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# EXISTENCE OF SOLUTIONS FOR NONLOCAL NABLA CONFORMABLE FRACTIONAL THERMISTOR PROBLEM ON TIME SCALES

**Abstract.** In this paper, we study the existence of solution for the nonlocal nabla conformable fractional thermistor problem of order $\alpha$ on an arbitrary bounded time scale. By a result of the work, the thermistor equation has been generalized on the time scale by using conformable fractional. The study is carried out by using the tube solution (a generalization of lower and upper solutions) and Schauder's fixed-point theorem. Finally, an example is given to illustrate the results of this work.

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**Key words and phrases.** Fractional nonlocal thermistor problems, nabla conformable fractional calculus on time scales, solution-tube, Schauder's fixed-point.

**რეზიუმე.** ნაშრომში ჩვენ ვსწავლობთ $\alpha$ რიგის არალოკალური ნაბლა კონფორმული წილადური ხარისხის თერმისტორის ამოცანის ამონახსნის არსებობას ნებისმიერ შემოსაზღვრულ დროის სკალაზე. ნაშრომის შედეგის თანახმად, თერმისტორის განტოლება განზოგადებულია დროის სკალაზე კონფორმული წილადური ხარისხის გამოყენებით. კვლევა ტარდება მილისებრი ამონახსნისა (ქვედა და ზედა ამონახსნების განზოგადება) და შაუდერის უძრავი წერტილის თეორემის გამოყენებით. ნაშრომის ბოლოს მოყვანილია მაგალითი მიღებული შედეგების საილუსტრაციოდ.

# 1 Introduction

Classical fractional differential operators have a group of known deficiencies. Although local operators appeared in the 60s of the past century, these disadvantages were overcome only in 2014, when Khalil et al. [20] defined and formalized the operators using the classic idea of the limit of a certain incremental quotient, and obtained a derivative that was called conformable. In 2018, a new direction of work was opened when what was called non-conformable was introduced (see [25,26]). These differential (local) operators have proven their usefulness in many applications (for example, see [10,16,22,23,25–27]).

In 2014, Khalil et al. [20] proposed another type of fractional derivative named "conformable fractional derivative". In particular, Benkhettou et al. [8] extended this definition to an arbitrary time scale, which is a natural extension of the conformable fractional calculus, satisfying the standard formulas of the product derivative and quotient derivative of two functions. B. Bendouma et al. [5] introduced a nabla conformable fractional derivative of order $\alpha$ for a function defined on $\mathbb{T}$ which reduces to the nabla derivative (see Definition 2.2) when $\alpha = 1$. For the recent results on conformable fractional derivatives on time scales, we refer the reader to [5,6,13,14,29,39]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. The theory of time scales was introduced by Stefan Hilger in his PhD thesis in 1988, in order to unify and generalize continuous and discrete analysis. For more detailed discussions, we refer the reader to [18] and the references therein.

A thermistor is an element with an electrical resistance that changes in response to temperature. The term is a combination of thermal and resistor. There are generally two types of thermistors: negative temperature coefficient thermistors (NTC) and positive temperature coefficient thermistors (PTC). With (NTC), the resistance variation is inverse to the temperature change. (NTC) thermistors are nonlinear, and their resistance decreases as temperature increases. Thermistors can be found in computers, digital thermostats, airplanes, portable heaters, cars, medical equipment, electrical outlets, chemical industries, etc. The study of thermistor problems (existence, uniqueness, stability, and multiplicity of solutions) can be found, for example, in [1,11,12,19,24,31–38,40].

In [36], Sidi Ammi et al. used the solution-tube method for the nonlocal conformable fractional thermistor problem

$$\begin{cases} x^{(\beta)}(s) = \dfrac{\lambda f(s,x(s))}{\left(\int_c^d f(\tau,x(\tau))\,d\tau\right)^2}, \quad s \in [c,d], \\ x(c) = x_c, \end{cases} \tag{1.1}$$

where $x^{(\beta)}$ is a new definition called the conformable fractional derivative of $x$ of order $\beta$.

Sidi Ammi et al. [35] discussed the existence and uniqueness results for a fractional Riemann–Liouville nonlocal thermistor problem on arbitrary time scales

$$\begin{cases} {}_c^{\mathbb{T}}\mathcal{D}_t^{2\beta} u(t) = \dfrac{\lambda f(u)}{\left(\int_c^d f(u)\,\triangle u\right)^2}, \quad t \in (c,d), \\ {}_c^{\mathbb{T}}I_t^{\beta} u(c) = 0 \ \text{ for all } \ \beta \in (0,1), \end{cases}$$

where $f : [0,\infty) \to [0,\infty)$ is Lipschitz continuous, ${}_{t_0}^{\mathbb{T}}\mathcal{D}_t^{2\beta}$ is the left Riemann–Liouville fractional derivative operator of order $2\alpha$ on $\mathbb{T}$ and ${}_{t_0}^{\mathbb{T}}I_t^{\beta}$ is the left Riemann–Liouville fractional integral operator of order $\beta$ defined on $\mathbb{T}$ by Benkhettou et al. [7].

In this paper, we are concerned with the existence of a solution for the following nonlocal nabla conformable fractional thermistor problem on a time scale $\mathbb{T}$:

$$\begin{cases} x_{\nabla}^{(\alpha)}(t) = \dfrac{\lambda f(t,x^{\rho}(t))}{\left(\int_a^b f(\tau,x^{\rho}(\tau))\,\nabla\tau\right)^2} \ \text{ for all } \ t \in \mathbb{T}_{\kappa}, \\ x(b) = x_b. \end{cases} \tag{1.2}$$

Here, $\mathbb{T}$ is an arbitrary bounded time scale such that $a = \min \mathbb{T} > 0$ and $b = \max \mathbb{T}$, $\mathbb{T}_0 = \mathbb{T} \setminus \{a\}$, $\lambda$ is a fixed positive real, $f : \mathbb{T}_\kappa \times [0, \infty) \to (0, \infty)$ is a continuous function, $x$ describes the temperature of the conductor and $x_\nabla^{(\alpha)}(t)$ denotes the nabla conformable fractional derivative of $x$ at $t$ of order $\alpha \in (0, 1)$.

In order to obtain the existence result for problem (1.2), we introduce the notion of solution-tube of (1.2). This is inspired by a notion of solution-tube for conformable fractional nonlocal thermistor problem introduced in [36]. The results of this paper were motivated by the results of [2–4, 36].

The paper is organized as follows. In Section 2, we introduce the definition of nabla conformable fractional calculus on time scales and their important properties. In Section 3, we prove the existence of a solution to problem (1.2) by using the solution-tube method and Schauder's fixed point theorem.

# 2 Preliminaries

In this section, we recall some notions and results that will be used in this paper.

## 2.1 Calculus on time scales

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $s \in \mathbb{T}$, we define the backward and forward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(s) := \sup\{\tau \in \mathbb{T} : \ \tau < s\} \quad \text{and} \quad \sigma(s) := \inf\{\tau \in \mathbb{T} : \ \tau > s\},$$

respectively. We present that classification point in a time scale: if $\sigma(s) > s$ (resp., if $\rho(s) < s$), we say that $s$ is right-scattered (resp., left-scattered); if $s$ is right-scattered and left-scattered, we say that $s$ is isolated; also, if $\sigma(s) = s < \sup \mathbb{T}$, we say that $s$ is right-dense. If $\rho(s) = s > \inf$, we say that $s$ is left-dense. Points that are right-dense and left-dense are called dense. The graininess function and backward graininess $\mu, \nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(s) := \sigma(s) - s$ and $\nu(s) := s - \rho(s)$, respectively. If $\mathbb{T}$ has a left-scattered maximum $M$, then $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$, otherwise, $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ is bounded, then $\mathbb{T}_0 \subseteq \mathbb{T}_k$, where $\mathbb{T}_0 = \mathbb{T} \setminus \{\min \mathbb{T}\}$. For $a, b \in \mathbb{T}$, we define the closed interval $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}$.

**Definition 2.1** ([9])**.** We say that $f : \mathbb{T} \to \mathbb{R}$ belongs to the ld-continuous function space, noted by $\mathcal{C}_{ld}(\mathbb{T}, \mathbb{R})$, if $f$ is continuous at a left-dense point in $\mathbb{T}$ and has a right-sided limits existing at the right-dense points in $\mathbb{T}$.

**Definition 2.2** ([9])**.** For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$, the $\nabla$-derivative of f at $t$, denoted by $f^\nabla(t)$, is defined to be the number (provided it exists) with the property given for any $\varepsilon > 0$: there is a neighborhood $U$ of $t$ such that

$$\left| f^\rho(t) - f(s) - f^\nabla(t)(\rho(t) - s) \right| \le \varepsilon |\rho(t) - s| \ \text{ for all } \ s \in \mathcal{U}.$$

We say that $f$ is $\nabla$-differentiable if $f^\nabla(t)$ exists for every $t \in \mathbb{T}_k$. The function $f^\nabla : \mathbb{T} \to \mathbb{R}$ is then called the $\nabla$-derivative of $f$ on $\mathbb{T}_k$.

We say that $f : \mathbb{T} \to \mathbb{R}$ belongs to $\mathcal{C}^1_{ld}(\mathbb{T}, \mathbb{R})$ if $f$ is $\nabla$-differentiable and $f^\nabla \in \mathcal{C}_{ld}(\mathbb{T}_k, \mathbb{R})$.

**Definition 2.3** ([9])**.** We say that $\varphi : \mathbb{T} \to \mathbb{R}$ is a $\nu$-regressive function if

$$1 - \nu(t)\varphi(t) \neq 0 \ \text{ for all } \ t \in \mathbb{T}_k.$$

We say that $\varphi$ belongs to the set $\mathcal{R}^+_\nu(\mathbb{T}, \mathbb{R})$ if $\varphi \in \mathcal{C}_{ld}(\mathbb{T}, \mathbb{R})$ and $\varphi$ is $\nu$-regressive. We define the set

$$\mathcal{R}^+_\nu(\mathbb{T}, \mathbb{R}) = \left\{ \varphi \in \mathcal{C}_{ld}(\mathbb{T}, \mathbb{R}) : \ \ 1 - \nu(t)\varphi(t) > 0 \ \text{ for all } \ t \in \mathbb{T}_k \right\}.$$

**Definition 2.4** ([9])**.** If $p \in \mathcal{R}_\nu(\mathbb{T}, \mathbb{R})$, then we define the nabla exponential function $\widehat{e}_p$ by

$$\widehat{e}_p(a, b) = \exp\left( \int_a^b \widehat{\xi}_{\nu(\tau)}(p(\tau)) \, \nabla\tau \right)$$

for $a, v \in \mathbb{T}$, where the $\nu$-cylinder transformation

$$\widehat{\xi}_h(z) = \begin{cases} -\dfrac{1}{h}\log(1-zh) & \text{if } h>0, \\ z & \text{if } h=0, \end{cases}$$

where log is the principal logarithm function.

## 2.2 Nabla conformable fractional calculus on time scales

Fractional differential equations play an important role in describing many phenomena and processes in various fields of science such as physics, control systems, aerodynamics and electrodynamics, etc. (see, e.g., [30]).

We introduce the definition of nabla conformable fractional calculus on time scales and their important properties.

**Definition 2.5** (Conformable fractional derivative, [20])**.** Given a function $\varphi:[0,\infty)\to\mathbb{R}$ and a real constant $\alpha\in(0,1]$, the conformable fractional derivative of $f$ of order $\alpha$ is defined by

$$\varphi^{(\alpha)}(t) := \lim_{\varepsilon\to 0}\frac{\varphi(t+\varepsilon t^{1-\alpha})-\varphi(t)}{\varepsilon} \quad \text{for all } t>0.$$

If $\varphi^{(\alpha)}(t)$ exists and is finite, we say that $\varphi$ is $\alpha$-differentiable at $t$.

If $\varphi$ is $\alpha$-differentiable in some interval $(0,a)$, $a>0$, and $\lim_{t\to0^+}\varphi^{(\alpha)}(t)$ exists, then the conformable fractional derivative of $\varphi$ of order $\alpha$ at $t=0$ is defined as

$$\varphi^{(\alpha)}(0) = \lim_{t\to 0^+}\varphi^{(\alpha)}(t).$$

**Definition 2.6** (Nabla conformable fractional derivative, [5])**.** Let $\varphi:\mathbb{T}\to\mathbb{R}$, $t\in\mathbb{T}_\kappa$, and $\alpha\in]0,1]$. For $t>0$, we define $\varphi_\nabla^{(\alpha)}(t)$ to be a number (provided it exists) with the property that, given any $\varepsilon>0$, there is a $\delta$-neighborhood $\mathcal{V}_t\subset\mathbb{T}$ (i.e., $\mathcal{V}_t=]t-\delta,t+\delta[\cap\mathbb{T}$) of $t$, $\delta>0$, such that

$$\left|(\varphi(\rho(t))-\varphi(s))t^{1-\alpha}-\varphi_\nabla^{(\alpha)}(t)(\rho(t)-s)\right| \le \varepsilon|\rho(t)-s| \quad \text{for all } s\in\mathcal{V}_t.$$

We call $\varphi_\nabla^{(\alpha)}(t)$ the nabla conformable fractional derivative of $\varphi$ of order $\alpha$ at $t$ and define the nabla conformable fractional derivative at 0 as $\varphi_\nabla^{(\alpha)}(0)=\lim_{t\to0^+}\varphi_\nabla^{(\alpha)}(t)$. The function $\varphi$ is nabla conformal fractional differentiable of order $\alpha$ on $\mathbb{T}_\kappa$, provided $\varphi_\nabla^{(\alpha)}(t)$ exists for all $t$ in $\mathbb{T}_\kappa$.

Note that if $\alpha=1$, we have $\varphi_\nabla^{(\alpha)}(t)=\varphi^\nabla(t)$, and if $\mathbb{T}=\mathbb{R}$, then $\varphi_\nabla^{(\alpha)}=\varphi^{(\alpha)}$ is the conformable fractional derivative of $\varphi$ of order $\alpha$.

We denote

(i) $\mathcal{C}^\alpha([a,b]_\mathbb{T},\mathbb{R}) := \Big\{\varphi:\ [a,b]_\mathbb{T}\to\mathbb{R},\ \varphi$ is nabla conformal fractional differentiable of order $\alpha$ on $[a,b]_\mathbb{T}$ $\alpha$ and $\varphi_\nabla^{(\alpha)}\in\mathcal{C}([a,b]_\mathbb{T},\mathbb{R})\Big\}$.

(ii) $\mathcal{C}_{ld}^\alpha([a,b]_\mathbb{T},\mathbb{R}) := \Big\{\varphi:\ [a,b]_\mathbb{T}\to\mathbb{R},\ \varphi$ is nabla conformal fractional differentiable of order $\alpha$ on $[a,b]_\mathbb{T}$ and $\varphi_\nabla^{(\alpha)}\in\mathcal{C}_{ld}([a,b]_\mathbb{T},\mathbb{R})\Big\}$.

Some useful properties of the nabla conformable fractional derivative of $\varphi$ of order $\alpha$ are given in the following theorems.

**Theorem 2.1** ([5])**.** *Let $\alpha\in(0,1]$ and $\mathbb{T}$ be a time scale. Assume $\varphi:\mathbb{T}\to\mathbb{R}$ and let $t\in\mathbb{T}_\kappa$. The following properties hold.*

(i) *If $\varphi$ is nabla conformal fractional differentiable of order $\alpha$ at $t>0$, then $\varphi$ is continuous at $t$.*

(ii) *If $\varphi$ is continuous at $t$ and $t$ is left-scattered, then $\varphi$ is nabla conformable fractional differentiable of order $\alpha$ at $t$ with*

$$\varphi_{\nabla}^{(\alpha)}(t)=\frac{\varphi(t)-\varphi(\rho(t))}{\nu(t)}\,t^{1-\alpha}. \tag{2.1}$$

(iii) *If $t$ is left-dense, then $\varphi$ is nabla conformable fractional differentiable of order $\alpha$ at $t$ if and only if the limit $\lim\limits_{s\to t}\frac{\varphi(t)-\varphi(s)}{(t-s)}t^{1-\alpha}$ exists as a finite number. In this case,*

$$\varphi_{\nabla}^{(\alpha)}(t)=\lim_{s\to t}\frac{\varphi(t)-\varphi(s)}{t-s}\,t^{1-\alpha}. \tag{2.2}$$

(iv) *If $\varphi$ is nabla conformable fractional differentiable of order $\alpha$ at $t$, then*

$$\varphi(\rho(t))=\varphi(t)-(\nu(t))t^{\alpha-1}\varphi_{\nabla}^{(\alpha)}(t).$$

**Theorem 2.2** ([5])**.** *Assume $\varphi,\psi:\mathbb{T}\to\mathbb{R}$ are nabla conformable fractional differentiable of order $\alpha$. Then*

(i) *for any $\lambda\in\mathbb{R}$, $\lambda\varphi+\psi:\mathbb{T}\to\mathbb{R}$ is nabla conformable fractional differentiable with*

$$(\lambda\varphi+\psi)_{\nabla}^{(\alpha)}=\lambda\varphi_{\nabla}^{(\alpha)}+\psi_{\nabla}^{(\alpha)},$$

(ii) *if $\varphi$ and $\psi$ are continuous, then the product $\varphi\psi:\mathbb{T}\to\mathbb{R}$ is nabla conformable fractional differentiable with*

$$(\varphi\psi)_{\nabla}^{(\alpha)}=\varphi_{\nabla}^{(\alpha)}\psi+\varphi^{\rho}\psi_{\nabla}^{(\alpha)}=\varphi_{\nabla}^{(\alpha)}\psi^{\rho}+\varphi\psi_{\nabla}^{(\alpha)},$$

(iii) *if $\varphi$ and $\psi$ are continuous, then $\varphi/\psi$ is nabla conformable fractional differentiable with*

$$\left(\frac{\varphi}{\psi}\right)_{\nabla}^{(\alpha)}=\frac{\varphi_{\nabla}^{(\alpha)}\psi-\varphi\psi_{\nabla}^{(\alpha)}}{\psi\psi^{\rho}},$$

*valid at all points $t\in\mathbb{T}_{\kappa}$ for which $\psi(t)\psi^{\rho}(t)\neq 0$.*

**Example.** Let $\alpha\in(0,1]$, $\lambda\in\mathbb{R}$, fix $t_0\in\mathbb{T}$ and $p\in\mathcal{R}_{\nu}$. The functions $\varphi(t)\equiv\lambda$, $\psi(t)=t$ and $\phi(t)=\widehat{e}_p(t,t_0)$ are nabla conformable fractional differentiable of order $\alpha$ with

$$\varphi_{\nabla}^{(\alpha)}(t)=0,\quad \psi_{\nabla}^{(\alpha)}(t)=t^{1-\alpha}\ \text{ and }\ \phi_{\nabla}^{(\alpha)}(t)=t^{1-\alpha}p\widehat{e}_p(t,t_0).$$

Now we introduce the nabla conformable fractional integral (or nabla $\alpha$-fractional integral) on time scales. The $\nabla$-measure and $\nabla$-integration are defined in the same way as those in [15].

**Definition 2.7** ([5])**.** Let $\varphi:\mathbb{T}\to\mathbb{R}$ be a regulated function. Then the nabla $\alpha$-fractional integral of $\varphi$, $0<\alpha\leq 1$, is defined by

$$\int\varphi(t)\nabla_{\alpha}t=\int\varphi(t)t^{\alpha-1}\,\nabla t.$$

Note that if $\alpha=1$, then

$$\int\varphi(t)\,\nabla_{\alpha}t=\int\varphi(t)\,\nabla t.$$

If $\mathbb{T}=\mathbb{R}$, then

$$\int\varphi(t)\,\nabla_{\alpha}t=\int t^{\alpha-1}\varphi(t)\,dt.$$

**Definition 2.8** ([5])**.** Assume $\varphi:\mathbb{T}\to\mathbb{R}$ is a function. Let $A$ be a $\nabla$-measurable subset of $\mathbb{T}$, then $\varphi$ is nabla $\alpha$-integrable on $A$ if and only if $t^{\alpha-1}\varphi(t)$ is integrable on $A$, and

$$\int_A\varphi(t)\,\nabla_{\alpha}t=\int_A t^{\alpha-1}\varphi(t)\,\nabla t.$$

**Theorem 2.3** ([5]). *Let $\alpha \in (0,1]$, $a,b,c \in \mathbb{T}$, $\lambda,\gamma \in \mathbb{R}$, and $\varphi$, $\psi$ be two ld-continuous functions. Then*

(i) $\displaystyle\int_a^b \left[\lambda\varphi(t)+\gamma\psi(t)\right] \nabla_\alpha t = \lambda \int_a^b \varphi(t)\,\nabla_\alpha t + \gamma \int_a^b \psi(t)\,\nabla_\alpha t;$

(ii) $\displaystyle\int_a^b \varphi(t)\,\nabla_\alpha t = \int_a^c \varphi(t)\,\nabla_\alpha t + \int_c^b \varphi(t)\,\nabla_\alpha t;$

(iii) *if there exists $\psi : \mathbb{T} \to \mathbb{R}$ with $|\varphi(t)| \le \psi(t)$ for all $t \in [a,\ b]$, then*

$$\left|\int_a^b \varphi(t)\,\nabla_\alpha t\right| \le \int_a^b \psi(t)\,\nabla_\alpha t;$$

(iv) *if $\varphi(t) > 0$ for all $t \in [a,\ b]$, then*

$$\int_a^b \varphi(t)\,\nabla_\alpha t \ge 0.$$

**Theorem 2.4** ([5]). *If $\varphi : \mathbb{T}_\kappa \to \mathbb{R}$ is an ld-continuous function and $t \in \mathbb{T}_\kappa$, then*

$$\int_{\rho(t)}^t \varphi(s)\,\nabla_\alpha s = \nu(t)\varphi(t)t^{\alpha-1}.$$

# 3 Main results

In this section, we establish an existence result for problem (1.2). A solution of problem (1.2) will be a function $x \in \mathcal{C}_{ld}^\alpha(\mathbb{T},\mathbb{R})$ for which (1.2) is satisfied. We introduce the notion of solution-tube of this problem as follows.

**Definition 3.1.** Let $(v,M) \in \mathcal{C}_{ld}^\alpha(\mathbb{T},\mathbb{R}) \times \mathcal{C}_{ld}^\alpha(\mathbb{T},[0,\infty))$. We say that $(v,M)$ is a solution-tube of (1.2) if

(i) $(x - v^\rho(t))(F(t,x) - v_\nabla^{(\alpha)}) \ge M^\rho(t)M_\nabla^{(\alpha)}(t)$ for every $t \in \mathbb{T}_\kappa$ and for every $x \in \mathbb{R}$ such that $|x - v^\rho(t)| = M^\rho(t)$,

(ii) $v_\nabla^{(\alpha)}(t) = F(t, v^\rho(t))$ and $M_\nabla^{(\alpha)}(t) = 0$ for all $t \in \mathbb{T}_\kappa$ such that $M^\rho(t) = 0$,

(iii) $|x_b - v(b)| \le M(b)$,

where

$$F(t,x) = \frac{\lambda f(t,x)}{\left(\int_a^b f(\tau,x)\,\nabla\tau\right)^2}. \tag{3.1}$$

We denote

$$\mathcal{T}(v,M) := \left\{x \in \mathcal{C}_{ld}^\alpha(\mathbb{T},\mathbb{R}) : \ |x(t) - v(t)| \le M(t) \ \text{ for all } \ t \in \mathbb{T}\right\}.$$

**Remark.** Our notion of solution-tube is equivalent to the notion of upper and lower solutions.

Consider the following modified problem:

$$\begin{cases} x_{\nabla}^{(\alpha)}(t) - \alpha t^{1-\alpha} x(\rho(t)) = F(t, \overline{x}(\rho(t))) - \alpha t^{1-\alpha}\overline{x}(\rho(t)), \ \ t \in \mathbb{T}_\kappa, \\ x(b) = x_b, \end{cases} \tag{3.2}$$

where

$$\overline{x}(t) = \begin{cases} \dfrac{M(t)}{|x - v(t)|}\,(x - v(t)) + v(t) & \text{if } \ |x - v(t)| > M(t), \\ x(t) & \text{if } \ |x - v(t)| \le M(t). \end{cases} \tag{3.3}$$

**Proposition 3.1.** *Let $\alpha \in (0,1]$ and $x : \mathbb{T} \to \mathbb{R}$ be nabla conformable fractional differentiable of order $\alpha$ at $t > 0$. Then the function $|\cdot| : \mathbb{R} \setminus \{0\} \to [0,\infty)$ is nabla conformal fractional differentiable of order $\alpha$ at $t$. If $t = \rho(t)$, we have*

$$|x(t)|_{\nabla}^{(\alpha)} = \frac{x(t)x_{\nabla}^{(\alpha)}(t)}{|x(t)|}\,.$$

*Proof.* From Definition 2.6, Theorem 2.1 and Theorem 2.2, we obtain

$$\begin{aligned} |x(t)|_{\nabla}^{(\alpha)} &= \lim_{s\to t} \frac{|x(\rho(t))| - |x(s)|}{\rho(t) - s} t^{1-\alpha} = \lim_{s\to t} \frac{|x(t)|^2 - |x(s)|^2}{t - s}\, \frac{1}{(|x(t)| + |x(s)|)}\, t^{1-\alpha} \\ &= \lim_{s\to t} \frac{|x(\rho(t))|^2 - |x(s)|^2}{\rho(t) - s}\, \frac{1}{(|x(t)| + |x(s)|)}\, t^{1-\alpha} = [x^2(t)]_{\nabla}^{(\alpha)}\, \frac{1}{2|x(t)|} = \frac{x(t)x_{\nabla}^{(\alpha)}(t)}{|x(t)|}\,. \end{aligned}$$

□

Similarly to [4, Lemma 3.4], we give the following maximum principle.

**Lemma 3.1.** *Let $r \in \mathcal{C}_{ld}^{\alpha}(\mathbb{T}, \mathbb{R})$ such that $r_{\nabla}^{(\alpha)}(t) > 0$ on $\{t \in \mathbb{T}_\kappa : \ r(\rho(t)) > 0\}$. If $r(b) \le 0$, then $r(t) \le 0$ for all $t \in \mathbb{T}$.*

*Proof.* Suppose the conclusion is false. Then there exists $t_0 \in \mathbb{T}$ such that $r(t_0) = \max_{t\in\mathbb{T}} r(t) > 0$, since $r$ is continuous on $\mathbb{T}$. If $\sigma(t_0) > t_0$, then $r_{\nabla}^{(\alpha)}(\sigma(t_0))$ exists, since $\nu(\sigma(t_0)) = \sigma(t_0) - t_0 > 0$ and because $r \in \mathcal{C}_{ld}^{\alpha}(\mathbb{T}, \mathbb{R})$. Then

$$r_{\nabla}^{(\alpha)}(\sigma(t_0)) = \frac{r(\sigma(t_0)) - r(t_0)}{\sigma(t_0) - t_0}\,(\sigma(t_0))^{1-\alpha} \le 0,$$

which is a contradiction, since $r(t_0) = r(\rho(\sigma(t_0))) > 0$.

If $t_0 = \sigma(t_0) < b$, then there exists an interval $(\sigma(t_0), t_1]$ such that $r(\sigma(t)) > 0$ for all $t \in (\sigma(t_0), t_1] \cap \mathbb{T}$. Thus

$$0 \le r(t_1) - r(t_0) = r(t_1) - r(\sigma(t_0)) = \int\limits_{(\sigma(t_0),t_1]\cap\mathbb{T}} r_{\nabla}^{(\alpha)}(s)\,\nabla_\alpha s < 0,$$

which contradicts the fact that $r(t_0)$ is a maximum. The case $t_0 = b$ is impossible from the hypothesis. If we take $t_0 = a$, by using previous steps of this proof, one can check that $r(a) \le 0$ and then the lemma is proved. □

We need the following auxiliary lemma.

**Lemma 3.2.** *The terminal problem*

$$\begin{cases} x_{\nabla}^{(\alpha)}(t) - \alpha t^{1-\alpha} x(\rho(t)) = g(t), \quad \nabla\text{-a.e. } \ t \in \mathbb{T}_0, \\ x(b) = x_b. \end{cases} \tag{3.4}$$

*with $\alpha \in (0,1]$, $x_b \in \mathbb{R}$, and $g \in L_{\alpha,\nabla}^1(\mathbb{T}_0, \mathbb{R})$, has a unique solution $x$ given by the following expression:*

$$x(t) := \int_a^b G(t,s)g(s)\,\nabla_\alpha s + x_b \widehat{e}_{-\alpha}(b,t), \ \ t \in \mathbb{T}, \tag{3.5}$$

*where*

$$G(t,s) = -\widehat{e}_{-\alpha}(s,t)\begin{cases}0, & a \le s \le t \le b,\\ 1, & a \le t \le s \le b.\end{cases} \tag{3.6}$$

*Proof.* Let $x$ be a solution to (3.4). By Theorem 2.2, we have

$$\begin{aligned}\big[x(t)\widehat{e}_{-\alpha}(t,b)\big]_{\nabla}^{(\alpha)} &= x_{\nabla}^{(\alpha)}(t)\widehat{e}_{-\alpha}(t,b) - \alpha t^{1-\alpha}\widehat{e}_{-\alpha}(t,b)x(\rho(t)),\\ &= \widehat{e}_{-\alpha}(t,b)g(t)\end{aligned}$$

and hence integrating the above on $(t,b]_{\mathbb{T}}$, we obtain

$$x(b) - x(t)\widehat{e}_{-\alpha}(t,b) = \int_{(t,b]_{\mathbb{T}}} \widehat{e}_{-\alpha}(s,b)g(s)\,\nabla_\alpha s.$$

Here,

$$x(t) = -\int_{(t,b]_{\mathbb{T}}} \widehat{e}_{-\alpha}(s,t)g(s)\,\nabla_\alpha s + x_b\widehat{e}_{-\alpha}(b,t) = \int_a^b G(t,s)g(s)\,\nabla_\alpha s + x_b\widehat{e}_{-\alpha}(b,t). \qquad \square$$

Let us define the operator $\mathcal{A} : \mathcal{C}(\mathbb{T},\mathbb{R}) \to \mathcal{C}(\mathbb{T},\mathbb{R})$ by

$$\mathcal{A}(x)(t) := \int_a^b G(t,s)\big(F(s,\overline{x}(\rho(s))) - \alpha s^{1-\alpha}\overline{x}(\rho(s))\big)\,\nabla_\alpha s + x_b\widehat{e}_{-\alpha}(b,t),$$

where $G$ is Green's function related to the terminal problem (3.4). From expression (3.6), we have $G \le 0$ on $\mathbb{T}\times\mathbb{T}$.

Clearly, from Lemma 3.2, the fixed point of the operator $\mathcal{A}$ is a solution of problem (3.2).

**Proposition 3.2.** *Let $f : \mathbb{T}_\kappa \times [0,\infty) \to (0,\infty)$ be a continuous function. Assume that there exists $(v,M) \in \mathcal{C}_{ld}^{\alpha}(\mathbb{T},\mathbb{R}) \times \mathcal{C}_{ld}^{\alpha}(\mathbb{T},[0,\infty))$, a solution-tube of (1.2), then the operator $\mathcal{A}$ is compact.*

*Proof.* The proof will be given in several steps.

***Step 1:*** $\mathcal{A}$ is continuous.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{C}(\mathbb{T},\mathbb{R})$ converging to $x \in \mathcal{C}(\mathbb{T},\mathbb{R})$. By Theorem 2.3, we have

$$\begin{aligned}&\big|\mathcal{A}(x_n(t)) - \mathcal{A}(x(t))\big|\\ &\quad\le \int_a^b |G(t,s)|\Big(\big|F(s,\overline{x}_n(\rho(s))) - F(s,\overline{x}(\rho(s)))\big| + \alpha|s^{1-\alpha}|\,|\overline{x}_n(\rho(s)) - \overline{x}(\rho(s))|\Big)\,\nabla_\alpha s\\ &\le \theta\int_a^b \left(\left|\frac{\lambda f(s,\overline{x}_n(\rho(s)))}{\big(\int_a^b f(\tau,\overline{x}_n(\rho(\tau)))\,\nabla\tau\big)^2} - \frac{\lambda f(s,\overline{x}(\rho(s)))}{\big(\int_a^b f(\tau,\overline{x}(\rho(\tau)))\,\nabla\tau\big)^2}\right| + b^{1-\alpha}\big|\overline{x}_n(\rho(s)) - \overline{x}(\rho(s))\big|\right)\nabla_\alpha s.\end{aligned}$$

Then

$$\begin{aligned}\big|\mathcal{A}(x_n(t)) - \mathcal{A}(x(t))\big| \le \theta\int_a^b \Bigg(&\frac{\lambda}{\big(\int_a^b f(\tau,\overline{x}_n(\rho(\tau)))\,\nabla\tau\big)^2\big(\int_a^b f(\tau,\overline{x}(\rho(\tau)))\,\nabla\tau\big)^2}\\ &\times\left[\big|f(s,\overline{x}_n(\rho(s))) - f(s,\overline{x}(\rho(s)))\big|\left(\int_a^b f(\tau,\overline{x}(\rho(\tau)))\,\nabla\tau\right)^2\right.\end{aligned}$$

$$+\big|f(s,\overline{x}(\rho(s)))\big|\Big(\int_a^b \big|f(\tau,\overline{x}_n(\rho(\tau)))-f(\tau,\overline{x}(\rho(\tau)))\big|\,\nabla\tau\Big)$$

$$\times\Big(\int_a^b \big|f(\tau,\overline{x}_n(\rho(\tau)))+f(\tau,\overline{x}(\rho(\tau)))\big|\,\nabla\tau\Big)\Big]+b^{1-\alpha}\big|\overline{x}_n(\rho(s))-\overline{x}(\rho(s))\big|\Big)\,\nabla_\alpha s,$$

where

$$\theta=\widehat{e}_{-\alpha}(b,a)\ \text{ and }\ \max_{s\in\mathbb{T}}|s^{1-\alpha}|=b^{1-\alpha}.$$

Since there is a constant $R>0$ such that $\|\overline{x}\|_{\mathcal{C}(\mathbb{T},\mathbb{R})}<R$, there exists an index $N$ such that $\|\overline{x}_n\|_{\mathcal{C}(\mathbb{T},\mathbb{R})}\leq R$ for all $n>N$. Thus $f$ is uniformly continuous and, consequently, uniformly bounded on $\mathbb{T}\times\overline{B}(0,R)$. Then there exist the constants $A>0$ and $B>0$ such that $A\leq f(s,x)\leq B$ for all $(s,x)\in\mathbb{T}_\kappa\times\overline{B}(0,R)$. Therefore, for $\varepsilon>0$ given, there is $\delta>0$ such that for all $x,y\in\mathbb{R}$, where

$$|y-x|\leq\delta<\frac{\varepsilon a^{1-\alpha}}{2\theta(b-a)b^{1-\alpha}}\,,$$

one has

$$|f(s,y)-f(s,x)|\leq\frac{\varepsilon A^4(b-a)a^{1-\alpha}}{6\lambda\theta B^2}\ \text{ for all }\ s\in\mathbb{T}_\kappa.$$

By assumption, one can find an index $\widehat{N}>N$ such that $\|\overline{x}_n-\overline{x}\|_{\mathcal{C}(\mathbb{T},\mathbb{R})}<\delta$ for $n>\widehat{N}$. In this case,

$$\big|\mathcal{A}(x_n(t))-\mathcal{A}(x(t))\big|\leq\theta a^{\alpha-1}\int_a^b\Big(\frac{\lambda}{A^4(b-a)^4}\Big[\big|f(s,\overline{x}_n(\rho(s)))-f(s,\overline{x}(\rho(s)))\big|B^2(b-a)^2$$

$$+2B^2(b-a)\int_a^b\big|f(\tau,\overline{x}_n(\rho(\tau)))-f(\tau,\overline{x}(\rho(\tau)))\big|\,\nabla\tau\Big]+b^{1-\alpha}\big|\overline{x}_n(\rho(s))-\overline{x}(\rho(s))\big|\Big)\,\nabla s$$

$$\leq\theta a^{\alpha-1}\int_a^b\Big(\frac{3\lambda B^2}{A^4(b-a)^2}\,\frac{\varepsilon A^4(b-a)a^{1-\alpha}}{6\lambda\theta B^2}+b^{1-\alpha}\,\frac{\varepsilon a^{1-\alpha}}{2\theta(b-a)b^{1-\alpha}}\Big)\,\nabla s\leq\varepsilon.$$

This proves the continuity of $\mathcal{A}$.

***Step 2:*** The set $\mathcal{A}(\mathcal{C}(\mathbb{T},\mathbb{R}))$ is uniformly bounded.

Let $(x_n)_{n\in\mathbb{N}}\in\mathcal{C}(\mathbb{T},\mathbb{R})$. We have, for each $t\in\mathbb{T}$,

$$|\mathcal{A}(x_n)(t)|\leq\int_a^b s^{1-\alpha}|G(t,s)|\Big(\big|F(s,\overline{x}_n(\rho(s)))\big|+\alpha s^{1-\alpha}|\overline{x}_n(\rho(s))|\Big)\,\nabla s+\widehat{e}_{-\alpha}(a,t)|x_b|$$

$$\leq\int_a^b s^{\alpha-1}|G(t,s)|\left(\frac{\lambda|f(s,\overline{x}_n(\rho(s)))|}{\big(\int_a^b f(\tau,\overline{x}_n(\rho(\tau)))\,\nabla\tau\big)^2}+\alpha s^{1-\alpha}|\overline{x}_n(\rho(s))|\right)\nabla s+\widehat{e}_{-\alpha}(b,t)|x_b|$$

$$\leq\theta\Big[a^{\alpha-1}\Big(\frac{\lambda}{A^2(b-a)^2}\int_a^b\big|f(s,\overline{x}_n(\rho(s)))\big|\,\nabla s+b^{1-\alpha}\int_a^b|\overline{x}_n(\rho(s))|\,\nabla s\Big)+|x_b|\Big].$$

By definition, there is $R>0$ such that $|\overline{x}_n(\rho(s))|\leq R$ for all $s\in\mathbb{T}$ and all $n\in\mathbb{N}$. The function $f$ is compact on $\mathbb{T}_\kappa\times\overline{B}(0,R)$ and we can deduce the existence of a constant $K>0$ such that

$$\big|f(s,\overline{x}_n(\rho(s)))\big|\leq K\ \text{ for all }\ s\in\mathbb{T}_\kappa.$$

So, $\mathcal{A}(\mathcal{C}(\mathbb{T},\mathbb{R}))$ is uniformly bounded.

***Step 3:*** The set $\mathcal{A}(\mathcal{C}(\mathbb{T},\mathbb{R}))$ is equicontinuous.

Let $t_1, t_2 \in \mathbb{T}$, $t_1 < t_2$, and let $x_n \in \mathcal{C}(\mathbb{T},\mathbb{R})$. Then

$$\begin{aligned}
&\big|\mathcal{A}(x_n)(t_2) - \mathcal{A}(x_n)(t_1)\big| \\
&\qquad \leq \int_a^{t_1} |G(t_2,s) - G(t_1,s)| \left|\Big(F(s,\overline{x}_n(\rho(s))) - \alpha s^{1-\alpha}\overline{x}_n(\rho(s))\Big)\right| \nabla_\alpha s \\
&\qquad\quad + \int_{t_2}^{b} |G(t_2,s) - G(t_1,s)| \left|\Big(F(s,\overline{x}_n(\rho(s))) - \alpha s^{1-\alpha}\overline{x}_n(\rho(s))\Big)\right| \nabla_\alpha s \\
&+ \int_{[t_1,t_2]_{\mathbb{T}}} |G(t_2,s) - G(t_1,s)| \left|\Big(F(s,\overline{x}_n(\rho(s))) - \alpha s^{1-\alpha}\overline{x}_n(\rho(s))\Big)\right| \nabla_\alpha s + |x_b|\,\big|\widehat{e}_{-\alpha}(b,t_2) - \widehat{e}_{-\alpha}(b,t_1)\big| \\
&\qquad \leq \int_a^{b} s^{\alpha-1}\big|e_\alpha(s,t_2) - e_\alpha(s,t_1)\big| \Big(\big|F(s,\overline{x}_n(\rho(s)))\big| + b^{1-\alpha}\big|\overline{x}_n(\rho(s))\big|\Big) \nabla s \\
&\quad + \int_{t_1}^{t_2} s^{\alpha-1}|e_\alpha(s,t_2)| \Big(\big|F(s,\overline{x}_n(\rho(s)))\big| + b^{1-\alpha}\big|\overline{x}_n(\rho(s))\big|\Big) \nabla s + |x_b|\,|\widehat{e}_{-\alpha}(b,t_2) - \widehat{e}_{-\alpha}(b,t_1)| \\
&\leq \big|\widehat{e}_{-\alpha}(b,t_2) - \widehat{e}_{-\alpha}(b,t_1)\big| \Big[D a^{\alpha-1}\Big(\frac{\lambda K}{A^2(b-a)} + b^{1-\alpha} R\Big) + |x_b|\Big] + \theta a^{\alpha-1}\Big(\frac{\lambda K}{A^2(b-a)^2} + b^{1-\alpha}R\Big)|t_2 - t_1|,
\end{aligned}$$

where

$$D := \max_{s\in\mathbb{T}}\left\{\frac{1}{\widehat{e}_{-\alpha}(b,s)}\right\}.$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. This proves that the sequence $(\mathcal{A}(x_n))_{n\in\mathbb{N}}$ is equicontinuous. By the Arzelà–Ascoli theorem, we conclude that the set $\mathcal{A}(\mathcal{C}(\mathbb{T},\mathbb{R}))$ is relatively compact in $\mathcal{C}(\mathbb{T},\mathbb{R})$. Hence $\mathcal{A}$ is compact. □

Now, we obtain our main theorem.

**Theorem 3.1.** *Let $f : \mathbb{T}_\kappa \times [0,\infty) \to (0,\infty)$ be a continuous function. Assume that there exists $(v, M) \in \mathcal{C}^\alpha_{ld}(\mathbb{T},\mathbb{R}) \times \mathcal{C}^\alpha_{ld}(\mathbb{T},[0,\infty))$, a solution-tube of* (1.2). *Then problem* (1.2) *has a solution $x \in \mathcal{C}^\alpha_{ld}(\mathbb{T},\mathbb{R}) \cap \mathcal{T}(v, M)$.*

*Proof.* By Proposition 3.2, the operator $\mathcal{A}$ is compact. It has a fixed point by Schauder's fixed-point theorem. Lemma 3.2 implies that this fixed point is a solution to problem (3.2). Then it suffices to show that for every solution $x$ of (3.2), $x \in \mathcal{T}(v, M)$.

Consider the set

$$\mathcal{B} := \big\{t \in \mathbb{T}_\kappa :\ |x(\rho(t)) - v(\rho(t))| > M(\rho(t))\big\}.$$

If $t \in \widehat{\mathcal{B}} = \{t \in \mathcal{B} :\ t = \rho(t)\}$, then by Proposition 3.1, one has

$$\big(|x(t) - v(t)| - M(t)\big)^{(\alpha)}_{\nabla} = \frac{(x(\rho(t)) - v(\rho(t)))(x^{(\alpha)}_{\nabla}(t) - v^{(\alpha)}_{\nabla}(t))}{|x(\rho(t)) - v(\rho(t))|} - M^{(\alpha)}_{\nabla}(t).$$

If $t \in \mathcal{B}$ is left-scattered, then $\nu(t) = t - \rho(t) > 0$ and

$$\begin{aligned}
\big(|x(t) - v(t)| - M(t)\big)^{(\alpha)}_{\nabla} &= \frac{|x(t) - v(t)| - |x(\rho(t)) - v(\rho(t))|}{\nu(t)}\, t^{1-\alpha} - M^{(\alpha)}_{\nabla}(t) \\
&= \frac{|x(\rho(t)) - v(\rho(t))||x(t) - v(t)| - |x(\rho(t)) - v(\rho(t))|^2}{\nu(t)|x(\rho(t)) - v(\rho(t))|}\, t^{1-\alpha} - M^{(\alpha)}_{\nabla}(t) \\
&\geq \frac{(x(\rho(t)) - v(\rho(t)))((x(t) - v(t)) - (x(\rho(t)) - v(\rho(t))))}{\nu(t)|x(\rho(t)) - v(\rho(t))|}\, t^{1-\alpha} - M^{(\alpha)}_{\nabla}(t) \\
&= \frac{(x(\rho(t)) - v(\rho(t)))(x^{(\alpha)}_{\nabla}(t) - v^{(\alpha)}_{\nabla}(t))}{|x(\rho(t)) - v(\rho(t))|} - M^{(\alpha)}_{\nabla}(t).
\end{aligned}$$

Since $(v, M)$ is a solution-tube of problem (1.2), we have on $\{t \in \mathcal{B} :\ M(\rho(t)) > 0\}$ that

$$\begin{aligned}
&\big(|x(t)-v(t)|-M(t)\big)^{(\alpha)}_{\nabla}\\
&\quad= \frac{(x(\rho(t))-v(\rho(t)))(F(t,\overline{x}(\rho(t)))-\alpha t^{1-\alpha}(\overline{x}(\rho(t))-x(\rho(t)))-v^{(\alpha)}_{\nabla}(t))}{|x(\rho(t))-v(\rho(t))|}-M^{(\alpha)}_{\nabla}(t)\\
&= \frac{(\overline{x}(\rho(t))-v(\rho(t)))(F(t,\overline{x}(\rho(t)))-v^{(\alpha)}_{\nabla}(t))}{M(\rho(t))}-\alpha t^{1-\alpha}\Big(M(\rho(t))-\big|x(\rho(t))-v(\rho(t))\big|\Big)-M^{(\alpha)}_{\nabla}(t)\\
&\qquad > \frac{M(\rho(t))M^{(\alpha)}_{\nabla}(t)}{M(\rho(t))}-M^{(\alpha)}_{\nabla}(t) > 0.
\end{aligned}$$

On the other hand, we have on $\{t \in \mathcal{B} :\ M(\sigma(t)) = 0\}$ that

$$\begin{aligned}
&\big(|x(t)-v(t)|-M(t)\big)^{(\alpha)}_{\nabla}\\
&\quad\geq \frac{(x(\rho(t))-v(\rho(t)))(F(t,\overline{x}(\rho(t)))-\alpha t^{1-\alpha}(\overline{x}(\rho(t))-x(\rho(t)))-v^{(\alpha)}_{\nabla}(t))}{|x(\rho(t))-v(\rho(t))|}-M^{(\alpha)}_{\nabla}(t)\\
&= \frac{(x(\rho(t))-v(\rho(t)))(F(t,v(\rho(t)))-v^{(\alpha)}_{\nabla}(t))}{|x(\rho(t))-v(\rho(t))|}+\alpha t^{1-\alpha}\big(|x(\rho(t))-v(\rho(t))|\big)-M^{(\alpha)}_{\nabla}(t) > -M^{(\alpha)}_{\nabla}(t)=0.
\end{aligned}$$

If we set $r(t) := |x(t)-v(t)|-M(t)$, then $r^{(\alpha)}_{\nabla} > 0$ on $\mathcal{B} := \{t \in \mathbb{T}_{\kappa} :\ r(\rho(t)) > 0\}$. Moreover, since $(v, M)$ is a solution-tube to problem (1.2) and $x(b) = x_b$, we have $r(b) < 0$ and, as a consequence, Lemma 3.1 implies that $\mathcal{B} = \varnothing$. So, $x \in \mathcal{T}(v, M)$ and the theorem is proved. □

# 4 Conclusion

In this paper, we study the existence of a solution to the nonlocal nabla conformal fractional thermistor problem on time scales by using solution-tube and Schauder's fixed point theorem, in particular, we study the existence of a solution to a nonlocal nabla conformal fractional thermistor problem of alpha order on an arbitrary bounded time scale (that is, this is an extension of continuous and discrete analysis). As a result of the work, a thermistor equation has been generalized on the time scale by using conformal fractions. Finally, an example is presented to illustrate the obtained results.

**Final remark.** The results presented can be generalized, using the operators defined in [17,27], which contain, as a particular case, the derivative and conformable integral of [20].

**Definition 4.1** ([27])**.** Let $f : [0, +\infty) \to \mathbb{R}$, $\alpha \in (0, 1)$ and $F(\,\cdot\,, \alpha)$ be some function. Then the $\mathcal{N}$-derivative of $f$ of order $\alpha$ is defined by

$$\mathcal{N}^{\alpha}_{F} f(t) = \lim_{\varepsilon\to 0}\Big[\frac{f(t+\varepsilon F(t,\alpha))-f(t)}{\varepsilon}\Big],\ \ t > 0.$$

Here, we will use some cases of $F$ defined in function of $E_{a,b}(\,\cdot\,)$, the classic definition of the Mittag–Leffler function with $\mathrm{Re}(a)$, $\mathrm{Re}(b) > 0$. Also, we consider $E_{a,b}(t^{-\alpha})_k$, the $k$-th term of $E_{a,b}(\,\cdot\,)$.

If $f$ is $\alpha$-differentiable in some $0 < \alpha \leq 1$ and $\lim\limits_{t\to 0^+} \mathcal{N}^{\alpha}_{F} f(t)$ exists, then we define

$$\mathcal{N}^{\alpha}_{F} f(0) = \lim_{t\to 0^+}\big[\mathcal{N}^{\alpha}_{F} f(t)\big].$$

**Definition 4.2** ([17])**.** Let $I \subseteq \mathbb{R}$ be an interval; $a$, $t \in I$ and $\alpha \in \mathbb{R}$. The integral operator $J^{\alpha}_{F,a+}$, right and left, is defined for every locally integrable function $f$ on $I$ as

$$J^{\alpha}_{F,a+} f(t) = \int_a^t \frac{f(s)}{F(t-s,\alpha)}\,ds,\ \ t > a,$$

and

$$J^{\alpha}_{F,b-}f(t) = \int_t^b \frac{f(s)}{F(s-t,\alpha)}\,ds, \quad b > t.$$

We will also use the "central" integral operator defined by

$$J^{\alpha}_{F,a}f(b) = \int_a^b \frac{f(t)}{F(t,\alpha)}\,dt, \quad b > a.$$

The presented results complement a number of results reported in the literature. Furthermore, the findings of this paper can be extended to study the existence of a solution for the following nonlocal $\mathcal{N}$-conformable fractional thermistor problem:

$$\begin{cases} \mathcal{N}^{\alpha}_{F}x(t) = \dfrac{\lambda f(t,x(t))}{\big(\int_a^b f(\tau,x^{\rho}(\tau))\,\nabla\tau\big)^2} & \text{for all } [0,a], \\ x(0) = x_0. \end{cases} \tag{4.1}$$

From the method given in this paper, one can obtain some oscillation criteria for (4.1). This means the obtaining generalizations of Theorem 3.1. The details are left to the reader.

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