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SEMICLASSICAL RESONANCES, THEORY AND APPLICATION TO A GENERAL DIATOMIC MOLECULAR HAMILTONIAN

Abstract. We study in this paper resonances of Schrödinger operators. Resonance energies are accessible from a general class of complex distortions, they also coincide with the poles of the meromorphic continuation of the resolvent. We prove that in the Born–Oppenheimer approximation for diatomic molecules, this study can be reduced to the one of a matrix of semiclassical pseudodifferential operators with operator-valued symbols, without modifying the Hamiltonian near the collision set of nuclei. We consider here the case where two electronic levels cross, and where molecular resonances appear and can be well located. We also investigate the action of the effective Hamiltonian on WKB solutions and show that these resonances have an imaginary part exponentially small.

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რეზიუმე. ნაშრომში შესწავლილია შროდინგერის ოპერატორების რეზონანსები. რეზონანსული ენერგიები ხელმისაწვდომია კომპლექსური დამახინჯებების ზოგადი კლასიდან, ისინი ასევე ემთხვევა რეზოლვენტის მერომორფული გაგრძელების პოლუსებს. ჩვენ ვამტკიცებთ, რომ დიატომური მოლეკულების ბორნ–ოპენჰეიმერის მიახლოებისას, კვლევა შეიძლება დავიყვანოთ ოპერატორ-მნიშვნელობებიანი სიმბოლოების მქონე კვაზიკლასიკური ფსევდოდიფერენციალური ოპერატორების მატრიცის შესწავლამდე, ბირთვების შეჯახების სიმრავლის მახლობლად ჰამილტონიანის მოდიფიცირების გარეშე. აქ ჩვენ განვიხილავთ შემთხვევას, როცა ორი ელექტრონული დონე იკვეთება და როდესაც ჩნდება მოლეკულური რეზონანსები და ისინი შეიძლება კარგად იყოს ლოკალიზებული. ჩვენ ასევე ვიკვლევთ ეფექტური ჰამილტონიანის მოქმედებას WKB ამონახსნებზე და ვაჩვენებთ, რომ ამ რეზონანსებს აქვს ექსპონენტურად მცირე წარმოსახვითი ნაწილი.

1 Introduction

In this paper, we investigate the resonances in the semiclassical approximation for Hamiltonians of the form $P = -h^2\Delta + V$ and $P(h) = -h^2\Delta_x - \Delta_y + V(x, y)$ defined on $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n \times \mathbb{R}^p)$, respectively, where h tends to 0^+ and the potential V is assumed to be analytic in a complex strip. We use analytic dilation, analytic distorsion methods and meromorphic continuation of the resolvent to define the resonances.

The resonances of the Schrödinger operator P defined on $L^2(\mathbb{R}^n)$ with domain $D(P) = H^2(\mathbb{R}^n) \cap D(V)$ are the eigenvalues of P_θ (respectively of P_μ) in the lower complex half-plane, where P_θ (respectively P_μ) is a dilation (respectively a distorsion) of P, they are independent of θ (respectively of μ) in the sense that replacing θ (respectively μ) by a larger value will not change the set of resonances in the corresponding complex sector.

The theory developed by Hunziker [13], identifies the resonances of P with the poles of the meromorphic extension from the upper complex half-plane of the resolvent of P, see also [1, 2, 13, 16, 17, 19, 22]. In order to prove the existence of resonances, we operate an explicit construction assuming appropriate conditions on the potential V.

This paper is also devoted to the study of resonances for Born–Oppenheimer Hamiltonians. We show that one can reduce the problem to a finite matrix of regular semiclassical pseudodifferential operators for diatomic molecules in the physically interesting case of Coulombic interactions, near energy levels where resonances may appear.

The Born–Oppenheimer approximation separates the fast electronic motion from the slower motion of the nuclei. As usual, in the Born–Oppenheimer approximation, the Schrödinger operator $P(h)$ for a polyatomic molecule in the semiclassical limit, where the mass ratio h^2 of electronic to nuclear mass tends to zero, is given by

$$P(h) = -h^2\Delta_x + Q(x), \quad Q(x) = -\Delta_y + V(x, y),$$

where $x \in \mathbb{R}^n$ denotes the nuclear and $y \in \mathbb{R}^p$ the electronic coordinates and V is the potential of nuclei-nuclei, nuclei-electron and electron-electron interactions. The operator $Q(x)$ is the so-called electronic Hamiltonian and its eigenvalues are the so-called electronic levels.

Assume first that $V \in C^\infty(\mathbb{R}^n \times \mathbb{R}^p; \mathbb{R})$ is bounded together with all its derivatives, and $Q(x)$ admits a gap in its spectrum. Let us denote by $\lambda_1(x) < \lambda_2(x) \leq \cdots \leq \lambda_N(x)$ the first N eigenvalues of $Q(x)$ and assume that there exists a gap between them and the rest of the spectrum of $Q(x)$:

$$\inf_{x\in\mathbb{R}^n} dist\Big(\sigma(Q(x)) \setminus \big\{\lambda_1(x), \dots, \lambda_N(x)\big\}, \big\{\lambda_1(x), \dots, \lambda_N(x)\big\}\Big) \geq \delta > 0, \tag{1.1}$$

σ stands for the spectrum, and $dist$ is the set-to-set distance. The resolvent set $\rho(\,\cdot\,)$ of an operator is the complement of its spectrum in the complex plane $\mathbb{C}$. I denotes the identity operator and I_k is the identity matrix of C^k, $k \in \mathbb{N}$, $k \geq 1$. Let T^* denote the adjoint of a linear operator T.

(1.1) implies that the spectral projection $\Pi(x)$ of $Q(x)$ associated to $\{\lambda_1(x), \dots, \lambda_N(x)\}$ is C^2-regular with respect to $x \in \mathbb{R}^n$ (see [5]).

Let us remember that, by using symbolic calculus, the spectral study of $P(h)$ on $L^2(\mathbb{R}^n_x \times \mathbb{R}^p_y)$ can be reduced to that of a semiclassical pseudodifferential matrix operator $P_{eff} = P_{eff}(x, D_x)$ on $\bigoplus\limits_N L^2(\mathbb{R}^n_x)^{\oplus N}$ where $N > 0$ depends on the energy level,

$$\lambda \in \sigma(P(h)) \Longleftrightarrow \lambda \in \sigma(P_{eff}).$$

The reduction for Coulomb-type interactions is treated in [15] and [17] for resonant states when h tends to 0, a regularization of the Hamiltonian is constructed far from the collision set of the nuclei, and this gives rise to an effective pseudodifferential Hamiltonian.

In quantum mechanics, a particle is described by a wave function $\varphi(x, y)$ which is normalized in L^2, $\|\varphi\|_{L^2} = 1$. The probability of finding the particle in a region $\Omega \subset \mathbb{R}^n \times \mathbb{R}^p$ is given by $\int\limits_\Omega |\varphi(x, y)|^2\, dx\, dy$. The time evolution of a semiclassical molecular system is determined by the time-dependent Schrödinger equation

$$\begin{cases} ih\,\dfrac{\partial\varphi}{\partial t}\,(x,y,t) = P(h)\varphi(x,y,t), \\ \varphi(x,y,0) = \varphi_0(x,y), \end{cases} \tag{1.2}$$

where $\varphi_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^p)$ is the initial state.

The following representation result on the solution of system (1.2) was established by Martinez and Sordoni in [18].

Theorem 1.1. *Let $(\mathcal{E}_\lambda)_{\lambda\in\mathbb{R}}$ be the family of spectral projections of $P(h)$, and $E \in \mathbb{R}$. Then there exists an orthogonal projection $\pi(x)$ on $L^2(\mathbb{R}^n \times \mathbb{R}^p)$ such that*

$$\pi(x) = \Pi(x) + \mathcal{O}(h)$$

and such that any solution φ of (1.2) with initial data $\varphi_0 \in \mathrm{Im}\ \mathcal{E}_{]-\infty,E]}$ satisfies

$$\varphi = e^{-\frac{it}{h}\,P_1}\pi(x)\varphi_0 + e^{-\frac{it}{h}\,P_2}(1-\pi(x))\varphi_0 + \mathcal{O}\big(|t|h^\infty\|\varphi_0\|_{L^2}\big)$$

uniformly with respect to h small enough, $t \in \mathbb{R}$ and φ_0, where

$$P_1 = \pi(x)P(h)\pi(x) \ \ \text{and} \ \ P_2 = (1-\pi(x))P(h)(1-\pi(x)).$$

If $\dim \mathrm{Im}\, \Pi(x) = k$ is finite for all $x \in \mathbb{R}^n$, then there exists a semiclassical pseudodifferential operator $W : L^2(\mathbb{R}^n \times \mathbb{R}^p) \to (L^2(\mathbb{R}^n_x))^{\oplus k}$ with an operator-valued symbol and a $k \times k$ self-adjoint matrix A of semiclassical pseudodifferential operators on $L^2(\mathbb{R}^n_x)$ such that the restriction U of W to $\mathrm{Im}\ \pi(x)$, $U : \mathrm{Im}\ \pi(x) \to (L^2(\mathbb{R}^n_x))^{\oplus k}$ is a unitary operator which satisfies $UP_1\pi(x) = AU\pi(x)$ (thus $e^{-\frac{it}{h}\,P_1}\pi(x) = U^*e^{-\frac{it}{h}\,A}U\pi(x)$ for all $t \in \mathbb{R}$). In addition, the symbol of A is

$$a(x,\xi) = \xi^2 I_k + \mu(x) + \sum_{j\geq 0} h^{j+1}r_j(x,\xi),$$

where $\partial^\alpha r_j(x,\xi) = \mathcal{O}(\xi^2)$ for any multi-index α and uniformly with respect to $(x,\xi) \in \mathbb{R}^{2n}$ and $h > 0$ small enough, and $\mu(x)$ is the matrix of $\Pi(x)P_{eff}$ in a smooth orthonormal basis of $\mathrm{Im}\ \Pi(x)$.

If $k = 1$, then for any $t \in \mathbb{R}$, there exists a semiclassical Fourier integral operator F_t on $L^2(\mathbb{R}^n_x)$:

$$F_t f(x) = (2\pi h)^{-n} \int e^{\frac{i}{h}\,\phi(t,x,y,\eta)} b(t,x,y,\eta;h) f(y)\, dy\, d\eta,$$

where b is a semiclassical symbol of order 0 and ϕ is a smooth phase function with nonnegative imaginary part such that any solution φ of (1.2) with initial data φ_0 satisfying

$$\|(1-\pi(x))\varphi_0\| + \|\mathcal{E}_{[\lambda_0,+\infty[}\varphi_0\| = \mathcal{O}(h^\infty)$$

is given by $\varphi = W^*F_tW\varphi_0 + \mathcal{O}(h^\infty)$, $\mathcal{O}(h^\infty)$ can be replaced by $\mathcal{O}(e^{-\varepsilon/h})$ for some $\varepsilon > 0$ when V is analytic with respect to x and bounded in a complex strip.

In that way, the evolution of the molecule reduces to that of an effective electric potential created by the electrons. So, there may be an even closer relation between the complete quantum evolution $e^{-\frac{it}{h}\,P(h)}$ and the reduced quantum evolution $e^{-\frac{it}{h}\,P_{eff}}$.

In [26], the authors find an approximation of $e^{-\frac{it}{h}\,P(h)}$ in terms of $e^{-\frac{it}{h}\,P_{eff}}$, and prove an error estimate in $\mathcal{O}(h)$. A whole perturbation of P_{eff} is constructed in [18] allowing an error estimate in $\mathcal{O}(h^\infty)$ for the quantum evolution. In [26] and [18], the interaction potential is assumed to be smooth, and then this situation excludes the physically interesting case of Coulomb interactions.

However, the solution of (1.2) with the initial condition $\varphi(x,y,t_0)$ is given exactly at time t by $\varphi(x,y,t) = U(t-t_0)\varphi(x,y,t_0)$, $t_0 \in \mathbb{R}$, where $U(t-t_0) = e^{-\frac{it}{h}\,P(h)}$ is the evolution operator. Precisely, if ψ is an eigenstate of $P(h)$, $P(h)\psi = E\psi$, the time evolution state is given by $\varphi(x,y,t) = e^{-\frac{it}{h}\,P(h)}\psi =$

$e^{-\frac{itE}{h}}\psi$, where $\varphi(x,y,0)=\psi(x,y)$. In particular, $\|\varphi\|_{L^2}=\|\psi\|_{L^2}$, so the probability density does not change when the state is propagated and the system is stable.

Turning now to the methods which lead directly to the complex energies, $E=a-ib$, $b>0$, then $\varphi(x,y,t)=e^{-\frac{ita}{h}}e^{-\frac{bt}{h}}\psi$ and the probability of survival beyond time t is $\|\varphi\|_{L^2}=e^{-\frac{bt}{h}}\|\psi\|_{L^2}$. In particular, $\lim\limits_{t\to\infty}\|\varphi\|_{L^2}=0$, such states are metastable and their corresponding energies are called resonances, or resonances encounter complexes. We can therefore associate with a resonance a lifetime ($\approx \frac{1}{b}$). A metastable state of a molecular system has a longer lifetime than the ordinary excited states and that generally has a shorter lifetime than the lowest, often stable, energy state, called the ground state. There are many examples of metastable states in atomic and nuclear systems. Thus the lifetime of a metastable system is important if $b=\operatorname{Im}E$ is small enough, which means that the resonance is quite close to the real axis. Whatever definition is adopted for a resonance, there is always the idea that a complex energy is involved, thus analyticity occurs in a natural way, in view of the method of complex scaling initiated by Aguilar–Combes [1] and Balslev–Combes [2] and further developed by many authors. In many instances, as in the case of shape resonances, such a complex eigenvalue can be viewed as arising from the perturbation of an eigenvalue embedded in the continuous spectrum. All these methods require an indirect procedure for the evaluation of the imaginary part of the resonance energy.

Here, our goal is to study the resonances, when h tends to 0, of $P(h)$ with potential having Coulomb-type singularities and when the electronic Hamiltonian admits a local gap in its spectrum:

$$V(x,y)=\frac{a}{|x|}+\sum_{j=1}^{p}\Big(\frac{b_j^-}{|y_j-x|}+\frac{b_j^+}{|y_j+x|}\Big)+\sum_{j\neq k}\frac{c_{j,k}}{|y_j-y_k|} \tag{1.3}$$

with $a>0$ constant and $b_j^-,b_j^+,c_{j,k}\in\mathbb{R}$, $b_j^{\pm}<0$. It is well known that $P(h)$ is selfadjoint on $L^2(\mathbb{R}^3\times\mathbb{R}^{3p})$ with domain in the Sobolev space $H^2(\mathbb{R}^3\times\mathbb{R}^{3p})$.

Resonances of $P(h)$ will be accessible via analytic distorsion introduced by Hunziker in [13], for their description we use the arguments developed in [17] and [18] in order to include possible singularities of the potentials. The reduction for resonant states with Coulomb-type interactions was treated by Martinez and Messirdi in [17], where a regularization of the Hamiltonian $P(h)$ is constructed far from the collision set of the nuclei and when the singularities coming from the collision set of the nuclei are avoided. In [20], an effective Hamiltonian, for the exact molecular operator, is constructed as the sum of a semibounded operator and a semiclassical pseudodifferential operator localized respectively near and far from the collision set of the nuclei.

In addition, to obtain a suitable approximation of the imaginary part of the resonances, we mainly define and study semiclassical Fourier integral operators, of which we give a complete description.

Precisely, we investigate resonances of diatomic molecular Hamiltonians and give estimates on their widths, when the second electronic level forms a well at some energy E, while the first one is non-trapping at E and when the second and third levels cross on a compact subset of $\mathbb{R}^3$. This situation was considered in [20], where they indicate that their method makes possible to determine the resonances near E with exponentially small widths as $h\to 0^+$. Our study provides the necessary details and proofs concerning molecular predissociation with crossing levels, it also generalizes to the case of singular potentials the results of Messirdi [22].

We adopt the reduction without modifying $P(h)$ near the collision set of nuclei and study the resonances of $P(h)$ where two electronic levels cross. We provide a link between the resonances of $P(h)$ and the discrete spectrum of the pseudodifferential part of the effective Hamiltonian. We examine the action of the effective Hamiltonian on WKB functions that have an asymptotic expansion in powers of $h^{1/2}$ and give estimates on the widths of resonances as h tends to 0. The Grushin problems, Fourier integral operator theory and pseudodifferential calculus are necessary tools in this work.

Recall that the discrete spectrum of a densely defined, closed linear operator A on a Hilbert space is the set $\sigma_{disc}(A)$ of isolated eigenvalues of A of finite multiplicity. A spectral singularity is said to be in the essential spectrum of A if it is not an isolated eigenvalue of finite multiplicity, the essential spectrum of A is $\sigma_{ess}(A)=\sigma(A)\setminus\sigma_{disc}(A)$.

If $\lambda\in\sigma(A)$ and $\operatorname{Im}(\lambda-A)$ is closed, then $\lambda\in\sigma_{disc}(A)$ if and only if the resolvent operator $(z-A)^{-1}$ has a pole of order N at λ. In this case, $(\lambda-A)P_\lambda(A)$ is nilpotent of index N, where $P_\lambda(A)$

is the Riesz projection associated with λ and defined by the familiar Cauchy integral ([11])

$$P_\lambda(A) = \frac{1}{2\pi i} \oint_\Gamma (z - A)^{-1}\, dz,$$

$\Gamma = \partial D$, where D is a closed disk centered at λ, $z \in \Gamma$ and $D \cap \sigma(A) = \{\lambda\}$.

In addition, the resolvent operator can be expanded as a Laurent series:

$$(z - A)^{-1} = \sum_{j=1}^{m-1} \frac{1}{(z-\lambda)^{j+1}}\, D^j + \frac{1}{z-\lambda}\, P_\lambda(A) - \sum_{j=0}^{\infty} (z-\lambda)^j S^{j+1},$$

$$D = (\lambda - A) P_\lambda(A) \quad \text{and} \quad S = -\frac{1}{2\pi i} \oint_\Gamma \frac{1}{z-\lambda}\, (z-A)^{-1}\, dz.$$

This definition has the advantage that Weyl's theorem remains valid for closed non-selfadjoint operators, that is, an arbitrary commuting compact perturbation leaves the essential spectrum unchanged.

We consider the selfadjoint operator $P(h)$ and fix an energy level $E \in \sigma_{ess}(P(h))$ such that for all $x \in \mathbb{R}^3 \setminus \{0\}$,

$$\begin{cases} \sigma(Q(x)) \cap J = \sigma_{disc}(Q(x)) \cap J, \\ \#\sigma_{disc}(Q(x)) \geq 3, \\ \#\sigma(Q(x)) \cap J \leq 3, \end{cases} \tag{1.4}$$

where $J =]-\infty, E]$. Let us denote by $\lambda_1(x) < \lambda_2(x) \leq \lambda_3(x)$ the first three eigenvalues of $Q(x)$.

Suppose assumption (1.1) holds for $\{\lambda_1(x), \lambda_2(x), \lambda_3(x)\}$ and in order to avoid regularity problems at infinity, we also assume that $\lambda_1(x), \lambda_2(x)$ and $\lambda_3(x)$ are simple at infinity:

$$\inf_{j,k \in \{1,2,3\}} |\lambda_j(x) - \lambda_k(x)| \geq \frac{1}{C} \ \text{ for } \ |x| \geq C, \ \ C > 0. \tag{1.5}$$

In the following, we set

$$\widetilde{Q}(x) = Q(x) - \frac{a}{|x|} \ \text{ and } \ \widetilde{\lambda}_j(x) = \lambda_j(x) - \frac{a}{|x|}, \ \ j \in \{1,2,3\}.$$

So, since $b_j^{\pm} < 0$, there exists $C' > 0$ such that $\sup\limits_{x \in \mathbb{R}^3 \setminus \{0\}} \widetilde{\lambda}_j(x) \leq C'$, $j \in \{1,2,3\}$.

Suppose that the second and third levels cross on some sphere $|x| = r_0 \ll C$:

$$\big\{x \in \mathbb{R}^3 : \ \lambda_2(x) = \lambda_3(x)\big\} = \big\{x \in \mathbb{R}^3 : \ |x| = r_0\big\} \tag{1.6}$$

In fact, we can assume that $\{\lambda_1(x), \lambda_2(x), \lambda_3(x)\}$ can be re-indexed in such a way that they become smooth functions outside of $\{0\}$, and that $\lambda_2(x)$ creates a potential well at the energy E:

$$\lambda_2 > 0, \quad \inf_{x \in \mathbb{R}^3 \setminus \{0\}} \lambda_2(x) = \lambda_0 < E,$$

$$\sup_{x \in \mathbb{R}^3 \setminus \{0\}} \lambda_1(x) < 0,$$

$$\lambda_2^{-1}(\lambda_0) = \big\{x \in \mathbb{R}^3 : \ |x| = r_1\big\},$$

$$\lambda_2'' > 0 \ \text{ on } \lambda_2^{-1}(\lambda_0) \ \text{ with } \ 0 < r_1 < r_0.$$

In order to avoid resonances issue from $\lambda_1(x)$ and $\lambda_3(x)$, we put the virial conditions:

$$\sup_{x \in \mathbb{R}^3 \setminus \{0\}} \big(2\lambda_j(x) + x \cdot \nabla \lambda_j(x)\big) < 2E - C < 0, \ \ j \in \{1,3\}. \tag{1.7}$$

The paper is organized as follows. In Section 2, we recall some basic results on semiclassical Fourier integral operators and pseudodifferential operators with operator-valued symbols. Sections 3 and 4 deal with resonances for semiclassical Schrödinger operators and for Born–Oppenheimer Hamiltonians. In Section 5, we give a reduction result, similar to the Feshbach standard result where we find that the spectral study of $P(h)$ is equivalent to that of the associated effective Hamiltonian. The main goal of Section 6 is the construction of a regularization of the Hamiltonian $P(h)$ far from the collision set of the nuclei, the effective Hamiltonian is given in terms of a matrix of smooth pseudodifferential operators with operator-valued symbols. In Section 7, we obtain estimates on the widths of located resonances.

2 Semiclassical Fourier integral operators with operator symbol

In this section, we define semiclassical Fourier integral operators with operator symbol and present some of their properties. This notion was recently introduced in [6] and [9]. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces.

Definition 2.1. A positive function $g \in C^\infty(\mathbb{R}^d;\mathbb{R}_+^*)$ is called an order function if $\partial_X^\alpha g(X) = \mathcal{O}(g(X))$ for all $\alpha \in \mathbb{N}^d$ uniformly with respect to $(X,h) \in \mathbb{R}^d \times]0,1]$ (the most common example of order function is $g(X) = (1+|X|)^m$, $m \in \mathbb{R}$).

A family of functions $a(X;h)$ defined on $\mathbb{R}^d \times]0,1]$ is said to be an operator symbol if $a(\,\cdot\,;h) \in C^\infty(\mathbb{R}^d \times]0,1]; \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2))$ and if for each multi-index $\alpha \in \mathbb{N}^d$ there exists a constant $C_\alpha > 0$ such that $\|\partial_X^\alpha a(X;h)\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} \le C_\alpha g(X)$ uniformly with respect to $h \in]0,1]$.

We denote by $S_g^d(\mathcal{H}_1,\mathcal{H}_2)$ the space of all operator-valued symbols on $\mathbb{R}^d \times]0,1]$ into $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$, where $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ is the set of all bounded linear operators mapping $\mathcal{H}_1$ into $\mathcal{H}_2$.

Example 2.1. The operator-valued symbol of the semiclassical Schrödinger operator $P(h) = -h^2\Delta_x + Q(x)$ with $Q(x) = -\Delta_y + V(x,y)$ is given by $a(x,\xi;h) = |\xi|^2 + Q(x)$. It is clear that $a \in S_g^{2n}(H^2(\mathbb{R}^p);L^2(\mathbb{R}^p))$ if $V \in C^\infty(\mathbb{R}^n \times \mathbb{R}^p;\mathbb{R})$ is bounded together with all its derivatives with $g(x,\xi) = 1 + |x| + |\xi|$.

As a direct consequence of the Leibniz formula, we have $ba \in S_d^{gg'}(\mathcal{H}_1,\mathcal{H}_3)$ if $a \in S_d^g(\mathcal{H}_1,\mathcal{H}_2)$ and $b \in S_d^{g'}(\mathcal{H}_2,\mathcal{H}_3)$.

$a \in S_g^d(\mathcal{H}_1,\mathcal{H}_2)$ is said to be elliptic if there exists a positive constant $C_0 > 0$ independent of h such that

$$\|a(X;h)\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} \ge \frac{1}{C_0}\, g(X)$$

uniformly with respect to $(X;h) \in \mathbb{R}^N \times]0,1]$. Thus, if $a \in S_g^d(\mathcal{H}_1,\mathcal{H}_2)$ is elliptic, $\frac{1}{a} \in S_{g^{-1}}^d(\mathcal{H}_1,\mathcal{H}_2)$.

Let $a \in S_g^d(\mathcal{H}_1,\mathcal{H}_2)$ and $(a_j)_{j\in\mathbb{N}}$ be a sequence of operator-valued symbols of $S_g^d(\mathcal{H}_1,\mathcal{H}_2)$. Then we say that a is asymptotically equivalent to the formal series $\sum\limits_{j=0}^{\infty} h^j a_j$ in $S_g^d(\mathcal{H}_1,\mathcal{H}_2)$ and we denote $a \sim \sum\limits_{j=0}^{\infty} h^j a_j$ if and only if for any $N \in \mathbb{N}$ and for any $\alpha \in \mathbb{N}^d$ there exist $h_{N,\alpha} \in]0,1]$ and $C_{N,\alpha} > 0$ such that

$$\Big\|\partial_X^\alpha\Big(a - \sum_{j=0}^{N} h^j a_j\Big)\Big\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} \le C_{N,\alpha} h^N g(X)$$

uniformly on $\mathbb{R}^d \times]0, h_{N,\alpha}[$. In the particular case, where all a_j are identically zero, we write $a = \mathcal{O}(h^\infty)$ in $S_g^d(\mathcal{H}_1,\mathcal{H}_2)$.

In the semiclassical case, a Fourier integral operator on the Schwartz space $\mathcal{S}(\mathbb{R}^n,\mathcal{H}_1)$ of rapidly decreasing vector-valued functions in $\mathcal{H}_1$ with operator symbol a, has the following form:

$$A_h(a,\phi;h)u(x) = (2\pi h)^{-n} \int\limits_{\mathbb{R}_y^n \times \mathbb{R}_\theta^N} e^{ih^{-1}\phi(x,\theta,y)} a(x,\theta,y;h) u(y)\, dy\, d\theta, \quad u \in \mathcal{S}(\mathbb{R}^n,\mathcal{H}_1), \tag{2.1}$$

where $x \in \mathbb{R}^n$, $N \in \mathbb{N}^*$, $h \in]0,1]$ and for any $\alpha, \beta \in \mathbb{N}^n$, $\gamma \in \mathbb{N}^N$, there exists a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$\left\|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma a(x,\theta,y;h)\right\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} \leq C_{\alpha,\beta,\gamma} g(x,\theta,y) \text{ for all } (x,\theta,y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$$

uniformly with respect to $h \in]0,1]$, i.e., $a \in S_g^{2n+N}(\mathcal{H}_1,\mathcal{H}_2)$, and where ϕ is a phase function: $\phi : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}n \to \mathbb{R}$ is real-valued C^∞-function in $\mathbb{R}^n \times \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^n$ and ϕ is positive-homogeneous with respect to θ of degree one, $\phi(x,\lambda\theta,y) = \lambda\phi(x,\theta,y)$, $\lambda > 0$, $(x,\theta,y) \in \mathbb{R}^n \times \mathbb{R}^N \setminus \{0\} \times \mathbb{R}^n$.

Many authors make different hypothesis on phase functions in order to prove more properties about the related Fourier integral operators. In general, the integral defined in (2.1) is not absolutely convergent, so we use the technique of the oscillatory integral developed by Hörmander [12], where ϕ satisfies the following assumptions:

(I) for all $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^N$, there exists $C'_{\alpha,\beta,\gamma} > 0$,

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma \phi(x,\theta,y)| \leq C_{\alpha,\beta,\gamma} g(x,\theta,y) \text{ for all } (x,\theta,y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n;$$

(II) there exist real numbers $C_1, C_2 > 0$ such that

$$C_1 g(x,\theta,y) \leq g(\partial_y\phi, \partial_\theta\phi, y) \leq C_2 g(x,\theta,y) \text{ for all } (x,\theta,y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n;$$

(III) there exist real numbers $C_1^*, C_2^* > 0$ such that

$$C_1^* g(x,\theta,y) \leq g(x, \partial_\theta\phi, \partial_x\phi) \leq C_2^* g(x,\theta,y) \text{ for all } (x,\theta,y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n.$$

By using some results of [24], essentially the proof of Proposition II.2, we can easily establish the following

Theorem 2.1 ([24]). *Let ϕ be a phase function satisfying assumptions* (I), (II) *and* (III), *and $a \in S_g^{2n+N}(\mathcal{H}_1,\mathcal{H}_2)$ with $g(x,\xi) = (1+|x|+|\theta|+|y|)^m$, $m \in \mathbb{R}$. Then:*

(1) *for all $u \in \mathcal{S}(\mathbb{R}^n,\mathcal{H}_1)$, $\lim\limits_{\sigma\to\infty}[A_h(a_\sigma,\phi;h)u](x)$ exists for every $x \in \mathbb{R}^n$, where $a_\sigma(x,\theta,y;h) = g(\frac{x}{\sigma},\frac{\theta}{\sigma},\frac{y}{\sigma})a(x,\theta,y;h)$, $\sigma > 0$. We put*

$$A_h(a,\phi;h)u(x) = \lim_{\sigma\to\infty}[A_h(a_\sigma,\phi;h)u](x).$$

(2) *$A_h(a,\phi;h)$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^n,\mathcal{H}_1)$ to $\mathcal{S}(\mathbb{R}^n,\mathcal{H}_2)$ (respectively by duality from $\mathcal{S}'(\mathbb{R}^n,\mathcal{H}_2)$ to $\mathcal{S}'(\mathbb{R}^n,\mathcal{H}_1)$), where $\mathcal{S}'(\mathbb{R}^n,\mathcal{H}_j)$ is the Schwartz space of vector-valued temperate distributions on $\mathbb{R}^n$ in $\mathcal{H}_j$, $j \in \{1,2\}$.*

Example 2.2. The Basic examples of Fourier integral operators with phase functions satisfying (I) to (III) are the pseudodifferential operators

$$Op_h(a)u(x) = (2\pi h)^{-n} \int\limits_{\mathbb{R}^n} e^{-ih^{-1}(x-y)\cdot\theta} a(x,\theta,y;h)u(y)\,dy\,d\theta$$

with $a \in S_g^{2n+N}(\mathcal{H}_1,\mathcal{H}_2)$ and $\phi(x,\theta,y) = (x-y)\cdot\theta$;

and the h-Fourier transform

$$\widehat{u}(x) = (2\pi h)^{-n} \int\limits_{\mathbb{R}^n} e^{-ih^{-1}x\cdot y} u(y)\,dy,$$

where $u \in \mathcal{S}(\mathbb{R}^n,\mathcal{H}_1)$ and $h \in]0,1]$.

Remark 2.1. If $a \in S_g^{2n+N}(\mathcal{H}_1, \mathcal{H}_2)$ and $a \sim \sum_{j=0}^{\infty} h^j a_j$, we can write

$$Op_h(a) = \sum_{j=0}^{N} h^j Op_h(a_j) + h^N R_N(h)$$

where $R_N(h)$ is uniformly bounded on $L^2(\mathbb{R}^n, \mathcal{H}_1)$ into $L^2\mathbb{R}^n, \mathcal{H}_2)$ as $h \to 0^+$. The operator-valued function a_0 is called the principal symbol of $Op_h(a)$.

Let us observe that some papers deal with the L^2-boundedness and L^2-compactness for Fourier integral operators, we can particularly cite the works of Bekkara et al. [3], Senoussaoui [25], Harrat–Senoussaoui [9], and Habel–Senoussaoui [8].

Here, we present a recent result due to [8] for the special form of the phase function $\phi_S(x, \theta, y) = S(x, \theta) - y \cdot \theta$, where $S \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$, satisfying two conditions below

(IV) for any $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\theta^\beta S(x, \theta)\right| \leq C_{\alpha,\beta}\, g(x, \theta);$$

(V) there exists $\delta_0 > 0$ such that

$$\inf_{x,\theta \in \mathbb{R}^n} \left| \det\left(\frac{\partial^2 S}{\partial x \partial \theta}\right)(x, \theta)\right| \geq \delta_0.$$

Let $A_h(a, \phi_S; h)$ be the Fourier integral operator defined by (2.1) with the distribution kernel

$$K(x, y; h) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{ih^{-1}(S(x,\theta) - y.\theta)} a(x, \theta, y)\, d\theta,$$

i.e.,

$$A_h(a, \phi_S; h)u(x) = \int_{\mathbb{R}^n} K(x, y; h) u(y)\, dy,$$

where the operator-valued symbol $a \in S_g^{3n}(\mathcal{H}_1, \mathcal{H}_2)$ and $h \in]0, 1]$.

So, $K(x, y; h) \in C^0(\mathbb{R}^{2n}, \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2))$, and if $A_h^*(a, \phi_S; h)$ is the adjoint of $A_h(a, \phi_S; h)$, then $A_h(a, \phi_S; h)A_h^*(a, \phi_S; h)$ and $A_h^*(a, \phi_S; h)A_h(a, \phi_S; h)$ are the pseudodifferential operators with the symbols $\|a\|^2_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} |\det(\frac{\partial^2 S}{\partial x \partial \theta})(x, \theta)|$ given modulo $S_g^{3n}(\mathcal{H}_1, \mathcal{H}_2)$. Using the stationary phase theorem and the Caldéron–Vaillancourt theorem [24], Habel and Senoussaoui showed in [8] the boundedness (respectively compactness) of $A_h(a, \phi_S; h)$ on $L^2(\mathbb{R}^n)$ when the weight of the amplitude a is bounded (respectively tends to 0). Precisely,

Theorem 2.2.

(1) *$A_h(a, \phi_S; h)$ is bounded from $L^2(\mathbb{R}^n, \mathcal{H}_1)$ into $L^2(\mathbb{R}^n, \mathcal{H}_2)$ if the order function g is bounded on $\mathbb{R}^{2n}$.*

(2) *$A_h(a, \phi_S; h)$ can be extended to a compact operator from $L^2(\mathbb{R}^n, \mathcal{H}_1)$ into $L^2(\mathbb{R}^n, \mathcal{H}_2)$ if*

$$\lim_{|x|+|\theta| \to \infty} g(x, \theta) = 0.$$

3 Resonance theory for $-h^2\Delta + V$

The resonance theory for Schrödinger operators has been developed following several approaches. We can mention here the analytic dilation (see [1]) or the analytic distortion (see [13]) and the meromorphic continuation of the resolvent or scattering matrix (see [4]).

3.1 Resonances via analytic dilation

We give the definition of the deformation for the Schrödinger operator $P = -h^2\Delta + V$ by analytic dilation on $\mathbb{R}^n$ and calculate the essential spectrum of the dilation-analytic Schrödinger operator P_θ. The discrete eigenvalues of P_θ are independent of the dilation which justifies the definition of a resonance as a discrete eigenvalue of the operator P_θ.

For $\theta \in \mathbb{R}$, we set

$$\mathcal{U}_\theta : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad f \mapsto \mathcal{U}_\theta f(x) = e^{n\theta/2} f(xe^\theta), \quad f \in C_0^\infty(\mathbb{R}^n),$$

and

$$P_\theta = \mathcal{U}_\theta P \mathcal{U}_\theta^* = e^{-2\theta}(-h^2\Delta) + V(xe^\theta).$$

Definition 3.1. The function $V : \mathbb{R}^n \to \mathbb{R}$ is dilation-analytic when

$$\theta \mapsto V(xe^\theta)(-h^2\Delta + 1)^{-1}$$

extends as an analytic family of compact operators on $L^2(\mathbb{R}^n)$.

Example 3.1.

(1) Let $V \in C^\infty(\mathbb{R}^n)$ be such that V extends as an analytic function in the complex strip

$$D_\delta = \left\{x \in \mathbb{C}^n : \ |\operatorname{Im} x| \le \delta(1 + |\operatorname{Re} x|), \ \ \delta > 0\right\} \ \text{ and } \ V(z) \to 0 \ \text{ as } \ z \in D_\delta, \ \ |z| \to \infty.$$

By virtue of Rellich–Kondrachov's theorem, we see that $V(xe^\theta)(-h^2\Delta + 1)^{-1}$ are compact operators on $L^2(\mathbb{R}^n)$ for $\theta \in \mathbb{C}$, $|\theta|$ small enough. Then V is dilation-analytic.

(2) Let

$$V(x) = -\frac{1}{|x|}, \ \ x \in \mathbb{R}^n \setminus \{0\}, \quad V(xe^\theta) = -\frac{1}{|x||e^\theta|}$$

and

$$V(xe^\theta)(-h^2\Delta + 1)^{-1} = -\frac{1}{|x|\,|e^\theta|}\,(-h^2\Delta + 1)^{-1}.$$

Since the function $-\frac{1}{|e^\theta|}$ is analytic and $\frac{1}{|x|}\,(-h^2\Delta+1)^{-1}$ is compact on $L^2(\mathbb{R}^n)$, we deduce that the Coulomb potential $V(x) = -\frac{1}{|x|}$ is dilation-analytic.

Remark 3.1. If V is dilation-analytic, one can then define the operator P_θ on $L^2(\mathbb{R}^n)$, for θ complex, $\operatorname{Im}\theta > 0$ and $|\theta|$ small enough,

$$P_\theta = e^{-2\theta}(-h^2\Delta) + \left[V(xe^\theta)(-h^2\Delta + 1)^{-1}\right](-h^2\Delta + 1)$$

with domain $D(P_\theta) = H^2(\mathbb{R}^n)$.

As $[V(xe^\theta)(-h^2\Delta + 1)^{-1}](-h^2\Delta + 1)$ is a compact operator, we deduce from Weyl's theorem that

$$\sigma_{ess}(P_\theta) = \sigma_{ess}(e^{-2\theta}(-h^2\Delta)) = e^{-2\theta}\mathbb{R}_+.$$

Definition 3.2. Let V be dilation-analytic in $|\theta| < \delta$. The discrete eigenvalues of P_θ that are located in the complex sector $\{z \in \mathbb{C} : \ -2\operatorname{Im}\theta < \arg(z) \le 0\}$ are called the resonances of P.

The set $\Gamma(P)$ of resonances of P is

$$\Gamma(P) = \bigcup_{0<|\theta|<\delta} \sigma_{disc}(P_\theta) \cap \left\{z \in \mathbb{C} : \ -2\operatorname{Im}\theta < \arg(z) \le 0\right\}.$$

3.2 Analytic vectors

We consider here a large set of vectors ψ for which the map $\theta \mapsto \mathcal{U}_\theta\psi$ is analytic in some disc around 0.

We say that $\psi \in L^2(\mathbb{R}^n)$ is an analytic vector if $\psi(z)$ is an entire function on $\mathbb{C}^n$ and there exists $\varepsilon > 0$ such that

$$\lim_{|z|\to\infty,\ |\arg z|<2\delta} e^{\varepsilon z^2}\psi(z) = 0$$

with $\delta > 0$ and $z^2 = z_1^2 + \cdots + z_1^2$ if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

The set of analytic vectors is denoted by $\mathcal{A}_\delta$. It is clear that $\mathcal{A}_\delta$ is a linear subspace of $L^2(\mathbb{R}^n)$ and examples of functions $\psi \in \mathcal{A}_\delta$ are $\psi(z) = e^{-\alpha z^2}P(z)$ with $\alpha > 0$ and P an arbitrary polynomial: $\mathcal{A}_\delta \subset L^2(\mathbb{R}^n)$ and $\mathcal{U}_\theta\mathcal{A}_\delta \subset L^2(\mathbb{R}^n)$ for $\theta \in \mathbb{C}$, $|\operatorname{Im}\theta| < \delta$ and $|\theta|$ small enough.

Proposition 3.1.

(1) *For any $\psi \in \mathcal{A}_\delta$, $\theta \mapsto \mathcal{U}_\theta\psi$ is an $L^2(\mathbb{R}^n)$-valued analytic function in some disc around* 0.

(2) *$\mathcal{U}_\theta\mathcal{A}_\delta$ is dense in $L^2(\mathbb{R}^n)$, $\theta \in \mathbb{C}$, $|\operatorname{Im}\theta| < \delta$ and $|\theta|$ small enough.*

Proof. (1) See [13, Theorem 3].

(2) (i) $\theta = 0$. Let $\psi \in L^2(\mathbb{R}^n)$ and $\varepsilon > 0$. Thus there exists $\psi_\varepsilon \in C_0^0(\mathbb{R}^n)$, the set of continuous functions compactly supported in $\mathbb{R}^n$ such that $\|\psi - \psi_\varepsilon\|_{L^2(\mathbb{R}^n)} < \varepsilon$.

Let

$$\phi_\varepsilon(z,\lambda) = C_\lambda \int_{\mathbb{R}^n} e^{-\lambda(z-x)^2}\psi_\varepsilon(x)\,dx, \quad \lambda > 0 \text{ large enough},$$

$$C_\lambda^{-1} = \int_{\mathbb{R}^n} e^{-\lambda(z-x)^2}\,dx = \lambda^{-n/2}\int_{\mathbb{R}^n} e^{-x^2}\,dx.$$

We set $K_\eta = \{x \in \mathbb{R}^n :\ dist(x,K) \le \eta\}$, $\eta > 0$, and K_η^c the complement of K_η in $\mathbb{R}^n$, where K is the support of ψ_ε.

In particular, if $|z - x| < \eta$ and $x \in K_\eta^c$, one has $\psi_\varepsilon(x) = \psi_\varepsilon(z) = 0$ and

$$\int_{\{|z-x|<\eta\}\cap K_\eta^c} e^{-\lambda(z-x)^2}(\psi_\varepsilon(x) - \psi_\varepsilon(z))\,dx = 0.$$

Thus, splitting the integral, we write

$$\phi_\varepsilon(z,\lambda) - \psi_\varepsilon(z) = C_\lambda \int_{\mathbb{R}^n} e^{-\lambda(z-x)^2}(\psi_\varepsilon(x) - \psi_\varepsilon(z))\,dx = I_1(z) + I_2(z),$$

$$I_1(z) = C_\lambda \int_{|z-x|\ge\eta} e^{-\lambda(z-x)^2}(\psi_\varepsilon(x) - \psi_\varepsilon(z))\,dx,$$

$$I_2(z) = C_\lambda \int_{\{|z-x|<\eta\}\cap K_\eta} e^{-\lambda(z-x)^2}(\psi_\varepsilon(x) - \psi_\varepsilon(z))\,dx.$$

For any given $\widetilde{\varepsilon} > 0$ and sufficiently small $\eta > 0$,

$$|I_2(z)| \le \widetilde{\varepsilon}\, C_\lambda \int_{K_{2\eta}} e^{-\lambda(z-x)^2}\,dx$$

uniformly with respect to λ. Thus

$$|I_2(z)| \le \widetilde{\varepsilon}\,|K_{2\eta}|^{1/2} \le \varepsilon,$$

where $\widetilde{\varepsilon} \le |K_{2\eta}|^{-1/2}\varepsilon$ and $|K_{2\eta}|$ is the volume of the compact set $K_{2\eta}$.

Moreover,

$$|I_1(z)| \le C_\lambda \sup_{x\in K} |\psi_\varepsilon(x)| e^{-\lambda\eta^2/2} \int_K e^{-\lambda(z-x)^2/2}\, dx + C_\lambda |\psi_\varepsilon(z)| e^{-\lambda\eta^2/2} \int_{\mathbb{R}^n} e^{-\lambda(z-x)^2/2}\, dx.$$

Hence

$$|I_1(z)| \le C_\lambda e^{-\lambda\eta^2/2} \sup_{x\in K} |\psi_\varepsilon(x)| \theta_\lambda(z) + C_0 |\psi_\varepsilon(z)| e^{-\lambda\eta^2/2}$$

with $C_0 > 0$ and $\theta_\lambda(z) = \int\limits_K e^{-\lambda(z-x)^2/2}\, dx$.

Using the fact that

$$|\theta_\lambda(z)| = \mathcal{O}(1)$$

and

$$|I_1(z)| = \mathcal{O}\Big(C_\lambda e^{-\lambda\eta^2/2} \sup_{x\in K} |\psi_\varepsilon(x)| + C_0 \|\psi_\varepsilon\|_{L^2\mathbb{R}^n}\Big) e^{-\lambda\eta^2/2}),$$

we obtain

$$\|I_1(z)\|_{L^2(\mathbb{R}^n)} \le \varepsilon \ \text{ uniformly as } \ \lambda \to \infty.$$

Thus

$$\|\phi_\varepsilon - \psi\|_{L^2(\mathbb{R}^n)} \le \|\phi_\varepsilon - \psi_\varepsilon(z)\|_{L^2(\mathbb{R}^n)} + \|\psi_\varepsilon - \psi\|_{L^2(\mathbb{R}^n)} \le 3\varepsilon \ \text{ uniformly as } \ \lambda \to \infty.$$

That is, $\mathcal{A}_\delta$ is dense in $L^2(\mathbb{R}^n)$ for $\delta > 0$ small enough.

(ii) $\theta \in \mathbb{C}$, $|\operatorname{Im}\theta| < \delta$. It follows from (i) that for any $\psi \in L^2(\mathbb{R}^n)$ and $\varepsilon > 0$, there is $\phi_\varepsilon \in \mathcal{A}_\delta$ such that $\|\phi_\varepsilon - \psi\|_{L^2(\mathbb{R}^n)} \le \varepsilon$, thus $\widetilde{\phi_\varepsilon} = \mathcal{U}_{-\theta}\phi_\varepsilon \in \mathcal{A}_\delta$ and $\|\psi - \mathcal{U}_\theta \widetilde{\phi_\varepsilon}\|_{L^2(\mathbb{R}^n)} \le \varepsilon$, which proves the density of $\mathcal{U}_\theta \mathcal{A}_\delta$ in $L^2(\mathbb{R}^n)$. □

3.3 Resonances via meromorphic continuation of the resolvent

We will see here that the resonances of $P = -h^2\Delta + V$ can also be viewed as the poles of the meromorphic extension, from $\{\operatorname{Im} z > 0\}$ of some matrix elements of the resolvent $R(z) = (P - z)^{-1}$ on the set of analytic vectors (see, e.g., [21]).

Definition 3.3. Let Ω be a complex connected open set. $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces.

(1) $z \mapsto A(z) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is holomorphic on Ω if for any $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, the function $\Omega \ni z \mapsto \langle A(z)x, y\rangle_{\mathcal{H}_2} \in \mathbb{C}$ is holomorphic in Ω. Thus, for any $z_0 \in \Omega$, there exist operators $A_j(z_0) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mu > 0$ such that

$$\sum_{j=0}^{\infty} \|A_j(z_0)\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} \mu^j < \infty \ \text{ and } \ A(z) = \sum_{j=0}^{\infty} (z - z_0)^j A_j(z_0) \ \text{ for } |z - z_0| \text{ small enough.}$$

(2) We say that $z \mapsto A(z) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a finitely meromorphic family of operators on Ω if for any $z_0 \in \Omega$, there exist operators $A_{-j}(z_0) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $1 \le j \le N$, of finite rank and a family of operators $\widetilde{A}(z) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ holomorphic near z_0 such that

$$A(z) = \frac{A_{-N}(z_0)}{(z - z_0)^N} + \cdots + \frac{A_{-1}(z_0)}{(z - z_0)} + \widetilde{A}(z), \ \text{ near } z_0.$$

Then $A(z)$ has Laurent expansion around z_0 of the type

$$A(z) = \sum_{j=-N}^{\infty} (z - z_0)^j A_j(z_0),$$

$A_j(z_0) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $j \in \mathbb{Z}$, $j \ge -N$; $A_{-N}(z_0), \ldots, A_{-1}(z_0)$ are of finite rank, $0 < |z - z_0| < \varepsilon$, for some $N = N(z_0)$ and some $0 < \varepsilon = \varepsilon(z_0)$ sufficiently small.

(3) We say that $A(z) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a finitely meromorphic family of Fredholm operators in Ω if for every $z_0 \in \Omega$, $\widetilde{A}(z)$ is a Fredholm operator for z near z_0. For nonsingular z_0, $A(z) = \widetilde{A}(z)$.

Recall that the operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is Fredholm if the kernel of A, $\ker A$, and the cokernel of A, $co\ker A = \mathcal{H}_2/\operatorname{Im} A$, are both finite dimensional. The index of a Fredholm operator A is $i(A) = \dim\ker A - \dim co\ker A$. Many important Fredholm operators of index 0 have the form $A = I + K$, where K is a compact operator mapping a Hilbert space $\mathcal{H}$ to itself.

The Cauchy formula is valid for holomorphic families of operators

$$A(\xi) = \frac{1}{2\pi i}\oint\limits_{\gamma} \frac{A(z)}{z-\xi}\, dz,$$

the integral is over a positively oriented closed curve γ enclosing ξ.

One then recalls the analytic Fredholm theorem in the following form.

Theorem 3.1 ([23, 27]). *Let $\Omega \subset \mathbb{C}$ be open and connected and $A : \Omega \ni z \to A(z) \in \mathcal{B}(\mathcal{H})$ be a holomorphic family of Fredholm operators on a Hilbert space $\mathcal{H}$.*

Then either:

(1) *$A(z)$ is not boundedly invertible for any $z \in \Omega$,*

or else,

(2) *$A(\cdot)^{-1}$ is finitely meromorphic on Ω. More precisely, there exists a discrete subset $D \subset \Omega$ (possibly, $D = \varnothing$) such that $A(z)^{-1} \in \mathcal{B}(\mathcal{H})$ for all $z \in \Omega \setminus D$, $A(\cdot)^{-1}$ is holomorphic on $\Omega \setminus D$ and finitely meromorphic on Ω. In addition, $A(z)^{-1}$ is a Fredholm operator for all $z \in \Omega \setminus D$, and if $z_0 \in D$, then*

$$A(z)^{-1} = \sum_{j=-N(z_0)}^{\infty} (z-z_0)^j B_j(z_0), \quad 0 < |z - z_0| < \varepsilon(z_0),$$

where $B_{-j}(z_0)$, $1 \le j \le N(z_0)$, are finite rank operators, $B_0(z_0)$ is a Fredholm operator on $\mathcal{H}$ and $B_j(z_0) \in \mathcal{B}(\mathcal{H})$, $j \in \mathbb{N}$.

Moreover, if $(I - A(z))$ is compact on $\mathcal{H}$ for all $z \in \Omega$, then $(I - A(z)^{-1})$, $z \in \Omega \setminus D$, and $(I - B_0(z_0))$, $z_0 \in D$, are compact operators on $\mathcal{H}$.

If $A(z_0)^{-1}$ exists at some point $z_0 \in \Omega$, then $\Omega \ni\mapsto A(z)^{-1}$ is a meromorphic family of operators with poles of finite rank.

Let the potential $V(x)$ be a smooth real function on $\mathbb{R}^n$ such that $V(-h^2\Delta + 1)^{-1}$ is compact and V extends analytically in $|\operatorname{Im}\theta| < \delta_0$, $\delta_0 > 0$. It follows that $V(e^\theta x)(-\Delta+1)^{-1}$ is a compact operator-valued analytic function of θ in the strip $|\operatorname{Im}\theta| < \delta_0$. Then $P_\theta = \mathcal{U}_\theta P \mathcal{U}_\theta^*$ is an analytic family of non-selfadjoint operators, where θ runs in the strip $|\operatorname{Im}\theta| < \delta_0$.

It is well known (see, e.g., the works of Messirdi [21, 22]) that the resolvent operator $R(z) = (P-z)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, admits an analytic extension in $\mathbb{C}^+ = \{z \in \mathbb{C} :\ \operatorname{Im} z > 0\}$, and under the assumption that $V(e^\theta x)$ is analytic, we can extend $R(z)$ to a meromorphic function in a larger domain, the set of poles of this extension is precisely $\Gamma(P)$.

Effectively,

$$P_\theta - z = -h^2 e^{-2\theta}\Delta + V(e^\theta x) - z = \left[I + V(e^\theta x)(-h^2 e^{-2\theta}\Delta - z)^{-1}\right](-h^2 e^{-2\theta}\Delta - z).$$

But $[I + V(e^\theta x)(-h^2 e^{-2\theta}\Delta - z)^{-1}]$ is invertible for $|\operatorname{Im} z| \to \infty$, since

$$\lim_{|\operatorname{Im} z|\to\infty} V(e^\theta x)(-h^2 e^{-2\theta}\Delta - z)^{-1} = 0 \ \text{ in } \mathcal{B}(L^2(\mathbb{R}^n)),$$

and $(-h^2 e^{-2\theta}\Delta - z)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n))$ if $z \in \mathbb{C} \setminus e^{-2\theta}\mathbb{R}_+$.

Furthermore, for all $\varphi,\psi\in\mathcal{A}_\delta$, $\theta\in\mathbb{R}$ and $|\operatorname{Im} z|\gg 1$,

$$\langle R(z)\varphi,\psi\rangle_{L^2(\mathbb{R}^n)}=\langle\mathcal{U}_\theta R(z)\varphi,\mathcal{U}_\theta\varphi\rangle_{L^2(\mathbb{R}^n)}=\langle R_\theta(z)\mathcal{U}_\theta\varphi,\mathcal{U}_\theta\psi\rangle_{L^2(\mathbb{R}^n)},$$

where

$$R_\theta(z)=\mathcal{U}_\theta R(z)\mathcal{U}_\theta^*=(P_\theta-z)^{-1}.$$

Let us first show that $\langle R_\theta(z)\mathcal{U}_\theta\varphi,\mathcal{U}_{\overline{\theta}}\,\psi\rangle_{L^2(\mathbb{R}^n)}$ has a meromorphic extension with respect to $\theta\in\mathbb{C}$, $|\theta|$ small enough.

Lemma 3.1. *For each $\varphi,\psi\in\mathcal{A}_\delta$, the function $\theta\mapsto\langle R_\theta(z)\mathcal{U}_\theta\varphi,\mathcal{U}_{\overline{\theta}}\,\psi\rangle_{L^2(\mathbb{R}^n)}$ defined for $|\operatorname{Im} z|\gg 1$ is holomoprhic on $|\operatorname{Im}\theta|<\delta$.*

Proof. If $|\operatorname{Im} z|\gg 1$, then $z\in\rho(P_\theta)$, therewith $\mathcal{U}_\theta\varphi, U_{\overline{\theta}}\,\psi\in L^2(\mathbb{R}^n)$, since $\varphi,\psi\in\mathcal{A}_\delta$, so, $\theta\mapsto\langle R_\theta(z)\mathcal{U}_\theta\varphi,\mathcal{U}_{\overline{\theta}}\psi\rangle_{L^2(\mathbb{R}^n)}$ is well-defined.

For $\theta=\alpha+i\beta$, one has

$$\begin{aligned}\frac{\partial}{\partial\overline{\theta}}\left\langle R_\theta(z)\mathcal{U}_\theta\varphi,\mathcal{U}_{\overline{\theta}}\,\psi\right\rangle_{L^2(\mathbb{R}^n)}&=\frac{1}{2}\Big(\frac{\partial}{\partial\alpha}+i\,\frac{\partial}{\partial\beta}\Big)\left\langle R_\theta(z)\mathcal{U}_\theta\varphi,\mathcal{U}_{\overline{\theta}}\,\psi\right\rangle_{L^2(\mathbb{R}^n)}\\&=\frac{1}{2}\lim_{h_1,h_2\to0,\ \ h_1,h_2\in\mathbb{R}}\left\langle\Big[\frac{R_{\theta+h_1}(z)-R_\theta(z)}{h_1}+i\,\frac{R_{\theta+ih_2}(z)-R_\theta(z)}{h_2}\Big]\mathcal{U}_\theta\varphi,\mathcal{U}_{\overline{\theta}}\,\psi\right\rangle_{L^2(\mathbb{R}^n)}.\end{aligned}$$

But

$$\begin{aligned}\frac{R_{\theta+h_1}(z)-R_\theta(z)}{h_1}&=\frac{1}{h_1}\big((P_{\theta+h_1}-z)^{-1}-(P_\theta-z)^{-1}\big)=\frac{1}{h_1}(P_{\theta+h_1}-z)^{-1}(P_\theta-P_{\theta+h_1})(P_\theta-z)^{-1}\\&=\frac{1}{h_1}(P_{\theta+h_1}-z)^{-1}\Big[-(e^{-2\theta}-e^{-2(\theta+h_1)})h^2\Delta+(V(e^\theta x)-V(e^{\theta+h_1}x))\Big](P_\theta-z)^{-1}.\end{aligned}$$

Since V is dilation-analytic, we have

$$\lim_{h_1\to0}\frac{R_{\theta+h_1}(z)-R_\theta(z)}{h_1}=(P_\theta-z)^{-1}\Big[\frac{\partial}{\partial\alpha}(e^{-2\theta})h^2\Delta-\frac{\partial}{\partial\alpha}(V(e^\theta x))\Big](P_\theta-z)^{-1}\ \text{ in }\mathcal{B}(L^2(\mathbb{R}^n))$$

with

$$\Big[\frac{\partial}{\partial\alpha}(V(e^\theta x))\Big](P_\theta-z)^{-1}=\frac{\partial}{\partial\alpha}\big(V(e^\theta x)(-h^2\Delta+1)^{-1}\big)\big((-h^2\Delta+1)(P_\theta-z)^{-1}\big).$$

We also have

$$\lim_{h_2\to0}\frac{R_{\theta+ih_2}(z)-R_\theta(z)}{h_2}=(P_\theta-z)^{-1}\Big[\frac{\partial}{\partial\beta}(e^{-2\theta})h^2\Delta-\frac{\partial}{\partial\beta}(V(e^\theta x))\Big](P_\theta-z)^{-1}\ \text{ in }\mathcal{B}(L^2(\mathbb{R}^n)),$$

and

$$\Big[\frac{\partial}{\partial\beta}(V(e^\theta x))\Big](P_\theta-z)^{-1}=\frac{\partial}{\partial\beta}\big(V(e^\theta x)(-h^2\Delta+1)^{-1}\big)(-h^2\Delta+1)(P_\theta-z)^{-1}.$$

Then

$$\frac{\partial}{\partial\overline{\theta}}\langle R_\theta(z)\mathcal{U}_\theta\varphi,\mathcal{U}_{\overline{\theta}}\,\psi\rangle_{L^2(\mathbb{R}^n)}=\left\langle R_\theta(z)\Big[\frac{\partial}{\partial\overline{\theta}}(e^{-2\theta})h^2\Delta-\frac{\partial}{\partial\overline{\theta}}(V(e^\theta x))\Big]R_\theta(z)\mathcal{U}_\theta\varphi,\mathcal{U}_{\overline{\theta}}\,\psi\right\rangle_{L^2(\mathbb{R}^n)}=0,$$

since

$$\frac{\partial}{\partial\overline{\theta}}(e^{-2\theta})=0\ \text{ and }\ \frac{\partial}{\partial\overline{\theta}}(V(e^\theta x))=0.$$

□

Lemma 3.2. *For each $\varphi,\psi\in\mathcal{A}_\delta$, the function $z\mapsto\langle R(z)\varphi,\psi\rangle_{L^2(\mathbb{R}^n)}$, $\operatorname{Im} z>0$, admits a finitely meromorphic continuation to the set $\{z\in\mathbb{C}:\ -2\delta<\arg z<\frac{\pi}{2}\}$.*

Proof. Using Lemma 3.1 and the uniqueness of analytic continuation, we deduce for $\varphi, \psi \in \mathcal{A}_\delta$, Im $z > 0$ and $0 < \text{Im } \theta < \delta$ that

$$\langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)} = \langle R_\theta(z)\mathcal{U}_\theta\varphi, \mathcal{U}_{\overline{\theta}}\,\psi\rangle_{L^2(\mathbb{R}^n)}. \tag{3.1}$$

In particular, since $\sigma_{ess}(P_\theta) = e^{-2\theta}\mathbb{R}_+ \subset \{\text{Im } z \le 0\}$, one obtains from the analytic Fredholm theorem, Theorem 3.1, that $R_\theta(z)$ and $z \mapsto \langle R_\theta(z)\mathcal{U}_\theta\varphi, \mathcal{U}_{\overline{\theta}}\,\psi\rangle_{L^2(\mathbb{R}^n)}$ are finitely meromorphic on $\mathbb{C}\backslash e^{-2\theta}\mathbb{R}_+$. Consequently, using (3.1), we immediately obtain that $\langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$ is finitely meromorphic on $\mathbb{C} \setminus e^{-2\theta}\mathbb{R}_+$ and a fortiori in the complex band $\{z \in \mathbb{C} :\ -2\delta < \arg z < \frac{\pi}{2}\}$. $\square$

Remark 3.2. If f and g are finitely meromorphic continuations of the function $z \mapsto \langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$, $\varphi, \psi \in \mathcal{A}_\delta$, on the band $\{z \in \mathbb{C} :\ -2\delta < \arg z < \frac{\pi}{2}\}$, then $f = g$ throughout $\{z \in \mathbb{C} :\ -2\delta < \arg z < \frac{\pi}{2}\}$ and f and g are meromorphic continuations of each other.

Let us now show that the discrete eigenvalues of P_θ can also be viewed as the poles of the finitely meromorphic extension from $\{\text{Im } z > 0\}$ of $\langle R_\theta(z)\mathcal{U}_\theta\varphi, \mathcal{U}_{\overline{\theta}}\,\psi\rangle_{L^2(\mathbb{R}^n)}$ for all $\varphi, \psi \in \mathcal{A}_\delta$.

Theorem 3.2. *Let V be dilation-analytic in $|\theta| < \delta$. Then*

$$\sigma_{disc}(P_\theta(h)) = \bigcup_{\varphi,\psi} \{\textit{poles of } z \mapsto \langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}\} \cap \Big\{z\in\mathbb{C} :\ -2\,\text{Im }\theta < \arg z < \frac{\pi}{2}\Big\},$$

$$\Gamma(P) = \bigcup_{\text{Im }\theta>0,\ \ |\theta|<\delta} \sigma_{disc}(P_\theta(h)).$$

Proof. If the finitely meromorphic extension from $\{\text{Im } z > 0\}$ of $\langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$, $\varphi, \psi \in \mathcal{A}_\delta$, has a pole at ρ, then there is a number $N \in \mathbb{N}$ such that

$$\oint_\gamma (z-\rho)^N \langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}\, dz \neq 0,$$

where γ is a simple closed oriented curve surrounding ρ and $\gamma \subset \rho(P_\theta)$.

By (3.1), we have

$$\Big\langle \oint_\gamma (z-\rho)^N R_\theta(z)\mathcal{U}_\theta\varphi\, dz, \mathcal{U}_{\overline{\theta}}\,\psi\Big\rangle_{L^2(\mathbb{R}^n)} \neq 0.$$

However, since $(P_\theta-\rho)R_\theta(z) = I + (z-\rho)R_\theta(z)$ and the identity operator I on $L^2(\mathbb{R}^n)$ is holomorphic on and inside γ,

$$\oint_\gamma (z-\rho)^N R_\theta(z)\, dz = (P_\theta-\rho)^N \oint_\gamma R_\theta(z)\, dz,$$

$$\Big\langle (P_\theta-\rho)^N \oint_\gamma R_\theta(z)\mathcal{U}_\theta\varphi\, dz, \mathcal{U}_{\overline{\theta}}\psi\Big\rangle_{L^2(\mathbb{R}^n)} = \Big\langle \oint_\gamma R_\theta(z)\mathcal{U}_\theta\varphi\, dz, (P_{\overline{\theta}}-\overline{\rho})^N\mathcal{U}_{\overline{\theta}}\,\psi\Big\rangle_{L^2(\mathbb{R}^n)} \neq 0.$$

Therefore,

$$\Pi_\theta = \frac{1}{2i\pi}\oint_\gamma (z-P_\theta)^{-1}\, dz \neq 0, \tag{3.2}$$

Π_θ is the spectral projector associated to P_θ and the interior of γ, Π_θ is of finite rank, since $R_\theta(z)$ is finitely meromorphic.

Consequently, we deduce from (3.2) that there exists $\widetilde{\rho} \in \sigma_{disc}(P_\theta)$ and $\widetilde{\rho}$ is inside γ. We necessarily have $\rho = \widetilde{\rho}$, since γ is chosen sufficiently small in $\rho(P_\theta)$.

Conversely, let $\rho \in \sigma_{disc}(P_\theta)$. We denote by $u \in H^2(\mathbb{R}^n)$ a normalized eigenstate of P_θ associated to ρ, $P_\theta u = \rho u$. Then

$$\langle \Pi_\theta u, u\rangle_{L^2(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} = 1, \quad \Big\langle \oint_\gamma R_\theta(z) u\, dz, u\Big\rangle_{L^2(\mathbb{R}^n)} = -2i\pi,$$

γ is the closed contour defined above in (3.2). Using density of $\mathcal{U}_\theta\mathcal{A}_\delta$ in $L^2(\mathbb{R}^n)$, $|\operatorname{Im}\theta| < \delta$ and $|\theta|$ small enough (Proposition 3.1), we can find, for $\varepsilon > 0$ sufficiently small, $\varphi, \psi \in \mathcal{A}_\delta$ such that

$$\|u - U_\theta\varphi\|_{L^2(\mathbb{R}^n)} < \varepsilon \ \text{ and } \ \|u - U_{\overline{\theta}}\,\psi\|_{L^2(\mathbb{R}^n)} < \varepsilon.$$

Moreover, since $(R_\theta(z))_{z\in\gamma}$ is a family of uniformly bounded operators on $L^2(\mathbb{R}^n)$, we see that

$$\oint_\gamma \left\langle R_\theta(z)\mathcal{U}_\theta\varphi, \mathcal{U}_{\overline{\theta}}\,\psi\right\rangle_{L^2(\mathbb{R}^n)} dz \neq 0.$$

But, since $\langle R_\theta(z)\mathcal{U}_\theta\varphi, \mathcal{U}_{\overline{\theta}}\,\psi\rangle_{L^2(\mathbb{R}^n)}$ is the meromorphic extension of $\langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$, we deduce that $\langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$ admits a pole inside γ, this pole is necessarily ρ, since γ is chosen arbitrarily small. □

Corollary 3.1.

(1) *If $\theta_1, \theta_2 \in \mathbb{C}$ are such that $0 < \operatorname{Im}\theta_1 < \operatorname{Im}\theta_2$, then $\sigma_{disc}(P_{\theta_1}(h)) \subset \sigma_{disc}(P_{\theta_2}(h))$.*

(2) *The definition of the resonances is independent of the dilation $\mathcal{U}_\theta$ and the particular choice of $\mathcal{A}_\delta$.*

Theorem 3.3.

(1) *$\Gamma(P)$ is a discrete subset of $\mathbb{C}$, located in the lower half-plane $\{\operatorname{Im} z \le 0\}$.*

(2) *For any resonance $\rho \in \Gamma(P)$, there are two linear subspaces $F_{\rho,\theta}$ and $G_{\rho,\theta}$ of $L^2(\mathbb{R}^n)$, $0 < \operatorname{Im}\theta < \delta$ such that*

$$F_{\rho,\theta} \oplus G_{\rho,\theta} = L^2(\mathbb{R}^n),$$
$$P_\theta(F_{\rho,\theta} \cap H^2(\mathbb{R}^n)) \subset F_{\rho,\theta}, \quad P_\theta(G_{\rho,\theta} \cap H^2(\mathbb{R}^n)) \subset G_{\rho,\theta},$$

$(P_\theta - \rho) : G_{\rho,\theta} \cap H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is boundedly invertible operator, $\dim F_{\rho,\theta} < \infty$ and the restriction of $(P_\theta - \rho)$ to $F_{\rho,\theta}$ is nilpotent. $F_{\rho,\theta}$ is called the space of resonant states of P_θ.

Proof. (1) By construction, $\Gamma(P)$ is discrete. Indeed, if $\rho \in \Gamma(P)$, there exist $\theta \in \mathbb{C}$, $0 < \operatorname{Im}\theta < \delta$, and a neighborhood W of ρ such that

$$\Gamma(P) \cap W = \sigma_{disc}(P_\theta) \cap W.$$

Furthermore, for each $\varphi, \psi \in \mathcal{A}_\delta$, $z \mapsto \langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$ is holomorphic on $\{0 < \arg z < \frac{\pi}{2}\} \subset \{-2\delta < \arg z < \frac{\pi}{2}\}$, so, the meromorphic extension of $\langle R(z)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$ is in fact holomorphic on $\{z \in \mathbb{C} :\ 0 < \arg z < \frac{\pi}{2}\}$. This implies that

$$\Gamma(P) \cap \left\{z \in \mathbb{C} :\ 0 < \arg z < \frac{\pi}{2}\right\} = \varnothing,$$

and

$$\Gamma(P) \subset \{z \in \mathbb{C} :\ \operatorname{Im} z \le 0\}.$$

(2) We denote by

$$\Pi_{\rho,\theta} = \frac{1}{2i\pi}\oint_\gamma (z - P_\theta)^{-1}\, dz$$

the spectral projection of P_θ associated to ρ and a simple closed curve γ isolating ρ from the rest of the spectrum of P_θ, with $-2\delta < -2\operatorname{Im}\theta < \arg\rho$. Since $\rho \in \sigma_{disc}(P_\theta)$, the multiplicity of the resonance ρ is finite and is equal to the rank of $\Pi_{\rho,\theta}$.

Then, if we set

$$F_{\rho,\theta} = \operatorname{Im}\Pi_{\rho,\theta} \ \text{ and } \ G_{\rho,\theta} = \ker \Pi_{\rho,\theta},$$

we easily obtain the stated properties. □

3.4 Resonances via distortion analyticity

Here we define the resonances of P by using the analytic distortion. This approach to resonances was initiated by Hunziker [13] and then followed by many others.

Let $\omega : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field such that $\omega(x) = x$ outside a compact subset of $\mathbb{R}^n$. We deduce the existence of a constant $C > 0$ such that

$$|\omega(x) - \omega(y)| \leq C|x - y| \text{ for all } x, y \in \mathbb{R}^n. \tag{3.3}$$

For $\mu \in \mathbb{R}$ small enough, we define $\mathcal{U}_\mu : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as the unitary operator:

$$\mathcal{U}_\mu \phi(x) = \left| \det(1 + \mu d\omega(x)) \right|^{1/2} \phi(x + \mu\omega(x)),$$

and the distorted operator

$$P_\mu = \mathcal{U}_\mu P \mathcal{U}_\mu^{-1},$$

defined on $L^2(\mathbb{R}^n)$ with the domain $H^2(\mathbb{R}^n)$.

As before, when P_μ can be extended to small enough complex values of μ as an analytic family, eigenvalues of P_μ that are located in the complex sector $\{\mathrm{Re}\, z > 0,\ -2\delta < \arg z \leq 0\}$ are called the resonances of P, and the set of resonances of P is $\Gamma(P)$.

Note that in case $\omega(x) = x$ on $\mathbb{R}^n$, the distorsion is a dilation $\mathcal{U}_\mu = \mathcal{U}_\theta$ with $e^\theta = 1 + \mu$.

Definition 3.4. Let $V : H^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ be a compact multiplication operator and $V_\mu(x) = \mathcal{U}_\mu V(x) \mathcal{U}_\mu^{-1}$, $\mu \in \mathbb{R}$ small enough. V is called distortion-analytic if $(V_\mu(x)(-h^2\Delta + 1)^{-1})$ can be extended to an analytic family of compact operators on $L^2(\mathbb{R}^n)$ in the neighborhood of 0 in $\mathbb{C}$.

The following are typical examples of distortion-analytic potentials.

Example 3.2.

(1) Let V be a continuous function on $\mathbb{R}^n$ with compact support $K = \mathrm{Supp}\, V$. Let $K_\varepsilon = \{x \in \mathbb{R}^n :\ dist(x, K) < \varepsilon\}$, $\varepsilon > 0$ be fixed small enough, and $dist(x, K)$ be the distance from x to the compact subset K of $\mathbb{R}^n$. Consider $\omega \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\omega = 0$ on K_ε. Then V is istortion-analytic.

Indeed, $V_\mu(x) = V(x)$ for all $x \in \mathbb{R}^n$ and by using the Rellich–Kondrachov theorem, it is clear that V is compact from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

(2) $V(x) = \sum_{j=1}^{N} \frac{\alpha_j}{x - x_j|}$ is a potential energy in a field of N fixed nuclei $\{x_1, \ldots, x_N\}$, $x, x_1, \ldots, x_N \in \mathbb{R}^n$ and $\alpha_1, \ldots, \alpha_N$ are real constants.

In particular, we know that $\frac{1}{|x - x_j|}$ are compact operators from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ (see [14]). Let ω be a smooth vector field on $\mathbb{R}^n$ such that $\omega(x) = x$ for sufficiently large $|x|$ and $\omega(x_j) = 0$ for each $j \in \{1, \ldots, N\}$.

The last condition on ω states that the Coulomb singularities $\frac{1}{|x - x_j|}$ are not displaced under the transformation $x \longrightarrow x + \mu\omega(x)$. The transformed potential is then given by

$$V_\mu(x) = V(x + \mu\omega(x)) = \sum_{j=1}^{N} \frac{\alpha_j}{|x + \mu\omega(x) - x_j|} = \sum_{j=1}^{N} \frac{\alpha_j}{|x - x_j|} \frac{1}{|1 + \mu \frac{\omega(x) - \omega(x_j)}{|x - x_j|}|},$$

where

$$\left| \frac{\omega(x) - \omega(x_j)}{|x - x_j|} \right| \leq C, \quad C > 0,$$

and $1 + \mu \frac{\omega(x) - \omega(x_j)}{|x - x_j|}$ is analytic with respect $\mu \in \mathbb{C}$, $|\mu|$ small enough.

So,

$$V_\mu(x)(-h^2\Delta + 1)^{-1} = \sum_{j=1}^{N} \frac{\alpha_j}{|1 + \mu \frac{\omega(x) - \omega(x_j)}{|x - x_j|}|} \frac{1}{|x - x_j|} (-h^2\Delta + 1)^{-1}$$

is an analytic family of compact operators on $L^2(\mathbb{R}^n)$ in the neighborhood of 0 in $\mathbb{C}$. Therefore, V is distortion-analytic with respect to the vector field ω.

Nextm, we study the distorted operator

$$P_\mu = \mathcal{U}_\mu P \mathcal{U}_\mu^{-1} = -h^2 \mathcal{U}_\mu \Delta \mathcal{U}_\mu^{-1} + V_\mu(x)$$

when V is ω-distortion-analytic and $\omega(x) = x$ for $|x|$ large enough. We start with establishing the following result on the operator $\mathcal{U}_\mu \Delta \mathcal{U}_\mu^{-1}$.

Proposition 3.2.

$$\mathcal{U}_\mu \Delta \mathcal{U}_\mu^{-1} = \frac{1}{(1+\mu)^2}\,\Delta + R_\mu(x, D_x),$$

where

$$R_\mu(x, D_x) = \sum_{\alpha\in\mathbb{N}^n, |\alpha|\le 2} a_\alpha(x,\mu) D_x^\alpha$$

is a second order differential operator, where the coefficients $a_\alpha(x,\mu) \in C_0^\infty(\overset{\circ}{K})$, K is a compact subset of $\mathbb{R}^n$; $\mu \mapsto a_\alpha(x,\mu)$ is an analytic function with respect to μ and

$$\sup_{\beta\in\mathbb{N}^n,\, x\in\mathbb{R}^n} |D_x^\beta a_\alpha(x,\mu)| = \mathcal{O}(|\mu|) \ \text{ for all } \ \alpha \in \mathbb{N}^n, \ \ |\alpha| \le 2, \ \ \mu \in \mathbb{C},$$

$|\mu|$ small enough.

Proof. By (3.3), $\|d\omega(x)\| \le C$ as an operator on $\mathbb{R}^n$, then $|J_\mu(x)| \ge (1 - C|\mu|)^n > 0$ if μ is small enough, where $J_\mu(x) = \det(1 + \mu\, d\omega(x))$ is the Jacobian of the transformation $F_\mu(x) = x + \mu\omega(x)$. Thus $F_\mu(x)$ is a C^∞-diffeomorphism of $\mathbb{R}^n$. Let $G_\mu = F_\mu^{-1} = (G_{\mu,1}, \ldots, G_{\mu,n})$ be the inverse of F_μ.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
\mathcal{U}_\mu \frac{\partial^2}{\partial x_l} \mathcal{U}_\mu^{-1} f(x) &= \mathcal{U}_\mu \frac{\partial^2}{\partial x_l}\big[f(G_\mu(x)) J_\mu^{-1}(x)\big] \\
&= \mathcal{U}_\mu \frac{\partial}{\partial x_l}\Big(\sum_{j=1}^n \frac{\partial f}{\partial x_j}(G_\mu(x))\,\frac{\partial G_{\mu,j}}{\partial x_l}(x) J_\mu^{-1}(x) + f(G_\mu(x))\,\frac{\partial}{\partial x_l}(J_\mu^{-1}(x))\Big) \\
&= \mathcal{U}_\mu\Big[\sum_{j,k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k}(G_\mu(x))\,\frac{\partial G_{\mu,j}}{\partial x_l}(x)\,\frac{\partial G_{\mu,k}}{\partial x_l}(x) J_\mu^{-1}(x) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(G_\mu(x))\,\frac{\partial^2 G_{\mu,j}}{\partial x_l^2}(x) J_\mu^{-1}(x) \\
&\quad + 2\sum_{j=1}^n \frac{\partial f}{\partial x_j}(G_\mu(x))\,\frac{\partial G_{\mu,j}}{\partial x_l}(x)\,\frac{\partial J_\mu^{-1}}{\partial x_l}(x) + f(G_\mu(x))\,\frac{\partial^2 J_\mu^{-1}}{\partial x_l^2}(x)\Big] \\
&= \sum_{j,k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k}(x)\,\frac{\partial G_{\mu,j}}{\partial x_l}(F_\mu(x))\,\frac{\partial G_{\mu,k}}{\partial x_l}(F_\mu(x)) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x)\,\frac{\partial^2 G_{\mu,j}}{\partial x_l^2}(F_\mu(x)) \\
&\quad + 2\sum_{j=1}^n \frac{\partial f}{\partial x_j}(x)\,\frac{\partial G_{\mu,j}}{\partial x_l}(F_\mu(x))\,\frac{\partial J_\mu^{-1}}{\partial x_l}(F_\mu(x)) J_\mu(x) + f(x)\,\frac{\partial^2 J_\mu^{-1}}{\partial x_l^2}(F_\mu(x)) J_\mu(x).
\end{aligned}$$

Let K_0 be the closure of $\{x \in \mathbb{R}^n : \omega(x) \ne x\}$ and $K_\varepsilon = \{x \in \mathbb{R}^n : \ dist(x, K_0) \le \varepsilon\}$ for $\varepsilon > 0$ small enough. Let

$$R_\mu(x, D_x) = \mathcal{U}_\mu \Delta \mathcal{U}_\mu^{-1} - (1+\mu)^{-2}\Delta.$$

In particular, since $F_\mu(x) = (1+\mu)x$, $J_\mu(x) = (1+\mu)^n$ and $G_\mu x = (1+\mu)^{-1}x$ for all $x \in \mathbb{R}^n \setminus K_\varepsilon$, one obtains

$$\begin{aligned}
\mathcal{U}_\mu \Delta \mathcal{U}_\mu^{-1} f(x) &= \mathcal{U}_\mu \Delta\big[f((1+\mu)^{-1}x)|1+\mu|^{-n/2}\big] \\
&= \mathcal{U}_\mu (1+\mu)^{-2}(\Delta f)((1+\mu)^{-1}x)|1+\mu|^{-n/2} = (1+\mu)^{-2}\Delta f(x).
\end{aligned}$$

Thus $R_\mu(x, D_x) = 0$ on $\mathbb{R}^n \setminus K_\varepsilon$.

Therefore, $R_\mu(x, D_x)$ is a second order differential operator with smooth coefficients $a_\alpha(x,\mu)$ compactly supported in $\mathbb{R}^n$, analytic in μ, and it is easy to verify that for any $\alpha \in \mathbb{N}^n, |\alpha| \le 2$, one has $|D_x^\beta a_\alpha(x,\mu)| = \mathcal{O}(|\mu|)$ uniformly. $\square$

Corollary 3.2. $\mathcal{U}_\mu \Delta \mathcal{U}_\mu^{-1}$ *and*

$$P_\mu = \frac{-h^2}{(1+\mu)^2}\,\Delta - h^2 R_\mu(x, D_x) + V_\mu(x)$$

are analytic families of operators on $L^2(\mathbb{R}^n)$ *in some neighborhood of* $\mu = 0$.

Since $\mathcal{U}_\mu$ is unitary for $\mu \in \mathbb{R}$, P_μ has the same spectrum as P on the real line, but, for nonreal μ, the continuous part of the spectrum of P_μ is obtained from the one of P by some rotation in the complex plane. Let us observe that $R_\mu(x, D_x)$ is not necessarily Δ-compact, thus, in particular, the Weyl perturbation theorem does not hold to determine the essential spectrum of P_μ.

3.4.1 Essential spectrum of P_μ

We investigate now the essential spectrum $\sigma_{ess}(P_\mu)$ of the distorted operator P_μ, this will be used to construct the resonances of P. However, $\sigma_{ess}(P_\mu)$ cannot be described here using Weyl's theorem directly, since P_μ is not selfadjoint for $\mu \in \mathbb{C}$ small enough, with $\operatorname{Im} \mu > 0$.

Remember that the distorted potential $V_\mu(x)$ is a compact multiplication operator from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, $\mu \in \mathbb{C}$, $|\mu|$ small enough. Furthermore, with the notation as in Proposition 3.2, we can write

$$a_\alpha(x,\mu) D_x^\alpha = \left[a_\alpha(x,\mu), D_x^\alpha\right] + D_x^\alpha \circ a_\alpha(x,\mu),$$

where the commutator $[a_\alpha(x,\mu), D_x^\alpha]$ is a first order differential operator for $\alpha \in \mathbb{N}^n$, $|\alpha| = 2$, and

$$\begin{aligned} R_\mu(x, D_x) &= \sum_{\alpha\in\mathbb{N}^n,\ |\alpha|=2} a_\alpha(x,\mu) D_x^\alpha + \sum_{\alpha\in\mathbb{N}^n,\ |\alpha|\le 1} a_\alpha(x,\mu) D_x^\alpha \\ &= \sum_{\alpha\in\mathbb{N}^n,\ |\alpha|=2} (D_x^\alpha \circ a_\alpha(x,\mu)) + \sum_{\alpha\in\mathbb{N}^n,\ |\alpha|\le 1} \widetilde{a}_\alpha(x,\mu) D_x^\alpha, \end{aligned}$$

$\widetilde{a}_\alpha(x,\mu) \in C_0^\infty(\overset{\circ}{K}_\varepsilon)$ resulting from the sum of $[a_\alpha(x,\mu), D_x^\alpha]$ for $|\alpha| = 2$, and coefficients $a_\alpha(x,\mu)$ for $|\alpha| \le 1$.

As $\sum\limits_{\alpha\in\mathbb{N}^n,|\alpha|\le 1} \widetilde{a}_\alpha(x,\mu) D_x^\alpha$ is Δ-compact, it suffices to study the spectral behavior of operators

$$S_\mu(x, D_x) = \sum_{\alpha\in\mathbb{N}^n,\ |\alpha|=2} (D_x^\alpha \circ a_\alpha(x,\mu))$$

and

$$\widetilde{\Delta}_\mu = \Delta_\mu + S_\mu(x, D_x),$$

where $\Delta_\mu = \frac{1}{(1+\mu)^2}\,\Delta$, $\mu \in \mathbb{C}$, $|\mu|$ small enough.

Lemma 3.3. *For all* $\lambda \in \mathbb{R}$, $|\lambda| \gg 1$, *and* $\mu \in \mathbb{C}$, $|\mu|$ *small enough,* $(i\lambda) \in \rho(\widetilde{\Delta}_\mu)$.

Proof. If $\mu \in \mathbb{C}$ is small enough and $\lambda \in \mathbb{R}$, $|\lambda| \gg 1$, we have to prove that $(\Delta_\mu - i\lambda)$ is boundedly invertible on $L^2(\mathbb{R}^n)$ and $(\Delta_\mu - i\lambda)^{-1} = \mathcal{O}(|\lambda|^{-1})$ uniformly.

Indeed,

$$(\Delta_\mu - i\lambda)^{-1} = \int_{\mathbb{R}} f_\lambda(t)\, dE_t,$$

where $(E_t)_{t\in\mathbb{R}}$ is the spectral resolution of Δ (selfadjoint on $L^2(\mathbb{R}^n)$ with the domain $H^2(\mathbb{R}^n)$) and

$$f_\lambda(t) = \Big(\frac{t}{(1+\mu)^2} - i\lambda\Big)^{-1}.$$

Therefore,

$$\|f_\lambda\|_\infty = \mathcal{O}(|\lambda|^{-1}) \ \text{ and } \ \lim_{\lambda\to\infty} (\Delta_\mu - i\lambda)^{-1} = 0 \ \text{ uniformly.}$$

Moreover, let $v \in L^2(\mathbb{R}^n)$ and $u = (\Delta_\mu - i\lambda)^{-1}v$, we then have

$$(\Delta_\mu - i\lambda)u = v \text{ and } \Delta u = (1+\mu)^2 v + i\lambda(1+\mu)^2 u.$$

So,

$$\|\Delta u\|_{L^2(\mathbb{R}^n)} \leq |1+\mu|^2\|v\|_{L^2(\mathbb{R}^n)} + |\lambda|\,|1+\mu|^2\|u\|_{L^2(\mathbb{R}^n)} \leq C\|v\|_{L^2(\mathbb{R}^n)},$$

since

$$\|u\|_{L^2(\mathbb{R}^n)} \leq \frac{C'}{|\lambda|}\,\|v\|_{L^2(\mathbb{R}^n)}, \quad C, C' > 0.$$

As $S_\mu(x, D_x) = \mathcal{O}(|\mu|)$, it follows that

$$S_\mu(x, D_x)(\Delta_\mu - i\lambda)^{-1} = \mathcal{O}(|\mu|) \text{ uniformly on } L^2(\mathbb{R}^n) \text{ for } |\lambda| \gg 1.$$

Consequently,

$$1 + S_\mu(x, D_x)(\Delta_\mu - i\lambda)^{-1}$$

and

$$(\widetilde{\Delta}_\mu - i\lambda) = \left[1 + S_\mu(x, D_x)(\Delta_\mu - i\lambda)^{-1}\right](\Delta_\mu - i\lambda),$$

are the boundedly invertible operators for $\lambda \in \mathbb{R}$, $|\lambda| \gg 1$, and $\mu \in \mathbb{C}$, $|\mu|$ small enough. □

Lemma 3.4. *For all $\lambda \in \mathbb{R}$, $|\lambda| \gg 1$, and $\mu \in \mathbb{C}$, $|\mu|$ small enough,*

$$(\widetilde{\Delta}_\mu - i\lambda)^{-1}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} \in \mathcal{B}(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n)), \quad j, k \in \{1, \ldots, n\}.$$

Proof. From Lemma 3.3, we already have

$$(\widetilde{\Delta}_\mu - i\lambda)^{-1} = (\Delta_\mu - i\lambda)^{-1}\left[1 + S_\mu(x, D_x)(\Delta_\mu - i\lambda)^{-1}\right]^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n))$$

and

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}(\widetilde{\Delta}_\mu - i\lambda)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n)), \quad j, k \in \{1, \ldots, n\}.$$

On the other hand

$$\begin{aligned}
(\widetilde{\Delta}_\mu - i\lambda)^{-1}\frac{\partial}{\partial x_j} &= \frac{\partial}{\partial x_j}(\widetilde{\Delta}_\mu - i\lambda)^{-1} + \left[(\widetilde{\Delta}_\mu - i\lambda)^{-1}, \frac{\partial}{\partial x_j}\right], \qquad (3.4)\\
\left[(\widetilde{\Delta}_\mu - i\lambda)^{-1}, \frac{\partial}{\partial x_j}\right] &= (\widetilde{\Delta}_\mu - i\lambda)^{-1}\Big(\frac{\partial}{\partial x_j} - (\widetilde{\Delta}_\mu - i\lambda)\frac{\partial}{\partial x_j}(\widetilde{\Delta}_\mu - i\lambda)^{-1}\Big)\\
&= (\widetilde{\Delta}_\mu - i\lambda)^{-1}\left[\frac{\partial}{\partial x_j}, \widetilde{\Delta}_\mu\right](\widetilde{\Delta}_\mu - i\lambda)^{-1}.
\end{aligned}$$

Let us observe that $\frac{\partial}{\partial x_j}(\widetilde{\Delta}_\mu - i\lambda)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n))$. $[\frac{\partial}{\partial x_j}, \widetilde{\Delta}_\mu]$ is a second-order differential operator with bounded coefficients, it is therefore continuous from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Then $(\widetilde{\Delta}_\mu - i\lambda)^{-1}\frac{\partial}{\partial x_j} \in \mathcal{B}(L^2(\mathbb{R}^n))$.

Similarly,

$$\begin{aligned}
(\widetilde{\Delta}_\mu - i\lambda)^{-1}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} = \frac{\partial^2}{\partial x_j \partial x_k}(\widetilde{\Delta}_\mu - i\lambda)^{-1} + \left[(\widetilde{\Delta}_\mu - i\lambda)^{-1}, \frac{\partial^2}{\partial x_j \partial x_k}\right],\\
\frac{\partial^2}{\partial x_j \partial x_k}(\widetilde{\Delta}_\mu - i\lambda)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n))
\end{aligned}$$

and

$$\left[(\widetilde{\Delta}_\mu - i\lambda)^{-1}, \frac{\partial^2}{\partial x_j \partial x_k}\right] = (\widetilde{\Delta}_\mu - i\lambda)^{-1}\left[\frac{\partial^2}{\partial x_j \partial x_k}, \widetilde{\Delta}_\mu\right](\widetilde{\Delta}_\mu - i\lambda)^{-1},$$

where $[\frac{\partial^2}{\partial x_j \partial x_k}, \widetilde{\Delta}_\mu]$ is a third-order differential operator with smooth coefficients and bounded derivatives of any order. We set

$$\Big[\frac{\partial^2}{\partial x_j \partial x_k}, \widetilde{\Delta}_\mu\Big] = \sum_{\alpha\in\mathbb{N}^n,\, |\alpha|\le 3} D_x^\alpha \circ b_\alpha(x,\mu),$$

then

$$\begin{aligned}\Big[(\widetilde{\Delta}_\mu - i\lambda)^{-1}, \frac{\partial^2}{\partial x_j \partial x_k}\Big] &= (\widetilde{\Delta}_\mu - i\lambda)^{-1} \sum_{\alpha\in\mathbb{N}^n,\, |\alpha|\le 2} D_x^\alpha \circ b_\alpha(x,\mu)(\widetilde{\Delta}_\mu - i\lambda)^{-1} \\ &\quad + \sum_{\substack{\alpha\in\mathbb{N}^n,\, |\alpha|=3,\, \alpha=\alpha'+\beta,\\ \alpha'\le\alpha,\, |\beta|=1}} \big[(\widetilde{\Delta}_\mu - i\lambda)^{-1} D_x^\beta\big]\,\big[D_x^{\alpha'}\big(b_\alpha(x,\mu)(\widetilde{\Delta}_\mu - i\lambda)^{-1}\big)\big].\end{aligned}$$

From Lemma 3.3 and (3.4), we conclude that $[(\widetilde{\Delta}_\mu - i\lambda)^{-1}, \frac{\partial^2}{\partial x_j \partial x_k}] \in \mathcal{B}(L^2(\mathbb{R}^n))$. Finally,

$$(\widetilde{\Delta}_\mu - i\lambda)^{-1}\,\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_k} \in \mathcal{B}(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n)),\ \ j,k\in\{1,\dots,n\}. \qquad \square$$

Moreover, using Lemma 3.3, $(\Delta_\mu - i\lambda)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n), H^2(\mathbb{R}^n))$, and from Proposition 3.2 and the fact that $\mathrm{Supp}(a_\alpha) \subseteq K_\varepsilon$ for any $\alpha\in\mathbb{N}^n$, $|\alpha|\le 2$, we also obtain that the coefficients $a_\alpha(x,\mu)$ are bounded from $H^2(\mathbb{R}^n)$ into $H^2_{K_\varepsilon}(\mathbb{R}^n)$, where $H^2_{K_\varepsilon}(\mathbb{R}^n) = \{f\in H^2(\mathbb{R}^n) :\ \mathrm{Supp}(f)\subseteq K_\varepsilon\}$. As the embedding from $H^2_{K_\varepsilon}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ is compact, we get the following result.

Lemma 3.5. *For all $\lambda\in\mathbb{R}$, $|\lambda|\gg 1$, and $\mu\in\mathbb{C}$, $|\mu|$ small enough,*

$$qa_\alpha(x,\mu)(\Delta_\mu - i\lambda)^{-1},\ \ \alpha\in\mathbb{N}^n,\ \ |\alpha|\le 2,$$

is a compact operator on $L^2(\mathbb{R}^n)$.

We will also need the spectral mapping theorem for the essential spectrum of any closed linear operator with non-empty resolvent set, acting in a complex Hilbert space. We use the following extension of Weyl's theorem.

Lemma 3.6 (Weyl's theorem)**.** *Let A be a closed linear operator acting in a Hilbert space $\mathcal{H}$ with domain $D(A)$ and let T be A-relatively compact operator. If for every open connected component Ω of $\mathbb{C}\setminus\sigma_{ess}(A)$, there is $z\in\Omega$ such that $A+T-z$ is boundedly invertible from $D(A)$ into H, then $\sigma_{ess}(A+T)=\sigma_{ess}(A)$.*

Proposition 3.3. *For all $\lambda\in\mathbb{R}$, $|\lambda|\gg 1$, and $\mu\in\mathbb{C}$, $|\mu|$ small enough,*

$$\sigma_{ess}((\widetilde{\Delta}_\mu - i\lambda)^{-1}) = \sigma_{ess}((\Delta_\mu - i\lambda)^{-1}).$$

Proof. From the three preceding lemmas, we see that

$$\begin{aligned}(\widetilde{\Delta}_\mu - i\lambda)^{-1} - (\Delta_\mu - i\lambda)^{-1} &= -(\widetilde{\Delta}_\mu - i\lambda)^{-1} S_\mu(x,D_x)(\Delta_\mu - i\lambda)^{-1} \\ &= -\sum_{\alpha\in\mathbb{N}^n,\, |\alpha|=2} (\widetilde{\Delta}_\mu - i\lambda)^{-1}(D_x^\alpha\circ a_\alpha(x,\mu))(\Delta_\mu - i\lambda)^{-1}\end{aligned}$$

is a compact operator on $L^2(\mathbb{R}^n)$.

One can then apply Weyl's theorem if $\Omega\cap\rho((\widetilde{\Delta}_\mu - i\lambda)^{-1})\neq\varnothing$, for any connected component Ω of the resolvent set $\rho((\Delta_\mu - i\lambda)^{-1})$.

Using the classical spectral mapping theorem, we have

$$\sigma((\Delta_\mu - i\lambda)^{-1}) = \overline{\Big\{\Big(\frac{t}{(1+\mu)^2} - i\lambda\Big)^{-1} :\ t\in\mathbb{R}_+\Big\}},$$

hence $\rho((\Delta_\mu - i\lambda)^{-1})$ is a connected subset of $\mathbb{C}$.

Let $z \in \mathbb{C}^*$, $z = \frac{1}{z'-i\lambda}$, $|\operatorname{Im} z'| \gg 1$, then

$$\begin{aligned}(\widetilde{\Delta}_\mu - i\lambda)^{-1} - zI &= -(\widetilde{\Delta}_\mu - i\lambda)^{-1}(z' - i\lambda)^{-1}(\widetilde{\Delta}_\mu - z') \\ &= -(\widetilde{\Delta}_\mu - i\lambda)^{-1}(z' - i\lambda)^{-1}(\Delta_\mu - z')\big[I + (\Delta_\mu - z')^{-1}S_\mu(x, D_x)\big],\end{aligned}$$

where $I + (\Delta_\mu - z')^{-1}S_\mu(x, D_x)$ is boundedly invertible on $L^2(\mathbb{R}^n)$, for $|\mu|$ small enough, with

$$\big((\widetilde{\Delta}_\mu - i\lambda)^{-1} - z\big)^{-1} = -(z' - i\lambda)\big[I + (\Delta_\mu - z')^{-1}S_\mu(x, D_x)\big]^{-1}(\Delta_\mu - z')^{-1}(\widetilde{\Delta}_\mu - i\lambda) \in \mathcal{B}(L^2(\mathbb{R}^n)).$$

Thus, if $z' = i\lambda'$, $\lambda' \gg 1$, $\lambda' \neq \lambda$,

$$z = \frac{1}{i(\lambda' - \lambda)} \in \rho((\widetilde{\Delta}_\mu - i\lambda)^{-1}) \cap \rho((\Delta_\mu - i\lambda)^{-1}).$$

Now, by Weyl's theorem, we get the result. □

Lemma 3.7 (Spectral mapping theorem)**.** *Let A be a closed linear operator on a Hilbert space $\mathcal{H}$ and let $z \in \rho(A)$. Then*

$$\Big\{\frac{1}{t-z} : \ t \in \sigma_*(A)\Big\} \subset \sigma_*((A-z)^{-1}) \subset \Big\{\frac{1}{t-z} : \ t \in \sigma_*(A)\Big\} \cup \{0\}, \quad \sigma_* \in \{\sigma, \sigma_{ess}, \sigma_{disc}\}.$$

Proof. First, we prove that

$$\Big\{\frac{1}{t-z} : \ t \in \sigma(A)\Big\} \subset \sigma((A-z)^{-1}) \subset \Big\{\frac{1}{t-z} : \ t \in \sigma(A)\Big\} \cup \{0\}. \tag{3.5}$$

Indeed, let $z' \in \sigma((A-z)^{-1}) \setminus \{0\}$. Using the spectral mapping theorem for unbounded closed linear operators with non-empty resolvent [7], we have $z' = (t-z)^{-1}$ with $t \in \sigma(A) \setminus \{z\}$. Moreover,

$$(A-z)^{-1} - (s-z)^{-1} = (A-z)^{-1}(s-z)^{-1}(s-A) \ \text{ for all } \ s \in \mathbb{C}.$$

Thus $(s-A)$ is boundedly invertible if and only if so is $(A-z)^{-1} - (s-z)^{-1}$, this gives (3.5).

We also show that

$$\sigma_{ess}((A-z)^{-1}) \subset \Big\{\frac{1}{t-z} : \ t \in \sigma_{ess}(A)\Big\} \cup \{0\}.$$

Using (3.5), it suffices to prove

$$\Big\{\frac{1}{t-z} : \ t \in \sigma_{disc}(A)\Big\} \subset \sigma_{disc}((A-z)^{-1}).$$

Let $t \in \sigma_{disc}(A)$, then the corresponding Riesz projection

$$P_t(A) = \frac{1}{2\pi i}\oint_{\Gamma}(z-A)^{-1}\,dz$$

has finite rank, where Γ is a contour that enlaces only t as element of the spectrum of A. Notice that $\widetilde{\Gamma} = \{\frac{1}{z'-z} : z' \in \Gamma\}$ is a closed contour, which encloses $(t-z)^{-1}$ and lies entirely within the resolvent set $\rho((A-z)^{-1})$ of the resolvent operator $(A-z)^{-1}$, and define

$$\widetilde{\Pi} = \frac{1}{2\pi i}\oint_{\widetilde{\Gamma}}(s - (A-zI)^{-1})^{-1}\,ds.$$

Then

$$\widetilde{\Pi} = -\frac{1}{2\pi i}\oint_{\widetilde{\Gamma}}\Big(\frac{1}{(z'-z)} - (A-zI)^{-1}\Big)^{-1}\frac{dz'}{(z'-z)^2} = -\frac{1}{2\pi i}\Big(A-z\Big)\oint_{\widetilde{\Gamma}}\frac{(A-z')^{-1}}{z'-z}\,dz'.$$

$(A-z')^{-1}$ is a meromorphic function with Laurent series

$$(A-z')^{-1} = \sum_{j=-N}^{\infty} A_j (z'-z)^j,$$

where the coefficients $A_{-N}, \dots, A_{-1}$ are bounded, finite rank operators.

Then,

$$\widetilde{\Pi} = \frac{1}{t-z}(A - zI)A_{-1}$$

and

$$A_{-1} = -\frac{1}{2\pi i} P_t(A),$$

that is, $\widetilde{\Pi}$ is of finite rank.

Finally, it remains to show

$$\Big\{\frac{1}{t-z} : \ t \in \sigma_{ess}(A)\Big\} \subset \sigma_{ess}((A-z)^{-1}).$$

By taking complements in $\sigma(A)$, it suffices to verify that

$$\sigma_{disc}((A-z)^{-1}) \subset \Big\{\frac{1}{t-z} : \ t \in \sigma_{disc}(A)\Big\}.$$

Let $t_0 \in \sigma_{disc}((A-z)^{-1})$ and $\Gamma_0 = \partial D_0$, where D_0 is a closed disk centered at t_0 and $D_0 \cap \sigma(A) = \{t_0\}$, $t_0 \neq 0$ (if $t_0 = 0$, there exists $u_0 \neq 0$ in the domain of A such that $(A-z)^{-1}u_0 = 0$ and hence $u_0 = 0$, which is impossible).

Let $\Gamma' = \{\frac{1}{t'} + z : \ t' \in \Gamma_0\}$, then Γ' is a closed contour which encloses $\frac{1}{t_0} + z$ and lies entirely within $\rho((A-zI)^{-1})$. We deduce from (3.5) that $\frac{1}{t_0+z}$ is an isolated point of the spectrum $\sigma(A)$ of A. Furthermore,

$$\oint_{\Gamma'} (t-A)^{-1}\, dt = \oint_{\Gamma_0} \Big(\Big(\frac{1}{t'} + z\Big) - A\Big)^{-1}\Big(-\frac{dt'}{t'^2}\Big) = -\frac{1}{t_0^3}\,(A-z)^{-1} \oint_{\Gamma_0} \big((A-z)^{-1} - t'\big)^{-1}\, dt',$$

then $\oint_{\Gamma'} (t-A)^{-1}\, dt$ is of finite rank and $(\frac{1}{t_0} + z) \in \sigma_{disc}(A)$. □

Now we use Lemma 3.7 to deduce the following result.

Corollary 3.3.

$$\sigma_{ess}(\widetilde{\Delta}_\mu) = \sigma_{ess}(\Delta_\mu), \ \ \mu \in \mathbb{C}, \ \ |\mu| \ \textit{small enough.}$$

Proof. From Proposition 3.3 and Lemma 3.7, we know that if $t \in \sigma_{ess}(\Delta_\mu)$, then $\frac{1}{t-i\lambda} \in \sigma_{ess}((\widetilde{\Delta}_\mu - i\lambda)^{-1})$ for $\lambda \in \mathbb{R}$, $|\lambda| \gg 1$, and $\mu \in \mathbb{C}$, $|\mu|$ small enough. So, $t \in \sigma_{ess}(\widetilde{\Delta}_\mu)$.

The reverse inclusion follows in the same way. □

Thanks to these results, Proposition 3.2, Lemmas 3.3–3.6, Proposition 3.3, Lemma 3.7 and Corollary 3.3; we are now able to show the following theorem concerning the location of essential spectrum of $P_\mu(h)$.

Theorem 3.4. *For $\mu \in \mathbb{C}$ sufficiently small, we have*

$$\sigma_{ess}(P_\mu) = \big\{z \in \mathbb{C} : \ \arg(z) = -2\arg(1+\mu)\big\}.$$

Proof.

$$\begin{aligned}\sigma_{ess}(P_\mu) &= \sigma_{ess}(-h^2\widetilde{\Delta}_\mu) = \frac{1}{(1+\mu)^2}\,\sigma_{ess}(-h^2\Delta_x)\\ &= \frac{1}{(1+\mu)^2}\,\mathbb{R}_+ = \big\{z \in \mathbb{C} : \ \arg(z) = -2\arg(1+\mu)\big\}.\end{aligned}$$

□

In the same way as in the previous section, the resonances of P can be identified with the poles of the meromorphic extensions of $z \mapsto \langle (P-z)^{-1}\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$ for $\mathrm{Im}\, z > 0$ and $\varphi, \psi \in \mathcal{A}_\delta$, $\delta > 0$ small enough.

Theorem 3.5.

(1) *For each $\varphi, \psi \in \mathcal{A}_\delta$, the function $z \mapsto \langle (P-z)^{-1}\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$, $\mathrm{Im}\, z > 0$, admits a finitely meromorphic continuation to the set $\{z \in \mathbb{C} :\ -2\delta < \arg z < \frac{\pi}{2}\}$ and*

$$\bigcup_{\varphi,\psi\in\mathcal{A}_\delta} \left\{\textit{poles of } z \mapsto \langle R(z)\varphi,\psi\rangle_{L^2(\mathbb{R}^n)}\right\} \cap \left\{z \in \mathbb{C} :\ -2\arg(1+\mu) < \arg z < \frac{\pi}{2}\right\}$$
$$= \sigma_{disc}(P_\mu), \quad \mu \in \mathbb{C}, \quad \mathrm{Im}\, \mu > 0, \quad |\mu| < \delta,$$
$$\Gamma_\delta(P) = \bigcup_{\substack{0<\arg(1+\mu)<\delta \\ \mu\in\mathbb{C},\ \mathrm{Im}\ \mu>0,\ |\mu| \textit{ small enough}}} \sigma_{disc}(P_\mu)\,.$$

(2) *If $0 < \arg(1+\mu_1) < \arg(1+\mu_2)$, then*

$$\sigma_{disc}(P_{\mu_1}) \subset \sigma_{disc}(P_{\mu_2}) \subset \left\{z \in \mathbb{C} :\ \mathrm{Im}\ z \le 0\right\}.$$

(3) *For every $\rho \in \Gamma_\delta(P)$, there are two P_μ-invariant complementary subspaces $F_{\rho,\mu}$ and $G_{\rho,\mu}$ of $L^2(\mathbb{R}^n)$ such that*

$$(P_\mu - \rho) : G_{\rho,\mu} \cap H^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \ \textit{ is boundedly invertible,}$$
$$\dim F_{\rho,\mu} < \infty \ \textit{ and } \ (P_\theta - \rho) : F_{\rho,\mu} \cap H^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \ \textit{ is nilpotent.}$$

Since the meromorphic extension of $\langle (P-z)^{-1}\varphi,\psi\rangle_{L^2(\mathbb{R}^n)}$, $\varphi,\psi \in \mathcal{A}_\delta$, is unique, the poles of these functions are independent of the dilation- or of the distortion-analytic, then the two definitions of the resonances of the operator P necessarily coincide. So, if V is both dilation-analytic and distortion-analytic, then $\Gamma_\delta^{dilation}(P) = \Gamma_\delta^{distortion}(P)$.

Theorem 3.6 ([10])**.** *When their domain of validity overlap, these different definitions of resonance (as well as more sophisticated ones) coincide.*

4 Resonances theory for $P(h) = -h^2\Delta_x - \Delta_y + V(x,y)$

In general, resonances can be defined by dilation-analytic (see Aguilar–Combes [1]) or distortion-analytic (see Hunziker [13]) and by meromorphic continuation of the resolvent or scattering matrix. We introduce here the resonances for $P(h)$, with Coulomb-type potentials (1.3), as the discrete eigenvalues of the non-selfadjoint operators $P_\mu(h)$ obtained from the Schrödinger operator $P(h)$ by analytic distortion.

Let $\chi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ such that $\chi(t) = 1$ for $t \gg 1$ large enough, and $\chi(t) = 0$ when $t \le R$, $R > 0$ large enough. Let $\omega : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field defined by $\omega(x) = \chi(|x|)x$.

So,

$$\begin{cases} \omega(x) = 0 & \text{for} |x| \le R, \\ \omega(x) = x & \text{for} |x| \gg 1, \\ \omega(\mathcal{R}x) = \mathcal{R}\omega(x) & \text{for any rotation } \mathcal{R} \text{ on } \mathbb{R}^3. \end{cases} \tag{4.1}$$

Let us consider the analytic distortion operator $\mathcal{U}_\mu$ defined on $C_0^\infty(\mathbb{R}^3_x \times \mathbb{R}^{3p}_y)$ for $\mu \in \mathbb{R}$ small enough by

$$\mathcal{U}_\mu f(x,y) = J_\mu(x,y) f\big(x + \mu\omega(x), y_1 + \mu\omega(y_1), \ldots, y_p + \mu\omega(y_p)\big), \quad x \in \mathbb{R}^3, \quad y = (y_1,\ldots,y_p) \in \mathbb{R}^{3p},$$
$$J_\mu(x,y) = \Big|\det(1 + \mu d\omega(x)) \prod_{j=1}^{p} \det(1 + \mu d\omega(y_j))\Big|^{1/2}.$$

Note that $J_\mu(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^3 \times \mathbb{R}^{3p}$ and thus the map $x \mapsto F_\mu(x,y) = (x+\mu\omega(x), y_1 + \mu\omega(y_1), \ldots, y_p + \mu\omega(y_p))$ is invertible for real μ small enough, with inverse transformation $G_\mu = F_\mu^{-1}$.

Indeed, since $\sup\limits_{X\in\mathbb{R}^3} |d\omega(X)| \leq C$, $C > 0$, we have $|J_\mu(x,y)| \geq (1 - C|\mu|)^{3+3p} > 0$ if μ is small enough. Therefore, $\mathcal{U}_\mu$ can be extended to a unitary operator on $L^2(\mathbb{R}^3_x \times \mathbb{R}^{3p}_y)$. Let $P_\mu(h)$ be the family of distorted Hamiltonians:

$$P_\mu(h) = \mathcal{U}_\mu P(h) \mathcal{U}_\mu^{-1} = -h^2 \mathcal{U}_\mu \Delta_x \mathcal{U}_\mu^{-1} + \mathcal{U}_\mu Q(x) \mathcal{U}_\mu^{-1}, \ \ \mu \in \mathbb{R},$$

with domain $H^2(\mathbb{R}^3_x \times \mathbb{R}^{3p}_y)$; $P_\mu(h)$ will be used to construct the resonances.

We know from Proposition 3.2 that $\mathcal{U}_\mu \Delta_x \mathcal{U}_\mu^{-1}$ is an analytic family in some neighbourhood of $\mu = 0$,

$$P_\mu(h) = -h^2 \mathcal{U}_\mu \Delta_x \mathcal{U}_\mu^{-1} + Q_\mu(x),$$

where for $x \neq 0$ we have set

$$\begin{aligned} Q_\mu(x) &= \mathcal{U}_\mu Q(x) \mathcal{U}_\mu^{-1} = -\mathcal{U}_\mu \Delta_y \mathcal{U}_\mu^{-1} + V_\mu(x,y), \\ V_\mu(x,y) &= V\big(x + \mu\omega(x), y_1 + \mu\omega(y_1), \ldots, y_p + \mu\omega(y_p)\big) \end{aligned}$$

($\mathcal{U}_\mu$ should be considered as acting on $L^2(\mathbb{R}^{3p}_y)$ if x is fixed).

The distorted potential V_μ has the form

$$\begin{aligned} V_\mu(x,y) = \frac{a}{|x+\mu\omega(x)|} &+ \sum_{j=1}^{p} \Big(\frac{b_j^-}{|y_j - x|} \, \frac{1}{|1 + \mu \frac{\omega(y_j)-\omega(x)}{|y_j - x|}|} + \frac{b_j^+}{|y_j + x|} \, \frac{1}{|1 + \mu \frac{\omega(y_j)-\omega(-x)}{|y_j + x|}|} \Big) \\ &+ \sum_{j\neq k} \frac{c_{j,k}}{|y_j - y_k|} \, \frac{1}{|1 + \mu \frac{\omega(y_j)-\omega(y_k)}{|y_j - y_k|}|} \, . \end{aligned}$$

Note that by our choice of ω odd, the singularities of the potential are not changed under the action of the operator $\mathcal{U}_\mu$ on $L^2(\mathbb{R}^3_x \times \mathbb{R}^{3p}_y)$.

Since

$$\sup_{X,Y\in\mathbb{R}^3\setminus\{0\}} \Big| \frac{\omega(X)-\omega(Y)}{|X-Y|} \Big| \leq C, \ \ C > 0,$$

we have that

$$\frac{1}{|x+\mu\omega(x)|}, \ \ \frac{1}{|1 + \mu \frac{\omega(y_j)-\omega(x)}{|y_j - x|}|} \ \text{ and } \ \frac{1}{|1 + \mu \frac{\omega(y_j)-\omega(y_k)}{|y_j - y_k|}|}$$

are analytic in μ for $|\mu|$ small enough. So, Re $Q_\mu(x) > 0$, $Q_\mu(x)$ and $P_\mu(h)$ extend for smallenough complex values of μ to analytic families.

$\sigma(P_\mu(h)) = \sigma(P(h))$ for $\mu \in \mathbb{R}$, but, for nonreal μ, $\sigma_{ess}(P_\mu(h))$ is obtained from $\sigma_{ess}(P(h))$ by some rotation in the complex plane. By definition, the resonances of $P(h)$ are the discrete eigenvalues of $P_\mu(h)$ which are located between $\sigma_{ess}(P(h))$ and $\sigma_{ess}(P_\mu(h))$.

Definition 4.1. We say that a complex number ρ is a resonance of $P(h)$ if Re $\rho > \inf \sigma_{ess}(P(h))$ and there exists $\mu \in \mathbb{C}$ small enough, Im $\mu > 0$, such that $\rho \in \sigma_{disc}(P_\mu(h))$. We denote by

$$\Gamma(h) = \bigcup_{\text{Im } \mu>0, \ |\mu| \text{ small enough}} \sigma_{disc}(P_\mu(h))$$

the set of resonances of $P(h)$.

It is well known that when Im $\mu > 0$, the discrete spectrum of $P_\mu(h)$ satisfies $\sigma_{disc}(P_\mu(h)) \subset \{z \in \mathbb{C} : \text{ Im } z \leq 0\}$ (see [22]), we consider here just the resonances of $P(h)$ which are near the real axis.

Theorem 4.1 (Absence of resonances)**.** *We have put virial conditions on $\lambda_1(x)$ and $\lambda_3(x)$ such that the operators $-h^2\Delta_x + \lambda_1(x)$ and $-h^2\Delta_x + \lambda_3(x)$ do not admit resonance near the energy level E.*

Proof. Set $\widetilde{p}_{\mu,j} = \mathcal{U}_\mu \widetilde{p}_j \mathcal{U}_\mu^{-1}$ with $\widetilde{p}_j = -h^2\Delta_x + \lambda_j(x)$, $j \in \{1,3\}$, $\mathcal{U}_\mu$ acts only with respect to the variable x, and consider $\chi_1 \in C_0^\infty(\mathbb{R}^3)$, $\chi_2 \in C^\infty(\mathbb{R}^3)$ such that

$$\begin{gathered}\chi_1 = 1 \text{ on } \{|x| \leq R_1\},\ \ 0 < R < R_1,\\ \operatorname{Supp}\chi_2 \subset \{|x| > R_1\},\ \ \omega(x) = x \text{ on } |x| > R_1,\\ \{x \in \mathbb{R}^3 :\ \chi_1(x) = 1\} \cup \big\{x \in \mathbb{R}^3 :\ \chi_2(x) = 1\big\} = \mathbb{R}^3,\end{gathered}$$

where R is given in (4.1) such as the vector field $\omega = 0$ for $|x| \leq R$. $\lambda_1(x)$ and $\lambda_3(x)$ can be reindexed in such a way that they depend analytically on $x \neq 0$, and $\lambda_j(x + \mu\omega(x)) - \lambda_j(x) = \mathcal{O}(|\mu|)$ uniformly with respect to $x \in \mathbb{R}^3 \setminus \{0\}$ and $\mu \in \mathbb{C}$, $|\mu|$ small enough, $j \in \{1,3\}$ (see [17]).

For $u \in H^2(\mathbb{R}^3)$ and $j \in \{1,3\}$, we have

$$\begin{gathered}\big\langle \chi_1(\widetilde{p}_j - E)u, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)} = \big\langle (\widetilde{p}_j - E)\chi_1 u, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)} + \big\langle [\chi_1, \widetilde{p}_j]u, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)},\\ [\chi_1, \widetilde{p}_j]u = h^2[\Delta, \chi_1] = h^2(\Delta\chi_1) + 2h^2(\nabla\chi_1)\cdot\nabla,\end{gathered}$$

then

$$\begin{aligned}\big|\big\langle \chi_1(\widetilde{p}_j - E)u, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)}\big| &\geq h^2\|\nabla(\chi_1 u)\|^2_{L^2(\mathbb{R}^3)}\\ &\quad + C_{1,j}\|\chi_1 u\|^2_{L^2(\mathbb{R}^3)} - C_{1,j}\big(h^{3/2}\|\nabla u\|_{L^2(\mathbb{R}^3)} + h^{1/2}\|u\|_{L^2(\mathbb{R}^3)}\big)^2,\ \ C_{1,j} > 0.\end{aligned}$$

Furthermore,

$$\widetilde{p}_{\mu,j} - \widetilde{p}_j = \sum_{\alpha\in\mathbb{N}^3, |\alpha|\leq 2} |\mu| a_{\alpha,j}(x,\mu) h^{|\alpha|} D_x^\alpha$$

with $a_{\alpha,j} = \mathcal{O}(1)$, uniformly with respect to x and μ, so,

$$\big\langle \chi_1(\widetilde{p}_{\mu,j} - E)u, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)} = \big\langle \chi_1(\widetilde{p}_j - E)u, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)} + |\mu| \sum_{\alpha\in\mathbb{N}^3,\ |\alpha|\leq 2} \big\langle \chi_1 a_{\alpha,j}(x,\mu) h^{|\alpha|} D_x^\alpha, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)}$$

and

$$\begin{aligned}&\Big| \sum_{\alpha\in\mathbb{N}^3,\ |\alpha|\leq 2} \big\langle \chi_1 a_{\alpha,j}(x,\mu) h^{|\alpha|} D_x^\alpha, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)}\Big|\\ &\qquad = \mathcal{O}\Big(h^2\|\nabla(\chi_1 u)\|^2_{L^2(\mathbb{R}^3)} + \|\chi_1 u\|^2_{L^2(\mathbb{R}^3)} + h^4\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + h^2\|u\|^2_{L^2(\mathbb{R}^3)}\Big),\end{aligned}$$

$j \in \{1,3\}$. Thus for μ small enough,

$$\begin{aligned}\big|\big\langle \chi_1(\widetilde{p}_{\mu,j} - E)u, \chi_1 u\big\rangle_{L^2(\mathbb{R}^3)}\big| &\geq \frac{h^2}{2}\,\|\nabla(\chi_1 u)\|^2_{L^2(\mathbb{R}^3)}\\ &\quad + C'_{1,j}\|\chi_1 u\|^2_{L^2(\mathbb{R}^3)} - C'_{1,j}\big(h^{3/2}\|\nabla u\|_{L^2(\mathbb{R}^3)} + h^{1/2}\|u\|_{L^2(\mathbb{R}^3)}\big)^2,\ \ C'_{1,j} > 0,\ \ j \in \{1,3\}.\end{aligned}$$

On the other hand, from assumption (1.7) we have

$$\frac{\partial}{\partial\mu}\big((1+\mu)^2(\lambda_j((1+\mu)x) - E)\big)_{|\mu=0} = 2(\lambda_j(x) - E) + x\cdot\nabla\lambda_j(x) < -C_j < 0$$

and

$$\big|\operatorname{Im}\big((1+\mu)^2(\lambda_j((1+\mu)x) - E)\big)\big| \geq C_j|\operatorname{Im}\mu| + \mathcal{O}(|\mu|^2) \geq \frac{C_j}{2}\,|\operatorname{Im}\mu|,\ \ j \in \{1,3\}.$$

Since $\omega(x) = x$ on $\operatorname{Supp}\chi_2$, for $|\mu|$ small enough and $u \in H^2(\mathbb{R}^3)$, one has

$$\big|\operatorname{Im}\big\langle (1+\mu)^2(\widetilde{p}_{\mu,j} - E)\chi_2 u, \chi_2 u\big\rangle_{L^2(\mathbb{R}^3)}\big| \geq \frac{C_j}{2}\,|\operatorname{Im}\mu|\,\|\chi_2 u\|^2_{L^2(\mathbb{R}^3)},\ \ C_j > 0,\ \ j \in \{1,3\}.$$

Moreover,

$$\begin{aligned}&\mathrm{Re}\,\big\langle(1+\mu)^2(\widetilde{p}_{\mu,j}-E)\chi_2 u,\chi_2 u\big\rangle_{L^2(\mathbb{R}^3)}\\&\qquad=h^2\|\nabla(\chi_2 u)\|^2_{L^2(\mathbb{R}^3)}+\mathrm{Re}\,\big\langle(1+\mu)^2(\lambda_j((1+\mu)x)-E)\chi_2 u,\chi_2 u\big\rangle_{L^2(\mathbb{R}^3)}\end{aligned}$$

and

$$\begin{aligned}h^2\|\nabla(\chi_2 u)\|^2_{L^2(\mathbb{R}^3)}-D_j\|\chi_2 u\|^2_{L^2(\mathbb{R}^3)}&\le \mathrm{Re}\,\langle(1+\mu)^2(\widetilde{p}_{\mu,j}-E)\chi_2 u,\chi_2 u\rangle_{L^2(\mathbb{R}^3)}\\&\le h^2\|\nabla(\chi_2 u)\|^2_{L^2(\mathbb{R}^3)}+D_j\|\chi_2 u\|^2_{L^2(\mathbb{R}^3)},\quad D_j>0,\quad j\in\{1,3\}.\end{aligned}$$

Then,

$$\begin{aligned}&\big|\big\langle(1+\mu)^2(\widetilde{p}_{\mu,j}-E)\chi_2 u,\chi_2 u\big\rangle_{L^2(\mathbb{R}^3)}\big|^2\\&\qquad\ge\big(h^2\|\nabla(\chi_2 u)\|^2_{L^2(\mathbb{R}^3)}-D_j\|\chi_2 u\|^2_{L^2(\mathbb{R}^3)}\big)^2+\frac{C_j^2}{4}\,|\,\mathrm{Im}\ \mu|^2\|\chi_2 u\|^4_{L^2(\mathbb{R}^3)}\\&\qquad\ge\frac{1}{E_j}\,|\,\mathrm{Im}\ \mu|^2\big(h^2\|\nabla(\chi_2 u)\|^2_{L^2(\mathbb{R}^3)}+\|\chi_2 u\|^2_{L^2(\mathbb{R}^3)}\big),\quad E_j>0,\end{aligned}$$

and

$$\begin{aligned}\big|\big\langle\chi_2(\widetilde{p}_j-E)u,\chi_2 u\big\rangle_{L^2(\mathbb{R}^3)}\big|&\ge\frac{1}{F_j}\,|\,\mathrm{Im}\ \mu|\big(h^2\|\nabla(\chi_2 u)\|^2_{L^2(\mathbb{R}^3)}+\|\chi_2 u\|^2_{L^2(\mathbb{R}^3)}\big)\\&\quad-F_j\big(h^{3/2}\|\nabla u\|_{L^2(\mathbb{R}^3)}+h^{1/2}\|u\|_{L^2(\mathbb{R}^3)}\big)^2,\quad F_j>0,\quad j\in\{1,3\}.\end{aligned}$$

As a consequence, for $h>0$ and $\mu\in\mathbb{C}$ small enough, we obtain

$$\begin{aligned}&\big|\big\langle\chi_1(\widetilde{p}_{\mu,j}-E)u,\chi_1 u\big\rangle_{L^2(\mathbb{R}^3)}\big|+\big|\big\langle\chi_2(\widetilde{p}_{\mu,j}-E)u,\chi_2 u\big\rangle_{L^2(\mathbb{R}^3)}\big|\\&\quad\ge\frac{1}{G_j}\,|\,\mathrm{Im}\ \mu|\Big(\sum_{k=1}^{2}h^2\|\nabla(\chi_k u)\|^2_{L^2(\mathbb{R}^3)}+\|\chi_k u\|^2_{L^2(\mathbb{R}^3)}\Big)-G_j\big(h^{3/2}\|\nabla u\|_{L^2(\mathbb{R}^3)}+h^{1/2}\|u\|_{L^2(\mathbb{R}^3)}\big)^2\\&\quad\ge\frac{1}{H_j}\,.|\,\mathrm{Im}\ \mu|\Big(\sum_{k=1}^{2}h^2\|\chi_k\nabla u\|^2_{L^2(\mathbb{R}^3)}+\|\chi_k u\|^2_{L^2(\mathbb{R}^3)}\Big)-H_j\big(h^{3/2}\|\nabla u\|_{L^2(\mathbb{R}^3)}+h^{1/2}\|u\|_{L^2(\mathbb{R}^3)}\big)^2,\\&\qquad\qquad G_j,H_j>0,\quad j\in\{1,3\}.\end{aligned}$$

Since

$$\|\chi_1 u\|_{L^2(\mathbb{R}^3)}+\|\chi_2 u\|_{L^2(\mathbb{R}^3)}\ge\|u\|_{L^2(\mathbb{R}^3)},$$

we also have

$$\begin{aligned}&\big|\big\langle\chi_1(\widetilde{p}_{\mu,j}-E)u,\chi_1 u\big\rangle_{L^2(\mathbb{R}^3)}\big|+\big|\big\langle\chi_2(\widetilde{p}_{\mu,j}-E)u,\chi_2 u\big\rangle_{L^2(\mathbb{R}^3)}\big|\\&\qquad\ge\frac{1}{H_j}\,|\,\mathrm{Im}\ \mu|\big(h^2\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\|u\|^2_{L^2(\mathbb{R}^3)}\big)-H_j\big(h^{3/2}\|\nabla u\|_{L^2(\mathbb{R}^3)}+h^{1/2}\|u\|_{L^2(\mathbb{R}^3)}\big)^2\\&\qquad\ge\frac{1}{J_j}\,|\,\mathrm{Im}\ \mu|\big(h^2\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\|u\|^2_{L^2(\mathbb{R}^3)}\big),\quad J_j>0,\quad j\in\{1,3\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\chi_1(\widetilde{p}_{\mu,j}-E)u\|_{L^2(\mathbb{R}^3)}\|\chi_1 u\|_{L^2(\mathbb{R}^3)}+\|\chi_2(\widetilde{p}_{\mu,j}-E)u\|_{L^2(\mathbb{R}^3)}\|\chi_2 u\|_{L^2(\mathbb{R}^3)}\\&\qquad\qquad\ge\frac{1}{J_j}\,|\,\mathrm{Im}\ \mu|\big(h^2\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\|u\|^2_{L^2(\mathbb{R}^3)}\big),\end{aligned}$$

using the fact that

$$\|\chi_k u\|_{L^2(\mathbb{R}^3)}\le\|u\|_{L^2(\mathbb{R}^3)},\quad k\in\{1,2\},$$

for $h \ll |\operatorname{Im}\mu|$, we obtain

$$\|(\widetilde{p}_{\mu,j} - E)u\|_{L^2(\mathbb{R}^3)} \geq \frac{1}{J_j} |\operatorname{Im}\mu| \, \|u\|_{L^2(\mathbb{R}^3)}$$

and

$$\|(\widetilde{p}_{\mu,j} - z)u\|_{L^2(\mathbb{R}^3)} \geq \frac{1}{J_j} |\operatorname{Im}\mu| \, \|u\|_{L^2(\mathbb{R}^3)}$$

for $z \in \mathbb{C}$ such that $|z - E| \ll |\operatorname{Im}\mu|$, $j \in \{1, 3\}$.

This proves that $(\widetilde{p}_{\mu,j} - E)$ is invertible with bounded inverse satisfying

$$\|(\widetilde{p}_{\mu,j} - z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^3), H^2(\mathbb{R}^3))} \leq \frac{J_j}{|\operatorname{Im}\mu|}, \quad j \in \{1, 3\}. \qquad \square$$

5 Reduction of $P_\mu(h)$

It is shown in [17] that, in the Born-Oppenheimer approximation, the study of $P(h)$ can be reduced to the one of a finite matrix of regular semiclassical pseudodifferential operators on the base space $\mathbb{R}^3_x$. This reduction can be obtained, without modifying the distorded Hamiltonian near $\{0\}$, following ideas from [20].

For $x \neq 0$, we set

$$\widetilde{Q}_\mu(x) = Q_\mu(x) - \frac{a}{|x + \mu\omega(x)|}$$

and

$$\widetilde{Q}(x) = \widetilde{Q}_0(x) = -\Delta_y + \sum_{j=1}^{p} \Big(\frac{b_j^-}{|y_j - x|} + \frac{b_j^+}{|y_j + x|} \Big) + \sum_{j \neq k} \frac{c_{j,k}}{|y_j - y_k|}\, .$$

We need to recall some properties about the operators $\widetilde{Q}_\mu(x)$.

For $x \in \mathbb{R}^3$, let $\gamma(x)$ be a continuous family of simple loops of $\mathbb{C}$ enclosing $\{\widetilde{\lambda}_1(x), \widetilde{\lambda}_2(x), \widetilde{\lambda}_3(x)\}$ and having the rest of $\sigma(\widetilde{Q}(x))$ in its exterior. By the gap condition (1.1), we may assume that

$$\min_{x \in \mathbb{R}^3} dist(\sigma(\widetilde{Q}(x)), \;\; \gamma(x)) \geq \frac{\delta}{2} > 0.$$

Therefore, $\gamma(x)$ can be taken in a fixed compact set of $\mathbb{C}$ (see [17, Lemma 2.1]).

In particular, for all $x \in \mathbb{R}^3$ and $z \in \gamma(x)$,

$$(z - \widetilde{Q}(x))^{-1} \in \mathcal{B}(L^2(\mathbb{R}^{3p}), H^2(\mathbb{R}^{3p}))$$

and

$$\left\|(z - \widetilde{Q}(x))^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{R}^{3p}))} \leq \frac{2}{\delta}.$$

Proposition 5.1 ([17]).

(1) *$(\widetilde{Q}(x) + t)^{-1}$ exists for $t \in \mathbb{R}_+$ large enough independently of x and $(-\Delta + t)(\widetilde{Q}(x) + t)^{-1}$ is uniformly bounded.*

(2) *For any $j, k \in \{1, \dots, p\}$, $j \neq k$, $\alpha \in \mathbb{N}^{3p}$, $|\alpha| \leq 2$, the operators*

$$\frac{1}{|y_j \pm x|} (z - \widetilde{Q}(x))^{-1}, \;\; \frac{1}{|y_j - y_k|} (z - \widetilde{Q}(x))^{-1} \;\; \text{and} \;\; \partial^\alpha (z - \widetilde{Q}(x))^{-1}$$

are uniformly bounded on $L^2(\mathbb{R}^{3p})$ as $x \in \mathbb{R}^3$ and $z \in \gamma(x)$.

(3) *If $\mu \in \mathbb{C}$ is small enough, then for any $x \in \mathbb{R}^3$ and $z \in \gamma(x)$, the operator $(z - \widetilde{Q}_\mu(x))^{-1}$ exists and*

$$(z - \widetilde{Q}_\mu(x))^{-1} - (z - \widetilde{Q}(x))^{-1} = \mathcal{O}(|\mu|) \;\; \text{uniformly.}$$

Moreover, one can easily check that $\widetilde{\lambda}_1(x), \widetilde{\lambda}_2(x), \widetilde{\lambda}_3(x)$ depend on $|x|$ only, and can be reindexed in such a way that each of them depends analytically on $x \neq 0$ and $\widetilde{\lambda}_j(x + \mu\omega(x)) - \widetilde{\lambda}_j(x) = \mathcal{O}(|\mu|)$ uniformly with respect to x and $\mu \in \mathbb{C}$, $|\mu|$ small enough, $j \in \{1, 2, 3\}$.

We can now define, for μ complex small enough, the spectral projectors associated to $\widetilde{Q}_\mu(x)$ and the loops $\gamma(x)$,

$$\Pi_\mu(x) = \frac{1}{2\pi i} \oint_{\gamma(x)} (z - \widetilde{Q}_\mu(x))^{-1}\, dz.$$

$\Pi_\mu(x)$ is of rank 3, it helps us to construct the Grushin problem associated to $P_\mu(h)$. Furthermore, under the previous assumptions, we can use the constructions made in [17] and obtain an orthonormal basis $\{v_{1,\mu}(x), v_{2,\mu}(x), v_{3,\mu}(x)\}$ of $\mathrm{Im}\ \Pi_\mu(x)$, depending analytically on μ small enough, and normalized in $L^2(\mathbb{R}^{3p}_y)$ by

$$\langle v_{k,\mu}(x), v_{l,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)} = \delta_{k,l}.$$

So,

$$\Pi_\mu(x)u = \sum_{k=1}^{3} \langle u, v_{k,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)} v_{k,\mu}(x), \quad u \in L^2(\mathbb{R}^3 \times \mathbb{R}^{3p}), \quad x \in \mathbb{R}^3.$$

Using Proposition 5.1, and (1.1), one can easily prove that

$$\mathrm{Re}\left\langle \widehat{\Pi}_\mu(x)(P_\mu(h) - z)\widehat{\Pi}_\mu(x)u, \widehat{\Pi}_\mu(x)u\right\rangle \geq \frac{1}{C}\, \|\widehat{\Pi}_\mu(x)u\|^2,$$

where $\widehat{\Pi}_\mu(x) = 1 - \Pi_\mu(x)$. Thus the operator $\widehat{\Pi}_\mu(x)(P_\mu(h) - z)$ is invertible on $\{u \in H^2(\mathbb{R}^3 \times \mathbb{R}^{3p}) : \widehat{\Pi}_\mu(x)u = u\}$ and its inverse is bounded, denoted by $(P'_\mu(h) - z)^{-1}$ for $z \in \mathbb{C}$ close enough to E.

Observe that we can apply Theorem 2.1 of [20] with $-h^2\mathcal{U}_\mu\Delta_x\mathcal{U}_\mu^{-1}$, $Q_\mu(x)$,

$$P_\mu(h) = -h^2\mathcal{U}_\mu\Delta_x\mathcal{U}_\mu^{-1} + Q_\mu(x),$$

the two open subsets $W_0 = \{|x| < 2\delta_0\}$ and $W_1 = \{|x| > \delta_0\}$ and $\Pi_\mu(x)$ with $0 < \delta \leq r_1 < r_0 < \delta_0$,

$$W_0 \cup W_1 = \mathbb{R}^3,$$
$$\mathrm{Re}\ Q_\mu(x) \geq E + \delta_0, \quad \mu \text{ small enough.}$$

Let $\varphi_0, \varphi_1, \psi_0, \psi_1 \in C^\infty(\mathbb{R}^3, [0, 1])$ be cut-off functions such that

$$\mathrm{Supp}\,\varphi_j \cup \mathrm{Supp}\,\psi_j \subset W_j, \quad j \in \{0, 1\},$$
$$\varphi_0^2 + \varphi_1^2 = 1 \ \text{ on } \mathbb{R}^3,$$
$$\psi_j = 1 \ \text{ on } \mathrm{Supp}\,\varphi_j,\ j \in \{0, 1\}.$$

By construction, one has

$$[Q_\mu(x), \Pi_\mu(x)] = 0 \ \text{ everywhere,} \tag{5.1}$$
$$\mathrm{Re}\ P_{\mu,0}(h) \geq E + \delta_0, \ \text{ with } \ P_{\mu,0}(h) = P_\mu(h) + (E + \delta_0 + C)(1 - \psi_0(x)),$$
$$\mathrm{Re}\ \widehat{\Pi}_\mu(x)\Big(P_{\mu,1}(h) - E - \frac{r_1}{4}\Big)\widehat{\Pi}_\mu(x) \geq 0$$

with

$$P_{\mu,1}(h) = -h^2\mathcal{U}_\mu\Delta_x\mathcal{U}_\mu^{-1} + Q_\mu(x)\psi_1(x) + (E + \delta_0)(1 - \psi_1(x)).$$

Thus, for z in a small enough complex neighborhood of J, both $P_{\mu,0}(h) - z$ and the restriction of $\widehat{\Pi}_\mu(x)P_{\mu,1}(h)\widehat{\Pi}_\mu(x) - z$ to the range of $\widehat{\Pi}_\mu(x)$ are invertible, with bounded inverse.

Our first main result is the following

Theorem 5.1. *For $h > 0$ small enough and z in a small enough complex neighborhood of $J =]-\infty, E]$, we have $z \in \Gamma(h)$ if and only if there exists $\mu \in \mathbb{C}$ small enough,* $\mathrm{Im}\ \mu > 0$, *such that $0 \in \sigma_{disc}(E_\mu^{-+}(z))$, where*

$$E_\mu^{-+}(z) = \Pi_\mu(x)(z - P_\mu(h))\Pi_\mu(x) + \mathcal{O}(h^2).$$

Proof. We consider the Grushin problems that will lead to the Feshbach reduction. For $z \in \mathbb{C}$ near J, define the operators

$$\mathcal{P}_\mu(z) = \begin{pmatrix} P_\mu(h) - z & I \\ \widetilde{\Pi}_\mu(x) & 0 \end{pmatrix} : H^2(\mathbb{R}^3_x \times \mathbb{R}^{3p}_y) \oplus \operatorname{Im} \Pi_\mu(x) \to L^2(\mathbb{R}^3_x \times \mathbb{R}^{3p}_y) \oplus \operatorname{Im} \Pi_\mu(x)$$

and

$$\mathcal{P}_{\mu,j}(z) = \begin{pmatrix} P_{\mu,j}(h) - z & I \\ \widetilde{\Pi}_\mu(x) & 0 \end{pmatrix}, \quad j \in \{0,1\}. \tag{5.2}$$

Thanks to (5.1), the operator $\mathcal{P}_{\mu,j}(z)$ is invertible and its inverse is given by

$$\mathcal{P}^{-1}_{\mu,j}(z) = \begin{pmatrix} E_{\mu,j}(z) & E^+_{\mu,j}(z) \\ E^-_{\mu,j}(z) & E^{-+}_{\mu,j}(z) \end{pmatrix}, \quad j \in \{0,1\}$$

where

$$\begin{aligned} E_{\mu,0}(z) &= \widehat{\Pi}_\mu(x)(P_{\mu,0}(h) - z)^{-1}\widehat{\Pi}_\mu(x), \\ E_{\mu,1}(z) &= \widehat{\Pi}_\mu(x)\big(\widehat{\Pi}_\mu(x)(P_{\mu,1}(h) - z)\widehat{\Pi}_\mu(x)\big)^{-1}\widehat{\Pi}_\mu(x), \\ E^+_{\mu,j}(z) &= (1 - E_{\mu,j}(z)M_{\mu,j}), \quad E^-_{\mu,j}(z) = \Pi_\mu(x)(1 + M_{\mu,j}E_{\mu,j}(z)), \\ E^{-+}_{\mu,j}(z) &= \Pi_\mu(x)\big(z - P_{\mu,j}(h) - M_{\mu,j}E_{\mu,j}(z)M_{\mu,j}\big), \\ M_{\mu,j} &= \big[P_{\mu,j}(h), \Pi_\mu(x)\big]. \end{aligned}$$

Let

$$\mathcal{F}_\mu(z) = \varphi_0 \mathcal{P}^{-1}_{\mu,0}(h)\varphi_0 + \varphi_1 \mathcal{P}^{-1}_{\mu,1}(h)\varphi_1.$$

Then

$$\mathcal{F}_\mu(z) = \begin{pmatrix} G_\mu(z) & I - Y_{\mu,1}(z) \\ \Pi_\mu(x)(I + Y'_{\mu,1}(z)) & \Pi_\mu(x)(z - P_\mu(h) - Y_{\mu,2}(z)) \end{pmatrix}$$

with

$$\begin{aligned} G_\mu(z) &= \varphi_0 E_{\mu,0}(z)\varphi_0 + \varphi_1 E_{\mu,1}(z)\varphi_1, \\ Y_{\mu,1}(z) &= \varphi_0 E_{\mu,0}(z)M_{\mu,0}\varphi_0 + \varphi_1 E_{\mu,1}(z)M_{\mu,1}\varphi_1, \\ Y'_{\mu,1}(z) &= \varphi_0 M_{\mu,0}E_{\mu,0}(z)\varphi_0 + \varphi_1 M_{\mu,1}E_{\mu,1}(z)\varphi_1, \\ Y_{\mu,2}(z) &= \varphi_0 M_{\mu,0}E_{\mu,0}(z)M_{\mu,0}\varphi_0 + \varphi_1 M_{\mu,1}E_{\mu,1}(z)M_{\mu,1}\varphi_1. \end{aligned}$$

So,

$$\begin{aligned} \mathcal{F}_\mu(z)\mathcal{P}_\mu(z) &= \begin{pmatrix} I + Y_\mu(z) & 0 \\ G_{\mu,1}(z) & I \end{pmatrix}, \\ T_{\mu,j} &= \big[-h^2\mathcal{U}_\mu\Delta_x\mathcal{U}_\mu^{-1}, \varphi_j\big], \quad j \in \{0,1\}, \\ Y_\mu(z) &= \varphi_0 E_{\mu,0}(z)T_{\mu,0} + \varphi_1 E_{\mu,1}(z)T_{\mu,1}, \quad Y'_\mu(z) = T_{\mu,0}E_{\mu,0}(z)\varphi_0 + T_{\mu,1}E_{\mu,1}(z)\varphi_1, \\ G_{\mu,1}(z) &= -\Pi_\mu(x)M_{\mu,0} - Y_{\mu,3}(z) + \Pi_\mu(x)Y_{\mu,4}, \\ Y_{\mu,3}(z) &= \varphi_0 M_{\mu,0}E_{\mu,0}(z)T_{\mu,0} + \varphi_1 M_{\mu,1}E_{\mu,1}(z)T_{\mu,1}, \\ Y_{\mu,4} &= \varphi_0 M_{\mu,0}\varphi_0 + \varphi_1 M_{\mu,1}\varphi_1, \end{aligned}$$

since

$$\varphi_0\big[P_\mu(h), \Pi_\mu(x)\big] = \varphi_0 M_{\mu,0}, \quad \varphi_1\big[P_\mu(h), \Pi_\mu(x)\big] = \varphi_1 M_{\mu,1},$$
$$G_\mu(z)\Pi_\mu(x) = 0, \quad Y'_{\mu,1}(z)\Pi_\mu(x) = 0 \ \text{ and } \ Y_{\mu,4}\widehat{\Pi}_\mu(x) = \Pi_\mu(x)Y_{\mu,4}.$$

Note that $Y_\mu(z)$ and $Y'_\mu(z)$ are the bounded operators and they will actually be very small,

$$\|Y_\mu(z)\|_{L^2(\mathbb{R}^{3p}_y)} < 1 \text{ and } \|Y'_\mu(z)\|_{L^2(\mathbb{R}^{3p}_y)} < 1,$$

then for z in a small enough complex neighborhood of J and $\mu \in \mathbb{C}$ sufficiently small, $I + Y_\mu(z)$ and $I + Y'_\mu(z)$ are the boundedly invertible operators on $L^2(\mathbb{R}^{3p}_y)$.

Consequently,

$$\begin{aligned}\begin{pmatrix} I + Y_\mu(z) & 0 \\ G_{\mu,1}(z) & I \end{pmatrix}^{-1} \mathcal{F}_\mu(z) &= \begin{pmatrix} (I + Y_\mu(z))^{-1} & 0 \\ -G_{\mu,1}(z)(I + Y_\mu(z))^{-1} & I \end{pmatrix} \mathcal{F}_\mu(z) \\ &= \begin{pmatrix} (I + Y_\mu(z))^{-1} G_\mu(z) & (I + Y_\mu(z))^{-1}(I - Y_{\mu,1}(z)) \\ \left\{ \begin{matrix} -G_{\mu,1}(z)(I + Y_\mu(z))^{-1} G_\mu(z) \\ +\Pi_\mu(x)(1 + Y'_{\mu,1}(z)) \end{matrix} \right\} & \left\{ \begin{matrix} -G_{\mu,1}(z)(I + Y_\mu(z))^{-1}(I - Y_{\mu,1}(z)) \\ +\Pi_\mu(x)(z - P_\mu(h) - Y_{\mu,2}(z)) \end{matrix} \right\} \end{pmatrix}\end{aligned}$$

is a left-inverse for $\mathcal{P}_\mu(z)$.

We also have

$$\begin{gathered}\mathcal{P}_\mu(z)\mathcal{F}_\mu(z) = \begin{pmatrix} I + Y'_\mu(z) & G_{\mu,2}(z) \\ 0 & I \end{pmatrix}, \\ G_{\mu,2}(z) = M_{\mu,0} - T_{\mu,0}E_{\mu,0}(z)M_{\mu,0}\varphi_0 - T_{\mu,1}E_{\mu,1}(z)M_{\mu,1}\varphi_1 - Y_{\mu,4}\Pi_\mu(x), \\ \mathcal{F}_\mu(z) \begin{pmatrix} I + Y'_\mu(z) & G_{\mu,2}(z) \\ 0 & I \end{pmatrix}^{-1} = \mathcal{F}_\mu(z) \begin{pmatrix} (I + Y'_\mu(z))^{-1} & -(I + Y'_\mu(z))^{-1} G_{\mu,2}(z) \\ 0 & I \end{pmatrix}\end{gathered}$$

is a right-inverse for $\mathcal{P}_\mu(z)$.

Thus $\mathcal{P}_\mu(z)$ is invertible with the inverse given by

$$\begin{aligned}\mathcal{P}^{-1}_\mu(z) &= \begin{pmatrix} E_\mu(z) & E^+_\mu(z) \\ E^-_\mu(z) & E^{-+}_\mu(z) \end{pmatrix}, \\ E_\mu(z) &= (I + Y_\mu(z))^{-1} G_\mu(z), \\ E^+_\mu(z) &= (I + Y_\mu(z))^{-1}(I - Y_{\mu,1}(z)), \\ E^-_\mu(z) &= \Pi_\mu(x)(1 + Y'_{\mu,1}(z)) - G_{\mu,1}(z)(I + Y_\mu(z))^{-1} G_\mu(z), \\ E^{-+}_\mu(z) &= \Pi_\mu(x)(z - P_\mu(h) - Y_{\mu,2}(z)) - G_{\mu,1}(z)(I + Y_\mu(z))^{-1}(I - Y_{\mu,1}(z)).\end{aligned} \tag{5.3}$$

Then

$$\begin{aligned}E^{-+}_\mu(z) &= \Pi_\mu(x)(z - P_\mu(h))\Pi_\mu(x) + A_\mu(z), \\ A_\mu(z) &= -\Pi_\mu(x)Y_{\mu,2}(z) - G_{\mu,1}(z)(I + Y_\mu(z))^{-1}(I - Y_{\mu,1}(z)) \\ &= -\Pi_\mu(x)Y_{\mu,2}(z) - \big[-\Pi_\mu(x)M_{\mu,0} - Y_{\mu,3}(z) + \Pi_\mu(x)Y_{\mu,4}\big](I + Y_\mu(z))^{-1}(I - Y_{\mu,1}(z)) \\ &= \Pi_\mu(x)\big(-Y_{\mu,2}(z) + (M_{\mu,0} + Y_{\mu,3}(z) - Y_{\mu,4})(I + Y_\mu(z))^{-1}(1 - Y_{\mu,1}(z))\big)\Pi_\mu(x).\end{aligned}$$

If $H^s(\mathbb{R}^3, \cdot)$, $s \in \mathbb{R}$, is equipped with the semiclassical norm

$$\|u\|_{H^s} = \left\| h^{-3/2}(1 + |\xi|^2)^{s/2}\widehat{u}\Big(\frac{\xi}{h}\Big)\right\|_{L^2},$$

it is clear that

$$E_{\mu,j}(z) = \mathcal{O}(1) : H^{-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)) \to H^1(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)), \ \ j \in \{0, 1\} \tag{5.4}$$

and thus

$$Y_\mu(z); Y'_\mu(z) = \mathcal{O}(h) : H^{-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)) \to H^1(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)).$$

Moreover, since $-h^2\mathcal{U}_\mu\Delta_x\mathcal{U}_\mu^{-1}$ is a differential operator of degree 2 with respect to x and $Q_\mu(x)\Pi_\mu(x) = \Pi_\mu(x)Q_\mu(x)$ everywhere, one has

$$\begin{aligned} M_{\mu,j} &= \mathcal{O}(h) : L^2(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)) \to H^{-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)), \\ M_{\mu,j} &= \mathcal{O}(h) : H^1(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)) \to L^2(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)), \quad j \in \{0,1\}, \end{aligned} \tag{5.5}$$

we also have

$$Y_{\mu,j}(z) = \mathcal{O}(h^2), \quad j \in \{2,3\} \text{ uniformly.}$$

Finally, using the fact that

$$\Pi_\mu(x)M_{\mu,0}\Pi_\mu(x) = 0 \text{ and } \Pi_\mu(x)Y_{\mu,4}\Pi_\mu(x) = 0,$$

we deduce that $A_\mu(z) = \mathcal{O}(h^2)$ uniformly.

The spectral reduction of $P_\mu(h)$ comes from the following two series of algebraic identities:

$$\begin{aligned} &(P_\mu(h) - z)u = v \iff \mathcal{P}_\mu(z)(u \oplus 0) = (v \oplus \Pi_\mu(x)u) \\ &\qquad \iff \mathcal{P}_\mu^{-1}(z)(v \oplus \Pi_\mu(x)u) = (u \oplus 0) \iff \begin{cases} E_\mu(z)v + E_\mu^+(z)\Pi_\mu(x)u = u, \\ E_\mu^-(z)v + E_\mu^{-+}(z)\Pi_\mu(x)u = 0 \end{cases} \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} &E_\mu^{-+}(z)f = g \iff \mathcal{P}_\mu^{-1}(z)(0 \oplus f) = (E_\mu^+(z)f \oplus g) \\ &\qquad \iff \mathcal{P}_\mu(z)(E_\mu^+(z)f \oplus g) = (0 \oplus f) \iff \begin{cases} (P_\mu(h) - z)E_\mu^+(z)f + g = 0, \\ \Pi_\mu(x)E_\mu^+(z)f = f. \end{cases} \end{aligned} \tag{5.7}$$

If $z \notin \sigma(P_\mu(h))$, from (5.7) we obtain the following equivalence:

$$E_\mu^{-+}(z)f = g \iff f = -\Pi_\mu(x)(P_\mu(h) - z)^{-1}g,$$

thus $0 \notin \sigma(E_\mu^{-+}(z))$ and

$$E_\mu^{-+}(z)^{-1} = -\Pi_\mu(x)(P_\mu(h) - z)^{-1}.$$

Conversely, if $0 \notin \sigma(E_\mu^{-+}(z))$, then (5.6) gives the following equivalence:

$$(P_\mu(h) - z)u = v \iff \begin{cases} \Pi_\mu(x)u = -E_\mu^{-+}(z)^{-1}E_\mu^-(z)v, \\ u = E_\mu(z)v - E_\mu^+(z)E_\mu^{-+}(z)^{-1}E_\mu^-(z)v. \end{cases}$$

Therefore, $z \notin \sigma(P_\mu(h))$ and

$$(P_\mu(h) - z)^{-1} = E_\mu(z) - E_\mu^+(z)E_\mu^{-+}(z)^{-1}E_\mu^-(z). \qquad \square$$

6 A smooth reduction of $P_\mu(h)$

It is shown in [17] that, modulo change of variables, the use of the Feshbach method in the Coulombian case is still possible and one can reduce the problem to a finite matrix of regular pseudodifferential operators. The main idea is to consider x-dependent changes in the y-variables that will localize the singularities, regularize $Q_\mu(x)$ and permit an adaptable semiclassical pseudodifferential calculus with operator-valued symbols. The constructions made in [17, Proposition 5.1] show that there are three functions $v_{1,\mu}(x), v_{2,\mu}(x), v_{3,\mu}(x)$ in $C^0(\mathbb{R}^3, H^2(\mathbb{R}^{3p}))$ depending analytically on $\mu \in \mathbb{C}$ small enough such that $\langle v_{k,\mu}(x), v_{l,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)} = \delta_{k,l}$, $\{v_{1,\mu}(x), v_{2,\mu}(x), v_{3,\mu}(x)\}$ is a basis of $\operatorname{Im} \Pi_\mu(x)$ if $x \in W_1$, $v_{k,\mu}(x) \in C^\infty(W_0, H^2(\mathbb{R}^{3p}))$, $Q_\mu(x)v_{k,\mu}(x) = \lambda_k(x + \mu\omega(x))v_{k,\mu}(x)$, $k \in \{1,2,3\}$ for $|x|$ large enough, and for $j \in \{0,\dots,L\}$, $k \in \{1,2,3\}$, $U_j(x)v_{k,\mu}(x) \in C_b^\infty(\Omega_j, H^2(\mathbb{R}^{3p}))$, where $(\Omega_j)_{0\le j\le L}$ is a finite family of open subsets in $\mathbb{R}^3$, $\Omega_0 \subset \{\psi_1 = 0\}$, $\bigcup_{j=0}^{L} \Omega_j = \mathbb{R}^3$, and $(U_j)_{0\leq j\leq L}$ is a family of unitary operators defined on $L^2(\Omega_j, H^2(\mathbb{R}^{3p}))$ with $U_0 = I$, $U_j(-h^2\Delta_x)U_j^{-1}$ is a semiclassical differential operator with operator-valued symbol and $U_jQ_\mu(x)\psi_1U_j^{-1}$, $U_j(-\Delta_y+1)U_j^{-1} \in C^\infty(\Omega_j, \mathcal{B}(H^2(\mathbb{R}^{3p}), L^2(\mathbb{R}^{3p})))$. C_b^∞ denotes the space of C^∞ functions whose derivatives of any order are uniformly bounded.

Let $\widetilde{\Pi}_\mu(x)$ be defined on $L^2(\mathbb{R}^3 \times \mathbb{R}^{3p})$ by

$$\widetilde{\Pi}_\mu(x)u = \sum_{j=1}^{3} \langle u, v_{j,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)} v_{j,\mu}(x), \quad x \in \mathbb{R}^3.$$

Thus $\widetilde{\Pi}_\mu(x) = \Pi_\mu(x)$ on W_1 and Re $Q_\mu(x) \geq E + \delta_0$ for $\mu \in \mathbb{C}$ small enough and $x \in W_0 \setminus \{0\}$, $\delta_0 \geq \delta_1$ (δ_1 is chosen small enough in the definition of W_0). So, the operator $\widehat{\Pi}_\mu(x)(P_\mu(h)-z)\widehat{\Pi}_\mu(x)$ is invertible with $\widehat{\Pi}_\mu(x) = 1 - \widetilde{\Pi}_\mu(x)$.

Consider now the following operators:

$$R_\mu^- : \bigoplus_1^3 L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3 \times \mathbb{R}^{3p}),$$
$$u^- = (u_1^-, u_2^-, u_3^-) \mapsto R_\mu^- u^- = \sum_{k=1}^{3} u_k^- v_{k,\mu}(x),$$

and

$$R_\mu^+ = (R_\mu^-)^*,$$
$$R_\mu^+ : L^2(\mathbb{R}^3 \times \mathbb{R}^{3p}) \to \bigoplus_1^3 L^2(\mathbb{R}^3),$$
$$u \mapsto R_\mu^+ u = \Big(\langle u, v_{1,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)}, \langle u, v_{2,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)}, \langle u, v_{3,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)}\Big).$$

We immediately observe that

$$\begin{cases} R_\mu^+\widetilde{\Pi}_\mu(x) = \widetilde{\Pi}_\mu(x)R_\mu^+ = R_\mu^+, \\ R_\mu^+R_\mu^- = I, \\ R_\mu^-R_\mu^+ = \widetilde{\Pi}_\mu(x). \end{cases}$$

So, R_μ^+ is an isomorphism from Im $\widetilde{\Pi}_\mu(x)$ to $\bigoplus_1^3 L^2(\mathbb{R}^3)$ with inverse R_μ^-, and R_μ^+ sends $H^2(\mathbb{R}^3 \times \mathbb{R}^{3p})$ into $\bigoplus_1^3 H^2(\mathbb{R}^3)$.

In this case, the Grushin operator $\mathcal{P}_\mu(z)$ defined in (5.2) can be expressed as

$$\mathcal{P}_\mu(z) = \begin{pmatrix} P_\mu(h)-z & R_\mu^+R_\mu^- \\ R_\mu^-R_\mu^+ & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & R_\mu^+ \end{pmatrix} \begin{pmatrix} P_\mu(h)-z & R_\mu^- \\ R_\mu^+ & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R_\mu^- \end{pmatrix}.$$

Since $\begin{pmatrix} I & 0 \\ 0 & R_\mu^+ \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ 0 & R_\mu^- \end{pmatrix}$ are invertible operators, we deduce that the study of $\mathcal{P}_\mu(z)$ is equivalent to that of $\widetilde{\mathcal{P}}_\mu(z) = \begin{pmatrix} P_\mu(h)-z & R_\mu^- \\ R_\mu^+ & 0 \end{pmatrix}$ from $H^2(\mathbb{R}^3 \times \mathbb{R}^{3p}) \oplus (\bigoplus_1^3 L^2(\mathbb{R}^3))$ to $L^2(\mathbb{R}^3 \times \mathbb{R}^{3p}) \oplus (\bigoplus_1^3 H^2(\mathbb{R}^3))$, for z in a small enough complex neighborhood of J and $\mu \in \mathbb{C}$ small enough.

As a direct consequence of Theorem 5.1, we deduce that, for h small enough, $\widetilde{\mathcal{P}}_\mu(z)$ is invertible and its inverse is given by (5.3), furthermore, the spectral study of the operator $\widetilde{\mathcal{P}}_\mu(z)$ is reduced to that of 3×3-matrices of operators $\widetilde{F}_\mu(z) = z - \widetilde{E}_\mu^{-+}(z)$ acting on the variable x such that

$$z \in \Gamma(h) \Longleftrightarrow \exists\, \mu \in \mathbb{C} \text{ small enough, } \operatorname{Im} \mu > 0 \text{ and } z \in \sigma_{disc}(\widetilde{F}_\mu(z))$$

with

$$\widetilde{E}_\mu^{-+}(z) = R_\mu^+\big(z - P_\mu(h) + \widetilde{A}_\mu(z)\big)R_\mu^- : \bigoplus_1^3 H^2(\mathbb{R}^3) \to \bigoplus_1^3 L^2(\mathbb{R}^3),$$
$$\widetilde{A}_\mu(z) = -Y_{\mu,2}(z) + (M_{\mu,0} + Y_{\mu,3}(z) - Y_{\mu,4})(I + Y_\mu(z))^{-1}(1 - Y_{\mu,1}(z)),$$
$$\widetilde{F}_\mu(z) = z - \widetilde{E}_\mu^{-+}(z) = R_\mu^+(P_\mu(h) - \widetilde{A}_\mu(z))R_\mu^- : \bigoplus_1^3 H^2(\mathbb{R}^3) \to \bigoplus_1^3 L^2(\mathbb{R}^3).$$

6.1 Agmon-type estimates for $P_\mu(h)$

Now, it remains to prove that $\widetilde{F}_\mu(z)$ becomes a family of 3×3-matrices of pseudodifferential operators on $\mathbb{R}^3$, analytic with respect to $\mu \in \mathbb{C}$ small enough. We use the main idea of [20], considering the effective Hamiltonian $\widetilde{F}_\mu(z)$ as the sum of a semiclassical pseudodifferential operator and a semibounded operator localized near the origin.

We have just established that the study of the operator $P_\mu(h)$ is reduced to that of the Feshbach operator $\widetilde{F}_\mu(z)$. Using (5.1), we have

$$P_\mu(h) = P_{\mu,1}(h) + (Q_\mu(x) - E - \delta_0)(1 - \psi_1(x))\psi_0(x),$$
$$\widetilde{F}_\mu(z) = R_\mu^+ P_{\mu,1}(h)R_\mu^- + R_\mu^+\Big[(Q_\mu(x) - E - \delta_0)(1 - \psi_1(x))\psi_0(x) - \widetilde{A}_\mu(z)\Big]R_\mu^-.$$

In particular, $R_\mu^+(P_{\mu,1}(h))R_\mu^-$ is a matrix of smooth pseudodifferential operators on $\mathbb{R}^3$ depending analytically on μ, since $P_{\mu,1}(h)$ is a twisted pseudodifferential operator associated to the family $(\Omega_j, U_j)_{0\le j\le L}$ (see [15] and [19]). Moreover, its symbol is a second-order polynomial with respect to ξ, and its principal symbol is of the form

$$\big((I + \mu^t d\omega(x))^{-1}\xi\big)^2 I_3 + \mathcal{M}_\mu(x),$$

where $\mathcal{M}_\mu(x)$ is the matrix

$$\mathcal{M}_\mu(x) = R_\mu^+\big[Q_\mu(x)\psi_1(x) + (E + \delta_0)(1 - \psi_1(x))\big]R_\mu^-.$$

If $x \in \mathbb{R}^3 \setminus W_0$, $\psi_1(x) = 1$ and the eigenvalues of $\mathcal{M}_\mu(x)$ are those of $Q_\mu(x)\widetilde{\Pi}_\mu(x)$, so, these are $\lambda_1(x + \mu\omega(x))$, $\lambda_2(x + \mu\omega(x))$ and $\lambda_3(x + \mu\omega(x))$.

If $x \in W_0$, Re $Q_\mu(x) \ge E + \delta_0$ and Re $\mathcal{M}_\mu(x) \ge E + \delta_0$ for μ complex small enough.

Now let $\varphi \in C_b^\infty(\mathbb{R}^3, \mathbb{R})$ satisfying

$$|\nabla\varphi(x)|^2 \le \theta(x, z),$$
$$\theta(x, z) = \min\Big\{E + \delta_0 - \operatorname{Re} z; \inf\sigma\big(\operatorname{Re} \widehat{\Pi}_\mu(x)(Q_\mu(x) - z)\widehat{\Pi}_\mu(x)\big)\Big\} - \frac{\delta_1}{2}.$$

Since $R_\mu^+(Q_\mu(x) - E - \delta_0)(1 - \psi_1(x))\psi_0(x)R_\mu^-$, R_μ^+ and R_μ^- commute with $e^{\varphi(x)/h}$, we have

$$e^{\varphi(x)/h}\widetilde{F}_\mu(z)e^{-\varphi(x)/h} = R_\mu^+ e^{\varphi(x)/h}P_{\mu,1}(h)e^{-\varphi(x)/h}R_\mu^- + R_\mu^+(Q_\mu(x) - E - \delta_0)(1 - \psi_1(x))R_\mu^-\psi_0(x) - e^{\varphi(x)/h}\widetilde{A}_\mu(z)e^{-\varphi(x)/h}R_\mu^-,$$

$e^{\varphi(x)/h}R_\mu^+ P_{\mu,1}(h)R_\mu^- e^{-\varphi(x)/h}$ is a pseudodifferential operator on $\mathbb{R}^3$ with the principal symbol

$$\big((I + \mu^t\, d\omega(x))^{-1}(\xi + i\nabla\varphi)\big)^2 + \mathcal{M}_\mu(x).$$

Furthermore, Lemma 4.3 of [20] asserts that for $j \in \{0, 1\}$ and $\mu \in \mathbb{C}$, $|\mu|$ small enough,

$$\operatorname{Re}\, e^{\varphi(x)/h}E_{\mu,j}(z)e^{-\varphi(x)/h} \ge 0,$$

$$e^{\varphi(x)/h}E_{\mu,j}(z)e^{-\varphi(x)/h} = \mathcal{O}(1) : L^2(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)) \to H^{\mathbf{2}}(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)) \text{ uniformly as } h \to 0_+,$$

$E_{\mu,1}(z)$ is a pseudodifferential operator.

Let

$$X_{*,\varphi}(z) = e^{\varphi(x)/h}X_*(z)e^{-\varphi(x)/h}$$
$$\text{if } X_* \in \{E_{\mu,j}, M_{\mu,j}, T_{\mu,j} :\ j=0,1\} \cup \{Y_{\mu,j} :\ 0 \le j \le 4\} \cup \{Y_\mu, \widetilde{A}_\mu, \widetilde{F}_\mu\}.$$

Then from (5.4) and (5.5) we have

$$\begin{aligned}\widetilde{A}_{\mu,\varphi}(z) &= -Y_{\mu,2,\varphi}(z) + (M_{\mu,0,\varphi} - Y_{\mu,4,\varphi} + Y_{\mu,3,\varphi}(z))(I + Y_{\mu,\varphi}(z))^{-1}(1 - Y_{\mu,1,\varphi}(z))\\ &= -Y_{\mu,2,\varphi}(z) + \Big(M_{\mu,0,\varphi} - Y_{\mu,4,\varphi} - \big[M_{\mu,0,\varphi}Y_{\mu,\varphi}(z) + Y_{\mu,4,\varphi}\big]\\ &\qquad \times \sum_{j=0}^{\infty}(-Y_{\mu,\varphi}(z))^j + Y_{\mu,3,\varphi}(z)\Big)(1 - Y_{\mu,1,\varphi}(z)),\end{aligned}$$

where

$$\begin{aligned}&\big[M_{\mu,0,\varphi}Y_{\mu,\varphi}(z) + Y_{\mu,4,\varphi}\big](I + Y_{\mu,\varphi}(z))^{-1} + Y_{\mu,3,\varphi}(z)\\ &\qquad = M_{\mu,0,\varphi}e^{\varphi(x)/h}\big[\varphi_0 E_{\mu,0}(z)T_{\mu,0} + \varphi_1 E_{\mu,1}(z)T_{\mu,1}\big]e^{-\varphi(x)/h}\\ &\qquad\qquad + e^{\varphi(x)/h}\big[\varphi_0 M_{\mu,0}\varphi_0 + \varphi_1 M_{\mu,1}\varphi_1\big]e^{-\varphi(x)/h}\\ &\qquad = M_{\mu,0,\varphi}e^{\varphi(x)/h}\varphi_0 E_{\mu,0}(z)T_{\mu,0}e^{-\varphi(x)/h} + M_{\mu,0,\varphi}e^{\varphi(x)/h}\varphi_1 E_{\mu,1}(z)T_{\mu,1}e^{-\varphi(x)/h}\\ &\qquad\qquad + e^{\varphi(x)/h}\varphi_0 M_{\mu,0}\varphi_0 e^{-\varphi(x)/h} + e^{\varphi(x)/h}\varphi_1 M_{\mu,1}\varphi_1 e^{-\varphi(x)/h}.\end{aligned}$$

So,

$$\widetilde{A}_{\mu,\varphi}(z) = -Y_{\mu,2,\varphi}(z) + (M_{\mu,0,\varphi} - Y_{\mu,4,\varphi} - B_{\mu,0}T_{\mu,0,\varphi} + B_{\mu,1}T_{\mu,1,\varphi})(1 - Y_{\mu,1,\varphi}(z))$$

with

$$B_{\mu,0}; B_{\mu,1} = \mathcal{O}(h) : H^{-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)) \to L^{\mathbf{2}}(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)) \text{ uniformly as } h \to 0_+.$$

Finally,

$$\begin{aligned}\widetilde{A}_{\mu,\varphi}(z) &= -\varphi_1 M_{\mu,1,\varphi}E_{\mu,1,\varphi}(z)M_{\mu,1,\varphi}\varphi_1\\ &\quad + (M_{\mu,0,\varphi} - Y_{\mu,4,\varphi} + B'_{\mu,0}T_{\mu,0,\varphi} + B'_{\mu,1}T_{\mu,1,\varphi})\big(1 - \varphi_1 E_{\mu,1,\varphi}(z)M_{\mu,1,\varphi}E_{\mu,1,\varphi}(z)\varphi_1\big) + B_{\mu,2}\psi_0,\end{aligned}$$

where

$$B_{\mu,2} = -R_\mu E_{\mu,0,\varphi}(z)R_\mu + \mathcal{O}(h) \text{ and } R_\mu = \big[Q_\mu(x), \Pi_\mu(x)\big] = \varphi_0\big[Q_\mu(x), \Pi_\mu(x)\big]\varphi_0.$$

Thus

$$\begin{aligned}R^+_\mu \widetilde{A}_{\mu,\varphi}(z)R^-_\mu &= \Lambda_{\mu,1,\varphi}\\ &\quad + R^+_\mu(B'_{\mu,0}T_{\mu,0,\varphi} + B'_{\mu,1}T_{\mu,1,\varphi})\varphi_1 E_{\mu,1,\varphi}(z)M_{\mu,1,\varphi}E_{\mu,1,\varphi}(z)R^-_\mu\varphi_1 + R^+_\mu B'_{\mu,2}R^-_\mu\psi_0,\end{aligned}$$

where $\Lambda_{\mu,1,\varphi}$ is a semiclassical pseudodifferential operator.

Since $T_{\mu,j,\varphi}E_{\mu,1,\varphi}(z)M_{\mu,1,\varphi}(1-\varphi_0) = \mathcal{O}(h^\infty)$ uniformly as $h \to 0_+$, $j \in \{0,1\}$ (see [20, Lemma 4.4]), we obtain a representation of the effective Hamiltonian $\widetilde{F}_\mu(z)$ in terms of a matrix operator away from $x = 0$, and BKW solutions. A complete proof is given in Theorem 4.1, Proposition 5.1 and Corollary 5.2 of [20].

Theorem 6.1 ([20]).

(1) $z \in \Gamma(h) \Longleftrightarrow \exists\, \mu \in \mathbb{C}$ *small enough,* $\operatorname{Im} \mu > 0$ *and* $z \in \sigma_{disc}(\widetilde{F}_\mu(z))$,

$$e^{\varphi(x)/h}\widetilde{F}_\mu(z)e^{-\varphi(x)/h} = \Lambda_{\mu,\varphi}(z) + L_{\mu,\varphi}(z)\psi_0 + \Theta_{\mu,\varphi}(z),$$

$\Lambda_{\mu,\varphi}(z)$ is a 3×3 matrix of pseudodifferential operators on $\mathbb{R}^3_x$ with the principal symbol $((I+\mu^t\,d\omega(x))^{-1}(\xi+i\nabla\varphi))^2 I_3+\mathcal{M}_\mu(x)$, $\mathcal{M}_\mu(x)$ is a smooth 3×3 matrix on $\mathbb{R}^3$ with eigenvalues $\lambda_1(x+\mu\omega(x)),\lambda_2(x+\mu\omega(x))$ and $\lambda_3(x+\mu\omega(x))$ for $x\in\mathbb{R}^3\setminus W_0$, and $\operatorname{Re}\mathcal{M}_\mu(x)\geq E+\delta_0$ for $x\in W_0$ and μ complex small enough,

$$L_{\mu,\varphi}(z)=R^+_\mu[(P_\mu(h)-E-\delta_0)(1-\psi_1(x))+R_\mu E_{\mu,0,\varphi}(z)R_\mu]R^-_\mu+\mathcal{O}(h),$$

$$\Theta_{\mu,\varphi}(z)=\mathcal{O}(h^\infty):L^2(\mathbb{R}^3_x\times\mathbb{R}^{3p}_y)\to\bigoplus_1^3 L^2(\mathbb{R}^3_x)\ \text{ uniformly as }\ h\to 0_+.$$

(2) *Let $u=a(x;h)e^{-\varphi(x)/h}\in(L^2(\mathbb{R}^3))^{\oplus 3}$, where $a\in(C^\infty(\mathbb{R}^3))^3$ admitting an asymptotic expansion of the type $a\sim\sum_{k=0}^\infty h^{k/2}a_k(x)$ as $h\to 0_+$, with $a_k\in(C^\infty(\mathbb{R}^3))^3$, $\operatorname{Supp}(a_k)\subset\mathbb{R}^3\setminus W_0$, $k\in\mathbb{N}$. Then*

$$e^{\varphi(x)/h}\widetilde{F}_\mu(z)u\sim\sum_{k=0}^\infty h^{k/2}b_k(x;z),$$

$$\operatorname{Supp}(b_k(\,\cdot\,;z))\subset\mathbb{R}^3\setminus W_0,\quad k\in\mathbb{N},$$

$$b_0(\,\cdot\,;z)=\big((I+\mu^t d\omega(x))^{-1}\nabla\varphi(x)\big)^2 a_0+\mathcal{M}_\mu(x)a_0.$$

7 Width of resonances

In this situation, one can work in the same spirit as in [16] and [22] to prove the existence of resonances near E with exponentially small widths as $h\to 0_+$.

Let z be a resonance of $P(h)$, $z\in J+i[-\epsilon,0]$, $\epsilon>0$ and v_μ be a normalized eigenfunction of $P_\mu(h)$ associated to z, $\mu\in\mathbb{C}$ small enough, $\operatorname{Im}\mu>0$. It follows from Theorem 6.1 that one can associate to v_μ a normalized function $\beta_\mu=\beta_{1,\mu}\oplus\beta_{2,\mu}\oplus\beta_{3,\mu}\in\bigoplus_1^3 L^2(\mathbb{R}^3_x)$ such that

$$\widetilde{F}_\mu(z)\beta_\mu=z\beta_\mu,$$

where the effective operator $\widetilde{F}_\mu(z)=R^+_\mu(P_\mu(h)-\widetilde{A}_\mu(z))R^-_\mu$ can be written as

$$\widetilde{F}_\mu(z)\beta_\mu=\bigoplus_1^3\langle\Phi_\mu(z)(\beta_{k,\mu}v_{k,\mu}(x)),v_{l,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)}\ \text{ on }\bigoplus_1^3 L^2(\mathbb{R}^3_x)\tag{7.1}$$

with $\Phi_\mu(z)=P_\mu(h)-\widetilde{A}_\mu(z)$.

The first 3 eigenvalues are re-indexed in such a way that they become smooth functions of $r=|x|$ and satisfy hypotheses (1.1) and (1.4)–(1.7).

We see, as in [22], that for $m\in\mathbb{Z}$, one has

$$\|E_\mu(z)\|_{\mathcal{B}(H^m(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y)),H^{m+j}(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y)))}=\mathcal{O}(h^{-j}),\quad j\in\{0,1,2\},$$

$$\big[\Delta_x,\widehat{\Pi}_\mu(x)\big]=\mathcal{O}(1):H^m(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y))\to H^{m-1}(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y))\ \text{ uniformly as }\ h\to 0_+.$$

Let $\Phi^1_\mu(z)=\langle\Phi_\mu(z)(v_{1,\mu}(x)),v_{1,\overline{\mu}}(x)\rangle_{L^2(\mathbb{R}^{3p}_y)}$, $|\mu|$ small enough. Using the virial condition on $\lambda_1(x)$ and Theorem 4.1, one can easily show that the operator $\Phi^1_\mu(z)-z$ is invertible from $H^2(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y))$ into $L^2(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y))$ for $|\mu|$, $|z-E|$, and h small enough, and

$$\big\|(\Phi^1_\mu(z)-z)^{-1}\big\|_{\mathcal{B}(H^m(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y)),H^{m+j}(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y)))}=\mathcal{O}(h^{-j}),\quad j\in\{0,1,2\}.\tag{7.2}$$

Using estimations (7.2), we see that equation (7.1) is equivalent to

$$\begin{cases}\beta_{1,\mu}=-(\Phi^1_\mu(z)-z)^{-1}\displaystyle\sum_{k=2}^3\big\langle\Phi_\mu(z)(\beta_{k,\mu}v_{k,\mu}(x)),v_{1,\overline{\mu}}(x)\big\rangle_{L^2(\mathbb{R}^{3p}_y)}\\ \qquad=(S_{2,\mu}(z)\oplus S_{3,\mu}(z))(\beta_{2,\mu}\oplus\beta_{3,\mu}),\\ H_\mu(z)(\beta_{2,\mu}\oplus\beta_{3,\mu})=z(\beta_{2,\mu}\oplus\beta_{3,\mu})\end{cases}$$

with

$$S_{k,\mu}(z)\beta_{k,\mu} = -(\Phi^1_\mu(z) - z)^{-1}\Big[\big\langle \Phi_\mu(z)(\beta_{k,\mu}v_{k,\mu}(x)), v_{1,\overline{\mu}}(x)\big\rangle_{L^2(\mathbb{R}^{3p}_y)}\Big],$$
$$H_\mu(z) = \Big(\Big\langle \Phi_\mu(z)\big[(v_{k,\mu}(x)) + S_{l,\mu}(z)(\,\cdot\,)v_{1,\mu}\big], v_{1,\overline{\mu}}(x)\Big\rangle_{L^2(\mathbb{R}^{3p}_y)}\Big),\ \ k,l \in \{2,3\}.$$

So, the spectral study of $P_\mu(h)$ is now reduced to that of the 2×2 matrix of pseudodifferential operators $H_\mu(z)$ acting on $L^2(\mathbb{R}^3_x) \oplus L^2(\mathbb{R}^3_x)$.

Applying the calculus of the previous section, $H_\mu(z)$ can be written as

$$H_\mu(z) = -\frac{h^2}{(1+\mu)^2}\,\Delta_x I_2 + \mathcal{N}_\mu(x) + \mathcal{R}_\mu(z,h),$$
$$\|\mathcal{R}_\mu(z,h)\| = \mathcal{O}(h^2) : (H^m(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)))^{\oplus 2} \to (H^{m-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)))^{\oplus 2},\ \ m \in \mathbb{Z},$$
$$e^{\varphi(x)/h}\mathcal{R}_\mu(z,h)e^{-\varphi(x)/h} = \mathcal{R}_{1,\mu}(z,h) + \mathcal{R}_{2,\mu}(z,h),$$
$$\|\mathcal{R}_{1,\mu}(z,h)\|_{\mathcal{B}(H^m(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y)),H^m(\mathbb{R}^3_x,L^2(\mathbb{R}^{3p}_y)))} \le C\big(h^2 + h\|\nabla\varphi\|_{L^\infty}\big)$$

and

$$\|\mathcal{R}_{2,\mu}(z,h)\| = \mathcal{O}(h^2) : (H^m(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)))^{\oplus 2} \to (H^{m-1}(\mathbb{R}^3_x, L^2(\mathbb{R}^{3p}_y)))^{\oplus 2},\ \ m \in \mathbb{Z},$$

φ is a real-valued Lipschitz function on $\mathbb{R}^3$ such that

$$\|\nabla\varphi\|_{L^\infty} \le \frac{1}{C}\,,\ \ C > 0,$$

and $\mathcal{N}_\mu(x)$ is the diagonal matrix $\begin{pmatrix}\lambda_2(x+\mu\omega(x)) & 0\\ 0 & \lambda_3(x+\mu\omega(x))\end{pmatrix}$ for $x \in \mathbb{R}^3 \setminus W_0$.

Let the Agmon metric

$$d_\mu = \mathrm{Re}\ \big[(1+\mu)^2\lambda_2(x+\mu\omega(x))\big]\,dx^2$$

and consider $\theta_\mu(x) \in C^\infty(\mathbb{R}^3)$ such that

$$\theta_\mu(x) = \begin{cases}\dfrac{1}{2}\,d_\mu(x,0) & \text{if}\ \ \min\Big(\mathrm{Re}\ \big[(1+\mu)^2\lambda_2(x+\mu\omega(x))\big], \mathrm{Re}\ \big[(1+\mu)^2\lambda_3(x+\mu\omega(x))\big]\Big) \le \dfrac{|\mu|^2}{4C^2}\,,\\[2ex] \text{constant} & \text{if}\ \ \min\Big(\mathrm{Re}\ \big[(1+\mu)^2\lambda_2(x+\mu\omega(x))\big], \mathrm{Re}\ \big[(1+\mu)^2\lambda_3(x+\mu\omega(x))\big]\Big) \ge \dfrac{|\mu|^2}{2C^2}\,,\end{cases}$$

and

$$\|\nabla\theta\|_{L^\infty} \le \frac{|\mu|^2}{2C^2}\ \ \text{everywhere.}$$

As in [16, 17, 20], one can show the exponential decay of the eigenfunctions of $\widetilde{F}_\mu(z)$ and $P_\mu(h)$:

$$\|e^{\theta_\mu(x)/h}\beta_\mu\|_{(H^1(\mathbb{R}^3_x))^3} = \mathcal{O}(e^{\varepsilon/h}), \tag{7.3}$$
$$\|e^{\theta_\mu(x)/h}v_\mu\|_{(H^1(\mathbb{R}^3_x))^3} = \mathcal{O}(e^{\varepsilon/h}),\ \ \varepsilon > 0.$$

We then use these estimates to establish the exponential decay of the resonant functions of $P(h)$. Indeed,

$$v_\mu \in H_{-\,\mathrm{Re}\ \theta_\mu(x)+\varepsilon|\,\mathrm{Im}\ x|}(\Omega_\mu, L^2(\mathbb{R}^{3p}_y))\ \ \varepsilon > 0,$$

where

$$\Omega_\mu = \Big\{x \in \mathbb{C}^3 :\ |x| < \frac{|\mu|}{C}\Big\},\ \ C > 0,$$

and $H_\varphi(\Omega_\mu, L^2(\mathbb{R}^{3p}_y))$ denotes the space of holomorphic functions $v(x,h)$ in a complex neighborhood of the closure $\overline{\Omega_\mu}$ of Ω_μ with values in $L^2(\mathbb{R}^{3p}_y)$ such that

$$\forall\,\varepsilon > 0,\ \ \exists C_\varepsilon > 0,\ \ \|v(x,h)\|_{L^2(\mathbb{R}^{3p}_y)} \le C_\varepsilon e^{(\varphi(x)+\varepsilon)/h}.$$

Moreover, if we denote $W = \{x \in \mathbb{R}^3 : |x| \leq \frac{|\mu|}{2C}\}$, according to the estimates (7.3) we get

$$\|v_\mu(x,h)\|_{L^2(W\times\mathbb{R}^{3p}_y)} \geq 1 - C_1 e^{-\epsilon_1/h} \text{ uniformly as } h \to 0_+, \ C_1 > 0 \tag{7.4}$$

and

$$\epsilon_1 = \frac{1}{2} \inf_{x\in\mathbb{R}^3\setminus W} (\text{Re } \theta_\mu(x)).$$

Denote $v = \mathcal{U}_\mu^{-1} v_\mu$. Then v is a holomorphic function on Ω_μ with values in $L^2(\mathbb{R}^{3p}_y)$, satisfying $(P(h) - z)v = 0$.

Moreover, in view of Green's formula, for $P(h)$, we have

$$\text{Im } z\|v\|^2_{L^2(W\times\mathbb{R}^{3p}_y)} = \text{Im } \left(\langle P(h)v, v\rangle_{L^2(W\times\mathbb{R}^{3p}_y)}\right) = -h^2 \,\text{Im} \int\limits_{\partial W\times\mathbb{R}^{3p}_y} \overline{v}\, \frac{\partial v}{\partial n}\, ds, \tag{7.5}$$

where ds is the surface measure on ∂W and n stands for the outward pointing unit normal to W.

Using (7.3)–(7.5), we deduce that for $\epsilon_1 > 0$, one has

$$|\text{ Im } z| \leq C_2 \|v\|^{-2}_{L^2(W\times\mathbb{R}^{3p}_y)} e^{-\epsilon_1/h}, \ \ C_2 > 0$$

uniformly as $h \to 0_+$. In order to estimate $\|v\|^{-2}_{L^2(W\times\mathbb{R}^{3p}_y)}$, we plan to use the analytic pseudodifferential calculus and Fourier integral operators with complex phase functions.

So, we can write (see [16, 22]),

$$\begin{aligned}
&\langle v_\mu(x,h), \psi\rangle_{L^2(\mathbb{R}^{3p}_y)} \\
&\quad = (2\pi)^{-n} \int\limits_{|\xi|\leq\frac{\varepsilon}{h}} e^{i(x-x')\xi - |\xi|(x-x')^2/2} a(x-x',\xi) \langle v_\mu(x',h),\psi\rangle_{L^2(\mathbb{R}^{3p}_y)} \chi(x')\, dx'\, d\xi \\
&\qquad\qquad + r_\varepsilon(x,h), \ \ \varepsilon > 0,
\end{aligned} \tag{7.6}$$

with

$$\sup_{x\in\Omega_\mu} |r_\varepsilon(x,h)| \leq e^{-\varepsilon'/h}, \ \ \varepsilon' > 0$$

uniformly with respect to $\psi \in L^2(\mathbb{R}^{3p}_y)$, $\|\psi\|_{L^2(\mathbb{R}^{3p}_y)} = 1$, and $h > 0$ small enough. $\chi \in C_0^\infty(\{x' \in \mathbb{R}^3 : |x'| \leq \frac{|\mu|}{C}\})$ and $\chi = 1$ on W.

Thus $\|v\|_{L^2(W\times\mathbb{R}^{3p}_y)}$ can be estimated as in [16] and for all $\varepsilon > 0$ we obtain

$$\|v\|_{L^2(\mathbb{R}^{3p}_y)} \leq C_\varepsilon e^{\varepsilon/h} \|v\|_{L^2(W\times\mathbb{R}^{3p}_y)} + e^{-\varepsilon''(\varepsilon)/h}$$

with $C_\varepsilon > 0$ and $\varepsilon''(\varepsilon) > 0$.

Thanks to (7.4), we deduce that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|v\|_{L^2(W\times\mathbb{R}^{3p}_y)} \geq \frac{1}{C_\varepsilon} e^{-\varepsilon/h}. \tag{7.7}$$

Since

$$\sup_{x\in\partial W} (\text{Re } \theta_\mu(x)) \geq \frac{1}{C'} |\mu|^2, \ \ C' > 0,$$

representation (7.6) and estimate (7.7) imply that $|\text{ Im } z|$ is exponentially small,

$$|\text{ Im } z| \leq K_\varepsilon e^{-\epsilon_1/h} \leq K_\varepsilon e^{-|\mu|^2/C'h}, \ \ K_\varepsilon > 0.$$

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