \ x\in D(A_1)\cap D(A_2) \ \textit{ and } \ (s,t)\in\Omega_r\times\Omega_r.$$

Furthermore, if $x\in D(A_1)\cap D(A_2)$, we have for all $(a,b)\in\Omega_r\times\Omega_r$, we have

$$DT(s,t)\begin{pmatrix}a\\b\end{pmatrix}x=(A_1,A_2)\begin{pmatrix}a\\b\end{pmatrix}T(s,t)x.$$

Proof. Let $x\in D(A_1)\cap D(A_2)$ and $(s,t)\in\Omega_r\times\Omega_r$. Then $DT(s,t)|_{(s\ ,t)=(0\ ,0)}$, the derivative of $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ at $(0,0)$, as a function of two variables exists if there is a linear transformation L from $\Omega_r\times\Omega_r$ into $B(D(A_1)\cap D(A_2),E)$ such that $T(s,t)=L(s,t)+R(s,t)$ where

$$\lim_{(s,t)\to(0,0)}\frac{\|R(s,t)\|}{\|(s,t)\|}=0.$$

Let A_1, A_2 be infinitesimal generators of the one-parameter groups $(T(s,0))_{s\in\Omega_r}$ and $(T(0,t))_{t\in\Omega_r}$, respectively. Set

$$J=T(s,t)-T(0,0)-(A_1,\ A_2)\begin{pmatrix}s\\t\end{pmatrix}.$$

Then

$$\begin{aligned}\|Jx\|&=\big\|T(s,t)x-x-sA_1x-tA_2x\big\|\\&=\big\|T(s,t)x-x-sA_1x-tA_2x\big\|\\&=\big\|T(s,0)T(0,t)x-T(s,0)x-tA_2x+T(s,0)x-x-sA_1x\big\|\\&=\left\|tT(s,0)\Big(\frac{T(0,t)x-x}{t}-A_2x\Big)+t\Big(T(s,0)A_2x-A_2x\Big)+s\Big(\frac{T(s,0)x-x}{s}-A_1x\Big)\right\|\\&\le\max\left\{|t|\,\|T(s,0)\|\left\|\frac{T(0,t)x-x}{t}-A_2x\right\|;\right.\\&\qquad\qquad\left.|t|\,\big\|(T(s,0)A_2x-A_2x)\big\|;\ |s|\left\|\frac{T(s,0)x-x}{s}-A_1x\right\|\right\}.\end{aligned}$$

Divide both sides by $\|(s,t)\|=\max(|s|,|t|)$. Since $T(s,0)$ is bounded in a neighborhood of $(0,0)$ and using simple calculations, it follows that

$$\lim_{(s,t)\to(0,0)}\frac{\|Jx\|}{\|(s,t)\|}=0.$$

Hence $DT(s,t)|_{(s,t)=(0,0)}=(A_1,A_2)$, as a linear transformation on $\Omega_r\times\Omega_r$, is the derivative of the two-parameter group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$. Thus the linear transformation $L=(A_1,A_2)$ is the infinitesimal generator of the two-parameter group $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$. Let $u=\begin{pmatrix}a\\b\end{pmatrix}\in\Omega_r\times\Omega_r$ and $x\in D(A_1)\cap D(A_2)$. Then it is easy to see that

$$\begin{aligned}DT(s,t)\begin{pmatrix}a\\b\end{pmatrix}x&=\Big(\frac{\partial T(s,t)}{\partial s},\frac{\partial T(s,t)}{\partial t}\Big)\begin{pmatrix}a\\b\end{pmatrix}x\\&=\Big(a\,\frac{\partial T(s,t)}{\partial s}+b\,\frac{dT(s,t)}{dt}\Big)x=\Big(a\Big(\frac{\partial T(s,0)}{\partial s}\Big)T(0,t)+b\Big(\frac{\partial T(0,t)}{\partial t}\Big)T(s,0)\Big)x.\end{aligned}$$

By Remark 2.1, we get

$$\begin{aligned}DT(s,t)\begin{pmatrix}a\\b\end{pmatrix}x&=\big(aA_1T(s,0)T(0,t)+bA_2T(0,t)T(s,0)\big)x\\&=\big(aA_1T(s,t)+bA_2T(s,t)\big)x=(A_1,A_2)\begin{pmatrix}a\\b\end{pmatrix}T(s,t)x.\end{aligned}$$ □

Let E be an ultrametric Banach space over $\mathbb{K}$. Let $(T(s,t))_{s,t\in\Omega_r}$ be two-parameter C_0-groups of bounded linear operators on E. The problem is to find an analogue of the Hille–Yosida theorem in ultrametric case. We have the following questions.

Questions. In Theorem 2.1, is L a closed operator? Has L dense domain?

From now, we will think that $A=(A_1,A_2)$ is the infinitesimal generator of the two-parameter C_0-groups $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$. We have the following examples.

Example 2.1. Let E be an ultrametric Banach space over $\mathbb{Q}_p$. Let $A,B\in B(E)$ such that $AB=BA$ and $\max\{\|A\|,\|B\|\}<r$ with $r=p^{\frac{-1}{p-1}}$, then for each $(s,t)\in\Omega_r\times\Omega_r$, the operator

$$T(s,t)=\sum_{n\in\mathbb{N}}\frac{(sA)^n}{n!}\cdot\sum_{n\in\mathbb{N}}\frac{(tB)^n}{n!}$$

satisfies the conditions of Definition 3.1, and we will show that. For each $s\in\Omega_r$, set

$$T(s,0)=\sum_{n\in\mathbb{N}}\frac{(sA)^n}{n!}\,. \tag{2.1}$$

Clearly, the series given by (2.1) converges in norm by $|s|\|A\|<r$ and define a one-parameter family of bounded linear operators on E. It is easy to check that $T(0,0)=I$ and for all $s,t\in\Omega_r$, $T(s+t,0)=T(s,0)T(t,0)$. It remains to show that $(T(s,0))_{s\in\Omega_r}$ given above is a C_0- and uniformly continuous one-parameter group. Indeed, for all $s\in\Omega_r$,

$$T(s,0)-I=sA\Big(\sum_{n\in\mathbb{N}}\frac{(sA)^n}{(n+1)!}\Big),$$

then for every $x\in E$, $\|T(s,0)x-x\|\le|s|\|A\|\|\zeta_sx\|$, where

$$\zeta_s=\sum_{n\in\mathbb{N}}\frac{(sA)^n}{(n+1)!}$$

converges to zero as $s\to0$, and hence $(T(s,0))_{s\in\Omega_r}$ is a C_0-group on E. The uniformly continuous property follows from

$$(\forall\,s\in\Omega_r)\quad\|T(s,0)-I\|\le|s|\,\|A\|\,\|\zeta_s\|,$$

then $\lim\limits_{s\to0}\|T(s,0)-I\|=0$. Furthermore,

$$(\forall\,s\in\Omega_r^*)\quad\Big\|\frac{T(s,0)-I}{s}-A\Big\|\le|s|\,\|A\|\,\|\varphi(s)\|,$$

where

$$\varphi(s) = \sum_{n\in\mathbb{N}} \frac{s^n A^{n+1}}{(n+2)!}$$

converges. Consequently,

$$\lim_{s\to 0}\left\|\frac{T(s,\ 0)-I}{s} - A\right\| = 0.$$

Hence, $(T(s,0))_{s\in\Omega_r}$ is a C_0- and uniformly continuous one-parameter group of bounded linear operators of infinitesimal generator A. Similarly, $(T(0,t))_{t\in\Omega_r}$ is also one-parameter C_0-, uniformly continuous group of bounded linear operators of infinitesimal generator B. By Theorem 1.2, $T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ given above is a uniformly continuous group of bounded linear operators of infinitesimal generator (A,B).

Example 2.2. Assume that $\mathbb{K} = \mathbb{Q}_p$ and $r = p^{\frac{-1}{p-1}}$. Let E be an ultrametric free Banach space over $\mathbb{Q}_p$ and $(e_i)_{i\in\mathbb{N}}$ be an orthogonal base of E. For each $s,t\in\Omega_r$ and for each $x\in E$, define

$$T(s,t)x = \sum_{i\in\mathbb{N}} e^{s\mu_i + t\beta_i} x_i e_i,$$

where $(\mu_i)_{i\in\mathbb{N}}, (\beta_i)_{i\in\mathbb{N}} \subset \Omega_r$. It is easy to check that the family $(T(s,t))_{t\in\Omega_r\times\Omega_r}$ is well defined and defines a two-parameter C_0-group of bounded linear operators on E.

Our next result indicates that a bounded linear operator generates an uniformly continuous two-parameter group.

Theorem 2.2. *Let E be an ultrametric Banach space over $\mathbb{Q}_p$. Let $A_1, A_2 \in B(E)$ such that $A_1A_2 = A_2A_1$, $\|A_1\| < r$ and $\|A_2\| < r$ $(r = p^{\frac{-1}{p-1}})$. Then $A = (A_1, A_2)$ is the infinitesimal generator of some uniformly continuous two-parameter group of bounded linear operators $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$.*

Proof. Assume that $A_1, A_2 \in B(E)$ such that $A_1A_2 = A_2A_1$, $\|A_1\| < r$ and $\|A_2\| < r$, where $r = p^{\frac{-1}{p-1}}$. Define

$$(\forall\,(s,t)\in\Omega_r\times\Omega_r)\quad T(s,t) = e^{sA_1}e^{tA_2}. \tag{2.2}$$

By

$$\lim_{n\to\infty}|s|^n\|A_1\|^n = 0 \ \text{ and } \ \lim_{n\to\infty}|t|^n\|A_2\|^n = 0,$$

the family (2.2) is well-defined and defines a family of bounded linear operators on E, and it is easy to see that

- $T(0,0) = I$,
- For all (s_1,t_1) and $(s_2,t_2)\in\Omega_r\times\Omega_r$, we have $T(s_1+s_2, t_1+t_2) = T(s_1,t_1)T(s_2,t_2)$.

Hence $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is a two-parameter group on E and it is easy to see that $(T(s,t))_{(s,t)\in\Omega_r\times\Omega_r}$ is uniformly continuous, since $(T(s,0))_{s\in\Omega_r}$ and $(T(0,t))_{t\in\Omega_r}$ are uniformly continuous groups and for all $s,t\in\Omega_r$, $T(s,t) = T(s,0)T(0,t)$. It is easy to check that

$$\frac{\partial T(s,t)}{\partial s}\bigg|_{(s,t)=(0,0)} = \frac{\partial(e^{sA_1}e^{tA_2})}{\partial s}\bigg|_{(s,t)=(0,0)} = A_1e^{sA_1}e^{tA_2}\big|_{(s,t)=(0,0)} = A_1$$

and

$$\frac{\partial T(s,t)}{\partial t}\bigg|_{(s,t)=(0,0)} = \frac{\partial(e^{sA_1}e^{tA_2})}{\partial t}\bigg|_{(s,t)=(0,0)} = A_2e^{sA_1}e^{tA_2}\big|_{(s,t)=(0,0)} = A_2.$$

Hence

$$(A_1,A_2) = \Big(\frac{\partial T(s,t)}{\partial s}, \frac{\partial T(s,t)}{\partial t}\Big)\bigg|_{(s,t)=(0,0)}$$

is the infinitesimal generator of uniformly continuous two-parameter group $(T(s,t))_{s,t\in\Omega_r}$. $\square$

Question. Is the converse of Theorem 2.2 true?

3 p-Adic series of bounded linear operators

Proposition 3.1. *Let E be an ultrametric Banach space over $\mathbb{C}_p$, let $A \in B(E)$ with $\|I - A\| < 1$. Then for each $t \in \mathbb{Z}_p$ and for all $x \in E$, the series $\sum\limits_{n=0}^{\infty} \binom{t}{n}(I-A)^n x$ converges in E.*

Proof. Since for any $n \in \mathbb{N}$ and for each $t \in \mathbb{Z}_p$, $|\binom{t}{n}|_p \leq 1$, then for any $x \in E$,

$$\left|\binom{t}{n}\right|_p \|(I-A)^n x\| \leq \|(I-A)\|^n \|x\|,$$

hence

$$\lim_{n\to\infty} \left\|\binom{t}{n}(I-A)^n x\right\| = 0,$$

thus

$$\sum_{n=0}^{\infty} \binom{t}{n}(I-A)^n x$$

converges in E. □

From Proposition 3.1, we have the following definition.

Definition 3.1. Let E be a finite-dimensional Banach space over $\mathbb{C}_p$, let $A \in B(E)$ be a diagonal operator such that $\|A\| < 1$. For all $t \in \mathbb{Z}_p$, we define

$$(I+A)^t = \sum_{n=0}^{\infty} \binom{t}{n} A^n$$

with the convention $(I+A)^0 = I$.

Under the conditions of Definition 3.1, we have

Proposition 3.2. *The family $((I+A)^t)_{t\in\mathbb{Z}_p}$ given above is a group on E.*

Proof. For each $t \in \mathbb{Z}_p$, set $T(t) = (I+A)^t$, then
(i) $T(0) = I$.
(ii) For all $t, s \in \mathbb{Z}_p$,

$$\begin{aligned} T(t)T(s) &= (I+A)^t (I+A)^s, \\ &= \left(\sum_{n=0}^{\infty} \binom{t}{n} A^n\right)\left(\sum_{n=0}^{\infty} \binom{s}{n} A^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{t}{k}\binom{s}{n-k}\right) A^n. \end{aligned}$$

From the Vandermonde convolution identity, for the binomial coefficients

$$\sum_{k=0}^{n} \binom{t}{k}\binom{s}{n-k}, \tag{3.1}$$

the coefficient of A^n equals $\binom{t+s}{n}$. Hence

$$\text{for all } t, s \in \mathbb{Z}_p, \quad T(t)T(s) = T(t+s).$$ □

In the next proposition, we assume that for all $1 \leq k \leq n$, $\|A^n\| \leq \|A^k\|$ and $p \neq 2$.

Proposition 3.3. *Let $A \in B(E)$ be a diagonal operator such that $\|A\| < \frac{1}{p}$, then for any $t, s \in \mathbb{Z}_p$,*

$$\left\|(I+A)^t - (I+A)^s\right\| \leq |t-s|_p \|A\|.$$

Proof. For any $t, s \in \mathbb{Z}_p$,

$$(I+A)^t - (I+A)^s = \sum_{n=1}^{\infty} \left(\binom{t}{n} - \binom{s}{n} \right) A^n,$$

then

$$\left\| (I+A)^t - (I+A)^s \right\| \le \sup_{n \ge 1} \left| \binom{t}{n} - \binom{s}{n} \right|_p \|A^n\|.$$

Set $u = t - s$, by the Vandermonde identity (3.1),

$$\binom{t}{n} = \binom{u+s}{n} = \sum_{k=0}^{n} \binom{s}{k}\binom{u}{n-k} = \sum_{k=0}^{n-1} \binom{s}{k}\binom{u}{n-k} + \binom{s}{n}.$$

Using the identity $\binom{X}{m} = \frac{X}{m}\binom{X-1}{m-1}$ for all $m \in \mathbb{N}^*$, we get

$$\binom{t}{n} - \binom{s}{n} = \sum_{k=0}^{n-1} \frac{u}{n-k} \binom{s}{k}\binom{u-1}{n-k-1},$$

hence

$$\begin{aligned} \left| \binom{t}{n} - \binom{s}{n} \right|_p &= \left| \sum_{k=0}^{n-1} \frac{u}{n-k} \binom{s}{k}\binom{u-1}{n-k-1} \right|_p \\ &\le \max_{0 \le k \le n-1} \frac{|u|_p}{|n-k|_p} = |u|_p \max\left(\frac{1}{|1|_p}, \frac{1}{|2|_p}, \dots, \frac{1}{|n|_p} \right). \end{aligned}$$

Thus for all $A \in B(E)$ such that $\|A\| < \frac{1}{p}$, we have

$$\left| \binom{t}{n} - \binom{s}{n} \right|_p \|A^n\| \le |u|_p \max_{1 \le k \le n} \left(\frac{\|A^n\|}{|k|_p} \right) \le |u|_p \max_{1 \le k \le n} \left(\frac{\|A^k\|}{|k|_p} \right) \le |t-s|_p \max_{1 \le k \le n} \left(\frac{\|A^k\|}{|k|_p} \right).$$

From $\|A\| < \frac{1}{p} < p^{\frac{-1}{p-1}}$, then for all $k \ge 2$, $\frac{\|A^k\|}{|k|_p} < \|A\|$, hence for all $k \ge 1$, $\frac{\|A^k\|}{|k|_p} \le \|A\|$, then

$$\left| \binom{t}{n} - \binom{s}{n} \right|_p \|A^n\| \le |t-s|_p \max_{1 \le k \le n} \left(\frac{\|A^k\|}{|k|_p} \right) \le |t-s|_p \|A\|. \quad \square$$

We finish with the following proposition.

Proposition 3.4. *Let E be the finite-dimensional Banach space over $\mathbb{Q}_p$, let $A \in B(E)$ be a nilpotent operator of index n such that $\|A\| < 1$, then for all $x \in E$,*

$$\int_{\mathbb{Z}_p} (I+A)^t x \, dt = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} A^k x.$$

Proof. By definition, for all $A \in B(E)$ such that $\|A\| < 1$, $t \in \mathbb{Z}_p$,

$$(I+A)^t = \sum_{k=0}^{\infty} \binom{t}{k} A^k.$$

Since A is a nilpotent operator of index n, we have

$$(I+A)^t = \sum_{k=0}^{\infty} \binom{t}{k} A^k = \sum_{k=0}^{n-1,} \binom{t}{k} A^k.$$

Thus for all $x \in E$,

$$\int_{\mathbb{Z}_p} (I+A)^t x \, dt = \int_{\mathbb{Z}_p} \sum_{k=0}^{n-1} \binom{t}{k} A^k x \, dt = \sum_{k=0}^{n-1} \int_{\mathbb{Z}_p} \binom{t}{k} A^k x \, dt = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} A^k x. \quad \square$$

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(Received 05.11.2022; revised 22.12.2022; accepted 28.12.2022)

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